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SR-fuzzy set theory applied to Sheffer Stroke BG-Algebras

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ABSTRACT: The paper introduces and elucidates the concept of an SR-fuzzy SBG-subalgebra and a level set of an SR-fuzzy set within the framework of Sheffer stroke BG-algebras. These concepts play a pivotal role in comprehending the nuances of SR-logic within this algebraic setting. By establishing a correlation between subalgebras and level sets, the study unveils a fundamental relationship essential for understanding the algebraic structure. Specifically, it demonstrates that the level set of SR-fuzzy SBG-subalgebras corresponds to the subalgebra of the algebra, and vice versa, indicating a profound interconnection between these two concepts within this algebraic structure.

Key Words: Sheffer stroke (BG-algebra), BG-ideal, fuzzy SBG-subalgebra, fuzzy SBG-ideal.

Contents

1. Introduction

The notion of a B-algebra was introduced by J. Neggers and H.S. Kim [13], and they discussed some connections between B-algebras and groups using digraphs. P.J. Allen et al. [2] gave another proof of the close relationship of B-algebras with groups using the observation that the zero adjoint mapping is surjective. Jung R. Cho and H.S. Kim [9] discussed further relations between B-algebras and other topics, especially quasigroups. Then C.B. Kim and H.S. Kim [10] introduced a BG-algebra as a generalization of B-algebra. A. Walendziak [23] proved that BG/BF1/B/BM-algebras are congruence per-mutable. S.S. Ahn and H.D. Lee [1] fuzzified the concept of BG-algebra and investigated some of their properties. T. Senapati and G. Chen [19] introduced and investigated cubic closed ideals of BG-algebras, and in the article [24], the subject of study was the direct product of BG-algebras. Furthermore, since fuzzy concept ([25]) and their generalizations ([4,5,6,18]) are important, many researchers extended fuzzy BG-algebras e.g., interval-valued fuzzy BG-algebras ([7]), multi-fuzzy subalgebras of BG-algebras ([11]), fuzzy BG-ideals in BG-algebras ([12]), normalization of fuzzy BG-algebras ([17]), fuzzy dot structure of BG-algebras ([20]), interval-valued intuitionistic fuzzy closed ideals of BG-algebras ([21]). The theory of Sheffer stroke operation has been introduced by H.M. Sheffer [22]. This operation holds significance because it can be used on its own, without any other logical operators, to construct a logical system. This means that any axiom of a logical system can be restated using only the Sheffer operation. Because of this property, it becomes easier to control certain properties of the newly constructed logical system. Additionally, it's worth noting that the axioms of Boolean algebra, which are the algebraic counterpart of classical propositional calculus, can be expressed solely using the Sheffer operation. This highlights the fundamental nature and versatility of the Sheffer operation in logical and algebraic systems. It has combined with the theory of algebraic structures [8,14,16]. T. Oner et al. [14] discussed Sheffer stroke BG-algebras, and defined an (implicative) ideal of a Sheffer stroke BG-algebra. It is proved that every implicative ideal of a Sheffer stroke BG-algebra is its ideal. T. Oner and T. Kalkan [16] included several new Sheffer stroke BG-algebras by filters and homomorphisms. In the article [15], was considered using fuzzy techniques were considered for the Sheffer stroke BG-algebras. It is shown that every medial Sheffer stroke BG-algebra is an implicative Sheffer stroke BG-algebra, and the relationships between a fuzzy

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(completely) closed ideal and a fuzzy p-ideal are shown. The paper introduces and elucidates the concept of an SR-fuzzy SBG-subalgebra and a level set of an SR-fuzzy set within the framework of Sheffer stroke BG-algebras. These concepts play a pivotal role in comprehending the nuances of SR-logic within this algebraic setting. By establishing a correlation between subalgebras and level sets, the study unveils a fundamental relationship essential for understanding the algebraic structure. Specifically, it demonstrates that the level set of SR-fuzzy SBG-subalgebras corresponds to the subalgebra of the algebra, and vice versa, indicating a profound interconnection between these two concepts within this algebraic structure. This connection suggests a well-defined structure and order among these subalgebras, facilitating systematic analysis and investigation. The paper further explores the notion of an SR-fuzzy SBG-ideal of a Sheffer stroke BG-algebra, outlining its properties and characteristics. Notably, it reveals that every SR-fuzzy SBG-ideal of a Sheffer stroke BG-algebra also qualifies as its SR-fuzzy SBG-subalgebra, although the reverse is not universally true. This observation sheds light on the specific traits and behavior of SR-fuzzy SBG-ideals within the prescribed algebraic context, offering deeper insights into the algebraic framework under consideration.

2. Preliminaries

Definition 2.1 ([22]) *Let $\langle L, |\rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions: for all $x, y, z \in L$*

$$(S1)\ x|y = y|x$$
$$(S2)\ (x|x)|(x|y) = x$$
$$(S3)\ x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$$
$$(S4)\ (x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x.$$

Definition 2.2 ([10]) A BG-*algebra* is an algebra $\langle L, *, 0\rangle$ of type $(2, 0)$ which satisfies the following axioms: for all $x, y, z \in L$

$$(BG1)\ x * x = 0,$$
$$(BG2)\ x * 0 = x,$$
$$(BG3)\ ((x * y) * (0 * y)) = x.$$

Definition 2.3 ([1]) Let μ be a fuzzy set in a BG-algebra. Then μ is called a *fuzzy subalgebra* of X if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in L$.

Definition 2.4 ([14]) A *Sheffer stroke BG-algebra* (briefly, SBG-algebra) is a structure $\langle L, |\rangle$ of type (2) such that 0 is the fixed element in L and the following conditions are satisfied for all $x, y, z \in L$

$$\text{(SBG1)}\ (x|(x|x))|(x|(x|x)) = 0$$
$$\text{(SBG2)}\ (0|(y|y))|(x|(y|y))|(x|(y|y)) = x|x.$$

Proposition 2.5 ([14]) Let $\langle L, |\rangle$ be an SBG-algebra. Then the binary relation $x \leq y$ if and only if $(y|(x|x))|(y|(x|x)) = 0$ is a partial order on L.

Definition 2.6 ([14]) A nonempty subset G of a Sheffer stroke BG-algebra L is called an SBG-subalgebra of L if $(x|(y|y))|(x|(y|y)) \in G$ for all $x, y \in G$.

Definition 2.7 ([14]) A nonempty subset G of a Sheffer stroke BG-algebra L is called an SBG-ideal of L if for all $x, y \in G$

1. $0 \in G$,
2. $(x|(y|y))|(x|(y|y)) \in G$ and $y \in G \Rightarrow x \in G$.

Definition 2.8 ([15]) A fuzzy subset μ of a Sheffer stroke BG-algebra L is called a fuzzy ideal of L if it satisfies the following conditions: for all $x, y \in L$

1. $\mu(0) \geq \mu(x)$,
2. $\mu(x) \geq \min\{\mu(y), \mu((x|(y|y))|(x|(y|y)))\}$.

Definition 2.9 ([3]) Let L be a universal set. Then the pair $\mathcal{S} = (\alpha, \beta)$ of two fuzzy sets α and β in L is called an SR-fuzzy set in L if it satisfies $0 \leq (\alpha(a))^2 + \sqrt{\beta(a)} \leq 1$ for all $a \in L$.

Definition 2.10 ([3]) Let $\mathcal{S} = (\alpha, \beta)$ be an SR-fuzzy set in L and $(t, s) \in [0,1]^2$ be such that $0 \leq t^2 + \sqrt{s} \leq 1$. Consider the following sets:

$$S(\alpha, t) = \{x \in L | \alpha(x) \geq t\} \text{ and } R(\beta, s) = \{x \in L | \beta(x) \leq s\}.$$

3. SR-fuzzy sets in Sheffer stroke BG-algebras

In this section, the study introduces the concepts of SR-fuzzy SBG-subalgebras and SR-fuzzy SBG-ideals within the context of Sheffer stroke BG-algebras. It's worth noting that unless explicitly stated otherwise, L refers to a Sheffer stroke BG-algebra.

Definition 3.1 *An SR-fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ in L is called an SR-fuzzy SBG-subalgebra of $\mathcal{L} = (L, |)$ if for all $x, y \in L$*

$$\begin{pmatrix} (\alpha((x|(y|y))|(x|(y|y))))^2 \geq \min\{(\alpha(x))^2, (\alpha(y))^2\} \\ \sqrt{\beta((x|(y|y))|(x|(y|y)))} \leq \max\{\sqrt{\beta(x)}, \sqrt{\beta(y)}\} \end{pmatrix}. \tag{3.1}$$

Definition 3.2 *An SR-fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ of L is called an SR-fuzzy SBG-ideal of L if for all $x, y \in L$*

$$\begin{pmatrix} (\alpha(0))^2 \geq (\alpha(x))^2 \geq \min\{(\alpha((x|(y|y))|(x|(y|y))))^2, (\alpha(y))^2\} \\ \sqrt{\beta(0)} \leq \sqrt{\beta(x)} \leq \max\{\sqrt{\beta((x|(y|y))|(x|(y|y)))}, \sqrt{\beta(y)}\} \end{pmatrix}. \tag{3.2}$$

Theorem 3.3 *Every SR-fuzzy SBG-ideal of L is an SR-fuzzy SBG-subalgebra of L.*

Theorem 3.4 *An SR-fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ in L is an SR-fuzzy SBG-subalgebra of L if and only if the nonempty sets $S(\alpha, t)$ and $R(\beta, s)$ are SBG-subalgebras of L for all $(t, s) \in [0,1]^2$ be such that $0 \leq t^2 + \sqrt{s} \leq 1$.*

Proof: Assume that $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy SBG-subalgebra of L. For every $(t, s) \in [0,1]^2$ be such that $0 \leq t^2 + \sqrt{s} \leq 1$, let $x, y \in S(\alpha, t) \cap R(\beta, s)$. Then $(\alpha(x))^2 \geq t^2$, $(\beta(y))^2 \geq t^2$, $\sqrt{\beta(x)} \leq \sqrt{s}$, and $\sqrt{\beta(y)} \leq \sqrt{s}$. If $(x|(y|y))|(x|(y|y))) \notin S(\alpha, t)$ or $(x|(y|y))|(x|(y|y))) \notin R(\beta, s)$, then $\alpha((x|(y|y))|(x|(y|y)))) < t$ or $\beta((x|(y|y))|(x|(y|y)))) > s$. It follows that

$$(\alpha((x|(y|y))|(x|(y|y))))^2 < t^2 \leq \min\{(\alpha(x))^2, (\alpha(y))^2\},$$

or

$$\sqrt{\beta((x|(y|y))|(x|(y|y)))} > \sqrt{s} \geq \max\{\sqrt{\beta(x)}, \sqrt{\beta(y)}\},$$

a contradiction, and so $(x|(y|y))|(x|(y|y))) \in S(\alpha, t)$ and $(x|(y|y))|(x|(y|y))) \in R(\beta, s)$.

Conversely, let the nonempty sets $S(\alpha, t)$ and $R(\beta, s)$ are SBG-subalgebras of L for all $(t, s) \in [0,1]^2$ be such that $0 \leq t^2 + \sqrt{s} \leq 1$. Suppose that

$$(\alpha((a|(b|b))|(a|(b|b))))^2 < \min\{(\alpha(a))^2, (\alpha(b))^2\}$$

or

$$\sqrt{\beta((a|(b|b))|(a|(b|b))))} > \max\{\sqrt{\beta(a)}, \sqrt{\beta(b)}\}$$

for some $a, b \in L$.

If we take $t = \min\{(\alpha(a))^2, (\alpha(b))^2\}$ and $s = \max\{\sqrt{\beta(a)}, \sqrt{\beta(b)}\}$, then $a \in S(\alpha, t) \cap R(\beta, s)$ and $b \in S(\alpha, t) \cap R(\beta, s)$. However, $(a|(b|b))|(a|(b|b))) \notin S(\alpha, t) \cap R(\beta, s)$, which is a contradiction. Hence, $(\alpha((x|(y|y))|(x|(y|y))))^2 \geq \min\{(\alpha(x))^2, (\alpha(y))^2\}$ and $\sqrt{\beta((x|(y|y))|(x|(y|y)))} \leq \max\{\sqrt{\beta(x)}, \sqrt{\beta(y)}\}$ for all $x, y \in L$. Therefore, $\mathcal{L} = (L, \alpha, \beta)$ is a SR-fuzzy SBG-subalgebra of L. $\square$

Theorem 3.5 *For a subset D of L, let $\mathcal{L} = (L, \alpha, \beta)$ be an SR-fuzzy set in L, which is defined as follows:*

$$\alpha(x) = \begin{cases} t & if\ x \in D, \\ 0 & otherwise, \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} s & if\ x \in D, \\ 1 & otherwise, \end{cases}$$

where $(t, s) \in (0, 1] \times [0, 1)$ with $0 \leq t^2 + \sqrt{s} \leq 1$. Then $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy SBG-subalgebra of L if and only if D is an SBG-subalgebra of L.

Proof: Assume that $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy SBG-subalgebra of L. Then $S(\alpha, t)$ and $R(\beta, s)$ are nonempty SBG-subalgebra of L. Let $x, y \in L$ be such that $x \in D$ and $y \in D$. Then $\alpha(x) = t = \alpha(y)$ and $\beta(x) = s = \beta(y)$, and so

$$(\alpha((x|(y|y))|(x|(y|y))))^2 \geq \min\{(\alpha(x))^2, (\alpha(y))^2\} = t^2$$

and

$$\sqrt{\beta((x|(y|y))|(x|(y|y)))} \leq \max\{\sqrt{\beta(x)}, \sqrt{\beta(y)}\} = \sqrt{s}.$$

It follows that $(\alpha((x|(y|y))|(x|(y|y))))^2 = t$ and $\sqrt{\beta((x|(y|y))|(x|(y|y)))} = s$. This shows that $(x|(y|y))|(x|(y|y))) \in D$. Therefore, D is an SBG-subalgebra of L.

Conversely, let D be an SBG-subalgebra of L. Let $x, y \in L$. If $x \in D$ and $y \in D$, then $(x|(y|y))|(x|(y|y))) \in D$. Hence $(\alpha((x|(y|y))|(x|(y|y))))^2 = t^2 = \min\{(\alpha(x))^2, (\alpha(y))^2\}$ and $\sqrt{\beta((x|(y|y))|(x|(y|y)))} = \sqrt{s} = \max\{\sqrt{\beta(x)}, \sqrt{\beta(y)}\}$. If $x \notin D$ or $y \notin D$, then $(\alpha(x))^2 = 0$ and $\sqrt{\beta(x)} = 1$, or $(\alpha(x))^2 = 0$ and $\sqrt{\beta(y)} = 1$. Thus $(\alpha((x|(y|y))|(x|(y|y))))^2 \geq 0 = \min\{(\alpha(x))^2, (\alpha(y))^2\}$ and $\sqrt{\beta((x|(y|y))|(x|(y|y)))} \leq 1 = \max\{\sqrt{\beta(x)}, \sqrt{\beta(y)}\}$. Therefore, $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy SBG-subalgebra of L. □

Definition 3.6 *[16] Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SBG-algebras. Then a mapping $f : A \to B$ is called a homomorphism if $f(x|_A y) = f(x)|_B f(y)$ for all $x, y \in X$ and $f(0_A) = 0_B$.*

Theorem 3.7 *Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SBG-algebras, $f : A \to B$ be a surjective homomorphism and $\mathcal{B} = (B, \alpha, \beta)$ be an SR-fuzzy set on B. Then $\mathcal{B} = (B, \alpha, \beta)$ is an SR-fuzzy SBG-ideal of B if and only if $\mathcal{B}^f = (A, \alpha^f, \beta^f)$ is an SR-fuzzy SBG-ideal of A.*

Proof: Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SBG-algebras, $f : A \to B$ be a surjective homomorphism and $\mathcal{B} = (B, \alpha, \beta)$ be an SR-fuzzy SBG-ideal of B. Let $x_1, x_2, x_3 \in L$. Then

$$(\alpha^f(0_A))^2 = (\alpha(f(0_A)))^2 \geq (\alpha(f(x_1)))^2 = (\alpha^f(x_1))^2,$$

$(\alpha^f(x_1))^2$

$$\begin{aligned}
&= (\alpha(f(x_1)))^2 \\
&\geq \min\{(\alpha(f(x_2)))^2, (\alpha((f(x_1)|_B(f(x_2)|_B f(x_2)))|_B(f(x_1)|_B(f(x_2)|_B f(x_2)))))^2\} \\
&= \min\{(\alpha(f(x_2)))^2, (\alpha(f((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2)))))^2\} \\
&= \min\{(\alpha^f(x_2))^2, (\alpha^f((x_1|_A(x_2|_A x_1))|_A(x_1|_A(x_2|_A x_2))))^2\},
\end{aligned}$$

$$\sqrt{\beta^f(0_A)} = \sqrt{\beta(f(0_A))} \leq \sqrt{\beta(f(x_1))} = \sqrt{\beta^f(x_1)},$$

$\sqrt{\beta^f(x_1)}$

$$\begin{aligned}
&= \sqrt{\beta(f(x_1))} \\
&\leq \max\{\sqrt{\beta(f(x_2))}, \sqrt{\beta((f(x_1)|_B(f(x_2)|_B f(x_2)))|_B(f(x_1)|_B(f(x_2)|_B f(x_2))))}\} \\
&= \max\{\sqrt{\beta(f(x_2))}, \sqrt{\beta(f((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2))))}\} \\
&= \max\{\sqrt{\beta^f(x_2)}, \sqrt{\beta^f((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2)))}\}.
\end{aligned}$$

Hence $\mathcal{B}^f = (A, \alpha^f, \beta^f)$ is an SR-fuzzy SBG-ideal of L.

Conversely, let $\mathcal{B}^f = (A, \alpha^f, \beta^f)$ be an SR-fuzzy SBG-ideal of L. Let $y_1, y_2 \in B$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$ for $x_1, x_2 \in L$. Then

$$(\alpha(0_B))^2 = (\alpha(f(0_A)))^2 = (\alpha^f(0_A))^2 \geq (\alpha^f(x_1))^2 = (\alpha(f(x_1)))^2 = (\alpha(y_1))^2,$$

$(\alpha(y_1))^2$

$$\begin{aligned}
&= (\alpha(f(x_1)))^2 \\
&= (\alpha^f(x_1))^2 \\
&\geq \min\{(\alpha^f(x_2))^2, (\alpha^f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))))^2\} \\
&= \min\{(\alpha(f(x_2)))^2, (\alpha(f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))))^2\} \\
&= \min\{(\alpha(f(x_2)))^2, (\alpha((f(x_1)|_B(f(x_2)|_Bf(x_2)))|_B(f(x_1)|_B(f(x_2)|_Bf(x_2)))))^2\} \\
&= \min\{(\alpha(y_2))^2, (\alpha((y_1|_B(y_2|_By_2))|_B(y_1|_B(y_2|_By_2))))^2\},
\end{aligned}$$

$$\sqrt{\beta(0_B)} = \sqrt{\beta(f(0_A))} = \sqrt{\beta^f(0_A)} \leq \sqrt{\beta^f(x_1)} = \sqrt{\beta(f(x_1))} = \sqrt{\beta(y_1)},$$

$\sqrt{\beta(y_1)}$

$$\begin{aligned}
&= \sqrt{\beta(f(x_1))} \\
&= \sqrt{\beta^f(x_1)} \\
&\leq \max\{\sqrt{\beta^f(x_2)}, \sqrt{\beta^f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))}\} \\
&= \max\{\sqrt{\beta(f(x_2))}, \sqrt{\beta(f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))))}\} \\
&= \max\{\sqrt{\beta(f(x_2))}, \sqrt{\beta((f(x_1)|_B(f(x_2)|_Bf(x_2)))|_B(f(x_1)|_B(f(x_2)|_Bf(x_2))))}\} \\
&= \max\{\sqrt{\beta(y_2)}, \sqrt{\beta((y_1|_B(y_2|_By_2))|_B(y_1|_B(y_2|_By_2)))}\}.
\end{aligned}$$

Hence $\mathcal{B} = (B, \alpha, \beta)$ is an SR-fuzzy SBG-ideal of B. □

Theorem 3.8 *Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SBG-algebras, $f : A \to B$ be a surjective homomorphism and $\mathcal{B} = (B, \alpha, \beta)$ be an SR-fuzzy set on B. Then $\mathcal{B} = (B, \alpha, \beta)$ is an SR-fuzzy SBG-subalgebra of B if and only if $\mathcal{B}^f = (A, \alpha^f, \beta^f)$ is an SR-fuzzy SBG-subalgebra of A.*

Proof: Let $\langle A, |_A, 0_A\rangle$ and $\langle B, |_B, 0_B\rangle$ be SBG-algebras, $f : A \to B$ be a surjective homomorphism and $\mathcal{B} = (B, \alpha, \beta)$ be an SR-fuzzy SBG-subalgebra of B. Let $x_1, x_2 \in A$. Then
$(\alpha^f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))))^2$

$$\begin{aligned}
&= (\alpha(f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))))^2 \\
&= (\alpha(f(x_1)|_B(f(x_2)|_Bf(x_2))|_B(f(x_1)|_B(f(x_2)|_Bf(x_2)))))^2 \\
&\geq \min\{(\alpha(f(x_1)))^2, (\alpha(f(x_2)))^2\} \\
&= \min\{(\alpha^f(x_1))^2, (\alpha^f(x_2))^2\},
\end{aligned}$$

$\sqrt{\beta^f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))}$

$$\begin{aligned}
&= \sqrt{\beta(f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))))} \\
&= \sqrt{\beta(f(x_1)|_B(f(x_2)|_Bf(x_2))|_B(f(x_1)|_B(f(x_2)|_Bf(x_2))))} \\
&\leq \max\{\sqrt{\beta(f(x_1))}, \sqrt{\beta(f(x_2))}\} \\
&= \max\{\sqrt{\beta^f(x_1)}, \sqrt{\beta^f(x_2)}\}.
\end{aligned}$$

Hence $\mathcal{B}^f = (A, \alpha^f, \beta^f)$ is an SR-fuzzy SBG-subalgebra of A.
Conversely, let $\mathcal{B}^f = (A, \alpha^f, \beta^f)$ be an SR-fuzzy SBG-subalgebra of A. Let $y_1, y_2 \in B$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$ for $x_1, x_2 \in A$. Then
$(\alpha((y_1|_B(y_2|_By_2))|_B(y_1|_B(y_2|_By_2))))^2$

$$\begin{aligned}
&= (\alpha((f(x_1)|_B(f(x_2)|_Bf(x_2)))|_B(f(x_1)|_B(f(x_2)|_Bf(x_2)))))^2 \\
&= (\alpha(f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))))^2 \\
&= (\alpha^f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))))^2 \\
&\geq \min\{(\alpha^f(x_1))^2, (\alpha^f(x_2))^2\} \\
&= \min\{(\alpha(f(x_1)))^2, (\alpha(f(x_2)))^2\} \\
&= \min\{(\alpha(y_1))^2, (\alpha(y_2))^2\},
\end{aligned}$$

$$
\begin{aligned}
\sqrt{\beta((y_1|_B(y_2|_By_2))|_B(y_1|_B(y_2|_By_2)))} &= \sqrt{\beta((f(x_1)|_B(f(x_2)|_Bf(x_2)))|_B(f(x_1)|_B(f(x_2)|_Bf(x_2))))} \\
&= \sqrt{\beta(f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))))} \\
&= \sqrt{\beta^f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)))} \\
&\leq \max\{\sqrt{\beta^f(x_1)}, \sqrt{\beta^f(x_2)}\} \\
&= \max\{\sqrt{\beta(f(x_1))}, \sqrt{\beta(f(x_2))}\} \\
&= \max\{\sqrt{\beta(y_1)}, \sqrt{\beta(y_2)}\}.
\end{aligned}
$$

Hence $\mathcal{B} = (B, \alpha, \beta)$ is an SR-fuzzy SBG-subalgebra of B. □

Definition 3.9 *An SR-fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ of L is called an SR-fuzzy implicative SBG-ideal of L if for all $x, y \in L$*

$$
\left(\begin{array}{l}
(\alpha(0))^2 \geq (\alpha(x))^2 \geq \min\{(\alpha(((((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))| \\
\qquad\qquad (((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))))^2, (\alpha(z))^2\} \\
\sqrt{\beta(0)} \leq \sqrt{\beta(x)} \leq \max\left\{\begin{array}{c}\sqrt{\begin{array}{c}\beta(((((x|(y|(x|x)))|(x|(y|(x|x)))) \\ |(z|z))|(((x|(y|(x|x)))| \\ (x|(y|(x|x))))|(z|z)))\end{array}}, \sqrt{\beta(z)}\end{array}\right\}
\end{array}\right). \tag{3.3}
$$

Proposition 3.10 *Every SR-fuzzy implicative SBG-ideal of an SBG-algebra L is an SR-fuzzy SBG-ideal of L.*

Proof: Let $\mathcal{L} = (L, \alpha, \beta)$ be an SR-fuzzy implicative SBG-ideal of L. Then $(\alpha(0))^2 \geq (\alpha(x))^2$ and $\beta(0) \leq \beta(x)$. Also,

$$
\begin{aligned}
(\alpha(x))^2 &\geq \min\{(\alpha(y))^2, (\alpha((((x|(x|(x|x)))|(x|(x|(x|x))))|(y|y))| \\
&\quad (((x|(x|(x|x)))|(x|(x|(x|x))))|(y|y))))^2\} \\
&= \min\{(\alpha(y))^2, (\alpha((((x|(0|0))|(x|(0|0)))|(y|y))| \\
&\quad (((x|(0|0))|(x|(0|0)))|(y|y))))^2\} \\
&= \min\{(\alpha(y))^2, (\alpha((x|(y|y))|(x|(y|y))))^2\},
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{\beta(x)} &\leq \max\left\{\sqrt{\beta(y)}, \sqrt{\begin{array}{c}\beta((((x|(x|(x|x)))|(x|(x|(x|x))))|(y|y))| \\ (((x|(x|(x|x)))|(x|(x|(x|x))))|(y|y)))\end{array}}\right\} \\
&= \max\left\{\sqrt{\beta(y)}, \sqrt{\begin{array}{c}\beta((((x|(0|0))|(x|(0|0)))|(y|y))| \\ (((x|(0|0))|(x|(0|0)))|(y|y)))\end{array}}\right\} \\
&= \max\{\sqrt{\beta(y)}, \sqrt{\beta((x|(y|y))|(x|(y|y)))}\}.
\end{aligned}
$$

Therefore, $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy SPG-ideal of L. □

The following theorem can be proved similarly to Theorem 3.6.

Theorem 3.11 *An SR-fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ in L is an SR-fuzzy SBG-filter of L if and only if the nonempty sets $S(\alpha, t)$ and $R(\beta, s)$ are SBG-filters of L for all $(t, s) \in [0, 1]^2$ be such that $0 \leq t^2 + \sqrt{s} \leq 1$.*

The following theorem can be proved similarly to Theorem 3.7.

Theorem 3.12 *For a subset D of L, let $\mathcal{L} = (L, \alpha, \beta)$ be an SR-fuzzy set in L, which is defined as follows:*

$$
\alpha(x) = \begin{cases} t & \textit{if } x \in D, \\ 0 & \textit{otherwise}, \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} s & \textit{if } x \in D, \\ 1 & \textit{otherwise}, \end{cases}
$$

where $(t, s) \in (0, 1] \times [0, 1)$ with $0 \leq t^2 + \sqrt{s} \leq 1$. Then $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy SBG-filter of X if and only if D is an SBG-filter of L.

Definition 3.13 *An SR-fuzzy subset $\mathcal{L} = (L, \alpha, \beta)$ of a SBG-algebra L is called a SR-fuzzy sub-implicative SBG-ideal of L if for all $x, y, z \in L$*

$$\left(\begin{array}{l} (\alpha(0))^2 \geq (\alpha(x))^2, \\ \sqrt{\beta(0)} \leq \sqrt{\beta(x)}, \\ (\alpha((y|(y|(x|x)))|(y|(y|(x|x)))))^2 \geq \min\{(\alpha((((x|(x|(y|y)))| \\ (x|(x|(y|y))))|(z|z))|(((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))))^2, (\alpha(z))^2\}, \\ \sqrt{\beta((y|(y|(x|x)))|(y|(y|(x|x))))} \leq \\ \max\left\{\sqrt{\begin{array}{c}\beta((((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))| \\ (((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z)))\end{array}}, \sqrt{\beta(z)}\right\}, \end{array}\right). \tag{3.4}$$

Proposition 3.14 *Every SR-fuzzy subimplicative SBG-ideal of L is an SR-fuzzy SBG-ideal of L.*

Proof: Let $\mathcal{L} = (L, \alpha, \beta)$ be an SR-fuzzy sub-implicative SBG-ideal of L. Then $(\alpha(0))^2 \geq (\alpha(x))^2$, $\sqrt{\beta(0)} \leq \sqrt{\beta(x)}$,
$(\alpha(x))^2$

$$\begin{aligned} &= (\alpha((x|(0|0))|(x|(0|0))))^2 \\ &= (\alpha((x|(x|(x|x)))|(x|(x|(x|x)))))^2 \\ &\geq \min\{(\alpha((((x|(x|(x|x)))|(x|(x|(x|x))))|(z|z))|(((x|(x|(x|x)))| \\ &\quad (x|(x|(x|x))))|(z|z))))^2, (\alpha(z))^2\} \\ &= \min\{(\alpha((((x|(0|0))|(x|(0|0)))|(z|z))|(((x|(0|0))|(x|(0|0)))|(z|z))))^2, (\alpha(z))^2\} \\ &= \min\{(\alpha((x|(z|z))|(x|(z|z))))^2, (\alpha(z))^2\}, \end{aligned}$$

$\sqrt{\beta(x)}$

$$\begin{aligned} &= \sqrt{\beta((x|(0|0))|(x|(0|0)))} \\ &= \sqrt{\beta((x|(x|(x|x)))|(x|(x|(x|x))))} \\ &\leq \max\left\{\sqrt{\begin{array}{c}\beta((((x|(x|(x|x)))|(x|(x|(x|x))))|(z|z))| \\ (((x|(x|(x|x)))|(x|(x|(x|x))))|(z|z)))\end{array}}, \beta(z)\right\} \\ &= \max\{\sqrt{\beta((((x|(0|0))|(x|(0|0)))|(z|z))|(((x|(0|0))|(x|(0|0)))|(z|z)))}, \sqrt{\beta(z)}\} \\ &= \max\{\sqrt{\beta((x|(z|z))|(x|(z|z)))}, \sqrt{\beta(z)}\}. \end{aligned}$$

Therefore, $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy SBG-ideal of L. □

Theorem 3.15 *Let $\mathcal{L} = (L, \alpha, \beta)$ be an SR-fuzzy SBG-ideal of L. Then $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy sub-implicative SBG-ideal of L if and only if*

$$(\alpha((y|(y|(x|x)))|(y|(y|(x|x)))))^2 \geq (\alpha((x|(x|(y|y)))|(x|(x|(y|y)))))^2$$

and

$$\sqrt{\beta((y|(y|(x|x)))|(y|(y|(x|x))))} \leq \sqrt{\beta((x|(x|(y|y)))|(x|(x|(y|y))))}.$$

Proof: Let $\mathcal{L} = (L, \alpha, \beta)$ be an SR-fuzzy sub-implicative SBG-ideal of L. We have
$(\alpha(y|(y|(x|x)))|(y|(y|(x|x))))^2$

$$\begin{aligned} &\geq \min\{(\alpha((((x|(x|(y|y)))|(x|(x|(y|y))))|(0|0))|(((x|(x|(y|y)))| \\ &\quad (x|(x|(y|y))))|(0|0))))^2, (\alpha(0))^2\} \\ &= \min\{(\alpha(x|(x|(y|y)))|(x|(x|(y|y))))^2, (\alpha(0))^2\} \\ &= (\alpha(x|(x|(y|y)))|(x|(x|(y|y))))^2, \end{aligned}$$

$\sqrt{\beta(y|(y|(x|x)))|(y|(y|(x|x)))}$

$$\begin{aligned} &\leq \max\left\{\sqrt{\begin{array}{c}\beta((((x|(x|(y|y)))|(x|(x|(y|y))))|(0|0))| \\ (((x|(x|(y|y)))|(x|(x|(y|y))))|(0|0)))\end{array}}, \sqrt{\beta(0)}\right\} \\ &= \max\{\sqrt{\beta(x|(x|(y|y)))|(x|(x|(y|y)))}, \sqrt{\beta(0)}\} \\ &= \sqrt{\beta(x|(x|(y|y)))|(x|(x|(y|y)))}. \end{aligned}$$

Conversely, since $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy SBG-ideal, $\sqrt{\beta(0)} \leq \sqrt{\beta(x)}$, $(\alpha(0))^2 \geq (\alpha(x))^2$, $(\alpha((y|(y|(x|x)))|(y|(y|(x|x)))))^2$

$$\begin{aligned} &\geq (\alpha((x|(x|(y|y)))|(x|(x|(y|y)))))^2 \\ &\geq \min\{(\alpha((((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))|(((x|(x|(y|y)))| \\ &\quad (x|(x|(y|y))))|(z|z))))^2, (\alpha(z))^2\}, \end{aligned}$$

$\sqrt{\beta((y|(y|(x|x)))|(y|(y|(x|x))))}$

$$\begin{aligned} &\leq \sqrt{\beta((x|(x|(y|y)))|(x|(x|(y|y))))} \\ &\leq \max\left\{\sqrt{\begin{array}{c}\beta((((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))| \\ (((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z)))\end{array}}, \sqrt{\beta(z)}\right\}. \end{aligned}$$

Therefore, $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy sub-implicative SBG-ideal of L. □

Definition 3.16 ([14]) *An SBG-algebra is said to be SBG-implicative if it satisfies the condition $x|(x|(y|y)) = y|(y|(x|x))$ for all $x, y \in L$.*

Theorem 3.17 *Let L be an implicative SBG-algebra. Then every fuzzy SBG-ideal of L is an SR-fuzzy sub-implicative SBG-ideal of L.*

Proof: Assume that $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy SBG-ideal of L. Then $(\alpha(0))^2 \geq (\alpha(x))^2$, $\sqrt{\beta(0)} \leq \sqrt{\beta(x)}$,
$(\alpha((y|(y|(x|x)))|(y|(y|(x|x)))))^2$

$$\begin{aligned} &\geq \min\{(\alpha((((y|(y|(x|x)))|(y|(y|(x|x))))|(z|z))|(((y|(y|(x|x)))| \\ &\quad (y|(y|(x|x))))|(z|z))))^2, (\alpha(z))^2\} \\ &= \min\{(\alpha((((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))|(((x|(x|(y|y)))| \\ &\quad (x|(x|(y|y))))|(z|z))))^2, (\alpha(z))^2\}, \end{aligned}$$

$\sqrt{\beta((y|(y|(x|x)))|(y|(y|(x|x))))}$

$$\begin{aligned} &\leq \max\left\{\sqrt{\begin{array}{c}\beta((((y|(y|(x|x)))|(y|(y|(x|x))))|(z|z))| \\ (((y|(y|(x|x)))|(y|(y|(x|x))))|(z|z)))\end{array}}, \sqrt{\beta(z)}\right\} \\ &= \max\left\{\sqrt{\begin{array}{c}\beta((((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))| \\ (((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z)))\end{array}}, \sqrt{\beta(z)}\right\}. \end{aligned}$$

Thereby, $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy sub-implicative SBG-ideal of L. □

Definition 3.18 ([14]) *A SBG-algebra L is called medial if $x|(x|(y|y)) = y|y$, for all $x, y \in L$.*

Lemma 3.19 *In a SBG-algebra L, the following property holds:*

$$((x|(x|(y|y)))|(x|(x|(y|y))))|(y|(x|x)) = x|(x|(y|y))$$

for all $x, y \in L$.

Theorem 3.20 *Every SR-fuzzy SBG-ideal of a medial SBG-algebra L is an SR-fuzzy sub-implicative SBG-ideal of L.*

Proof: Let $\mathcal{L} = (L, \alpha, \beta)$ be an SR-fuzzy SBG-ideal of a medial SBG-algebra L. It is obtained that $(\alpha(0))^2 \geq (\alpha(x))^2$, $\sqrt{\beta(0)} \leq \sqrt{\beta(x)}$,
$(\alpha((y|(y|(x|x)))|(y|(y|(x|x)))))^2$

$$\begin{aligned}
&= (\alpha(x))^2 \\
&\geq \min\{(\alpha((x|(z|z))|(x|(z|z))))^2, (\alpha(z))^2\} \\
&= \min\{(\alpha((((y|(y|(x|x)))|(y|(y|(x|x))))|(z|z))|(((y|(y|(x|x)))| \\
&\quad (y|(y|(x|x))))|(z|z))))^2, (\alpha(z))^2\} \\
&= \min\{(\alpha(((((x|(x|(y|y)))|(x|(x|(y|y))))|(y|(x|x)))|(((x|(x|(y|y)))| \\
&\quad (x|(x|(y|y))))|(y|(x|x)))|(z|z))|((((x|(x|(y|y)))|(x|(x|(y|y))))| \\
&\quad (y|(x|x)))|(((x|(x|(y|y)))|(x|(x|(y|y))))|(y|(x|x)))|(z|z))))^2, (\alpha(z))^2\} \\
&= \min\{(\alpha((((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))|(((x|(x|(y|y)))| \\
&\quad (x|(x|(y|y))))|(z|z))))^2, (\alpha(z))^2\},
\end{aligned}$$

$\sqrt{\beta((y|(y|(x|x)))|(y|(y|(x|x))))}$

$$\begin{aligned}
&= \sqrt{\beta(x)} \\
&\leq \max\{\sqrt{\beta((x|(z|z))|(x|(z|z)))}, \sqrt{\beta(z)}\} \\
&= \max\left\{\sqrt{\begin{array}{c}\beta((((y|(y|(x|x)))|(y|(y|(x|x))))|(z|z))| \\ (((y|(y|(x|x)))|(y|(y|(x|x))))|(z|z)))\end{array}}, \sqrt{\beta(z)}\right\} \\
&= \max\left\{\sqrt{\begin{array}{r}\beta(((((x|(x|(y|y)))|(x|(x|(y|y))))|(y|(x|x)))| \\ (((x|(x|(y|y)))|(x|(x|(y|y))))| \\ (y|(x|x)))|(z|z))|((((x|(x|(y|y)))| \\ (x|(x|(y|y))))|(y|(x|x)))|(((x|(x|(y|y)))| \\ (x|(x|(y|y))))|(y|(x|x)))|(z|z)))\end{array}}, \sqrt{\beta(z)}\right\} \\
&= \max\left\{\sqrt{\begin{array}{c}\beta((((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))| \\ (((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z)))\end{array}}, \sqrt{\beta(z)}\right\}.
\end{aligned}$$

Hence, $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy sub-implicative SBG-ideal of L. $\square$

Theorem 3.21 *Let L be an SBG-algebra satisfying*

$$\left(\begin{array}{l}
(\alpha((y|(z|z))|(y|(z|z))))^2 \geq (\alpha((((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))| \\
\qquad\qquad\qquad\qquad\qquad (((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))))^2 \\
\sqrt{\beta((y|(z|z))|(y|(z|z)))} \leq \sqrt{\begin{array}{c}\beta((((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))| \\ (((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z)))\end{array}}
\end{array}\right). \tag{3.5}$$

for all $x, y, z \in L$. Then every SR-fuzzy SBG-ideal of L is an SR-fuzzy sub-implicative SBG-ideal of L.

Proof: It is obtained that
$(\alpha((y|(y|(x|x)))|(y|(y|(x|x)))))^2$

$$\begin{aligned}
&\geq (\alpha((((x|(x|(y|y)))|(x|(x|(y|y))))|(y|(x|x)))|(((x|(x|(y|y)))| \\
&\quad (x|(x|(y|y))))|(y|(x|x)))))^2 \\
&= (\alpha((x|(x|(y|y)))|(x|(x|(y|y)))))^2,
\end{aligned}$$

$\sqrt{\beta((y|(y|(x|x)))|(y|(y|(x|x))))}$

$$\begin{aligned}
&\leq \sqrt{\begin{array}{c}\beta((((x|(x|(y|y)))|(x|(x|(y|y))))|(y|(x|x)))| \\ (((x|(x|(y|y)))|(x|(x|(y|y))))|(y|(x|x))))\end{array}} \\
&= \sqrt{\beta((x|(x|(y|y)))|(x|(x|(y|y))))}.
\end{aligned}$$

Thus, $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy sub-implicative SBG-ideal of L. $\square$

Theorem 3.22 *Every medial SBG-algebra is an implicative SBG-algebra.*

Theorem 3.23 *Let $\mathcal{L} = (L, \alpha, \beta)$ be an SR-fuzzy SBG-ideal of L. Then $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy implicative SBG-ideal of L if and only if, for all $x, y \in L$*

$$\left(\begin{array}{c} (\alpha(x))^2 \geq (\alpha((x|(y|(x|x)))|(x|(y|(x|x)))))^2 \\ \sqrt{\beta(x)} \leq \sqrt{\beta((x|(y|(x|x)))|(x|(y|(x|x))))} \end{array} \right). \tag{3.6}$$

Proof: Let $\mathcal{L} = (L, \alpha, \beta)$ be an SR-fuzzy implicative SBG-ideal of L. Then

$$\begin{array}{rcl} (\alpha(x))^2 & \geq & \min\{(\alpha(0))^2, (\alpha((((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))|(((x|(y|(x|x)))| \\ & & (x|(y|(x|x))))|(0|0))))^2\} \\ & = & \min\{(\alpha(0))^2, (\alpha((x|(y|(x|x)))|(x|(y|(x|x)))))^2\} \\ & = & (\alpha((x|(y|(x|x)))|(x|(y|(x|x)))))^2, \end{array}$$

$$\begin{array}{rcl} \sqrt{\beta(x)} & \leq & \max\left\{ \sqrt{\beta(0)}, \sqrt{\begin{array}{l} \beta((((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))| \\ \quad (((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))) \end{array}} \right\} \\ & = & \max\{\sqrt{\beta(0)}, \sqrt{\beta((x|(y|(x|x)))|(x|(y|(x|x))))}\} \\ & = & \sqrt{\beta((x|(y|(x|x)))|(x|(y|(x|x))))}, \end{array}$$

for all $x, y \in L$. Conversely, let $\mathcal{L} = (L, \alpha, \beta)$ be an SR-fuzzy SBG-ideal of L satisfying the inequality (3.6). Then $(\alpha(0))^2 \geq (\alpha(x))^2$ and $\sqrt{\beta(0)} \leq \sqrt{\beta(x)}$ for all $x \in L$. Since

$$\begin{array}{rcl} (\alpha(x))^2 & \geq & (\alpha((x|(y|(x|x)))|(x|(y|(x|x)))))^2 \\ & \geq & \min\{(\alpha((((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))|(((x|(y|(x|x)))| \\ & & (x|(y|(x|x))))|(z|z))))^2, (\alpha(z))^2\}, \end{array}$$

$$\begin{array}{rcl} \sqrt{\beta(x)} & \leq & \sqrt{\beta((x|(y|(x|x)))|(x|(y|(x|x))))} \\ & \leq & \max\left\{ \sqrt{\begin{array}{l} \beta((((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))| \\ \quad (((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))) \end{array}}, \sqrt{\beta(z)} \right\}, \end{array}$$

for all $x, y, z \in L$, we have $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy implicative SBG-ideal of L. □

Theorem 3.24 *Let L be a medial SBG-algebra satisfying the following: for all $x, y \in L$*

$$\left(\begin{array}{c} (\alpha((x|(x|(y|y)))|(x|(x|(y|y)))))^2 \geq (\alpha((x|(y|(x|x)))|(x|(y|(x|x)))))^2 \\ \sqrt{\beta((x|(x|(y|y)))|(x|(x|(y|y))))} \leq \sqrt{\beta((x|(y|(x|x)))|(x|(y|(x|x))))} \end{array} \right). \tag{3.7}$$

Then every SR-fuzzy sub-implicative SBG-ideal of L is an SR-fuzzy implicative ideal of L.

Proof: Let $\mathcal{L} = (L, \alpha, \beta)$ be an SR-fuzzy sub-implicative SBG-ideal of a medial SBG-algebra L satisfying the inequality (3.7). Then we obtain that

$$\begin{array}{rcl} (\alpha(x))^2 & = & (\alpha((y|(y|(x|x)))|(y|(y|(x|x)))))^2 \\ & \geq & \min\{(\alpha((((x|(x|(y|y)))|(x|(x|(y|y))))|(0|0))| \\ & & (((x|(x|(y|y)))|(x|(x|(y|y))))|(0|0))))^2, (\alpha(0))^2\} \\ & = & \min\{(\alpha((x|(x|(y|y)))|(x|(x|(y|y)))))^2, \alpha(0))^2\} \\ & = & (\alpha((x|(x|(y|y)))|(x|(x|(y|y)))))^2 \\ & \geq & (\alpha((x|(y|(x|x)))|(x|(y|(x|x)))))^2, \end{array}$$

$$\begin{array}{rcl} \sqrt{\beta(x)} & = & \sqrt{\beta((y|(y|(x|x)))|(y|(y|(x|x))))} \\ & \leq & \max\left\{ \sqrt{\begin{array}{l} \beta((((x|(x|(y|y)))|(x|(x|(y|y))))|(0|0))| \\ \quad (((x|(x|(y|y)))|(x|(x|(y|y))))|(0|0))) \end{array}}, \sqrt{\beta(0)} \right\} \\ & = & \max\{\sqrt{\beta((x|(x|(y|y)))|(x|(x|(y|y))))}, \sqrt{\beta(0)}\} \\ & = & \sqrt{\beta((x|(x|(y|y)))|(x|(x|(y|y))))} \\ & \leq & \sqrt{\beta((x|(y|(x|x)))|(x|(y|(x|x))))}. \end{array}$$

Thus, $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy implicative SBG-ideal of L. □

Theorem 3.25 *Let L be an implicative SBG-algebra. Then every SR-fuzzy implicative SBG-ideal of L is an SR-fuzzy sub-implicative SBG-ideal of L.*

Proof: Let $\mathcal{L} = (L, \alpha, \beta)$ be an SR-fuzzy implicative SBG-ideal of an implicative SBG-algebra L. Then $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy SBG-ideal of L. So, it is obvious that $(\alpha(0))^2 \geq (\alpha(x))^2$ and $\sqrt{\beta(0)} \leq \sqrt{\beta(x)}$ for all $x \in L$. Thus

$$\begin{aligned}
&(\alpha((y|(y|(x|x)))|(y|(y|(x|x)))))^2 \\
&\quad = (\alpha((x|(x|(y|y)))|(x|(x|(y|y)))))^2 \\
&\quad \geq \min\{(\alpha((((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))|(((x|(x|(y|y)))| \\
&\qquad (x|(x|(y|y))))|(z|z))))^2, (\alpha(z))^2\},
\end{aligned}$$

$$\begin{aligned}
&\sqrt{\beta((y|(y|(x|x)))|(y|(y|(x|x))))} \\
&\quad = \sqrt{\beta((x|(x|(y|y)))|(x|(x|(y|y))))} \\
&\quad \leq \max\left\{\sqrt{\begin{array}{l}\beta((((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))| \\ \quad (((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z)))\end{array}}, \sqrt{\beta(z)}\right\},
\end{aligned}$$

for all $x, y, z \in L$. Hence $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy sub-implicative SBG-ideal of L. $\square$

Corollary 3.26 *Let L be a medial SBG-algebra. Then every SR-fuzzy implicative SBG-ideal of L is an SR-fuzzy sub-implicative SBG-ideal of L.*

Definition 3.27 *An SR-fuzzy SBG-ideal $\mathcal{L} = (L, \alpha, \beta)$ of L is said to be SR-fuzzy closed if for all $x \in L$*

$$\left(\begin{array}{l} (\alpha((0|(x|x))|(0|(x|x))))^2 \geq (\alpha(x))^2 \\ \sqrt{\beta((0|(x|x))|(0|(x|x)))} \leq \sqrt{\beta(x)} \end{array}\right). \tag{3.8}$$

Definition 3.28 *Let $\mathcal{L} = (L, \alpha, \beta)$ be an SR-fuzzy SBG-ideal of L. Then $\mathcal{L} = (L, \alpha, \beta)$ is called an SR-fuzzy completely closed SBG-ideal of L if for all $x, y, z \in L$*

$$\left(\begin{array}{l} (\alpha((x|(y|y))|(x|(y|y))))^2 \geq \min\{(\alpha(x))^2, (\alpha(y))^2\}, \\ \sqrt{\beta((x|(y|y))|(x|(y|y)))} \leq \max\{\sqrt{\beta(x)}, \sqrt{\beta(y)}\}, \end{array}\right). \tag{3.9}$$

Theorem 3.29 *Let L be an SBG-algebra satisfying*

$$\left(\begin{array}{l} ((((x|(y|y))|(x|(y|y)))|(x|(z|z)))|(((x|(y|y))|(x|(y|y)))| \\ (x|(z|z))))|(z|(y|y)) = 0|0, \end{array}\right). \tag{3.10}$$

for all $x, y \in L$. Then L is implicative if and only if every SR-fuzzy closed SBG-ideal of L is an SR-fuzzy implicative SBG-ideal of L.

Proof: Let L be an SBG-algebra satisfying (3.10). Assume that L is implicative and $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy closed SBG-ideal of L. Then $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy SBG-ideal of L. Thus, $(\alpha(0))^2 \geq (\alpha(x))^2$ and $\sqrt{\beta(0)} \leq \sqrt{\beta(x)}$. Also

$$\begin{aligned}
(\alpha(x))^2 &\geq \min\{(\alpha(z))^2, (\alpha((x|(z|z))|(x|(z|z))))^2\} \\
&= \min\{(\alpha(((((x|x)|(x|x))|((x|x)|y))|(z|z))| \\
&\quad ((((x|x)|(x|x))|((x|x)|y))|(z|z))))^2\} \\
&= \min\{(\alpha(z))^2, (\alpha((((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))| \\
&\quad (((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))))^2\},
\end{aligned}$$

$$\begin{aligned}
\sqrt{\beta(x)} &\leq \max\{\sqrt{\beta(z)}, \sqrt{\beta((x|(z|z))|(x|(z|z)))}\} \\
&= \max\left\{\sqrt{\beta(z)}, \sqrt{\begin{array}{l}\beta(((((x|x)|(x|x))|((x|x)|y))|(z|z))| \\ \quad ((((x|x)|(x|x))|((x|x)|y))|(z|z)))\end{array}}\right\} \\
&= \max\left\{\sqrt{\beta(z)}, \sqrt{\begin{array}{l}\beta((((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))| \\ \quad (((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z)))\end{array}}\right\},
\end{aligned}$$

which means that $\mathcal{L}=(L,\alpha,\beta)$ is an SR-fuzzy implicative SBG-ideal of L. Conversely, suppose that every SR-fuzzy closed SBG-ideal of L is an SR-fuzzy implicative SBG-ideal of L. So, it follows from the equation (3.10) that $z|(y|y)=(x|(z|z))|((x|(y|y))|(x|(y|y)))$. Since $z|(y|y)=(x|(z|z))|((x|(y|y))|(x|(y|y)))=((x|(x|(z|z)))|(x|(x|(z|z))))|(y|y)$ from (S1) and (S3), it is obtained from (S2) that $z=(x|(x|(z|z)))|(x|(x|(z|z)))$. Thus, we get from (S1)-(S3) that

$$\begin{array}{rcl} x|(x|(y|y)) & = & ((y|(y|(x|x)))|(y|(y|(x|x))))|(((y|(y|(x|x)))|(y|(y|(x|x))))|(y|y)) \\ & = & ((y|(y|(x|x)))|(y|(y|(x|x))))|(y|(((y|y)|(y|(x|x)))|((y|y)|(y|(x|x))))) \\ & = & ((y|(y|(x|x)))|(y|(y|(x|x))))|(y|(y|y)) \\ & = & (((y|(y|y))|((y|y)|(y|y)))|((y|(y|y))|((y|y)|(y|y))))|(y|(x|x)) \\ & = & y|(y|(x|x)), \end{array}$$

for all $x,y\in L$, which means that L is implicative. □

Proposition 3.30 *Let L be an implicative SBG-algebra satisfying the equation (3.10). Then every SR-fuzzy completely closed SBG-ideal of L is an SR-fuzzy implicative ideal of L.*

Proof: Let $\mathcal{L}=(L,\alpha,\beta)$ be an SR-fuzzy completely closed SBG-ideal of an implicative SBG-algebra L. Then $\mathcal{L}=(L,\alpha,\beta)$ is an SR-fuzzy SBG-ideal of L. Since

$$\sqrt{\beta((0|(y|y))|(0|(y|y)))}\le\max\{\sqrt{\beta(0)},\sqrt{\beta(y)}\}=\sqrt{\beta(y)}$$

and

$$(\alpha((0|(y|y))|(0|(y|y))))^2\ge\min\{(\alpha(0))^2,(\alpha(y))^2\}=(\alpha(y))^2,$$

we get f is an SR-fuzzy closed SBG-ideal of L. Therefore, $\mathcal{L}=(L,\alpha,\beta)$ is an SR-fuzzy implicative SBG-ideal of L. □

Corollary 3.31 *Let L be a medial SBG-algebra satisfying the equation (3.10). Then every SR-fuzzy completely closed SBG-ideal of L is an SR-fuzzy implicative SBG-ideal of L.*

Definition 3.32 *An SR-fuzzy set $\mathcal{L}=(L,\alpha,\beta)$ of L is called an SR-fuzzy p-ideal of L if for all $x,y\in L$*

$$\left(\begin{array}{l} (\alpha(0))^2\ge(\alpha(x))^2 \\ \sqrt{\beta(0)}\le\sqrt{\beta(x)} \\ (\alpha(x))^2\ge\min\{(\alpha((((x|(z|z))|(x|(z|z)))|(y|(z|z)))| \\ \qquad\qquad (((x|(z|z))|(x|(z|z)))|(y|(z|z)))))^2,(\alpha(y))^2\} \\ \sqrt{\beta(x)}\le\max\left\{\sqrt{\begin{array}{l}\beta((((x|(z|z))|(x|(z|z)))|(y|(z|z)))| \\ \quad(((x|(z|z))|(x|(z|z)))|(y|(z|z))))\end{array}},\sqrt{\beta(y)}\right\} \end{array}\right). \tag{3.11}$$

Definition 3.33 *Let L be an SBG-algebra. Then the set*

$$A+=\{x\in A:(0|(x|x))|(0|(x|x))=0\}$$

is called the BCA-part of L.

Theorem 3.34 *Let $A=A+$ be an SBG-algebra. Then every SR-fuzzy p-ideal of L is an intuitionistic fuzzy implicative SBG-ideal of L.*

Proof: Let $\mathcal{L}=(L,\alpha,\beta)$ be an SR-fuzzy p-ideal of L. Then

$$\begin{array}{rcl} (\alpha(x))^2 & \ge & \min\{(\alpha((((x|(y|(x|x)))|(x|(y|(x|x))))|(0|(y|(x|x))))| \\ & & (((x|(y|(x|x)))|(x|(y|(x|x))))|(0|(y|(x|x))))))^2,(\alpha(0))^2\} \\ & = & \min\{(\alpha((((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))| \\ & & (((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))))^2,(\alpha(0))^2\} \\ & = & \min\{(\alpha((x|(y|(x|x)))|(x|(y|(x|x)))))^2,(\alpha(0))^2\} \\ & = & (\alpha((x|(y|(x|x)))|(x|(y|(x|x)))))^2, \end{array}$$

$$\begin{aligned}
\sqrt{\beta(x)} &\leq \max\left\{\sqrt{\begin{array}{c}\beta((((x|(y|(x|x)))|(x|(y|(x|x))))|(0|(y|(x|x))))| \\ (((x|(y|(x|x)))|(x|(y|(x|x))))|(0|(y|(x|x)))))\end{array}}, \sqrt{\beta(0)}\right\} \\
&= \max\left\{\sqrt{\begin{array}{c}\beta((((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))| \\ (((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0)))\end{array}}, \sqrt{\beta(0)}\right\} \\
&= \max\{\sqrt{\beta((x|(y|(x|x)))|(x|(y|(x|x))))}, \sqrt{\beta(0)}\} \\
&= \sqrt{\beta((x|(y|(x|x)))|(x|(y|(x|x))))},
\end{aligned}$$

Hence $\mathcal{L} = (L, \alpha, \beta)$ is an SR-fuzzy implicative SBG-ideal of L. □

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