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Eulerian and Clique number of the Zero Divisor Graph $\Gamma[L(+)M]$

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ABSTRACT: In this article, we investigate $\Gamma[Z_n(+)Z_m]$, where n is equal to the product of $p_1^r q_1$ and $m = p_1$ for some prime numbers. To find out when these graphs are Eulerian and, more importantly, we are examining the clique number of $\Gamma[Z_n(+)Z_m]$ for $n = p_1^r q_1$ and $m = p_1$.

Key Words: Zero divisor graph, Euler tour, Euler graph, Clique number.

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1. Introduction

The idea of the zero divisor graph, denoted by Z.D, of the idealization ring $L(+)M$ was first introduced by Axtell and Stickles in 2006 [8] is defined as $L(+)M = \{(v_1,\ w_1) :\ v_1 \in L,\ w_1 \in M\}$ and let $(v_1,\ w_1)$ and $(v_2,\ w_2)$ be two elements of $R(+)M$, such that $(v_1,\ w_1) + (v_2, w_2) = (v_1 + r_2,\ w_1 + w_2)$ and $(v_1, w_1)(v_2, w_2) = (v_1 v_2,\ v_1 w_2 + v_2 w_1)$. I. Beck [11] in 1988 introduced the idea of the Z.D of the ring R. Subsequently, Anderson and Livingston [10], as well as Akbari and Mohammadian [9], extended the theory by focusing on the Z.D taking into consideration just the non-zero zero divisors. The notion of Euler graphs is explored in [12]. More properties of the Z.D are studied in [1-7]. In this article, we introduce the idea of Euler graphs to $\Gamma(Z_n(+)Z_m)$ and identify which $\Gamma(Z_n(+)Z_m)$ are Eulerian. A clique is a subgraph of G where every pair of vertices is connected by an edge. The size of the largest clique in a graph G is called its clique number, denoted as $\omega(G)$ [13-14]. A subgraph K_m with m nodes is called a clique of dimension m if every pair of distinct nodes in K_m is connected by an edge.
This article is structured ithe following way; first it covers the basics, such as definitions and notations related to the Z.D within a commutative ring denoted as $L(+)M$. Up is an exploration into the Euler graphs for the $\Gamma[Z_{p_1^r}(+)Z_{p_1}]$. Following that is a discussion on the Euler graphs $\Gamma[Z_n(+)Z_{p_1}]$ for any integer n equal to $p_1^r q_1$. The clique number for $\Gamma[Z_{p_1^r}(+)Z_{p_1}]$ and $\Gamma[Z_n(+)Z_{p_1}]$ for any integer n equal to $p_1^r q_1$.

Definition 1 ***Zero divisor Graph of idealization ring*** *[8]*
Consider the ring $L(+)M$ is a commutative ring with unity, and $Z[L(+)M]$ be the set of its zero divisors. Then the Z.D of $L(+)M$ denoted by $\Gamma[L(+)M]$, is the graph (undirected) with vertex set $Z^*[L(+)M] = Z[L(+)M] - \{(\mathbf{0}, \mathbf{0})\}$, the non-zero zero divisors of $L(+)M$, such that two vertices (v_1, w_1) and $(v_2, w_2) \in Z^*[L(+)M]$ are adjacent if $(v_1, w_1)(v_2, w_2) = (0, 0)$.

Definition 2 ***Euler trial*** *[12]*
An Euler trial, on a graph G, is a trial that covers every edge of the graph G once.

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Definition 3 ***Euler graph*** *[12]*
Eulerian gragraph that a graph G has an Euler trial.

Theorem 1 *[12] A connected graph is Euler if and only if the degree of every vertex is even.*

2. Eulerian of $\Gamma[Z_{p_1^r}(+)Z_{p_1}]$

We discuss the Eulerian of $\Gamma[Z_{p_1^r}(+)Z_{p_1}]$ where p_1 is a prime number and $r \geq 1$.
To start with, we consider $\Gamma[Z_{p_1}(+)Z_{p_1}]$.

Theorem 2 *The $\Gamma[Z_{p_1}(+)Z_{p_1}]$ is not Euler graph.*

Proof: Consider $\Gamma[Z_{p_1}(+)Z_{p_1}]$. The vertex set is $A = \{(0, m) : m \in Z^*_{p_1}\}$ and so $|A| = p_1 - 1$. For any two vertices in the set A, they are adjacent, which makes the graph a complete on $p_1 - 1$ vertices, that is, $\Gamma[Z_{p_1}(+)Z_{p_1}] = K_{p_1-1}$. As the graph is complete, each vertex of the degree is $p_1 - 2$.
If $p_1 > 2$, then the degree of each vertex is odd. Thus $\Gamma[Z_{p_1}(+)Z_{p_1}]$ is not Eulerian. □

Theorem 3 *For any prime $p_1 > 2$, $\Gamma[Z_{p_1^2}(+)Z_{p_1}]$ is not Euler graph.*

Proof: Consider $\Gamma[Z_{p_1^2}(+)Z_{p_1}]$. The vertex sets $A = \{(0, m) : m \in Z^*_{p_1}\}$, $|A| = p_1 - 1$ and $B = \{(kp, m) : m \in Z_{p_1}, gcd(k, p_1) = 1\}$, $|B| = p_1(p_1 - 1)$.
For any two vertices, in the set A and B, they are adjacent, which makes the graph complete on $p_1 - 1 + p_1(p_1 - 1) = (p_1 - 1)(p_1 + 1)$ vertices, that is, $\Gamma[Z_{p_1^2}(+)Z_{p_1}] = K_{(p_1-1)(p_1+1)}$. The degree of each vertex is $(p_1 - 1)(p_1 + 1) - 1$ since the graph is complete.
If $p_1 > 2$ then every prime greater than 2 is odd, and hence the degree of each vertex is odd. Thus $\Gamma[Z_{p_1^2}(+)Z_{p_1}]$ is not Eulerian.
If $p_1 = 2$, then $\Gamma[Z_{p_1^2}(+)Z_{p_1}]$ is Eulerian. □

Theorem 4 *For any prime number p_1, the $\Gamma[Z_{p_1^3}(+)Z_{p_1}]$ is not an Euler graph.*

Proof: *Consider $\Gamma[Z_{p_1^3}(+)Z_{p_1}]$. We separate the vertices of $\Gamma[Z_{p_1^3}(+)Z_{p_1}]$ into disjoint 3-sets, which are given by*

$$
\begin{aligned}
A &= \{(0, m) : \ m \in Z^*_{p_1}\}, \\
B &= \{(kp_1, m) : \ m \in Z_{p_1} \textit{ and } gcd(k, p_1^2) = 1\} \\
\textit{and} & \\
C &= \{(lp_1^2, m) : \ m \in Z_{p_1} \textit{ and } gcd(k, p_1) = 1\},
\end{aligned}
$$

cardinality $|A| = p_1 - 1$, $|B| = p_1(p_1^2 - p_1)$ and $|C| = p_1(p_1 - 1)$.
Each item in set A is connected to every item in sets A, B and C. Then each vertex in the set A is degree equal $p_1 - 1 + p_1(p_1^2 - p_1) + p_1(p_1 - 1) - 1 = (p_1 - 1)(p_1 + 1) + p_1(p_1^2 - p_1) - 1$, which is odd.
Each item in the set B is connected to every item in the sets A and C. Then each vertex in the set B is degree equal $p_1 - 1 + p_1(p_1^2 - p_1) = (p_1 - 1) + p_1^2(p_1 - 1) = (p_1 - 1)(p_1^2 + 1)$.
Each item in set A is connected to every item in sets A B, and C. Then each vertex in the set C is degree equal $p_1 - 1 + p_1(p_1^2 - p_1) + p_1(p_1 - 1) - 1 = (p_1 - 1)(p_1 + 1) + p_1(p_1^2 - p_1) - 1$, which is an odd.
Therefore, $\Gamma[Z_{p_1^3}(+)Z_{p_1}]$ is not an Eulerian graph. □

With similar techniques, we prove the more general case in the following theorem.

Theorem 5 *If p_1 is any prime number, then $\Gamma[Z_{p_1^r}(+)Z_{p_1}]$ is not Eulerian graph.*

Proof: *We separate the vertices of* $\Gamma[Z_{p_1^r}(+)Z_{p_1}]$ *into* $r-$*disjoint sets, given by*

$$A_1 = \{(k_1p_1, m) \ : \ m \in Z_{p_1}, \ gcd(k_1, p_1^{r-1}) = 1\},$$
$$A_2 = \{(k_2p_1^2, m) \ : \ m \in Z_{p_1}, \ gcd(k_2, p_1^{r-2}) = 1\},$$

and for general

$$A_i = \{(k_ip_1^i, m) \ : \ m \in Z_{p_1}, \ gcd(k_i, p_1^{r-i}) = 1, \ for \ i = 3, ..., r\},$$

cardinality $|A_i| = p_1(p_1^{r-i} - p_1^{r-i-1})$, *for* $i = 1, 2, ...r-1$ *and* $|A_r| = p_1 - 1$. *Now the degree of any element in* A_r *is* $p_1 - 1 + \sum_{j=1}^{r-1} p_1(p_1^{r-j} - p_1^{r-j-1}) - 1 = p_1(p_1^{r-1} - 1) + p_1 - 2$.
The degree of any element in the set A_i *is* $\sum_{j=1}^{i} p_1(p_1^j - p_1^{j-1}) + p_1 - 1$, *for all* $i \leq \lfloor \frac{r}{2} \rfloor$. *Also, the degree of any element in the set* A_i *is* $\sum_{j=1}^{i} p_1(p_1^j - p_1^{j-1}) + p_1 - 2$, *for all* $i > \lfloor \frac{r}{2} \rfloor$. *Hence* $\Gamma[Z_{p_1^r}(+)Z_{p_1}]$ *is not Eulerian graph.* □

3. Eulerian of $\Gamma[Z_n(+)Z_m]$

We discuss the Eulerian $\Gamma[Z_n(+)Z_m]$ where $n = p_1^r q_1$ and $m = p_1$.
Let $n = p_1q_1$ and $m = p_1$.

Theorem 6 *If* p_1 *and* q_1 *are distinct primes such that* $q_1 > p_1$, *then* $\Gamma[Z_{p_1q_1}(+)Z_{p_1}]$ *is not an Eulerian graph.*

Proof: *Consider* $\Gamma[Z_{p_1q_1}(+)Z_{p_1}]$. *The vertices of* $\Gamma[Z_{p_1q_1}(+)Z_{p_1}]$ *are divided into sets*

$$A = \{(0, m) : \ m \in Z_{p_1}^*\},$$
$$B_1 = \{(kp_1, m) : \ m \in Z_{p_1}^*, \ gcd(k, q_1) = 1\},$$
$$B_1^* = \{(kp_1, 0) : \ gcd(k, q_1) = 1\},$$

and

$$C = \{(lq_1, m) : \ m \in Z_{p_1}, \ gcd(l, p_1) = 1\},$$

the cardinality $|A| = p_1 - 1$, $|B_1| = (p_1 - 1)(q_1 - 1)$, $|B_1^*| = q_1 - 1$ *and* $|C| = p_1(p_1 - 1)$. *Now, the degree of any element in* A *is* $p_1 - 2 + p_1(q_1 - 1)$. *The degree of any element in the sets* B_1 *is* $p - 1$ *and* B_1^* *is* $p_1 - 1 + p_1(p_1 - 1) = (p_1 - 1)(p_1 + 1)$ *and the degree of any element in the sets* C *is* $q_1 - 1$.
If p_1 *and* q_1 *are odd, then* $\Gamma[Z_{p_1q_1}(+)Z_{p_1}]$ *is not Eulerian graph. Also, If* $p_1 = 2$, *then* $\Gamma[Z_{p_1q_1}(+)Z_{p_1}]$ *is not Eulerian graph.* □

Theorem 7 *If* p_1 *and* q_1 *are distinct primes such that* $q_1 > p_1$, *then* $\Gamma[Z_{p_1^2q_1}(+)Z_{p_1}]$ *is not Eulerian.*

Proof: Consider $\Gamma[Z_{p_1^2q_1}(+)Z_{p_1}]$. The vertices of $\Gamma[Z_{p_1^2q_1}(+)Z_{p_1}]$ are divided into sets

$$A = \{(0, m) : \ m \in Z_{p_1}^*\},$$
$$B_1 = \{(kp_1, m) : \ m \in Z_{p_1}, \ gcd(k, p_1q_1) = 1\},$$
$$B_2 = \{(kp_1^2, m) : \ m \in Z_{p_1^*}, \ gcd(k, q_1) = 1\},$$
$$B_2^* = \{(kp_1^2, 0) : \ gcd(k, q_1) = 1\},$$
$$C = \{(kq_1, m) : \ m \in Z_{p_1}, \ gcd(k, p_1^2) = 1\},$$

and

$$D = \{(kp_1q_1, m) : \ m \in Z_{p_1}, \ gcd(k, p_1) = 1\},$$

the cardinality $|A| = p_1 - 1$, $|B_1| = p_1(p_1 - 1)(q_1 - 1)$, $|B_2| = p_1(q_1 - 1)$, $|C| = p_1(p_1^2 - p_1)$ and $|D| = p_1(p_1 - 1)$. Now, every element v in the set A is adjacent to every element in A, B_1, B_2, B_2^* and D. So, the degree, $deg_A(v) = p_1 - 2 + p_1(p_1 - 1)(q_1 - 1) + p_1(q_1 - 1) + p_1(p_1 - 1)$.

Every element v in the set B_1 is adjacent to every element in A and D; $deg_{B_1}(v) = p_1 - 1 + p_1(p_1 - 1) = (p_1 - 1)(p_1 + 1)$.
Every element v in the set B_2 is adjacent to every element in A and D; $deg_{B_2}(v) = p_1 - 1 + p_1(p_1 - 1) = (p_1 - 1)(p_1 + 1)$.
Every element v in the set B_2^* is adjacent to every element in A, C, and D; $deg_{B_2^*}(v) = p_1 - 1 + p_1(p_1^2 - p_1) + p_1(p_1 - 1)$.
Every element v in the set C is adjacent to every element in B_2^*; $deg_C(v) = q_1 - 1$.
Every element v in the set D is adjacent to every element in A, B_1, B_2, B_2^* and D, $deg_D(v) = p_1 - 1 + p_1(p_1 - 1)(q_1 - 1) + p_1(q_1 - 1) + p_1(p_1 - 1) - 1$.
If p_1 and q_1 are odd, then $\Gamma[Z_{p_1^2 q_1}(+)Z_{p_1}]$ is not Eulerian graph.
If $p_1 = 2$, then $\Gamma[Z_{p_1 q_1}(+)Z_{p_1}]$ is not Eulerian graph. □

Theorem 8 *If p_1 and q_1 are distinct primes such that $q_1 > p_1$, then $\Gamma[Z_{p_1^r q_1}(+)Z_{p_1}]$ is not Eulerian.*

Proof: *Consider $\Gamma[Z_{p_1^r q_1}(+)Z_{p_1}]$. We divide the vertices of $\Gamma[Z_{p_1^r q_1}(+)Z_{p_1}]$ into sets*

$$A_1 = \{(k_1 p_1, m) \ : \ m \in Z_{p_1}, \ gcd(k_1, p_1^{r-1} q_1) = 1\},$$
$$A_2 = \{(k_2 p_1^2, m) \ : \ m \in Z_{p_1}, \ gcd(k_2, p_1^{r-2} q_1) = 1\},$$

and for general

$$A_i = \{(k_i p_1^i, m) \ : \ m \in Z_{p_1}, \ gcd(k_i, p_1^{r-i} q_1) = 1, \ for \ i = 3, ..., r\},$$
$$B_1 = \{(k_1 p_1 q_1, m) \ : \ m \in Z_{p_1}, \ gcd(k_1, p_1^{r-1}) = 1\},$$
$$B_2 = \{(k_2 p_1^2 q_1, m) \ : \ m \in Z_{p_1}, \ gcd(k_2, p_1^{r-2}) = 1\},$$

and for general

$$B_i = \{(k_i p_1^i q_1, m) \ : \ m \in Z_{p_1}, \ gcd(k_i, p_1^{r-i}) = 1, \ for \ i = 3, ..., r\},$$

cardinality $|A_i| = p_1(p_1^{r-i} - p_1^{r-i-1})(q_1 - 1)$, for $i = 1, 2, ...r - 1$ and $|A_r| = (p_1 - 1)(q_1 - 1)$. And cardinality $|B_i| = p_1(p_1^{r-i} - p_1^{r-i-1})$, for $i = 1, 2, ...r - 1$ and $|B_r| = p_1 - 1$.
Now, the degree of any element in B_r is $p_1 - 1 + \sum_{j=1}^{r-1} p_1(p_1^{r-j} - p_1^{r-j-1}) - 1 = p_1(p_1^{r-1} - 1) + p_1 - 2$. The degree of any element in the set B_i is $\sum_{j=1}^{i} p_1(p_1^j - p_1^{j-1}) + p_1 - 1$, for all $i \leq \lfloor \frac{r}{2} \rfloor$. Also, the degree of any element in the set B_i is $\sum_{j=1}^{i} p_1(p_1^j - p_1^{j-1}) + p_1 - 2$, for all $i > \lfloor \frac{r}{2} \rfloor$. Hence $\Gamma[Z_{p_1^r q_1}(+)Z_{p_1}]$ is not Eulerian graph. □

4. Clique number of $\Gamma[Z_{p_1^r}(+)Z_{p_1}]$

We discuss the clique number of $\Gamma[Z_{p_1^r}(+)Z_{p_1}]$ where p_1 is a prime number and $r \geq 1$.
We consider $\Gamma[Z_{p_1}(+)Z_{p_1}]$.

Theorem 9 *The clique number of $\Gamma[Z_{p_1}(+)Z_{p_1}]$ is $\omega(\Gamma[Z_p(+)Z_{p_1}]) = p_1 - 1$.*

Proof: Consider $\Gamma[Z_{p_1}(+)Z_{p_1}]$. The vertex set is $A = \{(0, m) : m \in Z_{p_1}^*\}$ and so $|A| = p_1 - 1$. For any two vertices, in the set A, they are adjacent, and so the graph is a complete graph with $p_1 - 1$ vertices, that is, $\Gamma[Z_{p_1}(+)Z_{p_1}] = K_{p_1 - 1}$. As the graph is complete, the $\omega(\Gamma[Z_{p_1}(+)Z_{p_1}]) = p_1 - 1$. See, the following figure. □

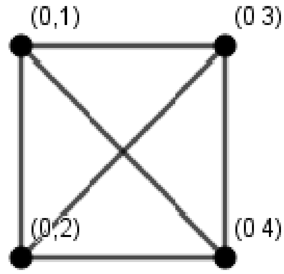

Figure 1: $\Gamma(\mathbf{Z}_5(+)\mathbf{Z}_5)$.

Theorem 10 *The clique number of Z.D is* $\omega(\Gamma[Z_{p_1^2}(+)Z_{p_1}]) = (p_1-1)(p_1+1)$.

Proof: Consider $\Gamma[Z_{p_1^2}(+)Z_{p_1}]$. The vertex sets $A = \{(0,m) : m \in Z_{p_1}^*\}$, $|A| = p_1 - 1$ and $B = \{(kp_1, m) : m \in Z_{p_1}, gcd(k, p_1) = 1\}$, $|B| = p_1(p_1-1)$.
For any two vertices is zero, they are adjacent, and so the graph is a complete graph on $p_1 - 1 + p_1(p_1 - 1) = (p_1-1)(p_1+1)$ vertices, that is, $\Gamma[Z_{p_1^2}(+)Z_{p_1}] = K_{(p_1-1)(p_1+1)}$. As the graph is complete, the $\omega(\Gamma[Z_{p_1}(+)Z_{p_1}]) = (p_1-1)(p_1+1)$. □

Theorem 11 *The clique number of* $\Gamma[Z_{p_1^3}(+)Z_{p_1}]$ *is* $\omega(\Gamma[Z_{p_1^3}(+)Z_{p_1}]) = p_1^2$.

Proof: *Consider* $\Gamma[Z_{p_1^3}(+)Z_{p_1}]$. *The vertices of* $\Gamma[Z_{p_1^3}(+)Z_{p_1}]$ *are divided into sets, which are given by*

$$A = \{(0,m) :\ m \in Z_{p_1}^*\},$$
$$B = \{(kp_1, m) :\ m \in Z_{p_1} \text{ and } gcd(k, p_1^2) = 1\}$$
and
$$C = \{(lp_1^2, m) :\ m \in Z_{p_1} \text{ and } gcd(k, p_1) = 1\}$$

cardinality $|A| = p_1 - 1$, $|B| = p_1(p_1^2 - p_1)$ *and* $|C| = p_1(p_1-1)$.
A subgraph with vertices of the sets A, C and one vertex of the set B is a complete subgraph with the largest number of vertices. Then the clique number is $\omega(\Gamma[Z_{p_1^3}(+)Z_{p_1}]) = p_1 - 1 + p_1(p_1-1) + 1 = p_1 + p_1(p_1-1) = p_1^2$. □

Using methods and strategies employed before we establish the broader scenario in the forthcoming theorem.

Theorem 12 *If* p_1 *is any prime number, then the clique number is* $\omega(\Gamma[Z_{p_1^r}(+)Z_{p_1}]) = \sum_{i=\lceil \frac{r}{2} \rceil}^{r} p_1\phi(p_1^{r-i}) + 1$, *where* ϕ *is an Euler function.*

Proof: *The vertices of* $\Gamma[Z_{p_1^r}(+)Z_{p_1}]$ *are divided into* r *disjoint sets, namely multiples of* p, *multiples of* p_1^2... *multiples of* p^r, *given by*

$$A_1 = \{(k_1p_1, m)\ :\ m \in Z_{p_1},\ gcd(k_1, p_1^{r-1}) = 1\},$$
$$A_2 = \{(k_2p_1^2, m)\ :\ m \in Z_{p_1},\ gcd(k_2, p_1^{r-2}) = 1\},$$
and for general
$$A_i = \{(k_ip_1^i, m)\ :\ m \in Z_{p_1},\ gcd(k_i, p_1^{r-i}) = 1,\ \text{for } i = 3, ..., r\}$$

cardinality $|A_i| = p_1(p_1^{r-i} - p_1^{r-i-1})$, *for* $i = 1, 2, ...r-1$ *and* $|A_r| = p_1 - 1$.
Now a subgraph with vertices of the sets $A_{\lceil \frac{r}{2} \rceil} \cup A_{\lceil \frac{r}{2} \rceil + 1} \cup A_r \cup \{(p_1^{\lfloor \frac{r}{2} \rfloor}, 0)\}$ *is a complete subgraph with the largest number of vertices. Then the clique number is* $\omega(\Gamma[Z_{p_1^r}(+)Z_{p_1}]) = \sum_{i=\lceil \frac{r}{2} \rceil}^{r} |A_i| + 1 = \sum_{i=\lceil \frac{r}{2} \rceil}^{r} p_1\phi(p_1^{r-i}) + 1$, *where* ϕ *is an Euler function.* □

5. Clique number of $\Gamma[Z_n(+)Z_m]$

We discuss the clique number of the Z.D $\Gamma[Z_n(+)Z_m]$ where $n = p_1^r q_1$ and $m = p_1$. We consider $n = p_1 q_1$ and $m = p_1$.

Theorem 13 *If p_1 and q_1 are distinct primes such that $q_1 > p_1$, then the clique number is $\omega(\Gamma[Z_{p_1q_1}(+)Z_{p_1}]) = p_1$.*

Proof: Consider $\Gamma[Z_{p_1q_1}(+)Z_{p_1}]$. The vertices of $\Gamma[Z_{p_1q_1}(+)Z_{p_1}]$ ared divided into sets

$$A = \{(0, m) : \ m \in Z^*_{p_1}\},$$
$$B_1 = \{(kp_1, m) : \ m \in Z^*_{p_1}, \ gcd(k, q_1) = 1\},$$
$$B^*_1 = \{(kp_1, 0) : \ gcd(k, q_1) = 1\},$$
and
$$C = \{(lq_1, m) : \ m \in Z_{p_1}, \ gcd(l, p_1) = 1\},$$

cardinality $|A| = p_1 - 1$, $|B_1| = (p_1 - 1)(q_1 - 1)$, $|B^*_1| = q_1 - 1$ and $|C| = p_1(p_1 - 1)$.
Now a subgraph with vertices of the sets $A \cup \{(p, 0)\}$ is a complete subgraph with the largest number of vertices. Then the clique number is $\omega(\Gamma[Z_{p_1q_1}(+)Z_{p_1}]) = p_1 - 1 + 1 = p_1$. □

Theorem 14 *If p_1 and q_1 are distinct primes such that $q_1 > p_1$, then clique number is $\omega(\Gamma[Z_{p_1^2q_1}(+)Z_{p_1}]) = p_1^2 - 1$.*

Proof: *Consider $\Gamma[Z_{p_1^2q_1}(+)Z_{p_1}]$. The vertices of $\Gamma[Z_{p_1^2q_1}(+)Z_{p_1}]$ are divided into sets*

$$A = \{(0, m) : \ m \in Z^*_{p_1}\}$$
$$B_1 = \{(kp_1, m) : \ m \in Z_{p_1}, \ gcd(k, p_1q_1) = 1\}$$
$$B_2 = \{(kp_1^2, m) : \ m \in Z_{p_1^*}, \ gcd(k, q_1) = 1\}$$
$$B^*_2 = \{(kp_1^2, 0) : \ gcd(k, q_1) = 1\}$$
$$C = \{(kq_1, m) : \ m \in Z_{p_1}, \ gcd(k, p_1^2) = 1\}$$
$$D = \{(kp_1q_1, m) : \ m \in Z_{p_1}, \ gcd(k, p_1) = 1\},$$

cardinality $|A| = p_1 - 1$, $|B_1| = p_1(p_1 - 1)(q_1 - 1)$, $|B_2| = p_1(q_1 - 1)$, $|C| = p_1(p_1^2 - p_1)$ and $|D| = p_1(p_1 - 1)$. A subgraph with vertices of the sets $A \cup D \cup \{(p_1, o)\}$ is a complete subgraph with the largest number of vertices. Then the clique number is $\omega(\Gamma[Z_{p_1^2q_1}(+)Z_{p_1}]) = |A| + |D| + 1 = p_1 - 1 + p_1(p_1 - 1) + 1 = (p_1 + 1)(p_1 - 1) = p_1^2 - 1$. □

Theorem 15 *If p_1 and q_1 are distinct primes such that $q_1 > p_1$, then clique number is $\omega(\Gamma[Z_{p_1^rq_1}(+)Z_{p_1}]) = \sum_{i=\lceil \frac{r}{2} \rceil}^{r} p_1\phi(p_1^{r-i}) + 1$, where ϕ is an Euler function.*

Proof: *Consider $\Gamma[Z_{p_1^rq_1}(+)Z_{p_1}]$. The vertices of $\Gamma[Z_{p_1^rq_1}(+)Z_{p_1}]$ are divided into sets*

$$A_1 = \{(k_1p_1, m) \ : \ m \in Z_{p_1}, \ gcd(k_1, p_1^{r-1}q_1) = 1\},$$
$$A_2 = \{(k_2p_1^2, m) \ : \ m \in Z_{p_1}, \ gcd(k_2, p_1^{r-2}q_1) = 1\},$$

and for general

$$A_i = \{(k_ip_1^i, m) \ : \ m \in Z_{p_1}, \ gcd(k_i, p_1^{r-i}q_1) = 1, \ for \ i = 3, ..., r\},$$
$$B_1 = \{(k_1p_1q_1, m) \ : \ m \in Z_{p_1}, \ gcd(k_1, p_1^{r-1}) = 1\},$$
$$B_2 = \{(k_2p_1^2q_1, m) \ : \ m \in Z_{p_1}, \ gcd(k_2, p_1^{r-2}) = 1\},$$

and for general

$$B_i = \{(k_ip_1^iq_1, m) \ : \ m \in Z_{p_1}, \ gcd(k_i, p_1^{r-i}) = 1, \ for \ i = 3, ..., r\},$$

cardinality $|A_i| = p_1(p_1^{r-i} - p_1^{r-i-1})(q_1 - 1)$, *for* $i = 1, 2, ...r - 1$ *and* $|A_r| = (p_1 - 1)(q_1 - 1)$. *And cardinality* $|B_i| = p_1(p_1^{r-i} - p_1^{r-i-1})$, *for* $i = 1, 2, ...r - 1$ *and* $|B_r| = p_1 - 1$.
Now a subgraph with vertices of the sets $B_{\lceil \frac{r}{2} \rceil} \cup B_{\lceil \frac{r}{2} \rceil+1} \cup B_r \cup \{(p_1^{\lfloor \frac{r}{2} \rfloor}, 0)\}$ *is a complete subgraph with the largest number of vertices. Then the clique number is* $\omega(\Gamma[Z_{p_1^r q_1}(+)Z_{p_1}]) = \sum_{i=\lceil \frac{r}{2} \rceil}^{r} |B_i| + 1 = \sum_{i=\lceil \frac{r}{2} \rceil}^{r} p_1\phi(p_1^{r-i}) + 1$, *where* ϕ *is an Euler function.* □

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