Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–18.
ISSN-0037-8712
doi:10.5269/bspm.76505

Optimal solution of Fractional Equal Width Equations

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ABSTRACT: That's a great summary of a research paper on the applications of the optimal homotopy asymptotic method (OHAM).In this article, applications of the OHAM modified for equal wave equations are studied. For Fractional Optimal Homotopy Asymptotic Method's (FOHAM) 3rd order solution is tried. For both fractional values Problems and integers the technique is tested. For the evaluation of fractional order, the Caputo and Caputo-Fabrizio operators are utilised. Validating the method by comparing with existing analytical solutions. By contrasting the approach's results with those of the earlier analytical method, the method's correctness is demonstrated. The convergence region is controlled by the convergence control parameters. The paper demonstrates the effectiveness and simplicity of foham in solving potentially linear and nonlinear problems, highlighting its potential as a valuable tool for researchers and scientists in a variety of fields. In this research, the time-fractional model of equal wave equations is examined using the optimal homotopy asymptotic approach and the Laplace Transformation with Caputo operator. The best outcomes in the Caputo meaning are demonstrated by comparing the numerical approximation produced by the suggested method to the exact solution. It was also looked at how the two fractional operators compared to one another.

Key Words: FOHAM, Fractional derivative operator, Caputo operator, CaputoFabrizio operator, non-linear PDE, Equal wave equations.

Contents

1. Introduction

Leibniz a renowned mathematician and one of the founder of calculus! proposed the idea of a fractional derivative in 1695. In equations Leibniz fractional order was able to solve derivatives or integrals with non-integer order. Fractional calculus is a mathematical concept that is used to illustrate a number of physical developments in engineering and science. As fractional-order models converge to integer-order equations, interest in fractional differential equations has grown significantly. Because they accurately describe non-linear behaviours, fractional differential equations are of great interest to researchers. In particular, in nano-hydrodynamics, fractional models are used by researchers as the greatest tool for finding solutions when the continuum assumption is violated. To identify non-linear earthquake oscillations and fix the flaw in a fluid dynamic, fractional derivatives models are used. created by the constant flow of traffic assumption in the traffic model. Such type of nonlinear equation has been extremely important in Applied Mathematics, Physics, and Engineering. Nonlinear partial differential equations (PDEs) make up the majority of issues in Applied Mathematics, Engineering Science, and Physics. These PDEs are difficult to solve using conventional analytic methods. It is dependable and expanding, combining ideas from theory and application, The controllability outcomes in fractional calculus[1-2], Based on fractional derivative, a new Rabotnov fractional exponential function [3,], On beta-time fractional biological population model with abundant solitary wave structures [4]. One of the main factors in the success of fractional calculus is that it is more realistic. Fluid mechanics's Numerous applications[5-7], viscoelasticity, existence, life sciences, physics, electrochemistry ,mathematical biology, and approximate controllability

Submitted April 23, 2025. Published July 02, 2025
2010 *Mathematics Subject Classification:* 26A33, 34A08, 45J05.

[8–10], as well as a numerical scheme based on fractional stochastic hemi variational inequality and an analysis of dengue transmission [12], have contributed to improvements in fractional calculus. The convergence of ADM, diffusion processes, and ecology has been investigated by numerous researchers [13–16]. Reaction diffusion is developed using the technique in many areas of science and engineering. Even in the medical sciences, scientists and researchers have demonstrated the approaches in an unsteady flow analysis [17]. Non-linear earthquake oscillations can be determined using fractional derivative models [18], and they can also be used to fix flaws in fluid dynamic traffic models brought on by the assumption of continuous traffic flow [19–21]. the covid-19 mathematical spread model [22]. Using experimental data from [23], fractional partial differential equations are proposed for seepage flow in porous media. Numerous analytical methods, such as the homotopy perturbation, Shehu transforms method, effective technique, Adomian decomposition method, reduced differential transform method, and fractional reduce differential transform method have recently been developed to solve fractional partial differential equations. By combining the Laplace transform approach and the homotopy perturbation method (HPM), Kumar et al. [33] were able to solve the nonlinear fractional Zakharov-Kuznetsov problem analytically (LTA). A novel technique termed the Optimal Homotopy Asymptotic Method is presented by Marinca et al. in chapter 34. (OHAM). The OHAM is more adaptable and has a greater convergence to the actual solution when compared to the HAM. The papers by Marinca et al. have been used to develop efficient and long-lasting solutions to scientific and engineering issues. Nonlinear differential equations can be solved using OHAM's fundamental theory [35–39]. OHAM shows the potential and reliability of addressing differential equation issues by using it to handle non-linear but also linear ODEs and PDEs. These methods are used to resolve nonlinear problems, including nonlinear wave equations [40], nonlinear boundary conditions problems [41], nonlinear fluid problems [42–43], the approximate solution of the Fornberg–Whitham equation by OHAM [44], and 2D–fuzzy Fredholm integral equations numerical solution[45]. Transmission of electric messages and the study of wave phenomena through cable transmission lines are two areas in which the telegraph equation is often used. A wide range of numerical and analytical methods have been used by different academics to solve the telegraph equation. The telegraph equation's high-order compact solutions were studied by Mohebbi and Dehaghan [46]. Regarding a one spatial-dimensional linear hyperbolic telegraph problem, Gao and Chi [47] employed an optimal control difference strategy. An analysis of the hyperbolic telegraph equation's fractional and integer-order equations is provided [48]. This is how such an equation is written: where is an unknown function, r and v are known constants, and are variables that are independent of space and time. For $r > 0$ and $= 0$, A damped wave equation is specified by the method. For $r > v > 0$, This equation is for the telegraph-The method is frequently applied to wave dynamics and signal analysis of electrical impulse propagation and transmission.This work investigates the time-fractional model of equal wave equations by means of the Laplace Transformation with Caputo operator and the optimum homotopy asymptotic method. By contrasting the actual solution with the numerical approximation generated by the proposed method, the best results in the Caputo meaning are demonstrated. The comparison between the two fractional operators was also examined. The outcomes of the previous method are contrasted graphically. The results and solution show how little computing is needed, how easy it is to use, and how successful the OHAM approach is at eliminating mistakes. In the future, OHAM can be used to address the majority of real-world problems. This research's primary objective is to use OHAM to an analytical solution of the time-fractional telegraph problem..The remainder of the essay provides some fundamental concepts and an overview of OHAM, along with a study of two cases. Through the use of graphics, the outcomes are contrasted. For the purpose of improving the paper, the discussion and conclusion sections are provided

2. Preliminaries

Definition 1.Riemann-Liouville Fractional Integral is define as [27-29]

$D\epsilon C_\mu$ of order $\sigma \geq 0$

$$L_d^\sigma D(d) = D(d), if \sigma = 0, \tag{2.1}$$

$$= \frac{1}{\Gamma(\sigma)} \int_0^d (d-s)^{\sigma-1} D(s) ds, if \sigma > 0 \tag{2.2}$$

where the gamma function is represent as Γ

Definition 2. The fractional derivative of $D \in C\mu$ has order $a \geq 0$ in the caputo-sence which is defined as [27-29]

$$L_d^{m-\sigma} D(d) = T_d^{m-\sigma} L_d D(d), if \sigma = 0, \tag{2.3}$$

$$= \frac{1}{\Gamma(m-\sigma)} \int_0^d (d-s)^{m-\sigma-1} T^m(s) ds, if \sigma > 0 \tag{2.4}$$

$$T_d^{\sigma} L_d^{\sigma} \xi(\varepsilon, D) = Y_0(\varepsilon, L) \Sigma_{i=1}^{1} L_i^{\sigma} T_d^{\sigma} \xi_i(\varepsilon, D) \tag{2.5}$$

Definition 3. If the order of the Caputo-Fabrizio fractional derivative of $D \in C\mu$ has $\sigma > 1$ then as [30,31]

$$L_d^{\sigma} D(d) = T_d^{m-\sigma} L_d^m D(d), \sigma = 0, \tag{2.6}$$

$$L_d^{\sigma} T(d) = \frac{1}{(m-\sigma)} \int_0^d -exp \frac{\sigma(d-s)^{m-\sigma-1}}{1-\sigma} D^m(s) ds, \sigma > 0 \tag{2.7}$$

where a normalization function $B(\sigma) > 0$ satisfies B(0)=B(1)=1

Definition 4 The Caputo (Caputo 1969) give the Caputo derivative of Laplace Transform;also et al.(2006) as [30,31]

$$T[L_d^{\sigma} D(d) = S^{\sigma} T[D(d)] - \sum_{k=0}^{n-1} S^{\sigma-k-1} D^{(k)}(0), n-1 < \sigma < n = 0, \tag{2.8}$$

3. OHAM Alogorithm

The boundary value problem

$$W(u(\emptyset, \mho) + z(\emptyset, \mho) = 0, \emptyset \in \Omega, \mho \geq 0, \emptyset \in \Gamma \tag{3.1}$$

$$B(u_0(\emptyset, \mho), \frac{\partial u_0(\emptyset, \mho)}{\partial \mho}) = 0 \tag{3.2}$$

The variables in this instance are the independent variables of space and time, x and t (12), respectively, the differential operator W, the boundary value B, the source solution $u(\emptyset, \mho)$,Γ the border of Ω and $z(\emptyset, \mho)$, the known analytic function.As a result, W split into differential operators D and N.

$$D(u(\emptyset, \mho) + N(u(\emptyset, \mho)) + z(\emptyset, \mho) = 0, \emptyset \in \Omega \tag{3.3}$$

The most simple and linear part of equation (12) is D, whose exact solution is easy to find.N, the challenging and nonlinear part of (12), might represent the exact solution.

now $u_0(\emptyset, \mho) : \Omega \to R$ are the solution of

$$D(u_0(\emptyset, \mho) + z(\emptyset, \mho) = 0, \tag{3.4}$$

$$B(u_0(\emptyset,\mho),\frac{\partial u_0(\emptyset,\mho)}{\partial\mho}) \tag{3.5}$$

and functions continuously. And the continuous solution to the above equation be let as. $u(\emptyset,\mho):\Omega\to R$.The homotopy defined as $M(\emptyset,\mho,q);\Omega\times[0,1]\to R$ which satisfies the specified technique is

$$(1-q)D(M(\emptyset,\mho,q))+z(\emptyset,\mho)-H(q)W(M(\emptyset,\mho,q)+z(\emptyset\mho)=0 \tag{3.6}$$

Where parameter embedded is $\emptyset\in\Omega$ and $q\in[0,1]$and the auxiliary function for (12) is

$H(q)\neq 0$,for all the q and h(0)=0. Clearly,

Equation 16 becames as $D(u_0(\emptyset,\mho))+z(\emptyset,\mho)=0$

if we put q=0 and q=1 in Eq.(16) becomes as

$$N(u(\emptyset,\mho))+D(u(\emptyset,\mho))+z(\emptyset,\mho)=0 \tag{3.7}$$

Now by the method provided

$M(\emptyset,\mho,q)=u_0(\emptyset,\mho), at$ q=0,

As q goes from 0 to 1, $M(\emptyset,\mho,q)$ Varies from $u_0(\emptyset,\mho)$ to $u(\emptyset,\mho)$; by merging equations (12), and (17). the equation (03) become as$u_0(\emptyset,\mho)$ derived at q=0 According to this approach, we use the auxiliary function H(q) of the differential equation as

$$H(q)=C_1q+C_2q^2+C_3q^3+....+C_kq^k+..... \tag{3.8}$$

Where the constants which is denoted by $c_1,c_2,c_3,......c_k,.....$ that have to be determined
To determine the approximate solution to (12) regarding expanding $M(x,t;q,c_1,c_2,c_3,.....)$
In Taylor,s series w.r.t

$$M(\emptyset,\mho,q,c_1,c_2,c_3,.....)=u_0(\emptyset,\mho)+\sum_{k=1}^{\infty}u_k(\emptyset,\mho,c_1,c_2,c_3,.....)q^k \tag{3.9}$$

We must extend equation (16) and compare the coefficient of the same power of the problem having zero order, which is provided in equation (16), as indicated by the value of C in equation (20). The first and second order problems are described by

$$D(u_k(\emptyset,\mho,))=C_1N_0(u_0(\emptyset,\mho,))$$
$$B(u_1(\emptyset,\mho),\frac{\partial u_1(\emptyset,\mho)}{\partial\mho})=0 \tag{3.10}$$

and $D(u_2(\emptyset,\mho))=C_2N_0(u_0(\emptyset,\mho,))+C_1N_1u_0(\emptyset,\mho),(u_1(\emptyset,\mho,))+(1+C_1)D(u_1(\emptyset,\mho,))$,

$$B(u_2(\emptyset,\mho),\frac{\partial u_2(\emptyset,\mho)}{\partial\mho})=0 \tag{3.11}$$

respectively The Caputo and Caputo-Fabrizio operators, as well as any additional operators specified in the definition above, are now applied to the ordered solution. Furthermore, for the analytical resolution of kth order problems $s_k(\lambda, y)$ are

$$D(u_k(\emptyset,\mho)) = D(u_{k-1}(\emptyset,\mho)) + C_k N_0(u_0(\emptyset,\mho)) + \sum_{j=1}^{k-1} C_j[D(u_{k-j}(\emptyset,\mho))$$

$$+N_{k-j}(u_0(\emptyset,\mho)(u_1(\emptyset,\mho),.....,u_{k-j}(\emptyset,\mho))], B(u_k(\emptyset,\mho,)\frac{\partial u_2(\emptyset,\mho)}{\partial \mho}) = 0 \tag{3.12}$$

where $N_{k-j}(u_0(\emptyset,\mho)(u_1(\emptyset,\mho),.....,u_{k-j}(\emptyset,\mho))]$ be coefficient of q^{k-j}in expansion of $N(M(\emptyset,\mho,q))$ with regard to embedded parameter q.

$$N(M(\emptyset,\mho,q,c_1,c_2,c_3,....)) = N_0(u_0(\emptyset,\mho)) + \sum_{k=1}^{\infty} N_k(u_0,u_1,u_2,....u_k)q^k \tag{3.13}$$

The linear equation of (12), which contains the boundary conditions and the solution $u_k(\emptyset,\mho)$, is easy to solve.The provided homotopy causes the series (19) to converge at q=1,

$$\tilde{u}(\emptyset,\mho,c_1,c_2,c_3,..) = u_0(\emptyset,\mho)) + \sum_{k=1}^{\infty} u_k(\emptyset,\mho,c_1,c_2,c_3,....,c_k) \tag{3.14}$$

The approximate solution of equation (25) can be found as

$$\tilde{u}(\emptyset,\mho,c_1,c_2,c_3,..) = u_0(\emptyset,\mho)) + \sum_{k=1}^{\infty} u_k(\emptyset,\mho,c_1,c_2,c_3,....,c_k) \tag{3.15}$$

By putting eq.(25)into (12) ,we get the residual of (12) which is

$$R(\emptyset,\mho,c_1,c_2,c_3,..) = L(\tilde{u}(\emptyset,\mho,c_1,c_2,c_3,..) + k(\emptyset,\mho). \tag{3.16}$$

If there is a residual R, then U is the exact solution (12). It never happens, particularly in the case of nonlinear DE. There are several methods for determining the ideal values of constants, such as the Ritz Method, Collocation Method, Least Squares Method, and Galerkin's Method.

In our method, we apply the Least Squares technique.

$$J(c_1,c_2,c_3,..) = \int_0^1 \int_0 R^2(x,t;c_1,c_2,c_3,..)dxdt, \tag{3.17}$$

as well as ideal constant values $c_1, c_2, c_3, ..$ can be computed.

Theorem; if $n-1 < a < N, n \in N, \chi(\varepsilon,T), nCk\chi\emptyset, \mho\varepsilon T \in k \geq 0$,then

$$(1-\delta)L_t^a(K(\emptyset,\mho,\delta)) + z(\emptyset,\mho) - A(\delta)U(K(\emptyset,\mho,q) + J(\emptyset,\mho)) \tag{3.18}$$

$(K(\emptyset,\mho,q) : \Omega \times [01] \to R$ is the nonlinear PDE, with Linear plus nonlinear term which is A and with analytic function $G(\emptyset,\mho)$.

Let $G(\emptyset, \mho) :\to R$ be the solution of Eq.(28) is be a continuous function. Considering the Homotopy $K(\emptyset, \mho : q) : \Omega \times [01] \to R$ boundary of the given PDE is Ω
It satisfies
$(1-\delta)[L_t^a(K(\emptyset, \mho, \delta))+z(\emptyset, \mho)-A(\delta)U(K(\emptyset, \mho, q) + J(\emptyset, \mho))]$ Where the embedded parameter is $\delta \in [0, 1]$. Since δ varies from 0 to 1,

$$K(\varepsilon, T; \delta, \lambda_1, \lambda_2.....) = \chi_0(\emptyset, \mho) + \sum_{i=1}^{\infty} \chi_1(\emptyset, \mho, \lambda_1, \lambda_2.....\lambda_i)\delta,$$

varies from
$L_t^a(\chi_i(\emptyset, \mho,) = L(\chi_{i=1}(\emptyset, \mho)) + \delta_i N_0 \chi_0(\emptyset, \mho) + \sum_{j=1}^{i=1} \delta_0[L(\chi_{i=j}(\emptyset, \mho)$
$+ N_{i=j}(\chi_0(\emptyset, \mho), (\chi_1(\emptyset, \mho), ..., (\chi_{i=j}(\emptyset, \mho)$ to $(\chi(\beta, \mho)$ as δ varies from 0 to 1 , $K(\emptyset, \mho : \delta)$ varies from $\chi_0(\emptyset, \mho) to \chi(\emptyset, \mho)$ where the solution of Eq.(28) is $(\chi_0(\emptyset, \mho)$ at $\delta = 0$ In the Caputo sense,this leads to the solution as follows:

$$D_t^a L_t^a(\xi(\emptyset, \mho) = (\xi(\emptyset, \mho)\lambda \tag{3.19}$$

On the other hand, utilising the auxiliary function of the approximate solution to Equation (28) $A(\delta) = \lambda_1\delta + \lambda_2\delta^2 + \lambda_3\delta^3 +, ...\lambda_i\delta^i + ...$ by expanding $K(\emptyset, \mho, \delta, \lambda_1, \lambda_2...)$ in Taylor's series, w.r.t δ as

$$K(\emptyset, \mho, \delta, \lambda_1, \lambda_2...) = \chi_0(\emptyset, \mho) + \Sigma_{i=1}^{\infty} \chi_i(\emptyset, \mho, \delta, \lambda_1, \lambda_2...\lambda_i)\delta \tag{3.20}$$

First order, second order, zero order, etc. can be obtained by expanding Eq. (30), comparing the coefficient of the same power of δ, where $\lambda_1, \lambda_2...\lambda_i$ are the optimal constants

$$L_t^a \chi_i(\emptyset, \mho,) = L\chi_{i=1}(\emptyset, \mho)) + \delta_i N_0(\chi_0(\emptyset, \mho)) + \sum_{j=1}^{i=1} \delta_0[L(\chi_{i=j}(\psi, \mho) + N_{i=j}(\chi_0(\emptyset, \mho), (\chi_1(\emptyset, \mho), ..., (\chi_{i=j}(\emptyset, \mho))] \tag{3.21}$$

Where $N(K(\emptyset, \mho, \delta, \lambda_1, \lambda_2...) = N_0(\chi_0(\emptyset, \mho)) + \Sigma_{i=1}^{\infty} N_i(\varepsilon_0, \varepsilon_1, \varepsilon_2........\varepsilon_i)\delta$
To solve equation (31) having the given boundary condition, we obtained

$$w(\phi, 0) = 3 \sec h^2(\frac{\phi - 15}{2}) \tag{3.22}$$

Equation (32) is used with the Caputo operator we obtain

$$L_t^a D_t^a \chi(\emptyset, \mho) = \gamma_0(\emptyset, \mho) + \Sigma_{i=1}^{1} D_t^a L_t^a \chi_i(\emptyset, \mho), \tag{3.23}$$

The approximate solution can be obtained by using the auxiliary/optimal constants.

Corollary: Convergence Theorem: As mentioned in section 2, the ideal/auxiliary constants are obtained using the Least Squares method.

$$L_t^a D_t^a \chi(\emptyset, \mho) = \gamma_0(\emptyset, \mho) + \Sigma_{i=1}^{1} D_t^a L_t^a \chi_i(\emptyset, \mho), \tag{3.24}$$

the residual of the modal is

$$R(\emptyset, \mho, \lambda_1, \lambda_2...) = T(\chi(\emptyset, \mho, \lambda_1, \lambda_2....) + k(\emptyset, \mho) \tag{3.25}$$

$\chi(\emptyset, \mho, \lambda_1, \lambda_2....)$ be the exact solution of Eq.(28),If $R(\emptyset, \mho, \lambda_1, \lambda_2...)$, but it doesn't always happen, though, especially when dealing with nonlinear equations. Therefore, unknown/optimal constants are found as

$$\partial_{\lambda_1} K = 0, \partial_{\lambda_2} K = 0, ...\partial_{\lambda_i} K = 0, \tag{3.26}$$

By using the list of values for λ_s , we were able to determine the approximate value

$$\tilde{\chi}(\emptyset, \mho) = \chi_0(\emptyset, \mho) + \sum_{i=1}^{1} \chi_i(\emptyset, \mho), \tag{3.27}$$

For optimal constants, the approximate solution to Eq. (34) is well-known. Rapid convergence of the auxiliary function $A(\delta)'s$ and effective error minimization both depend on the optimal

Numerical Problem-:1

Consider the Fractional Equal-Width Equation given as

$$D^l_{\wp} w + ww_\phi + w_{\phi\phi\wp} = 0, \wp > 0, \phi \in R, 0 < l \leq 1, \tag{3.28}$$

With the initial condition

$$u(x,t) = 3sech[\frac{x-15}{2}]^2 \tag{3.29}$$

The zero order, first, second and third order,etc approximations can be obtained using the OHAM formulation, in which the zero order is

$$u_0[x,t] = 3sech[\frac{1}{2}(-15+x)]^2 \tag{3.30}$$

The mentiond model's first-order approximation is

$$(u_1)^{(0,1)}[x,t] = (u_0)^{(0,1)}[x,t] + c_1((u_0)^{(0,1)}[x,t] + c_1 u_0[x,t](u_0)^{(1,0)}[x,t] - c_1(u_0)^{(2,1)}[x,t] \tag{3.31}$$

The 1st order approximation's solution is

$$(u_1)[x,t] = -9Sech[\frac{1}{2}(-15+x)]^4 c_1 Tanh[\frac{1}{2}(-15+x)], \tag{3.32}$$

Using the Caputo-Fabrizio operator, the solution may be given as

$$(u_1)[x,t] = \frac{9t^{1-a} Sech[\frac{1}{2}(-15+x)]^4 c_1 Tanh[\frac{1}{2}(-15+x)]}{(-1+a)Gamma[1-a]}, \tag{3.33}$$

the approximate 2nd order is derived as

$$\begin{aligned}(u_2)^{(0,1)}[x,t] &= c_2(u_0)^{(0,1)}[x,t] + c_2 u_0[x,t](u_0)^{(1,0)}[x,t] - c_1(u_0)^{(2,1)}[x,t] - c_2(u_0)^{(2,1)}[x,t] \\ &+ c_1 u_1[x,t](u_0)^{(1,0)}[x,t] + (u_1)^{(0,1)}[x,t] + c_1(u_1)^{(0,1)}[x,t] + c_1 u_0[x,t](u_1)^{(1,0)}[x,t]\end{aligned} \tag{3.34}$$

the 2nd order approximation's solution is

$u_2[x,t] = \frac{9(1-a)t^{-a} Sech[\frac{1}{2}(-15+x)]^4}{(-1+a)Gamma[1-a]} c_1 Tanh[\frac{1}{2}(-15+x)]$

$-\frac{9(1-a)t^{-a}Sech[\frac{1}{2}(-15+x)]^4c_1^2Tanh[\frac{1}{2}(-15+x)]}{(-1+a)Gamma[1-a]}$ $-9Sech[\frac{1}{2}(-15+x)]^4c_2Tanh[\frac{1}{2}(-15+x)]$

$-\frac{27t^{-a}Sech[\frac{1}{2}(-15+x)]^6c_1^2Tanh[\frac{1}{2}(-15+x)]}{(-1+a)Gamma[1-a]}^2 + 3Sech[\frac{1}{2}(-15+x)]^2c_1(\frac{9t^{1-a}Sech[\frac{1}{2}(-15+x)]^6c_1}{2(-1+a)Gamma[1-a]}$

$$-\frac{18t^{1-a}Sech[\frac{1}{2}(-15+x)]^4c_1Tanh[\frac{1}{2}(-15+x)]^2}{(-1+a)Gamma[1-a]} \tag{3.35}$$

The approximation second order equation in the Caputo sense was found as

$u_2[x,t] = -\frac{1}{2(-1+a)^2Gamma[1-a]^2}9t^{1-2a}Sech[\frac{1}{2}(-15+x)]^4(2(-1+a)c_1Tanh[\frac{1}{2}(-15+x)]$

$+2(-1+a)t^aGamma[1-a]c_2Tanh[\frac{1}{2}(-15+x)] + c_1^2(-3tSech[\frac{1}{2}(-15+x)]^4$

$$+288tCosh[15-x]^4Sinh[\frac{1}{2}(-15+x)]^6 + 2(-1+a)Tanh[\frac{1}{2}(-15+x)])) \tag{3.36}$$

the result of the given equation is

$$u[x,t] = u_0[x,t] + u_1[x,t] + u_2[x,t] \tag{3.37}$$

Which become as

$u[x,t] = 3sech[\frac{1}{2}(-15+x)]^2 + \frac{9t^{1-a}Sech[\frac{1}{2}(-15+x)]^4c_1Tanh[\frac{1}{2}(-15+x)]}{(-1+a)Gamma[1-a]}$

$-\frac{1}{2(-1+a)^2Gamma[1-a]^2}9t^{1-2a}Sech[\frac{1}{2}(-15+x)]^4(2(-1+a)c_1Tanh[\frac{1}{2}(-15+x)]$

$+2(-1+a)t^aGamma[1-a]c_2Tanh[\frac{1}{2}(-15+x)] + c_1^2(-3tSech[\frac{1}{2}(-15+x)]^4$

$$+288tCosh[15-x]^4Sinh[\frac{1}{2}(-15+x)]^6 + 2(-1+a)Tanh[\frac{1}{2}(-15+x)])) \tag{3.38}$$

The residual of the problem is

$$R = \frac{1}{Gamma[1-a])}\int_0^t -(t-r)^{-a}(\partial_t u[x,t])dr + u[x,t]\partial_x u[x,t] - \partial_{x,x,t}u[x,t] \tag{3.39}$$

There are several ways to get the optimal constant in the approximate equation above; in this case, we utilise the least squares method, which is explained in section 2.

$$J = \int_0^1\int_0^1 R^2 dxdt, \tag{3.40}$$

$$equ1 = \frac{\partial J}{\partial C_1}, equ2 = \frac{\partial J}{\partial C_2}, \tag{3.41}$$

for finding the constants of above equation we will solve as

$$NSolve[equ1 = 0, equ2 = 0, C_1, C_2], \tag{3.42}$$

as a result, the optimal constants of Caputo operator's are
$C_1 = -0.6212740736349955; C_2 = 0.29391298247218745,$

the approximated solution get with the help of optimal constants . Exact solution of the given problem is

$$exact = 3Sech[\frac{x - 15 - t}{2}]^2 \tag{3.43}$$

Table 1: Description of OHAM sol. via Caputo at,$\alpha = 0.98$ of above example is

X	OHAM	Exact	Close form
0.	3.67046×10^{-6}	3.67081×10^{-6}	3.54732×10^{-10}
0.1	4.05648×10^{-6}	4.05687×10^{-6}	3.90606×10^{-10}
0.2	4.48311×10^{-6}	4.48354×10^{-6}	4.29935×10^{-10}
0.3	4.9546×10^{-6}	4.95507×10^{-6}	4.73014×10^{-10}
0.4	5.47568×10^{-6}	5.4762×10^{-6}	5.20149×10^{-10}
0.5	6.05156×10^{-6}	6.05213×10^{-6}	5.71663×10^{-10}
0.6	6.68801×10^{-6}	6.68864×10^{-6}	6.27889×10^{-10}
0.7	7.39139×10^{-6}	7.39208×10^{-6}	6.89166×10^{-10}
0.8	8.16875×10^{-6}	8.16951×10^{-6}	7.55833×10^{-10}
0.9	9.02786×10^{-6}	9.02869×10^{-6}	8.28225×10^{-10}
1.	9.97733×10^{-6}	9.97824×10^{-6}	9.06658×10^{-10}

Table 2: Description of values of U at $X \in [1, 2]$,$\alpha = 0.78$.

X	OHAM	Exact	Close form
0.	3.67042×10^{-6}	3.67081×10^{-6}	3.91436×10^{-10}
0.1	4.05644×10^{-6}	4.05687×10^{-6}	4.3117×10^{-10}
0.2	4.48306×10^{-6}	4.48354×10^{-6}	4.74766×10^{-10}
0.3	4.95455×10^{-6}	4.95507×10^{-6}	5.22559×10^{-10}
0.4	5.475662×10^{-6}	5.4762×10^{-6}	5.74906×10^{-10}
0.5	6.0515×10^{-6}	6.05213×10^{-6}	6.32179×10^{-10}
0.6	6.68794×10^{-6}	6.68864×10^{-6}	6.94769×10^{-10}
0.7	7.39132×10^{-6}	7.39208×10^{-6}	7.63079×10^{-10}
0.8	8.16867×10^{-6}	8.16951×10^{-6}	8.3752×10^{-10}
0.9	9.02777×10^{-6}	9.02869×10^{-6}	9.18503×10^{-10}
1.	9.97723×10^{-6}	9.97824×10^{-6}	1.00643×10^{-9}

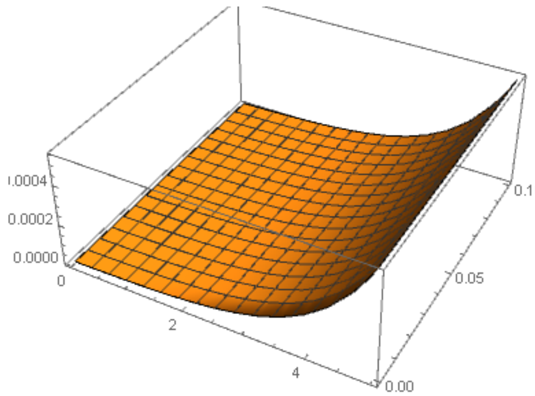

Fig.1:3D approximate solution of U at $X \in [1, 2]$,$\alpha = 0.0.001$.

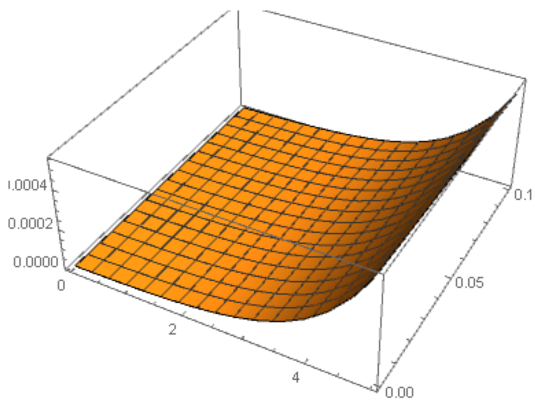

Fig.2:3D exact solution of U at $X \in [1, 2]$,$\alpha = 0.001$.

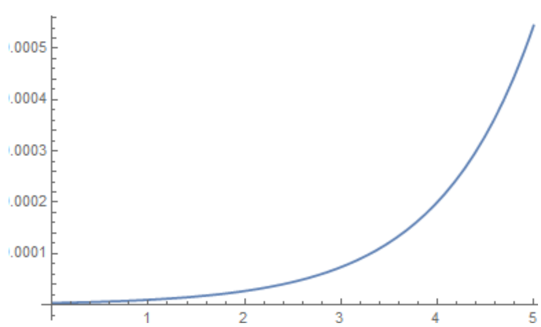

Fig.3:2D approximate solution of U at $X \in [0, 5]$,$\alpha = 0.001$.

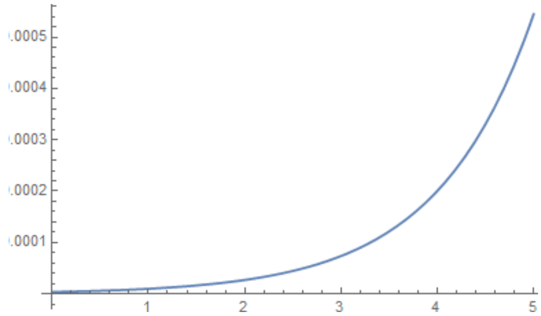

Fig.4:2D exact solution of U at $X \in [0, 5]$,$\alpha = 0.001$.

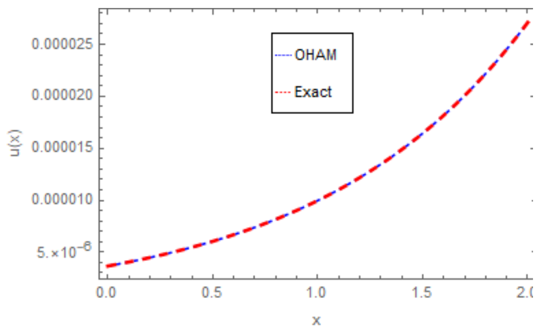

Fig.5:2D OHAM and exact solution of U at $X \in [0, 2]$,$\alpha = 0.001$.

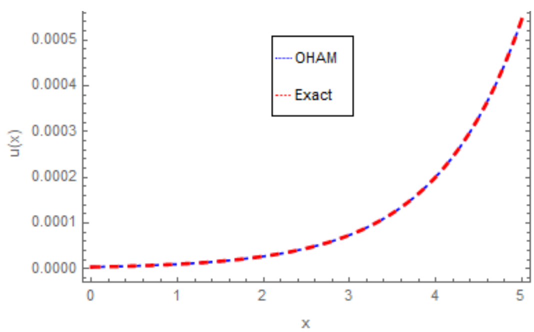

Fig.6:2D OHAM and exact solution of U at $X \in [0, 5]$,$\alpha = 0.001$.

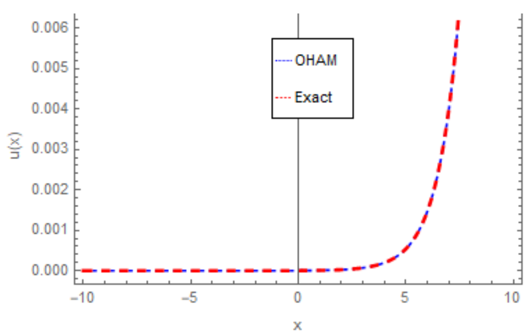

Fig.7:2D OHAM and exact solution of U at $X \in [-10, 10]$,$\alpha = 0.001$.

The aforementioned tables and graphs clearly show that the approximated solution approaches the actual solution as well as the fractional value approach to integer value. Additionally, the difference between the two operators is shown.

Numerical Problem-2.2:

The nonlinear fractional-order modified equal width equation is stated as follows.

$$D^l_\wp w + \frac{12}{7}(w^6)_\phi - \frac{3}{7}(w^6)_{\phi\phi\wp} = 0, \wp > 0, \phi \in R, 0 < l \leq 1, \tag{3.44}$$

having the initial condition

$$u(x,t) = Cosh[\frac{5x}{6}]^{2/5} \tag{3.45}$$

The zero order, first, second, third, etc. approximations can be obtained using the OHAM formulation. In which the zero order

$$u_0[x,t] = 0 \tag{3.46}$$

Laplace Transform allows us to obtain the answer with the initial condition specified in above equation

$$u_0[x,t] = Cosh[\frac{5x}{6}]^{2/5} \tag{3.47}$$

Using the specified method, the first order approximation equation is derived.

$$(u_1)^{0,1}[x,t] = (u_0)^{0,1}[x,t] + c_1(u_0)^{0,1}[x,t] + \frac{12}{7}c_1(u_0)[x,t]^6(u_0)^{1,0}[x,t] - \frac{3}{7}c_1(u_0)[x,t]^6(u_0)^{2,1}[x,t], \tag{3.48}$$

First order approximation's solution in the sense of Caputo is obtained by using Transformation.

$$(u_1)[x,t] = \frac{4}{7}Cosh[\frac{5x}{6}]^{9/5}Sinh[\frac{5x}{6}]c_1, \tag{3.49}$$

Using the Caputo operator, the answer can be found as

$$(u_1)[x,t] = \frac{4t^{1-a}Cosh[\frac{5x}{6}]^{9/5}Sinh[\frac{5x}{6}]c_1}{7(1-a)Gamma[1-a]} \tag{3.50}$$

Approximation of the second order is

$$(u_2)^{0,1}[x,t] = c_2(u_0)^{0,1}[x,t] + \tfrac{12}{7}c_2(u_0)[x,t]^6(u_0)^{1,0}[x,t] - \tfrac{18}{7}c_1(u_0)[x,t]^6(u_0)^{2,1}[x,t]$$
$$-\frac{3}{7}c_2(u_0)[x,t]^6(u_0)^{2,1}[x,t] + (u_1)^{0,1}[x,t] + c_1(u_1)^{0,1}[x,t] + \frac{72}{7}c_1(u_0)[x,t]^5(u_0)^{1,0}[x,t](u_1)^{1,0}[x,t], \tag{3.51}$$

the solution is

$$u_2[x,t] = \tfrac{4t^{-a}Cosh[\frac{5x}{6}]^{9/5}Sinh[\frac{5x}{6}]c_1}{7Gamma[1-a]} + \tfrac{4t^{-a}Cosh[\frac{5x}{6}]^{9/5}Sinh[\frac{5x}{6}]c_1^2}{7Gamma[1-a]} + \tfrac{24}{7}Cosh[\tfrac{5x}{6}]^{7/5}Sinh[\tfrac{5x}{6}]c_1$$
$$+\frac{10t^{1-a}Cosh[\frac{5x}{6}]^{14/5}]c_1}{21(1-a)Gamma[1-a]} + \frac{6t^{1-a}Cosh[\frac{5x}{6}]^{4/5}Sinh[\frac{5x}{6}]^2c_1}{7(1-a)Gamma[1-a]} + \frac{4}{7}Cosh[\frac{5x}{6}]^{9/5}Sinh[\frac{5x}{6}]c_2 \tag{3.52}$$

Moreover, the Caputo operator's result is

$$u_2[x,t] = \frac{1}{Gamma[1-a]}(4t^{1-2a}Cosh[\tfrac{5x}{6}]^{9/5}Sinh[\tfrac{5x}{6}])(-7(-1+a)c_1 + (7-7a+20tCosh[\tfrac{5x}{6}]^{12/5}$$

$$+36tCosh[\frac{5x}{6}]^{2/5}Sinh[\frac{5x}{6}]^2)c_1^2 - 7(-1+a)t^a Gamma[1-a]c_2/49(1-a)Gamma[1-a]^2) \quad (3.53)$$

and solution is

$$u[x,t] = u_0[x,t] + u_1[x,t] + u_2[x,t] \quad (3.54)$$

The residual of the problem is

$$R = \frac{1}{Gamma[1-a])}\int_0^t -(t-r)^{-a}(\partial_t u[x,t])dr + \frac{12}{7}(\partial_x(u[x,t])^6) - \frac{3}{7}\partial_{x,x,t}(u[x,t])^6 \quad (3.55)$$

the optimal constants of the approximate equation can find on many ways here we find the constants by least squares method.

$$J = \int_0^1 \int_0^1 R^2 dxdt, \quad (3.56)$$

$$equ1 = \frac{\partial J}{\partial C_1}, equ2 = \frac{\partial J}{\partial C_2}, \quad (3.57)$$

for finding the constants of above equation we will solve as

$$NSolve[equ1 = 0, equ2 = 0, C_1, C_2], \quad (3.58)$$

hence, optimal constants obtained through the Caputo sense is

$C_1 = -0.07834269353773295; C_2 = -1.7507600298033261,$

Exact solution of the given problem is

$$Exact = Cosh[\frac{5}{6}(x-t)]^{2/5}; \quad (3.59)$$

The tables that demonstrate the error approximation are provided below.

Table 3: Description of OHAM sol. via Caputo at $\tau = 0.1$,$\alpha = 0.0001$.

X	OHAM	Exact	Close form
0.	1.	1.	1.38889×10^{-11}
0.1	1.00139	1.00139	2.61627×10^{-6}
0.2	1.00555	1.00554	5.33561×10^{-6}
0.3	1.01246	1.01245	8.26679×10^{-6}
0.4	1.02207	1.02206	0.0000115304
0.5	1.03435	1.03434	0.0000152653
0.6	1.04924	1.04922	0.0000196367
0.7	1.06666	1.06664	0.0000248447
0.8	1.08656	1.08653	0.0000311351
0.9	1.10887	1.10883	0.0000388135
1.	1.13351	1.13346	0.0000482609

Table 4: Description of OHAM sol. via Caputo at $\tau = 0.1$,$\alpha = 0.00011$.

X	OHAM	Exact	Close form
0.	1.	1.	1.38889×10^{-11}
0.1	1.00139	1.00139	2.82956×10^{-6}
0.2	1.00555	1.00554	5.77172×10^{-6}
0.3	1.01246	1.01245	8.94531×10^{-6}
0.4	1.02207	1.02206	0.000012482
0.5	1.03435	1.03434	0.0000165336
0.6	1.04924	1.04922	0.0000212802
0.7	1.06666	1.06664	0.0000269402
0.8	1.08656	1.08653	0.0000337821
0.9	1.10887	1.10883	0.000042139
1.	1.13351	1.13346	0.000052427

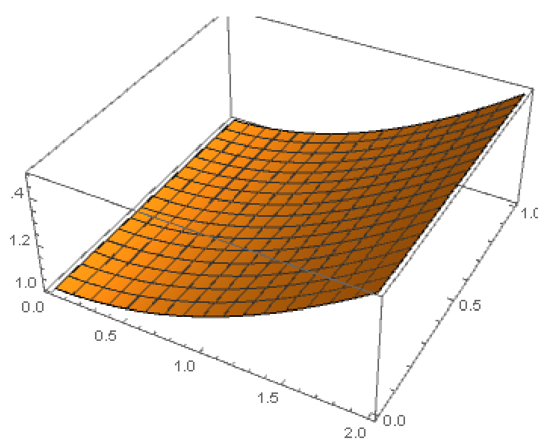

Fig.1:3D approximate solution of U at $X \in [0, 2]$,$\alpha = 0.001$.

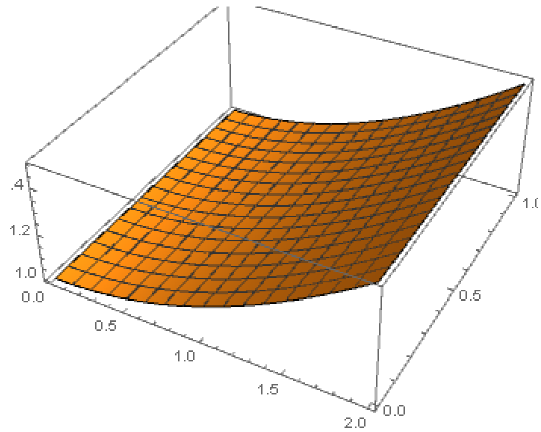

Fig.2:3D exact solution of U at $X \in [0, 2]$,$\alpha = 0.001$.

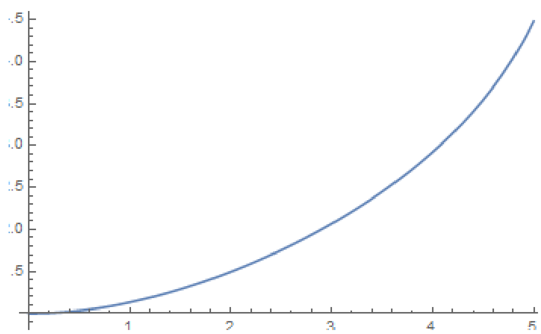

Fig.3:2D approximate solution of U at $X \in [0, 5]$,$\alpha = 0.001$.

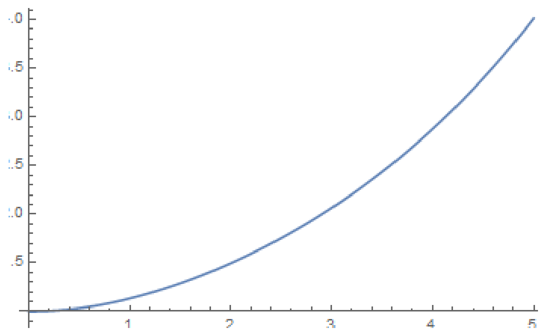

Fig.4:2D exact solution of U at $X \in [0, 5]$,$\alpha = 0.001$.

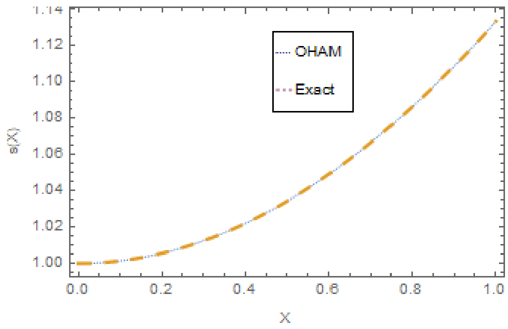

Fig.5:2D OHAM and exact solution of U at $X \in [0,1]$,$\alpha = 0.001$.

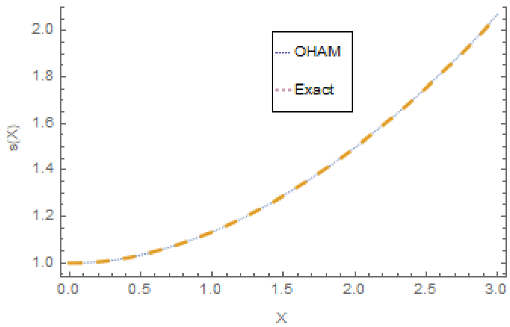

Fig.6:2D approximate and exact solution of U at $X \in [0,3]$,$\alpha = 0.001$.

4. Result and Discussion:

In the current paper, we use OHAM to solve the equal wave equation and employs the least squares method to determine optimal constants . For calculations, Mathematica 11 was utilised. For various values of α and $X \in [0,2]$ Table.1 and Table.2 of Examples.1 and 2 respectively represent the OHAM solution in Caputo operators. This made it very evident how the Caputo derivative operators differed from one another. We check to see if the solution produced by the Caputo operator agrees strongly with the closed-form solution. It also demonstrates how, as we approach the fractional number α =2, the absolute error is reducing and approaching the actual solution. It demonstrates that the suggested method quickly approaches the actual Caputo solution. The third order 3D and 2D approximate and exact solutions at X=[0,2], α =0.001 of case 2.1 are shown in Figs. 1 and 2. For the range of α=0.001 and X=[0,5], see Fig. 3 for the OHAM and exact solution. The third order 3D and 2D approximate and exact solutions at X=[0,2], α =0.0001, of case 2.2 are shown in Figs. 4 and 5. The OHAM and exact solution for the range of α=0.0001 and X=[0,2] are shown in Fig. 6. The tables and graphical display demonstrate the speedy convergence analysis of the current method. Over all,the above article demonstrates the effectiveness of OHAM in solving equal wave equation with Caputo derivative operators,highlighting its potential for application in various fields.

5. Conclusion:

In this research, the time-fractional model of equal wave equations is examined using the optimal homotopy asymptotic approach and the Laplace Transformation with Caputo operator. The best outcomes in the Caputo meaning are demonstrated by comparing the numerical approximation produced by the suggested method to the exact solution. It was also looked at how the two fractional operators compared to one another. With the earlier method, the results are graphically contrasted. The outcomes and solution demonstrate how effective the OHAM method is in reducing errors, how little computation is required, and how simple it is to use. In the future, most real-world phenomena can be solved by the method of OHAM. Its conclude that OHAM is a powerful method for solving real- world problems and has the potential to be widely applied in various fields.

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