Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–18.
ISSN-0037-8712
doi:10.5269/bspm.76485

Certain properties and numerical applications of generalized hybrid special polynomials associated with Hermite polynomials

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ABSTRACT: This paper introduces a novel class of generalized Hermite-based Apostol-type Frobenius-Euler polynomials and numbers of order ν and level α. We establish fundamental identities and properties by employing generating function techniques, including summation formulas, differential and integral relations, and addition theorems. Furthermore, we investigate their connections with Stirling numbers of the second kind and various other polynomial families. Additionally, we derive a corresponding differential equation and a recurrence relation for these newly defined polynomials. To visualize their behaviour, we utilize Maple to compute numerical values and illustrate the distribution of their zeros through surface plots.

Key Words: generalized Hermite-based Apostol type Frobenius-Euler polynomials, the generalized Apostol-Euler polynomials, differential equations, recurrence relations, numerical representation.

Contents

1. Introduction

For $\alpha, u \in \mathbb{C}$, with $u \neq 1$, the Frobenius-Euler polynomials $H_r^{(\alpha)}(\xi; u)$ of order α, are defined by the generating function as follows

$$\left(\frac{1-u}{e^z-u}\right)^{\alpha} e^{\xi z} = \sum_{r=0}^{\infty} H_r^{(\alpha)}(\xi; u)\frac{z^r}{r!}, \quad |z| < |\log(u)|.$$

When $\alpha = 1$, $H_r^{(1)}(\xi; u) := H_r(\xi; u)$ is called the r-th Frobenius-Euler polynomials. In the special case when $\alpha = 1$ and $u = -1$, $H_r(\xi; -1) := E_r(\xi)$ denotes the Euler polynomials. (see, [14]).

The Frobenius–Euler polynomials and numbers have been extensively investigated by numerous researchers and have garnered significant attention in the mathematical community. Several authors [5,7,13,16,17] have proposed novel generalizations of these polynomials, leading to the derivation of fundamental properties, recurrence relations, and differential equations. Moreover, their studies have established intricate connections between Frobenius–Euler polynomials and various other classes of polynomials and numerical sequences, further enriching the field of mathematical analysis.

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Submitted April 21, 2025. Published July 11, 2025
2010 *Mathematics Subject Classification:* 11B68; 11B83; 11B39; 05A19.

This paper introduces and investigates a novel class of generalized Hermite–based Apostol–type Frobenius–Euler polynomials and numbers, characterized by order ν and level α. By leveraging generating function techniques, we derive fundamental identities and key structural properties, including summation formulas, differential and integral relations, and addition theorems. Furthermore, we explore their interconnections with Stirling numbers of the second kind and various well-known polynomial families. Additionally, we establish a corresponding differential equation and a recurrence relation for these polynomials. To further analyze their behaviour, we employ computational tools such as Maple to compute numerical values and visualize the distribution of their zeros through surface plots.

The paper is organized as follows:

Section 2 introduces essential definitions and reviews relevant background results, including Stirling numbers of the second kind, generalised Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials of level α, as well as Fubini, Bernstein, and Hermite polynomials. Section 3 explores the fundamental properties of the generalized Hermite–based Apostol–type Frobenius–Euler polynomials, focusing on their recurrence relations and differential equations. Additionally, it presents explicit formulas that establish connections between these polynomials and other significant families of numbers and polynomials. Section 4 is dedicated to the numerical investigation of the newly introduced polynomials, including the computation of numerical values and an analysis of the distribution of their zeros using surface plots. Section 5 provides concluding remarks, summarizing the key findings and potential directions for future research.

2. Background and previous results

In this paper, we follow standard mathematical notations: $\mathbb{N} = \{1, 2, 3, \dots\}$ represents the set of natural numbers, while $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denotes the set of non-negative integers. The set of all integers is denoted by $\mathbb{Z}$, the set of real numbers by $\mathbb{R}$, and the set of complex numbers by $\mathbb{C}$. For the complex logarithm, we consider the principal branch. Additionally, when dealing with expressions of the form $w = z^{\nu}$, we assume a single-valued interpretation such that $1^{\nu} = 1$, ensuring consistency in our computations.

The Stirling numbers of the second kind, denoted by $S(r, s)$, are defined through the generating function (see [14, p. 78]):

$$\frac{(e^z - 1)^s}{s!} = \sum_{s=0}^{\infty} S(r, s)\frac{z^r}{r!}, \tag{2.1}$$

$$z^r = \sum_{s=0}^{r} S(r, s) z(z-1)\cdots(z-s+1),$$

which satisfies the conditions:

$$S(r, 1) = S(r, r) = 1, \quad S(r, r-1) = \binom{r}{2}.$$

Furthermore, Açíkgöz et al. [1, Eq. 2.12] introduced a generalized form of the (p, q)-Stirling numbers of the second kind, denoted by $S^{[\alpha-1]}_{p,q}(r; \upsilon; \gamma)$, through the generating function:

$$\frac{\left(\gamma e^z_{p,q} - \sum_{h=0}^{\alpha-1} \frac{z^h}{[h]_{p,q}!}\right)^{\upsilon}}{[\upsilon]_{p,q}!} = \sum_{r=0}^{\infty} S^{[\alpha-1]}_{p,q}(r; \upsilon; \gamma)\frac{z^r}{[r]_{p,q}!}.$$

By taking the limit $q \to p = 1$, this formulation reduces to

$$\frac{\left(\gamma e^z - \sum_{h=0}^{\alpha-1} \frac{z^h}{h!}\right)^{\upsilon}}{\upsilon!} = \sum_{r=0}^{\infty} S^{[\alpha-1]}(r; \upsilon; \gamma)\frac{z^r}{r!}.$$

Notably, when setting $\alpha=\gamma=1$, the above expression simplifies to equation (2.1).

The Bernstein polynomials $B_{s,r}(\xi)$, of degree r, are defined by employing the following generating function (see, [2]):

$$\frac{(z\xi)^s}{s!}e^{(1-\xi)z}=\sum_{r=s}^{\infty}B_{s,r}(\xi)\frac{z^r}{r!},\quad s\in\mathbb{N}_0,$$

where

$$B_{s,r}(\xi)=\binom{r}{s}\xi^s\left(1-\xi\right)^{r-s}.$$

For mathematical convention, we usually set $B_{s,r}(\xi)=0$ if $s>r$.

In recent years, notable progress has been made in developing various generalizations of special functions within mathematical physics. These advancements provide a robust analytical framework for solving various mathematical physics problems and have extensive practical applications across diverse domains. Notably, the significance of generalized Hermite polynomials has been underscored, as noted in previous studies [9,10]. These polynomials find utility in addressing challenges in quantum mechanics, optical beam transport, and a spectrum of problems spanning partial differential equations to abstract group theory.

The "2-variable Hermite Kampé de Feriet polynomials (2VHKdFP)", denoted as $\mathcal{Q}_n(\xi,\eta)$ [3], are expressed through the following generating function:

$$e^{\xi z+\eta z^2}=\sum_{n=0}^{\infty}\mathcal{Q}_n(\xi,\eta)\frac{z^n}{n!},$$

which for $\eta=0$, gives

$$e^{\xi z}=\sum_{n=0}^{\infty}\mathcal{Q}_n(\xi)\frac{z^n}{n!}.$$

The generating function of the ordinary Hermite polynomials is defined by (see, [11]):

$$e^{2\xi z-z^2}=\sum_{r=0}^{\infty}H_r(\xi)\frac{z^r}{r!},$$

so that

$$H_r(\xi)=r!\sum_{s=0}^{[\frac{r}{2}]}\frac{(-1)^s(2\xi)^{r-2s}}{(r-2s)!s!},$$

where $\left[\frac{r}{2}\right]$ is the truncated part of $\frac{r}{2}$.

Let $\gamma,\nu\in\mathbb{C}$, $\alpha\in\mathbb{N}$ and $a,c\in\mathbb{R}^+$ the generalized Apostol–Euler $\mathfrak{E}_r^{[\alpha-1,\nu]}(\xi;c,a;\gamma)$ polynomials of order ν, are defined respectively (cf. [4,6,12]):

$$\sum_{r=0}^{\infty}\mathfrak{E}_r^{[\alpha-1,\nu]}(\xi;c,a;\gamma)\frac{z^r}{r!}=\left(\frac{2^\alpha}{\gamma c^z+\sum_{k=0}^{\alpha-1}\frac{(z\ln a)^k}{k!}}\right)^{\nu}c^{\xi z}.$$

When $c=a=e$, we arrive at the following:

$$\mathfrak{E}_r^{[\alpha-1,\nu]}(\xi;e,e;\gamma):=\mathfrak{E}_r^{[\alpha-1,\nu]}(\xi;\gamma).$$

Recently, Quintana et al. [8] introduced some properties and recurrence formula of the generalized Euler polynomials $E_r^{[\alpha-1]}(\xi)$ of level α. Also, they provided the following expression

$$e^z+\sum_{k=0}^{\alpha-1}\frac{z^k}{k!}=\sum_{n=0}^{\infty}(1+a_{r,\alpha})\frac{z^r}{r!},$$

where

$$a_{r,\alpha}=\begin{cases}1, & \text{if } 0\leq r<\alpha,\\ 0, & \text{if } r\geq\alpha.\end{cases}$$

Motivated by these papers, we define the generalized Hermite-based Apostol type Frobenius-Euler Polynomials of order ν and level α.

3. The generalized Hermite based Apostol type Frobenius-Euler polynomials and their Properties

Definition 3.1 *For a fixed $\alpha\in\mathbb{N}$, $r\in\mathbb{N}_0$, $\nu,\gamma\in\mathbb{C}$, $u\in\mathbb{C}\setminus\{1\}$, the generalized Hermite based Apostol-type Forbenius-Euler polynomials of order ν and level α and variable $\xi,\eta\in\mathbb{R}$, re defined through the following generating function, in a suitable neighbourhood of $z=0$:*

$$\left(\frac{(1-u)^\alpha}{\gamma e^z-u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^\nu e^{\xi z+\eta z^2}=\sum_{r=0}^{\infty}{}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)\frac{z^r}{r!}. \tag{3.1}$$

Setting $\eta=0$, the above expression reduces to generalized Apostol-type Frobenius-Euler polynomials:

$$\left(\frac{(1-u)^\alpha}{\gamma e^z-u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^\nu e^{\xi z}=\sum_{r=0}^{\infty}{}_F\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi;u;\gamma)\frac{z^r}{r!}.$$

Upon setting $\xi=0=\eta$ in (3.1), we have

$${}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(0,0;u;\gamma):={}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(u;\gamma),$$

called the generalized Hermite based Apostol-type Frobenius-Euler numbers of order ν and level α, and for $\nu=0$ the generalized Hermite based Apostol-type Forbenius-Euler polynomials reduces to the generalized Hermite based Apostol-type Forbenius-Euler polynomials reduces to 2-variable Hermite Kampé de Feriet polynomials (2VHKdFP), denoted as $\mathcal{Q}_n(\xi,\eta)$.

Performing some manipulations on the generating function (3.1), we have

$$\left(\frac{(1-u)^\alpha}{\gamma e^z-u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^\nu e^{\xi z+\eta z^2}=\left(\frac{(1-u)^\alpha}{2^\alpha(-u)}\right)^\nu\left(\frac{2^\alpha}{\left(\frac{-\gamma}{u}\right)e^z+\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^\nu e^{\xi z+\eta z^2},$$

and thus,

$$\sum_{r=0}^{\infty}{}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)\frac{z^r}{r!}=\left(\frac{(1-u)^\alpha}{2^\alpha(-u)}\right)^\nu\sum_{r=0}^{\infty}\mathfrak{E}_r^{[\alpha-1,\nu]}\left(\xi,\eta;\frac{-\gamma}{u}\right)\frac{z^r}{r!}.$$

The proposition 3.1, provides some properties of the generalized Hermite based Apostol type Frobenius-Euler polynomials ${}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)$ whiteout proofs since they can easily be proved through Definition 3.1.

Proposition 3.1 *For a fixed $\alpha\in\mathbb{N}$, let $\left\{{}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)\right\}_{r=0}^{\infty}$ be the sequence of the generalized Hermite based Apostol type Frobenius-Euler polynomials, of order ν and level α. Then the following identities hold true:*

1. *Summation formula. For every $r\in\mathbb{N}_0$*

$$ {}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)=\sum_{s=0}^{r}\binom{r}{s}{}_H\mathcal{E}_{r-s}{}^{[\alpha-1,\nu]}(u;\gamma)\ \mathcal{Q}_s(\xi,\eta). $$

2. *Differential relation. For a fixed $\alpha\in\mathbb{N}$, $\nu,\gamma\in\mathbb{C}$ and $r,j\in\mathbb{N}_0$ with $0\leq j\leq r$, we have*

$$ D_\xi\,{}_H\mathcal{E}_{r+1}{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)=(r+1)\,{}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma), $$

$$ D_\xi^{(j)}\,{}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)=\frac{r!}{(r-j)!}{}_H\mathcal{E}_{r-j}{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma). \tag{3.2} $$

3. *Integral formula. For a fixed $\alpha\in\mathbb{N}$, $\nu,\gamma\in\mathbb{C}$, we have*

$$ \int_{\xi_0}^{\xi_1}{}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)d\xi=\frac{{}_H\mathcal{E}_{r+1}{}^{[\alpha-1,\nu]}(\xi_1,\eta;u;\gamma)-{}_H\mathcal{E}_{r+1}{}^{[\alpha-1,\nu]}(\xi_0,\eta;u;\gamma)}{(r+1)}. $$

4. *Addition formula.*

$$ {}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)=\sum_{s=0}^{r}\binom{r}{s}{}_H\mathcal{E}_{r-s}{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)y^s, \tag{3.3} $$

$$ {}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)=\sum_{s=0}^{r}\binom{r}{s}{}_H\mathcal{E}_s{}^{[\alpha-1,\nu]}(\eta;u;\gamma)\,(\xi+y)^{r-s}, \tag{3.4} $$

$$ {}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi,\eta+y;u;\gamma)=\sum_{s=0}^{[\frac{r}{2}]}\binom{r}{2s}{}_H\mathcal{E}_{r-2s}{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)y^{2s}, \tag{3.5} $$

$$ {}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi,\eta+y;u;\gamma)=\sum_{s=0}^{[\frac{r}{2}]}\binom{r}{2s}{}_H\mathcal{E}_s{}^{[\alpha-1,\nu]}(\eta;u;\gamma)\,(\eta+y)^{r-2s}. $$

Setting $y=1$ in (3.3), we have

$$ {}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi+1,\eta;u;\gamma)=\sum_{s=0}^{r}\binom{r}{s}{}_H\mathcal{E}_{r-s}{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma). $$

5. *Addition formula of the argument.*

$$ {}_H\mathcal{E}_r^{[\alpha-1,\nu\pm\beta]}(\xi+y,\eta;u;\gamma)=\sum_{s=0}^{r}\binom{r}{s}{}_H\mathcal{E}_{r-s}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)\,{}_H\mathcal{E}_s^{[\alpha-1,\pm\beta]}(y;u;\gamma), $$

$$ {}_H\mathcal{E}_r^{[\alpha-1,\nu\pm\beta]}(\xi,\eta+y;u;\gamma)=\sum_{s=0}^{[\frac{r}{2}]}\binom{r}{2s}{}_H\mathcal{E}_{r-2s}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)\,{}_H\mathcal{E}_s^{[\alpha-1,\pm\beta]}(y;u;\gamma). $$

6. *We have*

$$ \begin{aligned} \mathfrak{B}_r^{[\alpha-1]}(\xi,\eta;\gamma)&=\frac{r!\gamma^{\alpha-1}}{(\gamma-1)^{\alpha}(r-\alpha)!}\,{}_H\mathcal{E}_{r-\alpha}^{[\alpha-1]}(\xi,\eta;\gamma^{-1};1);\quad r\geq\alpha,\\ \mathfrak{E}_r^{[\alpha-1]}(\xi,\eta;\gamma)&=\frac{2^{\alpha}\gamma^{\alpha-1}}{(\gamma+1)^{\alpha}}\,{}_H\mathcal{E}_r^{[\alpha-1]}(\xi,\eta;-\gamma^{-1};1),\\ \mathfrak{G}_r^{[\alpha-1]}(\xi,\eta;\gamma)&=\frac{2^{\alpha}\gamma^{\alpha-1}r!}{(\gamma+1)^{\alpha}(r-\alpha)!}\,{}_H\mathcal{E}_{r-\alpha}^{[\alpha-1]}(\xi,\eta;-\gamma^{-1};1);\quad r\geq\alpha. \end{aligned} $$

Since this proposition is a straightforward consequence of the Definition (3.1), we shall omit its proof. So, we focus our efforts on the proof of the Addition theorems (Equations (3.3), (3.4) and (3.5)).

Proof: To demonstrate identity (3.3), substituting ξ by $\xi+y$ in (3.1) we have

$$ \left(\frac{(1-u)^{\alpha}}{\gamma e^{z}-u\displaystyle\sum_{l=0}^{\alpha-1}\frac{z^{l}}{l!}}\right)^{\nu}e^{(\xi+y)z+\eta z^{2}}=\sum_{r=0}^{\infty}{}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)\frac{z^{r}}{r!}, \tag{3.6} $$

rewriting the left hand of (3.6)

$$ \sum_{r=0}^{\infty}{}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)\frac{z^{r}}{r!}\sum_{r=0}^{\infty}y^{r}\frac{z^{r}}{r!}=\sum_{r=0}^{\infty}{}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)\frac{z^{r}}{r!}, $$

applying the Cauchy product, we obtain

$$ \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}{}_H\mathcal{E}_{r-s}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)y^{s}\frac{z^{r}}{r!}=\sum_{r=0}^{\infty}{}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)\frac{z^{r}}{r!}. $$

By comparing the coefficients of $\frac{z^{r}}{r!}$ on both sides in above equation, we obtain (3.3). The equation (3.4) is obtain rewriting the equation (3.6) in the form:

$$ \sum_{r=0}^{\infty}{}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\eta;u;\gamma)\frac{z^{r}}{r!}\sum_{r=0}^{\infty}(\xi+y)^{r}\frac{z^{r}}{r!}=\sum_{r=0}^{\infty}{}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)\frac{z^{r}}{r!}. $$

Making the corresponding modifications, we can apply the same reasoning as in the proof of (3.5). □

Theorem 3.1 *The 2-variable Hermite Kampé de Feriet polynomials (2VHKdFP), $\mathcal{Q}_n(\xi,\eta)$ can be espres in term of the generalized Hermite based Apostol-type Frobenius-Euler polynomials.*

$$\mathcal{Q}_r(\xi,\eta)=\frac{1}{(1-u)^{\alpha}}\sum_{s=0}^{r}\binom{r}{s}(\gamma-ua_{s,\alpha})\,{}_H\mathcal{E}_{r-s}{}^{[\alpha-1]}(\xi,\eta;u;\gamma) \tag{3.7}$$

where

$$a_{s,\alpha}=\begin{cases}1, & \text{if } 0\leq s<\alpha,\\ 0, & \text{if } s\geq\alpha.\end{cases}$$

Proof: (3.7). Setting $\nu=1$ in (3.1), we have

$$\begin{aligned}(1-u)^{\alpha}e^{\xi z+\eta z^2}&=\left(\gamma e^z-u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}\right)\left(\sum_{r=0}^{\infty}{}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi,\eta;u;\gamma)\frac{z^r}{r!}\right)\\&=\left(\sum_{r=0}^{\infty}(\gamma-ua_{r,\alpha})\frac{z^r}{r!}\right)\left(\sum_{r=0}^{\infty}{}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi,\eta;u;\gamma)\frac{z^r}{r!}\right),\end{aligned}$$

therefore,

$$(1-u)^{\alpha}\sum_{r=0}^{\infty}\mathcal{Q}_r(\xi,\eta)\frac{z^r}{r!}=\sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}(\gamma-ua_{s,\alpha})\,{}_H\mathcal{E}_{r-s}{}^{[\alpha-1]}(\xi,\eta;u;\gamma)\frac{z^r}{r!}.$$

By comparing the coefficients of $\frac{z^r}{r!}$ on both sides, we obtain the result.

□

Theorem 3.2 *The generalized Hermite-based Apostol type Frobenius-Euler polynomials satisfy the following relations*

$$\gamma\,{}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi+1;u;\gamma)-u\sum_{s=0}^{\min(r,\alpha-1)}\binom{r}{s}{}_H\mathcal{E}_{r-s}{}^{[\alpha-1,\nu]}(\xi;u;\gamma)=(1-u)^{\alpha}\,{}_H\mathcal{E}_r{}^{[\alpha-1,\nu-1]}(\xi;u;\gamma), \tag{3.8}$$

$$\gamma\,{}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi;u;\gamma)-u\sum_{s=0}^{\min(r,\alpha-1)}\binom{r}{s}{}_H\mathcal{E}_{r-s}{}^{[\alpha-1,\nu]}(\xi-1;u;\gamma)=(1-u)^{\alpha}\,{}_H\mathcal{E}_r{}^{[\alpha-1,\nu-1]}(\xi-1;u;\gamma). \tag{3.9}$$

Proof: (3.8). Form (3.1)

$$\begin{aligned}-u\left(\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}\right)\left(\frac{(1-u)^{\alpha}}{\gamma e^z-u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^{\nu}e^{\xi z}+\gamma\left(\frac{(1-u)^{\alpha}}{\gamma e^z-u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^{\nu}e^{(\xi+1)z}&=-u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}\sum_{r=0}^{\infty}{}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi;u;\gamma)\frac{z^r}{r!}\\&+\gamma\sum_{r=0}^{\infty}{}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi;u;\gamma)\frac{z^r}{r!}\end{aligned} \tag{3.10}$$

operating in the l.h.s. of (3.10), we find

$$
\begin{aligned}
&- u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}\left(\frac{(1-u)^{\alpha}}{\gamma e^z - u\displaystyle\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^{\nu} e^{\xi z} + \gamma\left(\frac{(1-u)^{\alpha}}{\gamma e^z - u\displaystyle\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^{\nu} e^{(\xi+1)z} \\
&= \left(\frac{(1-u)^{\alpha}}{\gamma e^z - u\displaystyle\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^{\nu}\left(\gamma e^z - u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}\right) e^{\xi z} \\
&= (1-u)^{\alpha}\left(\frac{(1-u)^{\alpha}}{\gamma e^z - u\displaystyle\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^{\nu-1} e^{\xi z}, \\
&= (1-u)^{\alpha}\sum_{r=0}^{\infty} {}_H\mathcal{E}_r^{[\alpha-1,\nu-1]}(\xi;u;\gamma)\frac{z^r}{r!}.
\end{aligned}
$$

Operating in the r.h.s. of (3.10), we find

$$
\begin{aligned}
&\gamma\sum_{r=0}^{\infty} {}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi+1;u;\gamma)\frac{z^r}{r!} - u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}\sum_{r=0}^{\infty} {}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi;u;\gamma)\frac{z^r}{r!} \\
&= \gamma\sum_{r=0}^{\infty} {}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi+1;u;\gamma)\frac{z^r}{r!} - u\sum_{r=0}^{\infty}\sum_{s=0}^{\min(r,\alpha-1)}\binom{r}{s} {}_H\mathcal{E}_{r-s}^{[\alpha-1,\nu]}(\xi;u;\gamma)\frac{z^r}{r!} \\
&\sum_{r=0}^{\infty}\left(\gamma {}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi+1;u;\gamma) - u\sum_{s=0}^{\min(r,\alpha-1)}\binom{r}{s} {}_H\mathcal{E}_{r-s}^{[\alpha-1,\nu]}(\xi;u;\gamma)\right)\frac{z^r}{r!}.
\end{aligned}
$$

Comparing the coefficients of $\frac{z^r}{r!}$, we obtain (3.8). □

Proof: (3.9). Form (3.1)

$$
\begin{aligned}
\gamma\sum_{r=0}^{\infty} {}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi;u;\gamma)\frac{z^r}{r!} - u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}\sum_{r=0}^{\infty} {}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi-1;u;\gamma)\frac{z^r}{r!} &= \left(\frac{(1-u)^{\alpha}}{\gamma e^z - u\displaystyle\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^{\nu} e^{\xi z} \\
&- u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}\left(\frac{(1-u)^{\alpha}}{\gamma e^z - u\displaystyle\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^{\nu} e^{(\xi-1)z},
\end{aligned} \tag{3.11}
$$

operating in the r.h.s. of (3.11) we find

$$= \gamma \left(\frac{(1-u)^{\alpha}}{\gamma e^{z} - u \sum_{l=0}^{\alpha-1} \frac{z^{l}}{l!}} \right)^{\nu} e^{\xi z} - u \sum_{l=0}^{\alpha-1} \frac{z^{l}}{l!} \left(\frac{(1-u)^{\alpha}}{\gamma e^{z} - u \sum_{l=0}^{\alpha-1} \frac{z^{l}}{l!}} \right)^{\nu} e^{(\xi-1)z}$$

$$= \left(\frac{(1-u)^{\alpha}}{\gamma e^{z} - u \sum_{l=0}^{\alpha-1} \frac{z^{l}}{l!}} \right)^{\nu} e^{\xi z} \left(\gamma - u \sum_{l=0}^{\alpha-1} \frac{z^{l}}{l!} e^{-z} \right)$$

$$= \left(\frac{(1-u)^{\alpha}}{\gamma e^{z} - u \sum_{l=0}^{\alpha-1} \frac{z^{l}}{l!}} \right)^{\nu} e^{\xi z} \left(\gamma e^{z} - u \sum_{l=0}^{\alpha-1} \frac{z^{l}}{l!} \right) e^{-z}$$

$$= (1-u)^{\alpha} \left(\frac{(1-u)^{\alpha}}{\gamma e^{z} - u \sum_{l=0}^{\alpha-1} \frac{z^{l}}{l!}} \right)^{\nu-1} e^{(\xi-1)z}$$

$$= (1-u)^{\alpha} \sum_{r=0}^{\infty} {}_{H}\mathcal{E}_{r}{}^{[\alpha-1,\nu-1]}(\xi-1;u;\gamma) \frac{z^{r}}{r!}.$$

Operating in the l.h.s. of (3.11), we find

$$= \gamma \sum_{r=0}^{\infty} {}_{H}\mathcal{E}_{r}{}^{[\alpha-1,\nu]}(\xi;u;\gamma) \frac{z^{r}}{r!} - u \sum_{l=0}^{\alpha-1} \frac{z^{l}}{l!} \sum_{r=0}^{\infty} {}_{H}\mathcal{E}_{r}{}^{[\alpha-1,\nu]}(\xi-1;u;\gamma) \frac{z^{r}}{r!}$$

$$= \gamma \sum_{r=0}^{\infty} {}_{H}\mathcal{E}_{r}{}^{[\alpha-1,\nu]}(\xi;u;\gamma) \frac{z^{r}}{r!} - u \sum_{r=0}^{\infty} \sum_{s=0}^{\min(r,\alpha-1)} \binom{r}{s} {}_{H}\mathcal{E}_{r-s}{}^{[\alpha-1,\nu]}(\xi-1;u;\gamma) \frac{z^{r}}{r!}$$

$$\sum_{r=0}^{\infty} \left(\gamma {}_{H}\mathcal{E}_{r}{}^{[\alpha-1,\nu]}(\xi;u;\gamma) - u \sum_{s=0}^{\min(r,\alpha-1)} \binom{r}{s} {}_{H}\mathcal{E}_{r-s}{}^{[\alpha-1,\nu]}(\xi-1;u;\gamma) \right) \frac{z^{r}}{r!}.$$

By comparing the coefficients of $\frac{z^{r}}{r!}$, we obtain (3.9). □

Upon setting $\nu = 1$, in (3.8), we give the following corollary.

Corollary 3.1 *The generalized Hermite-based Apostol type Frobenius-Euler polynomials, for $\nu = 0$ satisfy the following relations*

$$\mathcal{Q}_{r}(\xi,\eta) = \frac{1}{(1-u)^{\alpha}} \left[\gamma \sum_{s=0}^{r} \binom{r}{s} {}_{H}\mathcal{E}_{k}{}^{[\alpha-1]}(\xi,\eta;u;\gamma) - u \sum_{s=0}^{\min(r,\alpha-1)} \binom{r}{s} {}_{H}\mathcal{E}_{r-s}{}^{[\alpha-1]}(\xi,\eta;u;\gamma) \right],$$

where $\mathcal{Q}_{r}(\xi,\eta)$ is the 2-variable Hermite Kampé de Feriet polynomials (2VHKdFP).

Theorem 3.3 *The generalized Hermite-based Apostol type Frobenius-Euler polynomials satisfy the following relation, with $u \neq 1$*

$$\frac{\gamma\ {}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi+1,\eta;u;\gamma) - u\ {}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)}{(1-u)} = \sum_{s=0}^{r}\binom{r}{s}{}_H\mathcal{E}_s^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)\ {}_H\mathcal{E}_{r-s}^{[-1]}(u;\gamma).$$

Proof:

$$\begin{aligned}
&\gamma\sum_{r=0}^{\infty}{}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi+1,\eta;u;\gamma)\frac{z^r}{r!} - u\sum_{r=0}^{\infty}{}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)\frac{z^r}{r!}\\
&= \gamma\left(\frac{(1-u)^\alpha}{\gamma e^z - u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^{\nu} e^{(\xi+1)z+\eta z^2} - u\left(\frac{(1-u)^\alpha}{\gamma e^z - u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^{\nu} e^{\xi z+\eta z^2}\\
&= \left(\frac{(1-u)^\alpha}{\gamma e^z - u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^{\nu} e^{\xi z+\eta z^2}\,(\gamma e^z - u)\\
&= \left(\frac{(1-u)^\alpha}{\gamma e^z - u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^{\nu} e^{\xi z+\eta z^2}\,(1-u)\left(\frac{(1-u)}{\gamma e^z - u}\right)^{-1}.
\end{aligned}$$

Hence by using (3.1) in the above equation and applying the Cauchy product

$$\begin{aligned}
&\sum_{r=0}^{\infty}\left(\gamma\ {}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi+1,\eta;u;\gamma) - u\ {}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)\right)\frac{z^r}{r!}\\
&= (1-u)\sum_{r=0}^{\infty}{}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)\frac{z^r}{r!}\sum_{r=0}^{\infty}{}_H\mathcal{E}_r^{[0,-1]}(u;\gamma)\frac{z^r}{r!}\\
&= (1-u)\sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}{}_H\mathcal{E}_s^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)\ {}_H\mathcal{E}_{r-s}^{[-1]}(u;\gamma)\frac{z^r}{r!},
\end{aligned}$$

comparing the coefficients of $\frac{z^r}{r!}$ on both sides, we obtain the proof.

□

Theorem 3.4 *The generalized Hermite-based Apostol type Frobenius-Euler polynomials satisfy the following identity*

$$\gamma_H\mathcal{E}_r^{[\alpha-1]}(\xi+1,\eta;u;\gamma) - u\sum_{s=0}^{\min(r,\alpha-1)}\binom{r}{s}{}_H\mathcal{E}_{r-s}^{[\alpha-1]}(\xi,\eta;u;\gamma) = \sum_{s=0}^{r}\binom{r}{s}(\gamma - ua_{s,\alpha}) \times {}_H\mathcal{E}_{r-s}^{[\alpha-1]}(\xi,\eta;u;\gamma),$$

where

$$a_{s,\alpha} = \begin{cases} 1, & \text{if } 0 \leq s < \alpha, \\ 0, & \text{if } s \geq \alpha. \end{cases} \tag{3.12}$$

Proof: Using Definition 3.1, we attain

$$\begin{aligned}
&\gamma \sum_{r=0}^{\infty} {}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi+1,\eta;u;\gamma)\frac{z^r}{r!} - u\sum_{r=0}^{\alpha-1}\frac{z^r}{r!}\sum_{r=0}^{\infty} {}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi,\eta;u;\gamma)\frac{z^r}{r!} \\
&= \left(\gamma e^z - u\sum_{r=0}^{\alpha-1}\frac{z^r}{r!}\right)\sum_{r=0}^{\infty} {}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi,\eta;u;\gamma)\frac{z^r}{r!} \\
&= \left(\sum_{r=0}^{\infty}\gamma\,\frac{z^r}{r!} - u\sum_{r=0}^{\alpha-1}\frac{z^r}{r!}\right)\sum_{r=0}^{\infty} {}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi,\eta;u;\gamma)\frac{z^r}{r!} \\
&= \left(\sum_{r=0}^{\infty}(\gamma - ua_{r,\alpha})\,\frac{z^r}{r!}\right)\sum_{r=0}^{\infty} {}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi,\eta;u;\gamma)\frac{z^r}{r!} \\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}(\gamma - ua_{s,\alpha})\,{}_H\mathcal{E}_{r-s}{}^{[\alpha-1]}(\xi,\eta;u;\gamma)\frac{z^r}{r!},
\end{aligned} \tag{3.13}$$

where $a_{s,\alpha}$ is defined in (3.12).
In the other hand of (3.13) by the Cauchy product we have

$$\begin{aligned}
&\gamma \sum_{r=0}^{\infty} {}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi+1,\eta;u;\gamma)\frac{z^r}{r!} - u\sum_{r=0}^{\alpha-1}\frac{z^r}{r!}\sum_{r=0}^{\infty} {}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi,\eta;u;\gamma)\frac{z^r}{r!} \\
&= \sum_{r=0}^{\infty}\gamma\,{}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi+1,\eta;u;\gamma)\frac{z^r}{r!} - u\sum_{r=0}^{\infty}\sum_{s=0}^{\min(r,\alpha-1)}\binom{r}{s}{}_H\mathcal{E}_{r-s}{}^{[\alpha-1]}(\xi,\eta;u;\gamma)\frac{z^r}{r!}
\end{aligned} \tag{3.14}$$

$$= \sum_{r=0}^{\infty}\left(\gamma\,{}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi+1,\eta;u;\gamma)\frac{z^r}{r!} - u\sum_{s=0}^{\min(r,\alpha-1)}\binom{r}{s}{}_H\mathcal{E}_{r-s}{}^{[\alpha-1]}(\xi,\eta;u;\gamma)\right)\frac{z^r}{r!}, \tag{3.15}$$

comparing the coefficients of $\frac{z^r}{r!}$ on (3.13) and (3.15) gives the desired result. □

Theorem 3.5 *The following implicit summation formula for the generalized Hermite based Apostol type Frobenius-Euler polynomials holds true*

$${}_H\mathcal{E}_{s+l}{}^{[\alpha-1,\nu]}(w+y,\eta;;u;\gamma) = \sum_{r,n=0}^{s,l}\binom{s}{r}\binom{l}{n}(w-\xi)^{r+n}\,{}_H\mathcal{E}_{s+l-r-n}{}^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma). \tag{3.16}$$

Proof: From (3.1), we have

$$\left(\frac{(1-u)^{\alpha}}{\gamma e^z - u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\right)^{\nu} e^{(\xi+y)z+\eta z^2} = \sum_{r=0}^{\infty} {}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)\frac{z^r}{r!},$$

substituting z by $z+a$ in above equation, we obtain

$$\left(\frac{(1-u)^{\alpha}}{\gamma e^{(z+a)}-u\sum_{l=0}^{\alpha-1}\frac{(z+a)^{l}}{l!}}\right)^{\nu} e^{\xi(z+a)}e^{y(z+a)}e^{\eta(z+a)^{2}}=\sum_{r=0}^{\infty}{}_{H}\mathcal{E}_{r}{}^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)\frac{(z+a)^{r}}{r!}.$$

Now, using the following formula [15, p.52]

$$\sum_{N=0}^{\infty}f(N)\frac{(\xi+y)^{N}}{N!}=\sum_{s,l}f(s+l)\frac{\xi^{s}y^{l}}{s!l!}, \tag{3.17}$$

we get

$$\left(\frac{(1-u)^{\alpha}}{\gamma e^{(z+a)}-u\sum_{l=0}^{\alpha-1}\frac{(z+a)^{l}}{l!}}\right)^{\nu} e^{y(z+a)}e^{\eta(z+a)^{2}}=e^{-\xi(z+a)}\sum_{s,l=0}^{\infty}{}_{H}\mathcal{E}_{s+l}{}^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)\frac{z^{s}a^{l}}{s!l!}.$$

Replacing ξ by w in the above equation and equating the resultant equation to the above equation, we obtain

$$e^{(w-\xi)(z+a)}\sum_{s,l=0}^{\infty}{}_{H}\mathcal{E}_{s+l}{}^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)\frac{z^{s}a^{l}}{s!l!}=\sum_{s,l=0}^{\infty}{}_{H}\mathcal{E}_{s+l}{}^{[\alpha-1,\nu]}(w+y,\eta;;u;\gamma)\frac{z^{s}a^{l}}{s!l!}$$

$$\sum_{N=0}^{\infty}(w-\xi)^{N}\frac{(z+a)^{N}}{N!}\sum_{s,l=0}^{\infty}{}_{H}\mathcal{E}_{s+l}{}^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)\frac{z^{s}a^{l}}{s!l!}=\sum_{s,l=0}^{\infty}{}_{H}\mathcal{E}_{s+l}{}^{[\alpha-1,\nu]}(w+y,\eta;;u;\gamma)\frac{z^{s}a^{l}}{s!l!}. \tag{3.18}$$

Recalling (3.17), the left-hand side of (3.18) becomes

$$\begin{aligned}
&\sum_{N=0}^{\infty}(w-\xi)^{N}\frac{(z+a)^{N}}{N!}\sum_{s,l=0}^{\infty}{}_{H}\mathcal{E}_{s+l}{}^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)\frac{z^{s}a^{l}}{s!l!}\\
&=\sum_{r,n=0}^{\infty}(w-\xi)^{r+n}\frac{z^{r}a^{n}}{r!n!}\sum_{s,l=0}^{\infty}{}_{H}\mathcal{E}_{s+l}{}^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)\frac{z^{s}a^{l}}{s!l!}\\
&=\sum_{s,l=0}^{\infty}\sum_{r,n=0}^{s,l}\frac{(w-\xi)^{r+n}}{r!n!}{}_{H}\mathcal{E}_{s-r+l-n}{}^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)\frac{z^{s}a^{l}}{(s-r)!(l-n)!}\\
&=\sum_{s,l=0}^{\infty}\sum_{r,n=0}^{s,l}\binom{s}{r}\binom{l}{n}(w-\xi)^{r+n}\,{}_{H}\mathcal{E}_{s+l-r-n}{}^{[\alpha-1,\nu]}(\xi+y,\eta;u;\gamma)\frac{z^{s}a^{l}}{s!l!}.
\end{aligned}$$

Comparing coefficients, we get the assertion (3.16). □

Proposition 3.2 *The following identity holds for the generalized Hermite-based Apostol-type Frobenius-Euler polynomials:*

$$(2u-1)\sum_{j=0}^{r}\sum_{s=0}^{\min(j,\alpha-1)}\binom{r}{j}\binom{j}{s}{}_{H}\mathcal{E}_{r-j}{}^{[\alpha-1]}(u;\gamma)\ {}_{H}\mathcal{E}_{j-s}{}^{[\alpha-1]}(\xi,\eta;1-u;\gamma)$$
$$=u^{\alpha}\,{}_{H}\mathcal{E}_{r}{}^{[\alpha-1]}(\xi,\eta;u;\gamma)-(1-u)^{\alpha}\,{}_{H}\mathcal{E}_{r}{}^{[\alpha-1]}(\xi,\eta;1-u;\gamma). \tag{3.19}$$

Proof: We begin by considering the identity

$$\frac{(2u-1)(1-u)^{\alpha}u^{\alpha}e^{\xi z+\eta z^2}\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}{\left(\gamma e^z-u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}\right)\left(\gamma e^z-(1-u)\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}\right)} = \frac{(1-u)^{\alpha}u^{\alpha}e^{\xi z+\eta z^2}}{\gamma e^z-u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}-\frac{(1-u)^{\alpha}u^{\alpha}e^{\xi z+\eta z^2}}{\gamma e^z-(1-u)\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}. \tag{3.20}$$

On the left-hand side of (3.20), we expand the generating functions:

$$\begin{aligned}&\frac{(2u-1)(1-u)^{\alpha}u^{\alpha}e^{\xi z+\eta z^2}\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}{\left(\gamma e^z-u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}\right)\left(\gamma e^z-(1-u)\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}\right)}\\&=(2u-1)\left(\sum_{n=0}^{\infty}{}_H\mathcal{E}_n{}^{[\alpha-1]}(u;\gamma)\frac{z^n}{n!}\right)\left(\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}\right)\left(\sum_{m=0}^{\infty}{}_H\mathcal{E}_m{}^{[\alpha-1]}(\xi,\eta;1-u;\gamma)\frac{z^m}{m!}\right)\\&=(2u-1)\sum_{r=0}^{\infty}\sum_{j=0}^{r}\binom{r}{j}{}_H\mathcal{E}_{r-j}{}^{[\alpha-1]}(u;\gamma)\sum_{s=0}^{\min(j,\alpha-1)}\binom{j}{s}{}_H\mathcal{E}_{j-s}{}^{[\alpha-1]}(\xi,\eta;1-u;\gamma)\frac{z^r}{r!}.\end{aligned}$$

For the right-hand side of (3.20), we expand:

$$\begin{aligned}&\frac{(1-u)^{\alpha}u^{\alpha}e^{\xi z+\eta z^2}}{\gamma e^z-u\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}-\frac{(1-u)^{\alpha}u^{\alpha}e^{\xi z+\eta z^2}}{\gamma e^z-(1-u)\sum_{l=0}^{\alpha-1}\frac{z^l}{l!}}\\&=u^{\alpha}\sum_{r=0}^{\infty}{}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi,\eta;u;\gamma)\frac{z^r}{r!}-(1-u)^{\alpha}\sum_{r=0}^{\infty}{}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi,\eta;1-u;\gamma)\frac{z^r}{r!}\\&=\sum_{r=0}^{\infty}\left(u^{\alpha}{}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi,\eta;u;\gamma)-(1-u)^{\alpha}{}_H\mathcal{E}_r{}^{[\alpha-1]}(\xi,\eta;1-u;\gamma)\right)\frac{z^r}{r!}.\end{aligned}$$

By comparing the coefficients of $\frac{z^r}{r!}$ on both sides, we obtain the desired identity (3.19). □

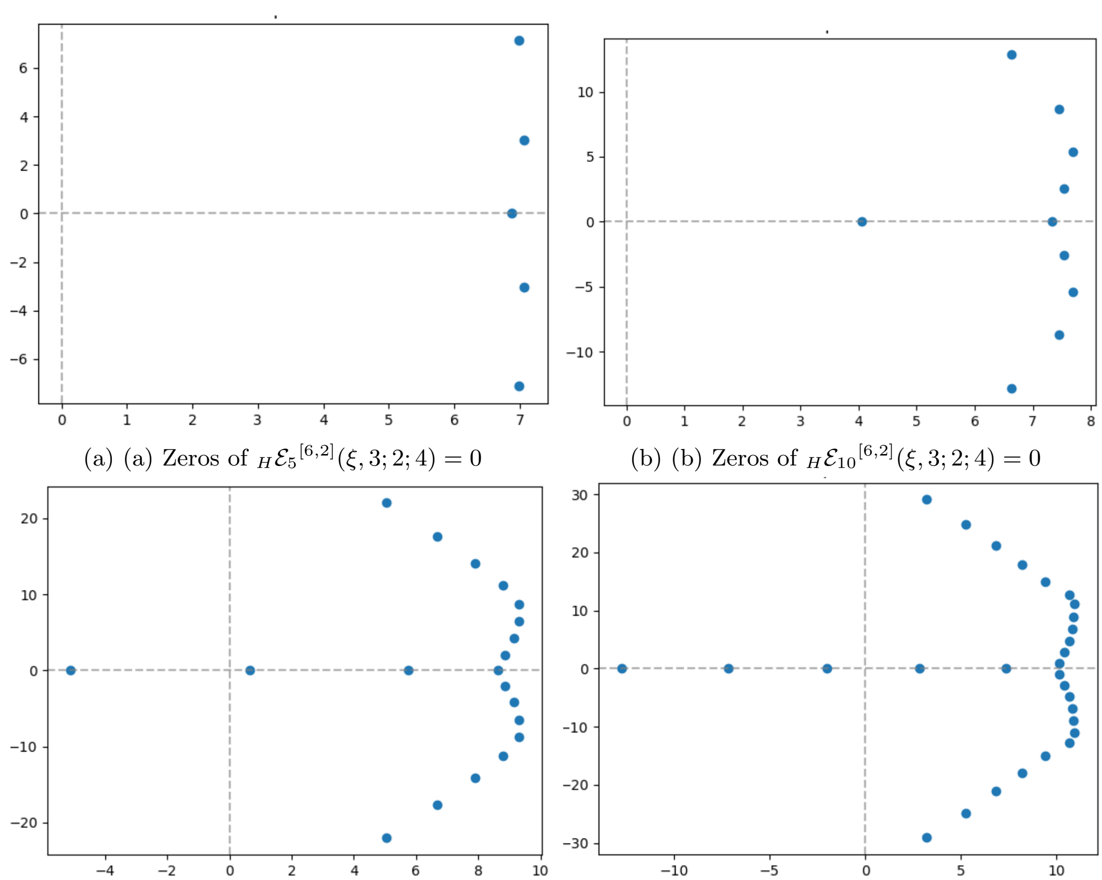
4. Distribution of Zeros and Graphical Representation

We present graphical representations of selected zeros for parametric families of generalized Hermite-based Apostol-type Frobenius-Euler polynomials, denoted by ${}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)$. Additionally, we utilize Wolfram Mathematica to provide illustrative examples that further support the existence and structure of these polynomial families.

For any $n\in\mathbb{N}_0$, and $\alpha=2$, $u=2$, $\nu=2$ and $\gamma=4$, the first few generalized Hermite based Apostol type Frobenius-Euler polynomials of order ν and level α are:

$$
\begin{aligned}
{}_H\mathcal{E}_0{}^{[1,2]}(\xi,\eta;2;4) &= \frac{1}{4}, \\
{}_H\mathcal{E}_1{}^{[1,2]}(\xi,\eta;2;4) &= \frac{1}{4}\xi - \frac{1}{2}, \\
{}_H\mathcal{E}_2{}^{[1,2]}(\xi,\eta;2;4) &= \frac{1}{4}\xi^2 - \xi + \frac{1}{2}\eta + 1, \\
{}_H\mathcal{E}_3{}^{[1,2]}(\xi,\eta;2;4) &= \frac{1}{4}\xi^3 - \frac{3}{2}\xi^2 + (3 + \frac{3}{2}\eta)\xi - 3\eta - \frac{5}{2}, \\
{}_H\mathcal{E}_4{}^{[1,2]}(\xi,\eta;2;4) &= \frac{1}{4}\xi^4 - 2\xi^3 + (6+3\eta)\xi^2 - (10+12\eta)\xi + 3\eta^2 + 12\eta + \frac{19}{2}, \\
{}_H\mathcal{E}_5{}^{[1,2]}(\xi,\eta;2;4) &= \frac{1}{4}\xi^5 - \frac{5\xi^4}{2} + 5\xi^3(2+\eta) - 5\xi^2(5+6\eta) + \frac{5}{2}\xi\left(19 + 24\eta + 6\eta^2\right) \\
&\quad -2\left(23 + 25\eta + 15\eta^2\right).
\end{aligned}
$$

Subsequently, we present numerical values of the zeros of these polynomial families, obtained by assigning specific values to the parameters. These computations allow us to illustrate their behavior and distribution. The corresponding graphical representations of ${}_H\mathcal{E}_r{}^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma) = 0$ for $\alpha = 7$, $u = 2$, $\nu = 2$, $\gamma = 4$ and $\eta = 3$ are provided in Figure1.



(a) (a) Zeros of ${}_H\mathcal{E}_5{}^{[6,2]}(\xi,3;2;4) = 0$

(b) (b) Zeros of ${}_H\mathcal{E}_{10}{}^{[6,2]}(\xi,3;2;4) = 0$

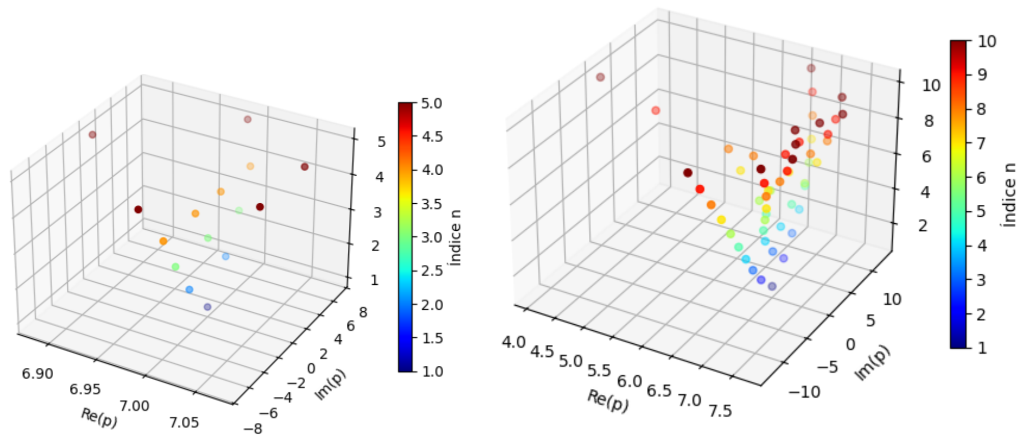
(c) (c) Zeros of ${}_H\mathcal{E}_{20}{}^{[6,2]}(\xi,3;2;4) = 0$

(d) (d) Zeros of ${}_H\mathcal{E}_{30}{}^{[6,2]}(\xi,3;2;4) = 0$

Figure 1: Zeros of ${}_H\mathcal{E}_n{}^{[6,2]}(\xi,3;2;4) = 0$, for $n = 5, 10, 20, 30$

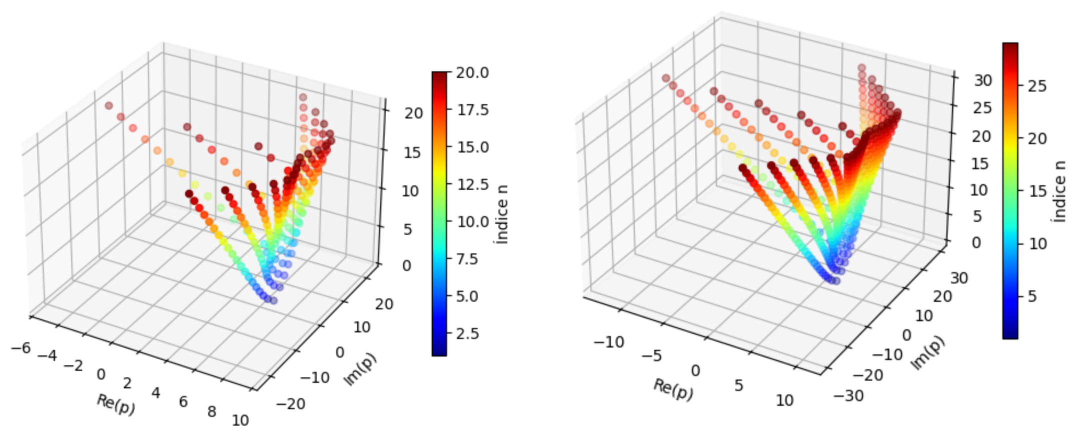
In Figure 1, we set $\alpha = 7$, $u = 2$, $\nu = 2$, $\gamma = 4$ and $\eta = 3$ while varying the order of the polynomial to examine the behavior of its zeros. Specifically, we considered different values of r: in the top-left panel, we set $r = 5$; in the top-right panel, $r = 10$; in the bottom-left panel, $r = 20$; and in the bottom-right panel, $r = 30$. This allows us to observe how the distribution of zeros evolves as the degree of the polynomial increases.

Figure 2 illustrates the evolution of the zeros' behavior as the polynomial order increases, specifically for ${}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi, \eta; u; \gamma) = 0$ with $0 \leq r \leq 30$, considering different values of the parameters α, u, ν, γ and η.



(a) Data visualization of zeros of ${}_H\mathcal{E}_5^{[6,2]}(\xi, 3; 2; 4) = 0$

(b) Data visualization of zeros of ${}_H\mathcal{E}_{10}^{[6,2]}(\xi, 3; 2; 4) = 0$



(c) Data visualization of zeros of ${}_H\mathcal{E}_{20}^{[6,2]}(\xi, 3; 2; 4) = 0$

(d) Data visualization of zeros of ${}_H\mathcal{E}_{30}^{[6,2]}(\xi, 3; 2; 4) = 0$

Figure 2: Data visualization of zeros of ${}_H\mathcal{E}_5^{[6,2]}(\xi, 3; 2; 4) = 0$, for $n \in [0, 30]$

In Figure 2, we vary r from 0 to 50 while adjusting the parameters α, u, ν, γ and η to analyze the behavior of zeros in 3D. This setup allows us to examine how the distribution of zeros evolves as the degree of the polynomial increases.

We then calculated an approximate solution of the generalized Hermite-based Apostol-type Frobenius-Euler polynomials ${}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi, \eta; u; \gamma)$. The results are given in Table 1

Order r	ξ
1	7
2	$7-2.44949i$, $7+2.44949i$
3	$7-4.24264i$, $7+4.24264i$, 7
4	$7-5.71812i$, $7+5.71812i$, $7-1.81743i$, $7+1.81743i$
5	$6.98327-3.32067i$, $6.98327+3.32067i$, $7.00377-6.99813i$, $7.00377+6.99813i$, 7.02592
6	$6.95087-4.6191i$, $6.95087+4.6191i$, $7.01459-8.14435i$, $7.01459+8.14435i$, $7.03454-1.52885i$, $7.03454+1.52885i$
7	$6.90851-5.76505i$, $6.90851+5.76505i$, $7.01432-2.8715i$, $7.01432+2.8715i$, $7.03457-9.19414i$, $7.03457+9.19414i$, 7.08519
8	$6.86416-6.78805i$, $6.86416+6.78805i$, $6.9633-4.07578i$, $6.9633+4.07578i$, $7.06443-10.1728i$, $7.06443+10.1728i$, $7.10811-1.36378i$, $7.10811+1.36378i$
9	$6.82838-7.70748i$, $6.82838+7.70748i$, $6.8795-5.16731i$, $6.8795+5.16731i$, $7.09977-2.60677i$, $7.09977+2.60677i$, $7.10338-11.0986i$, $7.10338+11.0986i$, 7.17793
10	$6.75836-6.15694i$, $6.75836+6.15694i$ $6.81485-8.54i$, $6.81485+8.54i$, $7.06097-3.75788i$, $7.06097+3.75788i$, $7.14941-11.9847i$, $7.14941+11.9847i$, $7.21641-1.25367i$, $7.21641+1.25367i$

Table 1: Values of ξ for different generalized Hermite-based Apostol-type Frobenius-Euler polynomials ${}_H\mathcal{E}_r^{[\alpha-1,\nu]}(\xi,\eta;u;\gamma)$ order r.

5. Conclusion

This study presents an innovative class of generalized Hermite-based Apostol-type Frobenius-Euler polynomials. It comprehensively examines their structural and analytical properties through the lens of generating function techniques. The paper successfully formulates an associated differential equation that governs this new family of polynomials by leveraging the factorisation approach. In addition, a recurrence relation is meticulously derived, offering a recursive framework for efficient computation and further theoretical exploration.

One of the key contributions of this work lies in establishing interconnection formulas, which reveal intricate correlations between the proposed family and well-known classical polynomials, including Bernstein, Fubini, and Hermite polynomials. These relationships not only enhance the theoretical richness of the study but also lay the groundwork for broadening the applicability of such polynomials in various branches of mathematical analysis.

Looking ahead, the results obtained in this paper open up promising avenues for future investigations. Potential directions include exploring orthogonality properties, zero distributions and constructing operational rules associated with these polynomials. Furthermore, the generalized Hermite-based Apostol-type Frobenius-Euler polynomials may find meaningful applications in approximation theory, analytic number theory, combinatorics, quantum mechanics, and solutions to partial differential equations. The analytical tools and relationships established herein are a robust foundation for extending this framework to multivariable or q-analogue settings, thereby enriching the ongoing development in the theory of special functions and mathematical physics.

6. Data availability.

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Funding

The research of Juan Hernández has been partially supported by the Fondo Nacional de Innovación y Desarrollo Científico y Tecnológico (FONDOCYT), Dominican Republic, under grant 2023-1-1D1-0490.

7. Conflict of interest.

The authors have no relevant financial or non-financial interests to disclose.

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