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Statistical Semi-convergence in Topological Framework

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ABSTRACT: This paper explains a novel concept of statistical semi-convergence under the context of topology. The classical idea of statistical convergence is expanded to encompass a larger class of sequences in the conventional topological sense by using semi-open sets. The connection between statistical semi-convergence and other recognized types of convergence is also examined. The Uniqueness of the limit of statistical semi-convergence and preservation under semi-continuous function has been established. In addition to providing new analytical methods for handling sequences in more extended topological spaces, this work expands the application of convergence theory. Lastly, the concept of statistical semi-limit point and statistical semi-cluster point has also been discussed.

Key Words: Natural density, semi-open set, semi-convergence, statistical semi-convergence, statistical semi-limit point, statistical semi-cluster point.

Contents

1. Introduction

In 1935, A. Zygmund [17] was among the first to propose the concept of a summability technique in a formal mathematical setting that contributes significantly to the understanding of convergence behavior. Building on these foundations, in 1951, H. Fast [9] introduced the concept of statistical convergence for sequences of real numbers. This marked a significant shift from classical convergence, as it allowed for more flexibility by tolerating a small set of terms that deviate from the limit. Around the same time, J.A. Schoenberg [16] also examined statistical convergence independently and contributed to its understanding as a summability method. Both Fast and Schoenberg outlined fundamental properties of statistical convergence and highlighted its role as a natural extension of summability theory.

Let $\mathbb{N}$ be the set of all natural numbers and $T \subseteq \mathbb{N}$. The symbol $\delta(T)$ is employed to denote the natural density (also called asymptotic density) of the set T, where

$$\delta(T) = \lim_{n\to\infty} \frac{|\{k \leq n : k \in T\}|}{n}$$

provided that the limit exists. Formally, a sequence $(t_n)_{n\in\mathbb{N}}$ is said to be statistically convergent to a real number t if, for every $\epsilon > 0$, the set $\{n \in \mathbb{N} : |t_n - t| \geq \epsilon\}$ has natural density zero.

Later, the idea was expanded to more abstract contexts. The study of convergence outside of metric and normed settings was made possible by the generalization of statistical convergence to topological spaces by G. Di Maio and L. Koĉinac [13] in 2008. Their research paved the way for further studies of convergence behavior in broader mathematical contexts.

The concept of semi-open sets was first introduced by Norman Levine in 1963 [12], marking a significant generalization of open sets within topology. Semi-open sets play a foundational role in the study

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of continuity, convergence, and separation axioms and have become a cornerstone in the development of generalized topological structures. Levine's pioneering work has inspired a range of further investigations into generalized openness, leading to the emergence of related notions such as semi-continuity, semi-compactness, and semi-separation axioms. These concepts have since been explored and expanded by numerous mathematicians [1,2,4,5,6,14,15], greatly enriching the theory and applications of generalized topology.

In parallel, topological convergence emerged as a natural extension of classical convergence, providing a broader framework for understanding the limiting behavior of sequences, nets, and filters in general topological spaces. Unlike classical convergence, which relies on distances defined in metric spaces, topological convergence is governed by open sets, neighborhood systems, and continuity structures. This broader perspective has proven essential across various fields, including algebraic topology, differential geometry, and functional analysis. Over time, many researchers [3,10,11] have contributed to the study of convergence, investigating diverse forms such as statistical convergence, ideal convergence, and weighted statistical convergence. These developments have significantly advanced the theory of convergence, with numerous publications refining and generalizing its definitions across mathematical disciplines.

Following on previous work, this paper uses the framework of semi-open sets to present a novel viewpoint on statistical convergence in topological spaces.

2. Preliminaries

Summability theory, functional analysis, and approximation theory are just a few of the mathematical disciplines that have been greatly impacted by statistical convergence. It offers a strong framework for examining convergence beyond traditional concepts and has applications in topology and measure theory. It is important to lay out some basic definitions and essential ideas before delving into the idea of statistical semi-convergence. The mathematical foundation required to understand the primary findings is highlighted in this section. For standard terminology, mathematical concepts, and nomenclature, this article adheres to [8].

Definition 2.1 [12] A set O in a topological space (X, τ) is semi-open if and only if there exists $P \in \tau$ such that $P \subseteq O \subseteq \bar{P}$, where $\bar{P}$ signifies the closure of the set P.

It is clear that every open set is a semi-open set [12].

Definition 2.2 [7] A sequence $\{\sigma_n : n \in \mathbb{N}\}$ in a topological space (X, τ) is said to be semi-convergent to a point $\sigma \in X$, if for every $A \in s\tau$ there exists a natural number $n_0 \in \mathbb{N}$ such that $\sigma_n \in A$ for all $n \geq n_0$. Symbolically, $\sigma_n \xrightarrow{semi} \sigma$.

$s\tau$ denotes the collection of all semi-open sets.

Theorem 2.1 *[7] Every semi-convergent sequence is convergent.*

Definition 2.3 [5] A function $f : X \longrightarrow Y$ between topological spaces is termed an irresolute function if the preimage of every semi-open set in Y is semi-open in X.

Definition 2.4 [13] A sequence $(Q_n)_{n\in\mathbb{N}}$ in a topological space X is said to be statistically convergent (or shortly s-convergent) to a point $q \in X$ if, for every neighborhood T of q, $\delta(\{n \in \mathbb{N} : Q_n \notin T\}) = 0$. Mathematically it can be written as,

$$S - \lim_{n\to\infty} Q_n = q.$$

Definition 2.5 [13] A point y is said to be a statistical limit point of a sequence $(y_n)_{n\in\mathbb{N}}$ in a space X if there exists a set $\{n_1 < n_2 < \cdots < n_p < \cdots\} \subseteq \mathbb{N}$ whose natural density is not equal to zero such that $\lim_{p\to\infty} y_{n_p} = y$.

Definition 2.6 [13] A point y is said to be a statistical cluster point of a sequence $(y_n)_{n\in\mathbb{N}}$ if for every open neighborhood V of y, the upper natural density of the set $\{n \in \mathbb{N} : y_n \in V\}$ is greater than zero.

3. On ${}_{semi}S$-Convergence

The objective of this study is to broaden and enhance the understanding of convergence behavior for sequences by employing semi-open sets to extend the concept of statistical convergence. To illustrate the characteristics and nature of this new form of convergence, the study presents essential definitions, theorems, and illustrative counterexamples.

Definition 3.1 A sequence $\{\beta_m : m \in \mathbb{N}\}$ from a topological space (X, τ) is considered as statistically semi convergent (or shortly ${}_{semi}S$-convergent) to a point β_0 of X if, for each $\xi \in s\tau$ of β_0, $\delta(\{m \in \mathbb{N} : \beta_m \notin \xi\}) = 0$.

It can be indicated mathematically as,

$$_{semi}S - \lim_{m\to\infty} \beta_m = \beta_0.$$

Definition 3.2 A sequence $\{\beta_m : m \in \mathbb{N}\}$ is considered as ${}_{semi}S'$-convergent to β_0 if, there exists a $P \subseteq \mathbb{N}$ with $\delta(P) = 1$ such that $semi - \lim_{m\to\infty, m\in P} \beta_m = \beta_0$.

It can be mathematically indicated as,

$$_{semi}S' - \lim_{m\to\infty} \beta_m = \beta_0.$$

Theorem 3.1 *If a sequence $\{\beta_m : m \in \mathbb{N}\}$ in a space X is ${}_{semi}S'$-converges to $\beta_0 \in X$, then the sequence $\{\beta_m\}$ is ${}_{semi}S$-converges to β_0.*

Proof: Let ${}_{semi}S' - \lim_{m\to\infty} \beta_m = \beta_0$. Therefore, there exists a $P \subseteq \mathbb{N}$ with $\delta(P) = 1$ such that $semi - \lim_{m\to\infty, m\in P} \beta_m = \beta_0$.
So, for every semi-open neighborhood ξ of β_0, $\{m \in P : \beta_m \notin \xi\}$ is finite. i.e., $\delta(\{m \in P : \beta_m \notin \xi\}) = 0$, also $\delta(P^{\complement}) = 0$.
Now, $\{m \in \mathbb{N} : \beta_m \notin \xi\} \subseteq \{m \in P : \beta_m \notin \xi\} \cup P^{\complement}$.
i.e., $\delta(\{m \in \mathbb{N} : \beta_m \notin \xi\}) \le \delta(\{m \in P : \beta_m \notin \xi\}) + \delta(P^{\complement}) = 0$.
Hence, $\beta_m \xrightarrow{semi S-\lim} \beta_0$. □

It is evident from the fact that every semi-convergent sequence is a convergent sequence [7] and every convergent sequence is a statistically convergent sequence which can be seen in many of the earlier works. Now, the question being raised whether every semi-convergent sequence is statistically semi-convergent and every statistically semi-convergent sequence is statistically convergent sequence.

Theorem 3.2 *Every semi-convergent sequence is a statistical semi-convergent sequence.*

Proof: Let $\beta_m \xrightarrow{semi} \beta_0$. Therefore, for every semi-open neighborhood ξ containing β_0, $\{m \in \mathbb{N} : \beta_m \notin \xi\}$ is finite. Clearly, it is obvious that the statistical density of a finite set is 0. Thus, $\delta(\{m \in \mathbb{N} : \beta_m \notin \xi\}) = 0$. Hence, $\beta_m \xrightarrow{semi S-\lim} \beta_0$. □

But the converse may not exist, so a counterexample is depicted.

Example 3.1 *Let (X, τ) be a topological space where $X = \{p, q, r, s\}$ and $\tau = \{\emptyset, X, \{p, q\}\}$. Then $s\tau = \{\emptyset, X, \{p, q\}, \{p, q, r\}, \{p, q, s\}\}$.*
Consider the sequence,

$$\beta_m = \begin{cases} r, & \text{for } m = l^2 \\ s, & \text{otherwise} \end{cases}$$

so, semi-open neighborhoods of s are $W_1 = X$, $W_2 = \{p, q, s\}$.
For the neighborhood W_1 of s, $\delta(\{m \in \mathbb{N} : \beta_m \notin W_1\}) = \delta(\emptyset) = 0$ and for the semi-open neighborhood W_2 of s, $\delta(\{m \in \mathbb{N} : \beta_m \notin W_2\}) = \delta(\{1, 4, 9, \cdots\}) = 0$. Thus, for every semi-open neighborhood W of s, $\beta_m \xrightarrow{semi S-\lim} s$.
But for the semi-open neighborhood $\{p, q, s\}$, we can't find an n_0 such that $\beta_m \notin \{p, q, s\}$ for all $m \ge n_0$. Therefore, $\beta_m \overset{semi}{\not\to} s$.

Theorem 3.3 *Every statistical semi-convergent sequence is a statistical convergent.*

Proof: Let $\beta_m \overset{_{semi}S-\lim}{\longrightarrow} \beta_0$. Therefore, for every semi-open neighborhood ξ of β_0, $\delta(\{m \in \mathbb{N} : \beta_m \notin \xi\}) = 0$.
Since every open set is a semi-open set, $\tau \subseteq s\tau$. Then it is obvious that for every open neighborhood ξ containing β_0, $\delta(\{m \in \mathbb{N} : \beta_m \notin \xi\}) = 0$.
Hence, $\beta_m \overset{S-\lim}{\longrightarrow} \beta_0$. □

But the converse of the previous theorem may not exist, so another example is shown.

Example 3.2 *Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{c, d\}\}$ be the topology defined on X and $s\tau = \{\emptyset, X, \{c, d\}, \{b, c, d\}, \{a, c, d\}\}$.*
Consider the sequence,

$$\xi_n = \begin{cases} b, & \text{for } n = l \\ a, & \text{otherwise} \end{cases}$$

here the open neighborhood of a is only $V = X$, thus, for every open neighborhood V of a, $\delta(\{n \in \mathbb{N} : \xi_n \notin V\}) = \delta(\emptyset) = 0$. Therefore, $\xi_n \overset{S-\lim}{\longrightarrow} a$.
But semi-open neighborhoods of a are $S_1 = X$ and $S_2 = \{a, c, d\}$. For the neighborhood S_2 of a, $\delta(\{n \in \mathbb{N} : \xi_n \notin S_2\}) = \delta(\{1, 2, 3, \cdots\}) \neq 0$. Therefore, $\xi_n \overset{_{semi}S-\lim}{\not\longrightarrow} a$.

In the first section, the relation of $_{semi}S$-convergence with other convergence criteria has been established.

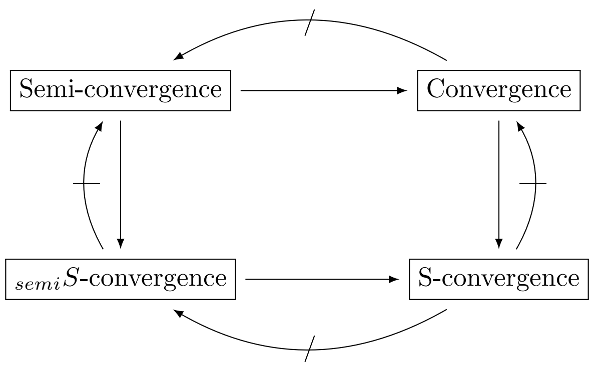

Figure 1: Relationship chart

Example 3.3 *The limit of a statistical semi-convergent sequence may have more than one limit point. Let $X = \{p, q, r, s\}$ and $\tau = \{\emptyset, X, \{p, r\}, \{q\}, \{p, r, q\}\}$ be the topology defined on X and the semi topology defined by $s\tau = \{\emptyset, X, \{q\}, \{p, r\}, \{q, s\}, \{p, q, r\}, \{p, r, s\}\}$.*
Consider the sequence,

$$\beta_m = \begin{cases} s, & \text{for } m = l^2 \\ p, & \text{for } m \neq l^2 \end{cases}$$

here, semi-open neighborhoods of p are $T_1 = X$, $T_2 = \{p, r\}$, $T_3 = \{p, q, r\}$ and $T_4 = \{p, r, s\}$.
For the neighborhood T_1 corresponding to the limit point p, $\delta(\{m \in \mathbb{N} : \beta_m \notin T_1\}) = \delta(\emptyset) = 0$.
For the neighborhood T_2 corresponding to the limit point p, $\delta(\{m \in \mathbb{N} : \beta_m \notin T_2\}) = \delta(\{1, 4, 9, \cdots\}) = 0$.
For the neighborhood T_3 corresponding to the limit point p, $\delta(\{m \in \mathbb{N} : \beta_m \notin T_3\}) = \delta(\{1, 4, 9, \cdots\}) = 0$.

And for the neighborhood T_4 corresponding to the limit point p, $\delta(\{m \in \mathbb{N} : \beta_m \notin T_4\}) = \delta(\emptyset) = 0$.
Thus, for every semi-open neighborhood T of p, $\delta(\{m \in \mathbb{N} : \beta_m \notin T\}) = 0$. So, $\beta_m \overset{semiS-\lim}{\longrightarrow} p$.
But the semi-open neighborhoods of p are the only semi-open neighborhoods of r. So, $\beta_m \overset{semiS-\lim}{\longrightarrow} r$.
Hence, the limit of a statistical semi-convergent sequence has more than one limit point.

Theorem 3.4 *The limit of any statistical semi-convergent sequence is always unique if every pair of distinct points in the space are strongly separated by two distinct semi-open sets.*

Proof: Let us assume $\{\beta_m : m \in \mathbb{N}\}$ is a statistical semi-convergent sequence that converges to two limit points x and y such that $x \neq y$.
But every pair of distinct points in the space (X, τ) are strongly separated by two distinct semi-open sets. So, fixing two semi-open neighborhoods $\alpha_1, \alpha_2 \in s\tau$ corresponds to the limit points x and y such that $x \in \alpha_1$, $y \in \alpha_2$ with $\alpha_1 \cap \alpha_2 = \emptyset$.
Now, as $\alpha_1 \cap \alpha_2 = \emptyset$ so that implies $\alpha_2 \subseteq \alpha_1^{\complement}$.
i.e., $\delta(\{m \in \mathbb{N} : \beta_m \in \alpha_2\}) \leq \delta(\{m \in \mathbb{N} : \beta_m \in \alpha_1^{\complement}\}) = 0$.
i.e., $\delta(\{m \in \mathbb{N} : \beta_m \in \alpha_2\}) = 0$ which contradicts the fact that $\delta(\{m \in \mathbb{N} : \beta_m \notin \alpha_2\}) = 0$.
Hence, the limit of the statistical semi-convergent sequence $\{\beta_m\}$ is unique in the space X. □

Example 3.4 *Sub sequence of a statistical semi-convergent sequence may not be a statistical semi-convergent sequence.*
Let $X = \{a_1, a_2, a_3, a_4\}$ and $\tau = \{\emptyset, X, \{a_1, a_2\}\}$. Then clearly (X, τ) is a topological space. Furthermore, $s\tau = \{\emptyset, X, \{a_1, a_2\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_4\}\}$. Now, consider the sequence

$$\xi_m = \begin{cases} a_3, & \text{for } n = l^3 \\ a_4, & \text{otherwise} \end{cases}$$

here semi-open neighborhoods of a_4 are $T_1 = X$ and $T_2 = \{a_1, a_2, a_4\}$. Clearly, for every semi-open neighborhood T of a_4, $\delta(\{m \in \mathbb{N} : \xi_m \notin T\}) = 0$. Therefore, $\xi_m \overset{semiS-\lim}{\longrightarrow} a_4$.
Let us consider the sub sequence $\{\xi_{m_n} : n \in \mathbb{N}\}$ of the sequence $\{\xi_m\}$ such that

$$\xi_{m_n} = \begin{cases} \xi_{n^3}, & \text{when } n \text{ is an odd number} \\ \xi_{n-1^3+1}, & \text{when } n \text{ is an even number} \end{cases}$$

for every semi-open neighborhood T of a_4, $\delta(\{n \in \mathbb{N} : \xi_{m_n} \notin T\}) \neq 0$. Thus, $\xi_{m_n} \overset{semiS-\lim}{\not\longrightarrow} a_4$.
Hence, the sub sequence $\{\xi_{m_n}\}$ is not a statistical semi-convergent sequence.

Definition 3.3 A function $f : (X, \tau) \longrightarrow (Y, \rho)$ is said to be a semi-continuous function if, for every semi-open set $\epsilon \subseteq Y$, the pre-image $f^{-1}(\epsilon)$ is open in X.

Theorem 3.5 *Let $f : (X, \tau) \longrightarrow (Y, \rho)$ be a semi-continuous function. If $\beta_m \overset{S-\lim}{\longrightarrow} \beta_0$ then $f(\beta_m) \overset{semiS-\lim}{\longrightarrow} f(\beta_0)$.*

Proof: Let ϵ be a semi-open neighborhood corresponding to the limit point $f(\beta_0) \in Y$. Since f is a semi-continuous function. So, $\beta_0 \in f^{-1}(\epsilon)$ is open in X.
Also, $\beta_m \overset{S-\lim}{\longrightarrow} \beta_0$. In that case, for every open neighborhood ϵ containing β_0, $\delta(\{m \in \mathbb{N} : \beta_m \notin \epsilon\}) = 0$.
Thus, for every semi-open neighborhood $f^{-1}(\epsilon)$, $\delta(\{m \in \mathbb{N} : \beta_m \notin f^{-1}(\epsilon)\}) = 0$.
Hence, $f(\beta_m) \overset{semiS-\lim}{\longrightarrow} f(\beta_0)$. □

4. Statistical semi-limit point & Statistical semi-cluster point

Definition 4.1 In a topological space (X, τ), a point $\beta_0 \in X$ will be termed a statistical semi-limit point of a sequence $\{\beta_m : m \in \mathbb{N}\}$ if $\{\beta_m\}$ has a sub sequence $\{\beta_{m_n}\}$ such that $\delta(\{m_n : n \in \mathbb{N}\}) \neq 0$. Also $\lim_{m\to\infty} \beta_{m_n} = \beta_0$.

Definition 4.2 In a topological space (X, τ), a point $\beta_0 \in X$ will be termed a statistical semi-cluster point of a sequence $\{\beta_m : m \in \mathbb{N}\}$ if, for every semi-open neighborhood ϵ containing β_0, $\delta(\{m \in \mathbb{N} : \beta_m \in \epsilon\}) \neq 0$.

Some notations are utilized for our research work.

Notations:-

${}_{semi}S\Xi(\beta_m)$: The set of all statistical semi-limit points of the sequence $\{\beta_m\}_{m\in\mathbb{N}}$

${}_{semi}S\Sigma(\beta_m)$: The set of all statistical semi-cluster points of the sequence $\{\beta_m\}_{m\in\mathbb{N}}$.

Theorem 4.1 *For any sequence $\{\beta_m\}_{m\in\mathbb{N}}$, ${}_{semi}S\Sigma(\beta_m)$ is a closed set.*

Proof: Let ϵ be an arbitrary semi-open neighborhood of $\beta_0 \in cl({}_{semi}S\Sigma(\beta_m))$, where $cl({}_{semi}S\Sigma(\beta_m))$ denotes the closure of the set ${}_{semi}S\Sigma(\beta_m)$. Then $\{(\epsilon \cap_{semi} S\Sigma(\beta_m)) \setminus \beta_0\} \neq \emptyset$. Consider another point $\beta \in \epsilon \cap_{semi} S\Sigma(\beta_m)$. Then there exists a semi-open neighborhood ξ of β such that $\xi \subset \epsilon$ and $\delta(\{m \in \mathbb{N} : \beta_m \in \xi\}) \neq 0$.
Since $\xi \subset \epsilon$ and β is a cluster point. So, $\{m \in \mathbb{N} : \beta_m \in \xi\} \subseteq \{m \in \mathbb{N} : \beta_m \in \epsilon\}$.
i.e., $0 \neq \delta(\{m \in \mathbb{N} : \beta_m \in \xi\}) \leq \delta(\{m \in \mathbb{N} : \beta_m \in \epsilon\})$
i.e., $\delta(\{m \in \mathbb{N} : \beta_m \in \epsilon\}) \neq 0$. Thus, $\beta_0 \in_{semi} S\Sigma(\beta_m)$. Hence ${}_{semi}S\Sigma(\beta_m) = cl({}_{semi}S\Sigma(\beta_m))$ and ${}_{semi}S\Sigma(\beta_m)$ is a closed set. □

Theorem 4.2 *For any sequence $\{\beta_m\}_{m\in\mathbb{N}}$, the set ${}_{semi}S\Xi(\beta_m) \subseteq_{semi} S\Sigma(\beta_m)$.*

Proof: Let $\beta_0 \in_{semi} S\Xi(\beta_m)$. Then there exists a sub sequence $\{\beta_{m_n}\}_{n\in\mathbb{N}}$ of $\{\beta_m\}_{m\in\mathbb{N}}$ such that $\delta(\{m_n : n \in \mathbb{N}\}) \neq 0$ and also $\lim_{m\to\infty} \beta_{m_n} = \beta_0$.
Consider a semi-open neighborhood ϵ containing the limit point β_0 and let H be any finite set with $\delta(H) = 0$.
Now, $\{m_n : n \in \mathbb{N}\} \setminus H \subseteq \{m \in \mathbb{N} : \beta_m \in \epsilon\}$.
i.e., $0 \neq \delta(\{m_n : n \in \mathbb{N}\} \setminus H) \leq \delta(\{m \in \mathbb{N} : \beta_m \in \epsilon\})$.
i.e., $\delta(\{m \in \mathbb{N} : \beta_m \in \epsilon\}) \neq 0$. Thus, $\beta_0 \in_{semi} S\Sigma(\beta_m)$.
Hence ${}_{semi}S\Xi(\beta_m) \subseteq_{semi} S\Sigma(\beta_m)$. □

Theorem 4.3 *For any two sequences $\{\beta_m\}$ and $\{\sigma_m\}$, if $\delta(\{m \in \mathbb{N} : \beta_m \neq \sigma_m\}) = 0$. Then ${}_{semi}S\Xi(\beta_m) = {}_{semi}S\Xi(\sigma_m)$ and ${}_{semi}S\Sigma(\beta_m) = {}_{semi}S\Sigma(\sigma_m)$.*

Proof: Let $\beta_0 \in_{semi} S\Xi(\beta_m)$. Then there exists a sub sequence $\{\beta_{m_n}\}$ of $\{\beta_m\}$ such that $\delta(\{\beta_{m_n} : n \in \mathbb{N}\}) \neq 0$ and $\lim_{m\to\infty} \beta_{m_n} = \beta_0$.
Now, $\{\sigma_{m_n} : n \in \mathbb{N}\} \supseteq \{\beta_{m_n} : n \in \mathbb{N}\} \setminus \{m \in \mathbb{N} : \beta_m \neq \sigma_m\}$.
i.e., $\delta(\{\sigma_{m_n} : n \in \mathbb{N}\}) \geq \delta(\{\beta_{m_n} : n \in \mathbb{N}\} \setminus \{m \in \mathbb{N} : \beta_m \neq \sigma_m\}) \neq 0$.
Since $\delta(\{m \in \mathbb{N} : \beta_m \neq \sigma_m\}) = 0$ so, $\delta(\{\sigma_{m_n} : n \in \mathbb{N}\}) \neq 0$. Thus $\beta_0 \in_{semi} S\Xi(\sigma_m)$ and ${}_{semi}S\Xi(\beta_m) \subseteq_{semi} S\Xi(\sigma_m)$; also by symmetric way ${}_{semi}S\Xi(\sigma_m) \subseteq {}_{semi}S\Xi(\beta_m)$ that implies ${}_{semi}S\Xi(\beta_m) = {}_{semi}S\Xi(\sigma_m)$.
Hence, in the same manner, it can be proven that ${}_{semi}S\Sigma(\beta_m) = {}_{semi}S\Sigma(\sigma_m)$. □

5. Conclusion

It has been shown that the limit of an ${}_{semi}S$-convergent sequence is not unique until every pair of distinct points of the space are strongly separated by semi-open sets. Although an infinite sub sequence of an ${}_{semi}S$-convergent sequence may not be ${}_{semi}S$-convergent, ${}_{semi}S$-convergence is preserved under semi-continuous function. The final section mainly deals with ${}_{semi}S$-limit point and ${}_{semi}S$-cluster point. It has been shown that the collection of all ${}_{semi}S$-cluster points is a super set of the collection of all ${}_{semi}S$-limit points and the collection of all ${}_{semi}S$-cluster points of a sequence is a closed set. This concept can further be applied for the analysis of convergence of functions and sets.

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