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Direct and Inverse Scattering Problems for Sturm-Liouville Operator with Discontinuous Coefficient Under Discontinuity Conditions

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ABSTRACT: This paper deals with the study of the direct and inverse problems of scattering theory for Sturm-Liouville operator with piecewise continuous coefficient under the discontinuity conditions at the interior point of the positive semi-axis. The scattering properties are examined and the eigenfunction expansion is obtained. The fundamental equation or Marchenko-type equation of the inverse scattering problem is derived and an algorithm of the reconstruction of the potential according to scattering data of this problem is given. Moreover, the continuity of scattering function of this problem is examined.

Key Words: Sturm-Liouville operator, direct and inverse scattering problems, fundamental equation, discontinuity conditions.

Contents

1. Introduction

In mathematics and physics, there has always been a great interest in scattering theory because of its applications of quantum mechanics (e.g. [6,9,14]). In particular, the study of the behavior of wave functions at infinity have made significant contributions to the advancement of the inverse problem of scattering theory. In brief, the inverse scattering problem is known as the problem of reconstructing the potential from the behavior of wave functions at infinity. In the mathematical literature, the answer of this problem was given in detail and comprehensively in [1,18]. Besides, the investigations of the properties of the scattering function, the eigenvalues and normalized numbers (the collection of these properties is defined as scattering data) of given problem and eigenfunction expansion are known as the direct problem of the scattering theory.

The purpose of this paper is to solve the direct and inverse problems of scattering theory. That is, we consider the following Sturm-Liouville equation with piecewise continuous coefficient

$$-v'' + q(u)v = \tau^2 \kappa(u)v, \quad u \in (0,\xi) \cup (\xi,+\infty), \tag{1.1}$$

and discontinuity conditions at the point $\xi \in (0,+\infty)$,

$$v(\xi-0) = \alpha v(\xi+0), \quad v'(\xi-0) = \alpha^{-1}v'(\xi+0) \tag{1.2}$$

and the boundary condition

$$v'(0) - hv(0) = 0, \tag{1.3}$$

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where $0<\alpha\neq 1$, τ is a complex parameter, $\kappa(u)$ is a piecewise continuous coefficient

$$\kappa(u)=\begin{cases}\beta^2, & 0<u<\xi,\\ 1, & \xi<u<\infty\end{cases}$$

with $0<\beta\neq 1$ and the real function $q(u)$ satisfies the following condition:

$$\int_0^\infty u|q(u)|du<\infty. \tag{1.4}$$

It is seen that the considered problem (1.1)-(1.3) has both the coefficient $\kappa(u)$ in equation (1.1) and the discontinuity conditions (1.2). So far, the problem involving discontinuous coefficient and the problem containing discontinuity conditions have been studied as two separate problems. In this study, we combine these two different problems to obtain a new generalized problem and study both the direct and inverse scattering problems of this generalized problem. To examine these problems, we need to use the Jost solution of equation (1.1) under discontinuity conditions (1.2) which is obtained in [2]. Especially, to solve the inverse scattering problem, we use the method of Marchenko that the fundamental equation according to the kernel of Jost solution plays a central role. The Jost solution of Sturm-Liouville equation in classical case is obtained in [18], the Jost solution of the Sturm-Liouville equation with discontinuous coefficient is constructed in [11] and the Jost solution of Sturm-Liouville equation under the discontinuity conditions is given in [13]. The direct and inverse scattering problems of Sturm-Liouville equation with discontinuous coefficient are examined in [7,11,16,17,19] and these problems of Sturm-Liouville equation under discontinuity conditions are worked in [3,4,5,8,12,13]. Furthermore, the continuity of scattering function and Levinson formula of Sturm-Liouville problems are investigated in [10,15,19]. Note that there are numerous studies on the direct and inverse scattering problems of Sturm-Liouville operator; therefore, we give the works close to the subject of this paper in the references.

This paper is organized as follows: in section 2, we give the Jost solution of equation (1.1) under the discontinuity condition (1.2). In section 3, firstly, we solve the direct scattering problem: the scattering data of problem (1.1)-(1.3) is examined; namely, we investigate the scattering function, the eigenvalues and normalized numbers of the problem (1.1)-(1.3). Moreover, we denote the normalized eigenfunctions and obtain the eigenfunctions expansion formula. Secondly, we solve the inverse scattering problem: we construct the fundamental equation of problem (1.1)-(1.3) and give an algorithm of the reconstruction of potential $q(u)$ from scattering data. Finally, using the fundamental equation, we show the continuity of the scattering function of the boundary value problem (1.1)-(1.3).

2. Preliminaries

In this section, we give the new Jost solution, which preserves the asymptotic of the solutions at infinity, of equation (1.1) with discontinuity condition (1.2). The triangular property of the Jost solution is lost due to the discontinuity coefficient $\kappa(u)$ and discontinuity conditions (1.2).

Theorem 2.1 *[2] Let conditions (1.4) holds. Then, for all τ from the closed upper half plane, the equation (1.1) with discontinuity conditions (1.2) has the Jost solution $J(u,\tau)$ that can be represented in the form*

$$J(u,\tau)=J_0(u,\tau)+\int_{\sigma(u)}^\infty k(u,z)e^{i\lambda z}dz, \tag{2.1}$$

where

$$J_0(u,\tau)=\begin{cases}e^{i\tau u}, & u>\xi,\\ \theta^+e^{i\tau(\beta(u-\xi)+\xi)}+\theta^-e^{i\tau(-\beta(u-\xi)+\xi)}, & 0<u<\xi,\end{cases} \tag{2.2}$$

with $\theta^\pm=\frac{1}{2}\left(\alpha\pm\frac{1}{\alpha\beta}\right)$, $\sigma(u)=\begin{cases}u, & u>\xi,\\ \beta(u-\xi)+\xi, & 0<u<\xi,\end{cases}$ the kernel $k(u,.)$ belongs to the space $L_1(\sigma(u),\infty)$ for each fixed $u\in(0,\xi)\cup(\xi,+\infty)$ and satisfies the inequality

$$\int_{\sigma(u)}^\infty |k(u,z)|dz\leq e^{cp(u)}-1$$

with $p(u)=\int_u^\infty z|q(z)|dz$ and $c=\theta^++|\theta^-|$.

Remark 2.1 *[2] The kernel function $k(u,z)$ has the following properties*

$$k(u,\sigma(u))=\begin{cases}\frac{1}{2}\int_u^\infty q(z)dz, & u>\xi,\\ \frac{\theta^+}{2}\int_u^\infty \frac{1}{\sqrt{\kappa(z)}}q(z)dz, & 0<u<\xi\end{cases}$$

$$k(u,\beta(\xi-u)+\xi+0)-k(u,\beta(\xi-u)+\xi-0)=\frac{\theta^-}{2}\left\{\int_\xi^\infty q(s)ds-\frac{1}{\beta}\int_u^\xi q(s)ds\right\},\quad 0<u<\xi.$$

Besides, if $q(u)$ is differentiable, then $k(u,z)$ satisfies the equation

$$\frac{\partial^2 k(u,z)}{\partial u^2}-\kappa(u)\frac{\partial^2 k(u,z)}{\partial z^2}=q(u)k(u,z),\quad u\in(0,\xi)\cup(\xi,+\infty),\quad z>\sigma(u),$$

and the conditions

$$\frac{d}{du}k(u,\sigma(u))=\begin{cases}-\frac{1}{2}q(u), & u>\xi,\\ -\frac{\theta^+}{2\beta}q(u), & 0<u<\xi,\end{cases}\tag{2.3}$$

$$\frac{d}{du}k(u,\beta(\xi-u)+\xi+0)-k(u,\beta(\xi-u)+\xi-0)=\frac{\theta^-}{2\beta}q(u),\tag{2.4}$$

$$k(\xi-0,z)=\alpha k(\xi+0,z),\quad k'_u(\xi-0,z)=\alpha^{-1}k'_u(\xi+0,z),$$

$$\lim_{u+z\to\infty}\frac{\partial k(u,z)}{\partial u}=\lim_{u+z\to\infty}\frac{\partial k(u,z)}{\partial z}=0,$$

which define it uniquely. Note that in order for $k(u,z)$ to be the kernel of representation (2.1), it is necessary and sufficient that it satisfies the relations in remark.

3. Main Results

3.1. Scattering Properties

In this section, we examine the scattering function $s_h(\tau)$, the eigenvalues τ_k and the normalized numbers υ_k of problem (1.1)-(1.3) and the collection of these properties is defined as scattering data that is

$$\left\{s_h(\tau),(-\infty<\tau<+\infty);\ \tau_k;\ \upsilon_k,\ (k=\overline{1,m})\right\}.$$

The solution $J(u,\tau)$ is regular with respect to τ in the half plane $\Im\tau>0$ and continuous for $\Im\tau\geq 0$. For real $\tau\neq 0$, the Wronskian of functions $J(u,\tau)$ and $J(u,-\tau)$ is as follows:

$$W\left\{J(u,\tau),J(u,-\tau)\right\}=J'(u,\tau)J(u,-\tau)-J(u,\tau)J'(u,-\tau)=2i\tau.\tag{3.1}$$

Denote $\vartheta(u,\tau)$ by the solution of equation (1.1) with discontinuity conditions (1.2) satisfying the initial conditions

$$\vartheta(0,\tau)=1,\quad \vartheta'(0,\tau)=h.$$

Lemma 3.1 *The relation*

$$\frac{2i\tau\vartheta(u,\tau)}{\omega(\tau)}=J(u,-\tau)-s_h(\tau)J(u,\tau)\tag{3.2}$$

holds for all real $\tau\neq 0$, where $\omega(\tau)=J'(0,\tau)-hJ(0,\tau)$ and

$$s_h(\tau)=\frac{\omega(-\tau)}{\omega(\tau)}=\overline{s_h(-\tau)}=\left[s_h(-\tau)\right]^{-1}.\tag{3.3}$$

Proof: Using the fundamental solutions $J(u,\tau)$ and $J(u,-\tau)$ of equation (1.1) with condition (1.2) for all real $\tau \neq 0$, we calculate

$$\vartheta(u,\tau) = \frac{1}{2i\tau}\left\{\omega(\tau)J(u,-\tau) - \omega(-\tau)J(u,\tau)\right\}. \tag{3.4}$$

Now, let us demonstrate that $\omega(\tau) = J'(0,\tau) - hJ(0,\tau) \neq 0$ for all real $\tau \neq 0$. Otherwise, there is a real $\tau^* \neq 0$ such that

$$\omega(\tau^*) = J'(0,\tau^*) - J(0,\tau^*) = 0.$$

Then, it can be written from (3.1) that

$$J'(0,\tau^*)J(0,-\tau^*) - J(0,\tau^*)J'(0,-\tau^*) = 2i\tau^*.$$

It is obtained from the last two equalities that $\tau^* = 0$ and we reach contradiction. We conclude that $\omega(\tau) \neq 0$, for all real $\tau \neq 0$. Thus, taking into account this result, it follows from (3.4) that the relations (3.2) and (3.3) are provided. □

Definition 3.1 *The function $s_h(\tau)$ is defined as the scattering function of the problem (1.1)-(1.3).*

Lemma 3.2 *In the half plane $\Im\tau > 0$, the function $\omega(\tau)$ may have only a finite number of zeros which are all simple and lie on the imaginary axis.*

Proof: Since $\omega(\tau) \neq 0$ for all real $\tau \neq 0$, the point $\tau = 0$ is the only possible real zero of the function $\omega(\tau)$. Taking into account that the function $\omega(\tau)$ is analytic in the upper half plane and the representation of the solution (2.1), it is obtained that the zeros of the function $\omega(\tau)$ form a bounded and at most countable set whose unique limit point may be only a zero.

Next, let us prove that all zeros of the function $\omega(\tau)$ lie on the imaginary axis. Assume that ρ_1 and ρ_2 are zeros of the function $\omega(\tau)$. Then, we have

$$\omega(\rho_i) = J'(0,\rho_i) - hJ(0,\rho_i), \quad i = 1,2. \tag{3.5}$$

Furthermore, regarding that the functions $J(u,\rho_1)$ and $J(u,\rho_2)$ satisfy the equation (1.1), we can write

$$\frac{d}{du}W\left\{J(u,\rho_1), \overline{J(u,\rho_2)}\right\} + \left(\rho_1^2 - \overline{\rho_2^2}\right)\kappa(u)J(u,\rho_1)\overline{J(u,\rho_2)} = 0.$$

Integrating this equality over the interval $(0,+\infty)$ and then using the discontinuity conditions (1.2) and the relation (3.5), we calculate

$$\left(\rho_1^2 - \overline{\rho_2^2}\right)\int_0^\infty J(u,\rho_1)\overline{J(u,\rho_2)}\kappa(u)du = 0. \tag{3.6}$$

Particularly, the choice $\rho_2 = \rho_1$ yields that $\rho_1^2 - \overline{\rho_2^2} = 0$ or $\rho_1 = i\tau_1$, where $\tau_1 \geq 0$. As a result, the zeros of the function $\omega(\tau)$ can lie only on the imaginary axis. It can be shown that the number of these zeros is finite. For this, it is obtained that the distance between neighboring zeros is bounded away from zero and this fact is similarly proved by using the method in [18] (see Lemma 3.1.6., pp. 186).

Now, we will demonstrate that the zeros of the function $\omega(\tau)$ are simple. Consider

$$-J''(u,\tau) + q(u)J(u,\tau) = \tau^2\kappa(u)J(u,\tau).$$

Differentiating this equation with respect to τ, we have

$$-\dot{J}''(u,\tau) + q(u)\dot{J}(u,\tau) = \tau^2\kappa(u)\dot{J}(u,\tau) + 2\tau\kappa(u)J(u,\tau),$$

where $\dot{J}(u,\tau) = \frac{\partial}{\partial\tau}J(u,\tau)$. It is obtained from these equations that

$$\frac{d}{du}W\left\{J(u,\tau), \dot{J}(u,\tau)\right\} = 2\tau\kappa(u)[J(u,\tau)]^2.$$

Integrating this equality over the interval $(0,+\infty)$ and then using the discontinuity conditions (1.2), we find

$$\omega(\tau)\dot{J}(0,\tau)-\dot{\omega}(\tau)J(0,\tau)+2\tau\int_0^\infty [J(u,\tau)]^2\kappa(u)du=0.$$

Let $\tau=i\rho$, $(\rho>0)$ be a zero of the function $\omega(\tau)$. Then, we get

$$2i\rho\int_0^\infty |J(u,i\rho)|^2\,\kappa(u)du=\dot{\omega}(i\rho)J(0,i\rho).$$

Thus, this equation yields $\dot{\omega}(i\rho)\neq 0$. Namely, the zeros of the function $\omega(\tau)$ are all simple. □

Denote the normalized numbers of the problem (1.1)-(1.3) by υ_k :

$$\upsilon_k^{-2}=\int_0^\infty |J(u,i\tau_k)|^2\,\kappa(u)du=\frac{\dot{\omega}(i\tau_k)J(0,i\tau_k)}{2i\tau_k}, \tag{3.7}$$

where $i\tau_k$, $\left(\tau_k>0,\ k=\overline{1,m}\right)$ are zeros of the function $\omega(\tau)$.

The following solutions are bounded solutions of the boundary value problem (1.1)-(1.3):

$$\varphi(u,\tau)=J(u,-\tau)-s_h(\tau)J(u,\tau),\quad (-\infty<\tau<\infty), \tag{3.8}$$

$$\varphi(u,i\tau_k)=\upsilon_k J(u,i\tau_k),\quad (k=\overline{1,m}). \tag{3.9}$$

These functions form a complete set of normalized eigenfunctions of the problem (1.1)-(1.3).

3.2. Eigenfunction Expansion

Consider the operator $L:f\to\ell(f)$ with the domain

$$\begin{aligned} D(L) \;=\; & \{f(u)\in L_{2,\kappa}(0,\infty): f(u),f'(u)\in AC[0,\xi]\cap AC[\xi,\infty],\\ & f(\xi-0)=\alpha f(\xi+0),\ f'(\xi-0)=\alpha^{-1}f(\xi+0),\\ & \ell(f)\in L_{2,\kappa}(0,\infty), f'(0)-hf(0)=0\}\,, \end{aligned}$$

where

$$\ell(f)=\frac{1}{\kappa(u)}\left\{-f''(u)+q(u)f(u)\right\}.$$

Now, we will construct the resolvent operator $R_{\tau^2}(L)$. Suppose that τ^2 is not a spectrum point of L.

Lemma 3.3 *The resolvent operator $R_{\tau^2}(L)$ is an integral operator as follows:*

$$v(u,\tau)=R_{\tau^2}(L)=\int_0^\infty g(u,z,\tau)f(z)\kappa(z)dz, \tag{3.10}$$

with the kernel

$$g(u,z,\tau)=-\frac{1}{\omega(\tau)}\begin{cases}\vartheta(u,\tau)J(z,\tau), & u\le z,\\ J(u,\tau)\vartheta(z,\tau), & z\le u.\end{cases} \tag{3.11}$$

Proof: To obtain the resolvent operator, we take into account the following non-homogenous boundary value problem:

$$-v''+q(u)y=\tau^2\kappa(u)v+f(u)\kappa(u),$$

$$v(\xi-0)=\alpha v(\xi+0),\quad v'(\xi-0)=\alpha^{-1}v'(\xi+0),$$

$$v'(0)-hv(0)=0.$$

As a result, applying the method of variation of parameters, we construct the resolvent operator (3.10) with kernel (3.11). □

Theorem 3.1 *The eigenfunction expansion of the problem (1.1)-(1.3) has the following form:*

$$\delta(z-u)\kappa(z)^{-1}=\sum_{k=1}^{m}\varphi(u,i\tau_k)\varphi(z,i\tau_k)+\frac{1}{2\pi}\int_0^\infty\varphi(u,\tau)\overline{\varphi(z,\tau)}d\tau, \tag{3.12}$$

where δ is a Dirac delta function.

Proof: Assume that $f(u)\in D(L)$ is a twice continuously differential function and finite at infinity. Then, for $\Im\tau>0$, it follows from (3.10) and (3.11) that

$$\begin{aligned} v(u,\tau) &= \int_0^\infty g(u,z,\tau)f(z)\kappa(z)dz \\ &= -\frac{J(u,\tau)}{\tau^2\omega(\tau)}\int_0^u\left[-\vartheta''(z,\tau)+q(z)\vartheta(z,\tau)\right]f(z)dz \\ &\quad -\frac{\vartheta(u,\tau)}{\tau^2\omega(\tau)}\int_u^\infty\left[-J''(z,\tau)+q(z)J(z,\tau)\right]f(z)dz. \end{aligned}$$

Integrating by parts separately for $u<\xi$ and $u>\xi$ cases, we obtain

$$v(u,\tau)=\int_0^\infty g(u,z,\tau)f(z)\kappa(z)dz=-\frac{f(u)}{\tau^2}+\frac{\phi(u,\tau)}{\tau^2}, \tag{3.13}$$

where

$$\phi(u,\tau)=\int_0^\infty g(u,z,\tau)\left[-f''(z)+q(z)f(z)\right]dz.$$

Denote Γ_R by the positively oriented contour formed by the circle of radius R and center at zero. Assume that $D_1=\{z:|z|\le R,\ |\Im z|\ge\epsilon\}$ and let $\Gamma^1_{R,\epsilon}$ be positive oriented boundary of D_1. Moreover, consider $D_2=\{z:|z|\le R,\ |\Im z|\le\epsilon\}$ and and let $\Gamma^2_{R,\epsilon}$ be negative oriented boundary of D_2. Now, multiplying both sides of the expression (3.13) by $\frac{\tau}{2\pi i}$ and integrating along the contour Γ_R with respect to τ, we have

$$\frac{1}{2\pi i}\int_{\Gamma_R}\tau v(u,\tau)d\tau=-\frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(u)}{\tau}d\tau+\phi_R(u), \tag{3.14}$$

where

$$\phi_R(u)=\frac{1}{2\pi i}\int_{\Gamma_R}\frac{\phi(u,\tau)}{\tau}d\tau.$$

On the other hand, we can write

$$\frac{1}{2\pi i}\int_{\Gamma^1_{R,\epsilon}}\tau v(u,\tau)d\tau=\frac{1}{2\pi i}\int_{\Gamma_R}\tau v(u,\tau)d\tau+\frac{1}{2\pi i}\int_{\Gamma^2_{R,\epsilon}}\tau v(u,\tau)d\tau. \tag{3.15}$$

As $R\to\infty$ and $\epsilon\to 0$, taking into account $\lim_{|\tau|\to\infty}\sup_{u\ge 0}|\phi(u,\tau)|=0$ which is obtained the expressions of the functions $J(u,\tau)$, $\vartheta(u,\tau)$ and $\omega(\tau)$, it follows from (3.14) and (3.15) that

$$\frac{1}{2\pi i}\int_{\Gamma^1_{R,\epsilon}}\tau v(u,\tau)d\tau=-f(u)+\frac{1}{2\pi i}\int_{-\infty}^\infty\tau\left[v(u,\tau+i0)-v(u,\tau-i0)\right]d\tau. \tag{3.16}$$

Besides, applying the residue theorem, we get

$$\frac{1}{2\pi i}\int_{\Gamma^1_{R,\epsilon}}\tau v(u,\tau)=\sum_{k=1}^{m}Res_{\tau=i\tau_k}\tau v(u,\tau)+\sum_{k=1}^{m}Res_{\tau=-i\tau_k}\overline{\tau v(u,\tau)}. \tag{3.17}$$

Then, using the relations (3.16) and (3.17), we have

$$\begin{aligned} f(u) &= -\sum_{k=1}^{m}Res_{\tau=i\tau_k}\tau v(u,\tau)-\sum_{k=1}^{m}Res_{\tau=-i\tau_k}\overline{\tau v(u,\tau)} \\ &\quad +\frac{1}{2\pi i}\int_{-\infty}^\infty\tau\left[v(u,\tau+i0)-v(u,\tau-i0)\right]d\tau. \end{aligned} \tag{3.18}$$

Denote $\mu(u,\tau)$ by the solution of equation (1.1) with the discontinuity conditions (1.2) under the initial conditions

$$\mu(0,\tau)=0,\qquad \mu'(0,\tau)=-1,$$

and $W\{\vartheta(u,\tau),\mu(u,\tau)\}=1$. Then, we can write

$$J(u,\tau)=J(0,\tau)\vartheta(u,\tau)-\omega(\tau)\mu(u,\tau). \tag{3.19}$$

It follows from (3.11) and (3.19) that

$$g(u,z,\tau)=-\frac{1}{\omega(\tau)}J(0,\tau)\vartheta(u,\tau)\vartheta(z,\tau)+\begin{cases}\vartheta(u,\tau)\mu(z,\tau), & u\le z,\\ \mu(u,\tau)\vartheta(z,\tau), & z\le u.\end{cases}$$

Then, using this expression in the relation (3.10), we obtain for $\Im\tau>0$

$$\begin{aligned}&\sum_{k=1}^{m}Res_{\tau=i\tau_k}\tau v(u,\tau)+\sum_{k=1}^{m}Res_{\tau=-i\tau_k}\overline{\tau v(u,\tau)}=\\ &=\sum_{k=1}^{m}\frac{-2i\tau_kJ(0,i\tau_k)}{\dot{\omega}(i\tau_k)}\int_0^\infty\vartheta(u,i\tau_k)\vartheta(z,i\tau_k)f(z)\kappa(z)dz.\end{aligned} \tag{3.20}$$

Now, we examine the relation $[v(u,\tau+i0)-v(u,\tau-i0)]$ in the equation (3.18). Since $v(u,\tau-i0)=\overline{v(u,\tau+i0)}$, we have

$$\begin{aligned}v(u,\tau+i0)-v(u,\tau-i0)&=\int_0^\infty\left[g(u,z,\tau+i0)-\overline{g(u,z,\tau+i0)}\right]f(z)\kappa(z)dz\\ &=\frac{2i\tau}{|\omega(\tau)|^2}\int_0^\infty\vartheta(u,\tau)\overline{\vartheta(z,\tau)}f(z)\kappa(z)dz.\end{aligned}$$

Then, using this relation, we calculate

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\tau\left[v(u,\tau+i0)-v(u,\tau-i0)\right]d\tau=\frac{2}{\pi}\int_0^\infty\frac{\tau^2}{|\omega(\tau)|^2}\int_0^\infty\vartheta(u,\tau)\overline{\vartheta(z,\tau)}f(z)\kappa(z)dzd\tau. \tag{3.21}$$

Thus, putting the relations (3.20) and (3.21) in the equation (3.18) and using the expressions (3.7)-(3.9), we obtain the eigenfunction expansion formula as follows:

$$f(u)=\sum_{k=1}^{m}\int_0^\infty\varphi(u,i\tau_k)\varphi(z,i\tau_k)f(z)\kappa(z)dz+\frac{1}{2\pi}\int_0^\infty\int_0^\infty\varphi(u,\tau)\overline{\varphi(z,\tau)}f(z)\kappa(z)dzd\tau.$$

□

3.3. Reconstruction of Potential Function

In this section, we will examine the inverse scattering problem of problem (1.1)-(1.3). That is, we give the reconstruction of potential $q(u)$ from scattering data $\left\{s_h(\tau),(-\infty<\tau<+\infty);\ \tau_k;\ \upsilon_k,\ (k=\overline{1,m})\right\}$.

Assume that $q(u)\equiv 0$ in equation (1.1) and denote $\zeta(u,\tau)$ by the solution of the equation (1.1) with discontinuity conditions (1.2) and initial conditions

$$\zeta(0,\tau)=1,\quad \zeta'(0,\tau)=0.$$

Then, the following relation is valid:

$$\frac{2i\tau\zeta(u,\tau)}{J_0'(0,\tau)}=J_0(u,-\tau)-s_0(\tau)J_0(u,\tau), \tag{3.22}$$

where

$$s_0(\tau)=e^{-2i\tau\xi(1+\beta)}\left(\frac{\theta^- -\theta^+e^{2i\tau\beta\xi}}{-\theta^- +\theta^+e^{-2i\tau\beta\xi}}\right) \tag{3.23}$$

and $s_h(\tau) = s_0(\tau) + O\left(\frac{1}{\tau}\right)$ as $|\tau| \to \infty$. Furthermore, $s_0(\tau) - s_h(\tau) \in L_2(-\infty,\infty)$ is Fourier transform of the function

$$F_s(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[s_0(\tau) - s_h(\tau)\right]e^{i\tau u}d\tau \tag{3.24}$$

and this function belongs to the space $L_2(-\infty,\infty)$.

Remark 3.1 *For the function $s_0(\tau)$, the following relation is valid:*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} s_0(\tau)e^{i\tau(y+z)}d\tau = \left(\frac{\theta^-}{\theta^+}\right)\delta(y+z-2\xi)$$
$$+\left[\left(\frac{\theta^-}{\theta^+}\right)^2 - 1\right]\sum_{m=0}^{\infty}\left(\frac{\theta^-}{\theta^+}\right)^m \delta\left[y+z-2\xi(1-\beta(m+1))\right]$$

here $\delta(y)$ is Dirac delta function. In fact, it follows from (3.23) that

$$\begin{aligned} s_0(\tau) &= e^{-2i\tau\xi(1+\beta)}\left(\frac{\theta^- - \theta^+ e^{2i\tau\beta\xi}}{-\theta^- + \theta^+ e^{-2i\tau\beta\xi}}\right) \\ &= e^{-2i\tau\xi}\left(\frac{\theta^-}{\theta^+} - e^{2i\tau\beta\xi}\right)\sum_{m=0}^{\infty}\left(\frac{\theta^-}{\theta^+}\right)^m e^{2i\tau\beta\xi m} \\ &= \left(\frac{\theta^-}{\theta^+}\right)e^{-2i\tau\xi} + \left[\left(\frac{\theta^-}{\theta^+}\right)^2 - 1\right]\sum_{m=0}^{\infty}\left(\frac{\theta^-}{\theta^+}\right)^m e^{-2i\tau\xi(1-\beta(m+1))}. \end{aligned}$$

Then, using this relation, we calculate

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} s_0(\tau)e^{i\tau(y+z)}d\tau = \frac{1}{2\pi}\left(\frac{\theta^-}{\theta^+}\right)\int_{-\infty}^{\infty} e^{-2i\tau\xi}e^{i\tau(y+z)}d\tau$$
$$+\frac{1}{2\pi}\left[\left(\frac{\theta^-}{\theta^+}\right)^2 - 1\right]\int_{-\infty}^{\infty}\left(\sum_{m=0}^{\infty}\left(\frac{\theta^-}{\theta^+}\right)^m e^{2i\tau\beta\xi m}\right)e^{i\tau(y+z)}d\tau$$
$$= \left[\left(\frac{\theta^-}{\theta^+}\right)^2 - 1\right]\sum_{m=0}^{\infty}\left(\frac{\theta^-}{\theta^+}\right)^m \delta\left[y+z-2\xi(1-\beta(m+1))\right] + \left(\frac{\theta^-}{\theta^+}\right)\delta(y+z-2\xi). \tag{3.25}$$

Theorem 3.2 *The kernel $k(u,y)$ of Jost solution (2.1) satisfies the following linear integral equation for each $0 \le u \ne \xi$,*

$$k(u,y) + \breve{F}(u,y) - \left(\frac{\theta^-}{\theta^+}\right)k(u,2\xi - y) + \int_{\sigma(u)}^{\infty} k(u,z)F(z+y)dz = 0, \quad y > \sigma(u), \tag{3.26}$$

where

$$F(u) = F_s(u) + \sum_{k=1}^{m} \upsilon_k^2 e^{-\tau_k u}, \tag{3.27}$$

$$\breve{F}(u,y) = \begin{cases} \theta^+ F(\beta(u-\xi)+\xi+y) + \theta^- F(-\beta(u-\xi)+\xi+y), & u < \xi, \\ F(u+y), & \xi < u. \end{cases} \tag{3.28}$$

Proof: Taking into account the relations (3.2) and (3.22), we calculate

$$\begin{aligned} &\frac{2i\tau\vartheta(u,\tau)}{J'(0,\tau) - hJ(0,\tau)} - \frac{2i\tau\zeta(u,\tau)}{J_0'(0,\tau)} = 2i\tau\vartheta(u,\tau)\left(\frac{1}{J'(0,\tau) - hJ(0,\tau)} - \frac{1}{J_0'(0,\tau) - hJ_0(0,\tau)}\right) \\ &\quad + \frac{2i\tau\left(\vartheta(u,\tau) - \zeta(u,\tau)\right)}{J_0'(0,\tau) - hJ_0(0,\tau)} + 2i\tau\zeta(u,\tau)\left(\frac{1}{J_0'(0,\tau) - hJ_0(0,\tau)} - \frac{1}{J_0'(0,\tau)}\right) \\ &= \left[s_0(\tau) - s(\tau)\right]\left\{J_0(u,\tau) + \int_{\sigma(u)}^{\infty} k(u,z)e^{i\tau z}dz\right\} \\ &\quad + \int_{\sigma(u)}^{\infty} k(u,z)e^{-i\tau z}dz - s_0(\tau)\int_{\sigma(u)}^{\infty} k(u,z)e^{i\tau z}dz. \end{aligned} \tag{3.29}$$

Let us multiply the equality (3.29) by $\frac{1}{2\pi}e^{i\tau y}$ and then integrate according to $\tau \in (-\infty, \infty)$. Taking into account the formula (3.25) and the relation $k(u,y) = 0$ for $y < u$, we find the right hand side of the equality (3.29) as follows:

$$k(u,y) + \breve{F}_s(u,y) - \left(\frac{\theta^-}{\theta^+}\right) k(u, 2\xi - y) + \int_{\sigma(u)}^{\infty} k(u,z)F_s(z+y)dz, \quad y > \sigma(u), \tag{3.30}$$

where

$$\breve{F}_s(u,y) = \begin{cases} \theta^+ F_s(\beta(u-\xi)+\xi+y) + \theta^- F_s(-\beta(u-\xi)+\xi+y), & u < \xi, \\ F_s(u+y), & u > \xi. \end{cases}$$

Now, applying Jordan's lemma for $y > \sigma(u)$, we can express the left hand side of the equality (3.29) in the form:

$$\sum_{k=1}^{m} \frac{-2i\tau_k \vartheta(u, i\tau_k)e^{-\tau_k y}}{\dot{\omega}(i\tau_k)}.$$

Using the expression of normalized numbers (3.7) and the relation $J(u,\tau) = J(0,\tau)\vartheta(u,\tau)$, we can write

$$\begin{aligned} \sum_{k=1}^{m} \frac{-2i\tau_k \vartheta(u, i\tau_k)e^{-\tau_k y}}{\dot{\omega}(i\tau_k)} &= \sum_{k=1}^{m} -\upsilon_k^2 J(u, i\tau_k)e^{-\tau_k y} \\ &= \sum_{k=1}^{m} -\upsilon_k^2 \left\{ J_0(u, i\tau_k)e^{-\tau_k y} + \int_{\sigma(u)}^{\infty} k(u,z)e^{-\tau_k(z+y)}dz \right\}. \end{aligned} \tag{3.31}$$

Thus, it follows from (3.30) and (3.31) that for $y > \sigma(u)$

$$\begin{aligned} &\sum_{k=1}^{m} -\upsilon_k^2 \left\{ J_0(u, i\tau_k)e^{-\tau_k y} + \int_{\sigma(u)}^{\infty} k(u,z)e^{-\tau_k(z+y)}dz \right\} \\ &= \quad k(u,y) + \breve{F}_s(u,y) - \left(\frac{\theta^-}{\theta^+}\right) k(u, 2\xi - y) + \int_{\sigma(u)}^{\infty} k(u,z)F_s(z+y)dz. \end{aligned}$$

As a result, we construct the fundamental equation or modified Marchenko equation (3.26) with the relations (3.27) and (3.28) for $y > \sigma(u)$. □

Lemma 3.4 *For each $u \geq 0$, $u \neq \xi$, the fundamental equation (3.26) has a unique solution $k(u,.) \in L_2(\sigma(u), \infty)$.*

Proof: The fundamental equation (3.26) can be written as follows:

$$A_u k(u,.) + T_u k(u,.) = -\breve{F}(u,.),$$

where

$$(A_u g)(y) = \begin{cases} g(y), & u > \xi, \\ g(y) - \frac{\theta^-}{\theta^+} g(2\xi - y), & u < \xi, \end{cases}$$

and

$$(T_u g) = \int_{\sigma(u)}^{\infty} F(z+y)g(z)dz, \quad y > \sigma(u).$$

It can be shown that the operator A_u is invertible in the space $L_2(\sigma(u), \infty)$. Moreover, the operator T_u is completely continuous in the space $L_2(\sigma(u), \infty)$ (see [18], Lemma 3.3.1). Then, the fundamental equation (3.26) is given as follows:

$$k(u,.) + A_u^{-1} T_u k(u,.) = -A_u^{-1} \breve{F}(u,.),$$

where the operator $A_u^{-1}T_u$ is completely continuous operator in $L_2(u,\infty)$. When $u>\xi$, we face the classical Marchenko equation, so for $u>\xi$ the proof of this lemma is as in [18]. When $u<\xi$, to prove this lemma, it is sufficient to show that the following homogeneous equation

$$g_u(y)-\frac{\theta^-}{\theta^+}g_u(2\xi-y)+\int_{\beta(u-\xi)+\xi}^{\infty}F(z+y)g_u(z)dz=0,\quad y>\beta(u-\xi)+\xi$$

has only trivial solution $g_u(y)=0$ in $L_2(\beta(u-\xi)+\xi,\infty)$ and this fact is similarly proved as in [13]. □

Now, we can give the reconstruction algorithm of the potential $q(u)$ from the scattering data

$$\left\{s_h(\tau),(-\infty<\tau<+\infty);\ \tau_k;\ \upsilon_k,\ (k=\overline{1,m})\right\}$$

as follows:

- Given scattering data $\left\{s_h(\tau),(-\infty<\tau<+\infty);\ \tau_k;\ \upsilon_k,\ (k=\overline{1,m})\right\}$, construct the functions $F_s(u)$, $F(u)$ and $\breve{F}(u,y)$ from the formulas (3.24), (3.27) and (3.28),
- From the constructed these functions, write the fundamental equation (3.26),
- Solve the fundamental equation, calculate $k(u,z)$,
- From the relations (2.3) and (2.4), find the potential $q(u)$.

3.4. Continuity of the Scattering Function

In this section, with the help of the fundamental equation (3.26), we will show that the scattering function $s_h(\tau)$ of the problem (1.1)-(1.3) is continuous at all real points τ.

Theorem 3.3 *The scattering function $s_h(\tau)$ is continuous at all real points τ and the relation is valid:*

$$s_h(0)=\begin{cases}1, & \omega(0)\neq 0,\\ -1, & \omega(0)=0.\end{cases}\tag{3.32}$$

Proof: The continuity of the function $s_h(\tau)$ at all real point $\tau\neq 0$ is given the Lemma 3.1. In case $\omega(0)\neq 0$, the function $s_h(\tau)$ is continuous at zero and $s_h(0)=1$. Then, it remains to examine the case $\omega(0)=0$. Using the representation of solution $J(u,\tau)$, we have

$$\omega(0)=J'(0,0)-hJ(0,0)=-h\left(\theta^++\theta^-\right)-\beta k(0,\xi(1-\beta))-\int_{\xi(1-\beta)}^{\infty}[hk(0,z)-k_u(0,z)]dz=0.\tag{3.33}$$

Now, putting $u=0$ in the fundamental equation (3.26), we get

$$k(0,z)+\theta^+F(\xi(1-\beta)+y)+\theta^-F(\xi(1+\beta)+y)-\left(\frac{\theta^-}{\theta^+}\right)k(0,2\xi-y)+\int_{\xi(1-\beta)}^{\infty}k(0,z)F(z+y)dz=0.\tag{3.34}$$

For $u<\xi$, integrating the fundamental equation (3.26) with respect to u and then writing $u=0$, we find

$$\begin{aligned}&k_u(0,y)+\beta(\theta^+F_u(\xi(1-\beta)+y)-\theta^-F_u(\xi(1+\beta)+y))-\left(\frac{\theta^-}{\theta^+}\right)k_u(0,2\xi-y)\\&-\beta k(0,\xi(1-\beta))F(\xi(1-\beta)+y)+\int_{\xi(1-\beta)}^{\infty}k_u(0,z)F(z+y)dz=0.\end{aligned}\tag{3.35}$$

After multiplying (3.34) by h and subtracting it from (3.35), we integrate the last obtained equation with respect to y from s to ∞ and obtain the following equation:

$$\begin{aligned}&\int_s^{\infty}[hk(0,y)-k_u(0,y)]dy+\theta^+\int_s^{\infty}[hF(\xi(1-\beta)+y)-\beta F_u(\xi(1-\beta)+y)]dy\\&+\theta^-\int_s^{\infty}[hF(\xi(1+\beta)+y)+\beta F_u(\xi(1+\beta)+y)]dy-\left(\frac{\theta^-}{\theta^+}\right)\int_s^{\infty}[hk(0,2\xi-y)+k_u(0,2\xi-y)]dy\\&+\beta\int_s^{\infty}k(0,\xi(1-\beta))F(\xi(1-\beta)+y)dy+\int_{\xi(1-\beta)}^{\infty}[hk(0,z)-k_u(0,z)]\left(\int_{z+s}^{\infty}F(x)dx\right)dz=0.\end{aligned}$$

In the last equation, using the following relation which is obtained from (3.33)

$$\theta^{+}h+\beta k(0,\xi(1-\beta))+\int_{\xi(1-\beta)}^{\infty}[hk(0,z)-k_u(0,z)]dz=-h\theta^{-},$$

we find

$$\tilde{k}(s)-\int_{\xi(1-\beta)}^{\infty}\tilde{k}(z)F(z+s)dz=\psi(s),\quad 0\le s<\infty \tag{3.36}$$

where

$$\tilde{k}(s)=\int_{s}^{\infty}[hk(0,x)-k_u(0,x)]dx \tag{3.37}$$

and

$$\begin{aligned}\psi(s) &= h\theta^{-}\int_{\xi(1-\beta)+s}^{\xi(1+\beta)+s}F(z)dz+\left(\frac{\theta^{-}}{\theta^{+}}\right)\int_{0}^{2\xi-s}[hk(0,z)-k_u(0,z)]dz\\ &\quad -\beta\int_{s}^{\infty}[\theta^{-}F_u(\xi(1+\beta)+y)-\theta^{+}F_u(\xi(1-\beta)+y)]dy.\end{aligned}$$

Thus, in case $\omega(0)=0$, it follows that $\tilde{k}(s)$ is bounded solution of the equation (3.36) and this solution is integrable on $[\xi(1-\beta),\infty)$. Besides, we can write

$$\omega(\tau)=J'(0,\tau)-hJ(0,\tau)=i\tau\hat{k}(\tau),$$

where

$$\hat{k}(\tau)=\beta\left(\theta^{+}e^{i\tau\xi(1-\beta)}-\theta^{-}e^{i\tau\xi(1+\beta)}\right)-\frac{h\theta^{-}}{i\tau}\left(e^{i\tau\xi(1+\beta)}-e^{i\tau\xi(1-\beta)}\right)-\int_{\xi(1-\beta)}^{\infty}\tilde{k}(z)e^{i\tau z}dz.$$

Similarly, we have

$$\omega(-\tau)=-i\tau\hat{k}(\tau).$$

Using this relation, we get

$$s_h(\tau)=\frac{\omega(-\tau)}{\omega(\tau)}=-\frac{\hat{k}(-\tau)}{\hat{k}(\tau)}.$$

Then, it follows from (3.2) that

$$2\vartheta(u,\tau)=\hat{k}(\tau)\left(J(u,-\tau)-s_h(\tau)J(u,\tau)\right).$$

This yields $\hat{k}(\tau)\neq 0$. As a result, the function $s_h(\tau)$ is continuous at $\tau=0$ and $s_h(0)=-1$. □

Remark 3.2 *Levinson formula which gives the correlation between the increment of logarithm of $s_h(\tau)$ and the number of negative eigenvalues of the problem (1.1)-(1.3) is as follows:*

$$m=\frac{\ln s_h(+0)-\ln s_h(+\infty)}{2\pi i}-\frac{1-s_h(0)}{4}.$$

This formula is obtained in a similar way as in [18].

4. Conclusion

In this paper, we investigated the direct and inverse scattering problem of the boundary value problem (1.1)-(1.3) on the positive half line. The scattering properties and eigenfunction expansion of this problem were examined as a direct problem and also, the inverse problem was solved by using the method of Marchenko. Furthermore, the continuity of the scattering function $s_h(\tau)$ of the problem (1.1)-(1.3) at all real points τ was shown using the fundamental equation.

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