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Impact of illegal logging and industrial effort on forestry biomass via a fractal-fractional model involving the Hattaf fractal-fractional derivative

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ABSTRACT: In this paper, we present a fractal-fractional model to examine the effects of illegal logging, tax policies and conservation efforts on forest biomass, as well as the complex interplay between biomass depletion and industrial growth. We demonstrate the existence and uniqueness of the model's solutions via fixed-point theory and apply Ulam-Hyers stability to evaluate the stability of the proposed model. It was found that increasing forest biomass densities can be achieved by mitigating illegal logging, implementing government taxes and reducing industrialization. Finally, some results from numerical simulations are presented.

Key Words: Forestry biomass model, harvesting, existence theory, Ulam-Hyers stability.

Contents

1. Introduction

Fractional calculus is a branch of mathematics that extends classical concepts of differentiation and integration to non-integer orders. Its applications have garnered significant attention in different scientific and engineering fields, particularly for modeling complex phenomena such as memory effects and inherited properties of materials and processes. Traditional approaches, including classical fractional calculus, often struggle to accurately model real-world complexities. To address these limitations, a new mathematical concept integrating fractal theory with fractional calculus has emerged. In this context, Antangana [1] introduced six types of fractal-fractional derivatives with exponential decay and Mittag-Leffler kernels, based on the Hausdorff fractal derivative. He also developed three kinds of fractal-fractional integrals with non-singular kernels that combine the integrals and fractional derivatives of Atangana-Baleanu and Caputo-Fabrizio. This framework was further extended by Hattaf [2] in 2023, who explored a range of new differential and integral operators. Hattaf's contributions include various special cases, such as the generalized Hattaf fractional derivative (GHF) [3], the Caputo-Fabrizio (CF) fractional derivative [4], the Atangana-Baleanu (AB) fractional derivative [5] and the weighted AB fractional derivative [6]. The fractal-fractional calculus framework has proven invaluable for modeling diverse systems such as in finance [7], chemistry [8], epidemiology [9,10,11], and ecology [12].

The stability of differential and functional equations is a crucial area that has become central in mathematical analysis. Finding exact solutions to these equations can be particularly challenging, leading to the development of numerous numerical methods. Once numerical solutions are obtained, assessing their stability becomes essential. The literature identifies different types of stability, including exponential

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stability, asymptotic stability and Ulam-Hyers (UH) stability. The letter type of stability is particularly notable because it examines the relationship between exact and numerical solutions. In [13], Ulam introduced the stability problem and the following year, Hyers [14], using Banach spaces, offered a partial answer to the problem of linear functional equations. In 1978, Rassias [15] expanded these results to linear applications. The work of [15] has motivated numerous researchers to apply these concepts to ordinary differential equations (ODEs) and functional differential equations, including first-order linear differential equations [16], multi-variable functional equations [17], advection-reaction-diffusion systems [18], as well as fields such as biology and economics [19], and evolution equations coupled with impulsive switching [20].

Forest biomass is an essential component of forest ecosystems and a key indicator of forest health and productivity. It plays an essential role in reducing climate change, in particular global warming, by removing carbon dioxide from the atmosphere via carbon sequestration in biomass and soil. Consequently, forest biomass equations will continue to be essential for carbon measurements and estimates in the future. Estimation models at various levels can have a substantial impact on biomass estimates at local, regional and global scales. In the litterature, most mathematical models have been used to describe biomass based on classical differential equations. In 2009, Dubey et al. [21] constructed a system of ordinary differential equations to model the densities of cumulative biomass, resources, population, and industrialization, as well as demographic pressure. The next year, Agrawal et al. [22] developed a mathematical model to analyze the impact of industrialization on forest biomass. In [23], Shukla et al. examined the effects of population growth, pollution, toxic substances and industrialization on forest biomass using a system of six differential equations. In addition, industries favor harvesting trees as their preferred source of raw material for use. However, this preference may shift depending on the availability of alternatives or the shortage of mature trees. Based on this assumption, Chaudhary et al. [24] developped a mathematical model for the preservation of forestry biomass using a substitute resource for industrialization. In 2022, the authors of [25] presented a mathematical model using differential equations to describe the interactions between nutrient density, forestry biomass, effort intensity and industrial density. Moreover, integrating fractal-fractional derivatives into such mathematical models describing forest biomass can provide a better understanding of the complex and self-similar nature of various elements of a forest such as branches, leaves, and roots, at different levels of observation. For this reason, Kumar et al. [26] improved the work in [25] by proposing a complex dynamic model using the Caputo-Fabrizio fractal-fractional derivative to examine the effects of illegal logging on forest biomass.

Inspired by the above discussions, we propose a fractal-fractional model to more describe the interactions between nutriment density, forest biomass, effort intensity and industrial density. The proposed model improves and generalizes the fractal-fractional forestry biomass models presented in [25,26].To do this, the structure of the present work is outlined as follows: Section 2 recalls the concepts related to Hattaf fractal and fractal-fractional derivatives. Section 3 presents the proposed fractal-fractional model for forestry biomass and establishes the existence and uniqueness of the model solution. The Ulam-Hyers stability is examined in Section 4. Section 5 deals with numerical simulations. The conclusion is discussed in Section 6.

2. Preliminary results

In this section, we give the essential definitions and results that are needed to prove the main findings.

Definition 2.1 *[2] Let $\mathbf{I}$ be an open interval in $\mathbb{R}$ and $\eta > 0$. The Hattaf fractal derivative of a function $g(x)$ in terms of a fractal measure $f(\eta, x)$ is defined as follows:*

$$\frac{d_f}{dt^\eta} g(x) = \lim_{\tau \to x} \frac{g(x) - g(\tau)}{f(\eta, x) - f(\eta, \tau)}. \tag{2.1}$$

Further, g is fractal differentiable on the interval $\mathbf{I}$ with order η if $\frac{d_f}{dt^\eta} g(x)$ exists for all $x \in \mathbf{I}$.

It obvious to see that the Hausdorff fractal derivative [27] is obtained from (2.1) when $f(\eta, x) = x^\eta$. Furthermore, we obtain the general derivative suggested by Yang [28] if $f(\eta, x) = h(x)$ with $h'(x) > 0$

and $g(x)$ is differentiable. In this case, (2.1) becomes

$$\frac{d_f}{dt^\eta}g(x) = \frac{1}{h'(x)}\frac{dg(x)}{dx}. \tag{2.2}$$

Definition 2.2 *[2] Let $\alpha \in [0,1)$, $\beta, \gamma, \eta > 0$ and $g(x)$ be differentiable in the interval (a,b) and fractal differentiable on (a,b) with order $0 < \eta \le 1$. The generalized Hattaf fractal-fractional derivative of $g(x)$ of order α in the sense of Caputo with respect to the weight function $w(x)$ is defined by:*

$${}^{FFC}D^{\alpha,\beta,\gamma,\eta}_{a,x,w}g(x) = \frac{\mathcal{GHF}(\alpha)}{1-\alpha}\frac{1}{w(x)}\int_a^x E_\beta[-\vartheta_\alpha(x-s)^\gamma]\frac{d_f}{ds^\eta}(wg)(s)ds, \tag{2.3}$$

where $w \in C^1(a,b)$, $w > 0$ on [a,b], $\mathcal{GHF}(\alpha)$ is a normalization function obeying $\mathcal{GHF}(0) = \mathcal{GHF}(1) = 1$, $\vartheta_\alpha = \dfrac{\alpha}{1-\alpha}$ and $E_\beta(x) = \displaystyle\sum_{k=0}^{+\infty}\frac{x^k}{\Gamma(\beta k+1)}$ is the Mittag-Leffler function of parameter β.

Many special instances are included in Definition 2.2. In specifically, the fractal-fractional derivative with an exponential decay kernel [1] is provided by: $g(x,\eta) = x^\eta$, $\gamma = \beta = 1$ and $w(x) = 1$.

$${}^{FFC}D^{\alpha,1,1,\eta}_{a,x,1}g(x) = \frac{\mathcal{GHF}(\alpha)}{1-\alpha}\int_a^x \exp[-\vartheta_\alpha(t-s)]\frac{d_f}{ds^\eta}g(s)ds, \tag{2.4}$$

where $\frac{d_f}{dt^\eta}g(x) = \lim\limits_{\tau\to x}\frac{g(x)-g(\tau)}{x^\eta - s^\eta}$.

The fractal-fractional derivative with an exponential decay kernel [1] is similarly obtained when $f(x,\eta) = x^\eta$, $\gamma = 2$, $w(x) = 1$, and $\beta = 1$.

$${}^{FFC}D^{\alpha,1,1,\eta}_{a,x,1}g(x) = \frac{\mathcal{GHF}(\alpha)}{1-\alpha}\int_a^x \exp[-\vartheta_\alpha(x-s)^2]\frac{d_f}{ds^\eta}g(s)ds, \tag{2.5}$$

When $w(x) = 1$, $f(x,\eta) = x^\eta$ and $\mathcal{GHF}(\alpha) = 1-\alpha+\frac{\alpha}{\Gamma(\alpha)}$. Using a generalized Mittag-Leffler kernel [1], the fractal-fractional derivative with $\beta = \alpha = \gamma$ is provided by:

$${}^{FFC}D^{\alpha,\alpha,\alpha,\eta}_{a,t,1}g(x) = \frac{\mathcal{GHF}(\alpha)}{1-\alpha}\int_a^x E_\alpha[-\vartheta_\alpha(x-s)^\alpha]\frac{d_f}{ds^\eta}g(s)ds. \tag{2.6}$$

The GHF derivative [3] is produced as follows when $g(t,\eta) = t$:

$${}^{C}D^{\alpha,\beta,\gamma}_{a,x,w}g(x) = \frac{\mathcal{GHF}(\alpha)}{1-\alpha}\frac{1}{w(x)}\int_a^x E_\beta[-\vartheta_\alpha(x-s)^\gamma]\frac{d}{ds}(gw)(s)ds. \tag{2.7}$$

The following definition introduces the Hattaf fractal-fractional derivative in the Riemann-Liouville sense.

Definition 2.3 *Let $\alpha \in [0,1)$, β, γ, $\eta > 0$ and $g(x)$ be continuous on the interval (a,b) and fractal differentiable on (a,b) with order $0 < \eta \le 1$. The generalized Hattaf fractal-fractional derivative of $g(x)$ of order α in the Riemann-Liouville sense with respect to the weight function $w(x)$ is defined as follows:*

$${}^{FFR}D^{\alpha,\beta,\gamma,\eta}_{a,x,w}g(x) = \frac{\mathcal{GHF}(\alpha)}{1-\alpha}\frac{1}{w(x)}\frac{d_f}{dx^\eta}\int_a^x E_\beta[-\vartheta_\alpha(x-s)^\gamma]g(s)w(s)ds. \tag{2.8}$$

Theorem 2.1 *[2] If $\frac{\partial f(\eta,x)}{\partial t}$ exists and not zero, then*

$${}^{FFR}D^{\alpha,\beta,\gamma,\eta}_{0,t,w}g(x) = \left(\frac{\partial f(\eta,x)}{\partial x}\right)^{-1}{}^{R}D^{\alpha,\beta,\gamma}_{0,t,w}g(x), \tag{2.9}$$

where ${}^{R}D^{\alpha,\beta,\gamma}_{0,t,w}$ represents the GHF derivative of the function $g(x)$ with respect to the weight function $w(x)$ in the Riemann-Liouville sense [3].

The following outcome is necessary to prove the existence and uniqueness of the solution to our fractal-fractional differential model.

Theorem 2.2 *(Leray-Schauder alternative) [29] Let Ω be a normed space, $\Phi : \Omega \to \Omega$ be a completely continuous operator (i.e., a map that restricted to any bounded set in Ω is compact), and $\mathfrak{B}(\Phi)$ be a subset of Ω defined by:*

$$\mathfrak{B}(\Phi) = \{A \in \Omega : A = \lambda\Phi(A) \text{ for some } \lambda \in [0,1]\}.$$

Then either the set $\mathfrak{B}(\Phi)$ is unbounded or the operator Φ has at least one fixed point.

3. Model formulation and basic properties

3.1. Model formulation

This subsection introduces a mathematical model for forestry biomass using the Hattaf fractal-fractional derivative. The dynamics of this model is described by the following nonlinear system:

$$\begin{cases} {}^{FFR}D_{0,t,w}^{\alpha,\beta,\beta,\eta}\mathcal{A}(t) = \lambda - \nu\mathcal{P}(t) - \eta_1\mathcal{A}(t), \\ {}^{FFR}D_{0,t,w}^{\alpha,\beta,\beta,\eta}\mathcal{P}(t) = r(A)\mathcal{P}(t)\left(1 - \dfrac{\mathcal{P}(t)}{K}\right) - a\mathcal{P}(t)E(t) - \eta_2\mathcal{P}(t)E(t) + \delta\mathcal{P}(t), \\ {}^{FFR}D_{0,t,w}^{\alpha,\beta,\beta,\eta}E(t) = a(b-\tau)\mathcal{P}(t)E(t) + b\eta_2\mathcal{P}(t)E(t) - cE(t), \\ {}^{FFR}D_{0,t,w}^{\alpha,\beta,\beta,\eta}\mathcal{I}(t) = dE(t) + \mu\mathcal{I}(t)\left(1 - \dfrac{\mathcal{P}(t)}{K}\right) - \eta_3\mathcal{I}(t), \end{cases} \tag{3.1}$$

where $\mathcal{A}(t)$, $\mathcal{P}(t)$, $E(t)$ and $\mathcal{I}(t)$ represent the nutrient density, forestry biomass, effort intensity and industrial density at time t, respectively. The parameter λ denotes the constant nutrient input rate, while ν signifies the rate at which nutrients are utilized by the forestry biomass. η_1 represents the nutrient washout rate. $r(A)$ is the modified intrinsic growth rate of the forestry biomass. The terms $a\mathcal{P}E$ and $\eta_2\mathcal{P}E$ describe legal and illegal tree-cutting, which directly contribute to the loss of forestry biomass. The parameters K and δ represent the carrying capacity of the forestry biomass and the rate of new plantation, respectively. The parameter a stands for the forestry biomass harvesting rate, while b and τ refer to the fixed price and tax per unit of forestry biomass, respectively. In addition, η_2 captures the illegal logging rate, and c represents the fixed cost per unit of effort expended to harvest forestry biomass. The parameter μ represents the maximum growth driven by alternative industries, and d indicates the rate of industrial growth due to harvesting efforts. Lastly, η_3 signifies the reduction rate of industrial activity.

Using Theorem 2.1, the system (3.1) can be reformulated as follows:

$$\begin{cases} {}^{C}D_{0,t,w}^{\alpha,\beta,\beta}\mathcal{A}(t) = \dfrac{\partial g(\eta,t)}{\partial t}\Big(\lambda - \nu\mathcal{P}(t) - \eta_1\mathcal{A}(t)\Big), \\ {}^{C}D_{0,t,w}^{\alpha,\beta,\beta}\mathcal{P}(t) = \dfrac{\partial g(\eta,t)}{\partial t}\left(r(A)\mathcal{P}(t)\left(1 - \dfrac{\mathcal{P}(t)}{K}\right) - a\mathcal{P}(t)E(t) - \eta_2\mathcal{P}(t)E(t) + \delta\mathcal{P}(t)\right), \\ {}^{C}D_{0,t,w}^{\alpha,\beta,\beta}E(t) = \dfrac{\partial g(\eta,t)}{\partial t}\Big(a(b-\tau)\mathcal{P}(t)E(t) + b\eta_2\mathcal{P}(t)E(t) - cE(t)\Big), \\ {}^{C}D_{0,t,w}^{\alpha,\beta,\beta}\mathcal{I}(t) = \dfrac{\partial g(\eta,t)}{\partial t}\left(dE(t) + \mu\mathcal{I}(t)\left(1 - \dfrac{\mathcal{P}(t)}{K}\right) - \eta_3 I(t)\right), \end{cases} \tag{3.2}$$

with the initial conditions:

$$\mathcal{A}(0) = \mathcal{A}_0, \mathcal{P}(0) = \mathcal{P}_0, E(0) = E_0, \mathcal{I}(0) = \mathcal{I}_0,$$

where $r(\mathcal{A}) = r + r_0\mathcal{A}$.

It important to note that when $g(\eta,t) = t^\eta$, $w(t) = 1$ and $\beta = 1$, we get the fractal-fractional model of Kumar et al. [26]. In addition, the ODE model presented by Chaudhary et al. [25] is a special case of our model, it suffices to take $g(\eta,t) = t$, $w(t) = 1$ and $\alpha = \beta = 1$.

3.2. Existence and uniqueness of solutions

This subsection investigates the existence and uniqueness of solutions for model (3.1).
System (3.2) can be written in the following form:

$$\begin{cases} {}^{C}D^{\alpha,\beta,\beta}_{0,t,w}Y(t) = \dfrac{\partial g(\eta,t)}{\partial t}H(t,Y(t)), \quad t \in [0,\mathsf{T}], \\ Y(0) = Y_0, \end{cases} \tag{3.3}$$

where $Y(t) = \Big(\mathcal{A}(t), \mathcal{P}(t), E(t), \mathcal{I}(t)\Big)$, $Y_0 = \Big(\mathcal{A}(0), \mathcal{P}(0), E(0), \mathcal{I}(0)\Big)$,
$H(t,Y(t)) = \Big(H_1(t,Y(t)), H_2(t,Y(t)), H_3(t,Y(t), H_4(t,Y(t))\Big)$ and

$$\begin{cases} H_1(t,Y(t)) = \lambda - \nu\mathcal{P}(t) - \eta_1\mathcal{A}(t), \\ H_2(t,Y(t)) = r(A)\mathcal{P}(t)\left(1 - \dfrac{\mathcal{P}(t)}{K}\right) - a\mathcal{P}(t)E(t) - \eta_2\mathcal{P}(t)E(t) + \delta\mathcal{P}(t), \\ H_3(t,Y(t)) = a(b-\tau)\mathcal{P}(t)E(t) + b\eta_2\mathcal{P}(t)E(t) - cE(t), \\ H_4(t,Y(t)) = dE(t) + \mu I(t)\left(1 - \dfrac{\mathcal{P}(t)}{K}\right) - \eta_3 I(t). \end{cases} \tag{3.4}$$

When the GHF integral is applied to both sides of (3.3), we obtain

$$\begin{aligned} Y(t) = {} & \frac{w(0)Y(0)}{w(t)} + \frac{1-\alpha}{\mathcal{GHF}(\alpha)}\frac{\partial g(\eta,t)}{\partial t}H(t,Y(t)) \\ & + \frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\frac{\partial g(\eta,s)}{\partial s}w(s)H(s,Y(s))ds. \end{aligned} \tag{3.5}$$

Then

$$\begin{aligned} \mathcal{A}(t) = {} & \frac{w(0)\mathcal{A}(0)}{w(t)} + \frac{1-\alpha}{\mathcal{GHF}(\alpha)}\frac{\partial g(\eta,t)}{\partial t}H_1(t,Y(t)) \\ & + \frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\frac{\partial g(\eta,s)}{\partial s}w(s)H_1(s,Y(s))ds. \\ \mathcal{P}(t) = {} & \frac{w(0)\mathcal{P}(0)}{w(t)} + \frac{1-\alpha}{\mathcal{GHF}(\alpha)}\frac{\partial g(\eta,t)}{\partial t}H_2(t,Y(t)) \\ & + \frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\frac{\partial g(\eta,s)}{\partial s}w(s)H_2(s,Y(s))ds. \\ E(t) = {} & \frac{w(0)E(0)}{w(t)} + \frac{1-\alpha}{\mathcal{GHF}(\alpha)}\frac{\partial g(\eta,t)}{\partial t}H_3(t,Y(t)) \\ & + \frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\frac{\partial g(\eta,s)}{\partial s}w(s)H_3(s,Y(s))ds. \\ \mathcal{I}(t) = {} & \frac{w(0)\mathcal{I}(0)}{w(t)} + \frac{1-\alpha}{\mathcal{GHF}(\alpha)}\frac{\partial g(\eta,t)}{\partial t}H_4(t,Y(t)) \\ & + \frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\frac{\partial g(\eta,s)}{\partial s}w(s)H_4(s,Y(s))ds. \end{aligned} \tag{3.6}$$

The functions $H_i(i=1,2,3,4) : [0,\mathsf{T}] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ are continuous. We define the space Ω as the product of continuous functions on $[0,\mathsf{T}]$ to $\mathbb{R}$ such as:

$$\Omega = \mathcal{C}([0,\mathsf{T}],\mathbb{R}) \times \mathcal{C}([0,\mathsf{T}],\mathbb{R}) \times \mathcal{C}([0,\mathsf{T}],\mathbb{R}) \times \mathcal{C}([0,\mathsf{T}],\mathbb{R}).$$

For any function $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ defined on Ω, the norm specified on this product space is $\|\varphi\| = \sum_{i=1}^{4} \max_{t\in[0,\mathsf{T}]} |\varphi_i(t)|$. Clearly, Ω is a Banach space. Let $\Phi : \Omega \to \Omega$ be an operator defined as follows:

$$\Phi(\mathcal{A},\mathcal{P},E,\mathcal{I}) = \begin{pmatrix} \Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I}) \\ \Phi_2(\mathcal{A},\mathcal{P},E,\mathcal{I}) \\ \Phi_3(\mathcal{A},\mathcal{P},E,\mathcal{I}) \\ \Phi_4(\mathcal{A},\mathcal{P},E,\mathcal{I}) \end{pmatrix},$$

where

$$\begin{aligned}
\Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I})(t) &= \frac{\mathcal{A}(0)w(0)}{w(t)} + \frac{1-\alpha}{\mathcal{GHF}(\alpha)}\frac{\partial g(\eta,t)}{\partial t}H_1(t,Y(t)) \\
&\quad + \frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\frac{\partial g(\eta,s)}{\partial s}w(s)H_1(s,Y(s))ds, \\
\Phi_2(\mathcal{A},\mathcal{P},E,\mathcal{I})(t) &= \frac{\mathcal{P}(0)w(0)}{w(t)} + \frac{1-\alpha}{\mathcal{GHF}(\alpha)}\frac{\partial g(\eta,t)}{\partial t}H_2(t,Y(t)) \\
&\quad + \frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\frac{\partial g(\eta,s)}{\partial s}w(s)H_2(s,Y(s))ds, \\
\Phi_3(\mathcal{A},\mathcal{P},E,\mathcal{I})(t) &= \frac{E(0)w(0)}{w(t)} + \frac{1-\alpha}{\mathcal{GHF}(\alpha)}\frac{\partial g(\eta,t)}{\partial t}H_3(t,Y(t)) \\
&\quad + \frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\frac{\partial g(\eta,s)}{\partial s}w(s)H_3(s,Y(s))ds, \\
\Phi_4(\mathcal{A},\mathcal{P},E,\mathcal{I})(t) &= \frac{\mathcal{I}(0)w(0)}{w(t)} + \frac{1-\alpha}{\mathcal{GHF}(\alpha)}\frac{\partial g(\eta,t)}{\partial t}H_4(t,Y(t)) \\
&\quad + \frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\frac{\partial g(\eta,s)}{\partial s}w(s)H_4(s,Y(s))ds.
\end{aligned}$$

Consider the following hypotheses:

$(\mathcal{H}_1)$ There exist $\mathcal{L}_{H_i}, \mathcal{K}_{H_i}, \mathcal{G}_{H_i}, \mathcal{J}_{H_i} > 0$, such that for $\mathcal{A}, \widetilde{\mathcal{A}}, \mathcal{P}, \widetilde{\mathcal{P}}, E, \widetilde{E}, \mathcal{I}, \widetilde{\mathcal{I}} \in \Omega$ and $t \in [0,\mathsf{T}]$, we have

$$\left|H_i(t,\mathcal{A},\mathcal{P},E,\mathcal{I}) - H_i(t,\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{E},\widetilde{\mathcal{I}})\right| \le \mathcal{L}_{H_i}\|\mathcal{A}-\widetilde{\mathcal{A}}\| + \mathcal{K}_{H_i}\|\mathcal{P}-\widetilde{\mathcal{P}}\| + \mathcal{G}_{H_i}\|E-\widetilde{E}\| + \mathcal{J}_{H_i}\|\mathcal{I}-\widetilde{\mathcal{I}}\|.$$

$(\mathcal{H}_2)$ There exists a $M_1 > 0$, such that

$$\left|\frac{\partial g(\eta,t)}{\partial t}\right| \le M_1, \quad \textit{for all} \quad t \in [0,\mathsf{T}].$$

Let $\aleph_{H_i} = \mathcal{L}_{H_i} + \mathcal{K}_{H_i} + \mathcal{G}_{H_i} + \mathcal{J}_{H_i}$ for $i = 1,2,3,4$ and $\Lambda_\Phi = \left(\frac{(1-\alpha)M_1}{\mathcal{GHF}(\alpha)} + \frac{\alpha M_1 \mathsf{T}^\beta}{\mathcal{GHF}(\alpha)\Gamma(\beta+1)}\right)$.

Theorem 3.1 *Assume that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. If $\Lambda_\Phi \sum_{i=1}^{4} \aleph_{H_i} < 1$, then system (3.1) has a unique solution.*

Proof: Let $\sup_{t\in[0,\mathsf{T}]} H_i(t,0,0,0,0) = \Omega_{H_i} < \infty$ for $i = 1,2,3,4$ and $\mathcal{B}_\mathcal{M} = \{(\mathcal{A},\mathcal{P},E,\mathcal{I}) \in \Omega : \|(\mathcal{A},\mathcal{P},E,\mathcal{I})\| \le \mathcal{M}\}$, with $\mathcal{M} \ge \frac{|\mathcal{A}_0|+|\mathcal{P}_0|+|E_0|+|\mathcal{I}_0|+\Lambda_\Phi \sum_{i=1}^{4}\Omega_{H_i}}{1-\Lambda_\Phi \sum_{i=1}^{4}\aleph_{H_i}}$. First, we prove that $\Phi(\mathcal{B}_\mathcal{M}) \subset \mathcal{B}_\mathcal{M}$.
For $(\mathcal{A},\mathcal{P},E,\mathcal{I}) \in \mathcal{B}_\mathcal{M}$, we have

$$
\begin{aligned}
\|\Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I})\| &= \max_{t\in[0,\mathsf{T}]} |\,\Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I})(t)\,| \\
&= \max_{t\in[0,\mathsf{T}]}\left\{\left|\frac{\mathcal{A}(0)w(0)}{w(t)}+\frac{1-\alpha}{\mathcal{GHF}(\alpha)}\frac{\partial g(\eta,t)}{\partial t}H_1(t,Y(t))\right.\right. \\
&\quad\left.\left.+\frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\frac{\partial g(\eta,s)}{\partial s}w(s)H_1(s,Y(s))ds\right|\right\} \\
&\le |\mathcal{A}(0)|+\frac{(1-\alpha)M_1}{\mathcal{GHF}(\alpha)}\max_{t\in[0,\mathsf{T}]}\Big(\,|\,H_1(t,\mathcal{A}(t),\mathcal{P}(t),E(t),\mathcal{I}(t))-H_1(t,0,0,0,0)\,| \\
&\quad+|\,H_1(t,0,0,0,0)\,|\Big)+\frac{\alpha M_1}{\mathcal{GHF}(\alpha)\Gamma(\beta)}\max_{t\in[0,\mathsf{T}]}\int_0^t(t-s)^{\beta-1}\frac{w(s)}{w(t)}H_1(s,Y(s))ds \\
&\le |\mathcal{A}_0|+\frac{(1-\alpha)M_1}{\mathcal{GHF}(\alpha)}\left(\mathcal{L}_{H_1}\|\mathcal{A}\|+\mathcal{K}_{H_1}\|\mathcal{P}\|+\mathcal{G}_{H_1}\|E\|+\mathcal{J}_{H_1}\|\mathcal{I}\|+\Omega_{H_1}\right) \\
&\quad+\frac{\alpha M_1\mathsf{T}^{\beta}}{\mathcal{GHF}(\alpha)\Gamma(\beta+1)}\left(\mathcal{L}_{H_1}\|\mathcal{A}\|+\mathcal{K}_{H_1}\|\mathcal{P}\|+\mathcal{G}_{H_1}\|E\|+\mathcal{J}_{H_1}\|\mathcal{I}\|+\Omega_{H_1}\right)
\end{aligned}
$$

Hence,

$$
\|\Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I})\| \le |\mathcal{A}_0|+\Lambda_\Phi\left(\aleph_{H_1}\mathcal{M}+\Omega_{H_1}\right). \tag{3.7}
$$

Similarly, we obtain

$$
\begin{aligned}
\|\Phi_2(\mathcal{A},\mathcal{P},E,\mathcal{I})\| &\le |\mathcal{P}_0|+\Lambda_\Phi\left(\aleph_{H_2}\mathcal{M}+\Omega_{H_2}\right), \\
\|\Phi_3(\mathcal{A},\mathcal{P},E,\mathcal{I})\| &\le |E_0|+\Lambda_\Phi\left(\aleph_{H_3}\mathcal{M}+\Omega_{H_3}\right), \\
\|\Phi_4(\mathcal{A},\mathcal{P},E,\mathcal{I})\| &\le |\mathcal{I}_0|+\Lambda_\Phi\left(\aleph_{H_4}\mathcal{M}+\Omega_{H_4}\right).
\end{aligned} \tag{3.8}
$$

By combining the inequalities (3.7) and (3.8), we get

$$
\begin{aligned}
\|\Phi(\mathcal{A},\mathcal{P},E,\mathcal{I})\| &\le |\mathcal{A}_0|+|\mathcal{P}_0|+|E_0|+|\mathcal{I}_0|+\Lambda_\Phi\sum_{i=1}^{4}\left(\mathcal{M}\aleph_{H_i}+\Omega_{H_i}\right) \\
&\le \mathcal{M}.
\end{aligned}
$$

Therefore, $\Phi(\mathcal{B}_\mathcal{M})\subset\mathcal{B}_\mathcal{M}$.
Secondly, we prove that Φ is a contraction. To do this, we consider $(\mathcal{A},\mathcal{P},E,\mathcal{I})$, $(\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{E},\widetilde{\mathcal{I}})\in\Omega$, for any $t\in[0,\mathsf{T}]$, we have

$$
\begin{aligned}
\left\|\Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I})-\Phi_1(\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{E},\widetilde{\mathcal{I}})\right\| &= \max_{t\in[0,\mathsf{T}]}\left|\Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I})(t)-\Phi_1(\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{E},\widetilde{\mathcal{I}})(t)\right| \\
&\le \frac{(1-\alpha)M_1}{\mathcal{GHF}(\alpha)}\max_{t\in[0,\mathsf{T}]}\Big[\Big|H_1(t,\mathcal{A}(t),\mathcal{P}(t),E(t),\mathcal{I}(t)) \\
&\quad-H_1(t,\widetilde{\mathcal{A}}(t),\widetilde{\mathcal{P}}(t),\widetilde{E}(t),\widetilde{\mathcal{I}}(t))\Big|\Big)+\frac{\alpha M_1}{\mathcal{GHF}(\alpha)\Gamma(\beta)} \\
&\quad\times\max_{t\in[0,\mathsf{T}]}\int_0^t(t-s)^{\beta-1}\frac{w(s)}{w(t)}\Big(\,|\,H_1(s,\mathcal{A}(s),\mathcal{P}(s),E(s),\mathcal{I}(s)) \\
&\quad-H_1(s,\widetilde{\mathcal{A}}(s),\widetilde{\mathcal{P}}(s),\widetilde{E}(s),\widetilde{\mathcal{I}}(s))\Big|\Big)ds\Big].
\end{aligned}
$$

Hence,

$$
\left\|\Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I})-\Phi_1(\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{E},\widetilde{\mathcal{I}})\right\| \le \Lambda_\Phi\aleph_{H_1}\left\|(\mathcal{A},\mathcal{P},E,\mathcal{I})-(\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{E},\widetilde{\mathcal{I}})\right\|. \tag{3.9}
$$

Similarly, we get

$$
\begin{aligned}
\left\|\Phi_2(\mathcal{A},\mathcal{P},E,\mathcal{I})-\Phi_2(\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{E},\widetilde{\mathcal{I}})\right\| &\le \Lambda_\Phi\aleph_{H_2}\left\|(\mathcal{A},\mathcal{P},E,\mathcal{I})-(\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{E},\widetilde{\mathcal{I}})\right\|, \\
\left\|\Phi_3(\mathcal{A},\mathcal{P},E,\mathcal{I})-\Phi_3(\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{E},\widetilde{\mathcal{I}})\right\| &\le \Lambda_\Phi\aleph_{H_3}\left\|(\mathcal{A},\mathcal{P},E,\mathcal{I})-(\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{E},\widetilde{\mathcal{I}})\right\|, \\
\left\|\Phi_4(\mathcal{A},\mathcal{P},E,\mathcal{I})-\Phi_4(\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{E},\widetilde{\mathcal{I}})\right\| &\le \Lambda_\Phi\aleph_{H_4}\left\|(\mathcal{A},\mathcal{P},E,\mathcal{I})-(\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{E},\widetilde{\mathcal{I}})\right\|.
\end{aligned} \tag{3.10}
$$

The inequalities (3.9) and (3.10) allow us to conclude that

$$\left\|\Phi\left(\mathcal{A},\mathcal{P},E,\mathcal{I}\right)-\Phi(\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{E},\widetilde{\mathcal{I}})\right\|\leq\left(\aleph_{H_1}+\aleph_{H_2}+\aleph_{H_3}+\aleph_{H_4}\right)\Lambda_\Phi\left\|\left(\mathcal{A},\mathcal{P},E,\mathcal{I}\right)-(\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{E},\widetilde{\mathcal{I}})\right\|.$$

Since $\Lambda_\Phi\sum_{i=1}^{4}\aleph_{H_i}<1$, we conclude that $\Phi\left(\mathcal{A},\mathcal{P},E,\mathcal{I}\right)$ is a contraction operator. According to the Banach contraction principle, Φ has a unique fixed point, which indicates that the solution to the system (3.1) is unique. □

Next, we consider the following hypotheses:
$(\mathcal{H}_3)$ Let $\mathcal{X}_{H_i},\mathcal{Y}_{H_i},\mathcal{Z}_{H_i},\mathcal{Q}_{H_i},\mathcal{W}_{H_i};(i=1,2,3,4):[0,\mathsf{T}]\to\mathbb{R}^+$ be functions, such that for all $\mathcal{A},\mathcal{P},E,\mathcal{I}\in\Omega$, we have

$$|H_i(t,\mathcal{A}(t),\mathcal{P}(t),E(t),\mathcal{I}(t))|\leq\mathcal{X}_{H_i}(t)+\mathcal{Y}_{H_i}(t)|\mathcal{A}(t)|+\mathcal{Z}_{H_i}(t)|\mathcal{P}(t)|+\mathcal{Q}_{H_i}(t)|E(t)|+\mathcal{Z}_{H_i}(t)|\mathcal{I}(t)|,$$

with $\sup_{t\in[0,\mathsf{T}]}\mathcal{X}_{H_i}(t)=\mathcal{X}^*_{H_i}$, $\sup_{t\in[0,\mathsf{T}]}\mathcal{Y}_{H_i}(t)=\mathcal{Y}^*_{H_i}$, $\sup_{t\in[0,\mathsf{T}]}\mathcal{Z}_{H_i}(t)=\mathcal{Z}^*_{H_i}$, $\sup_{t\in[0,\mathsf{T}]}\mathcal{Q}_{H_i}(t)=\mathcal{Q}^*_{H_i}$, $\sup_{t\in[0,\mathsf{T}]}\mathcal{W}_{H_i}(t)=\mathcal{W}^*_{H_i}$ and $\mathcal{X}^*_{H_i},\mathcal{Y}^*_{H_i},\mathcal{Z}^*_{H_i},\mathcal{Q}^*_{H_i},\mathcal{W}^*_{H_i}>0$.
$(\mathcal{H}_4)$ $\Lambda_\Phi\sum_{j=1}^{4}\psi_{H_j}<1$, where $\psi\in\{\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{Q},\mathcal{W}\}$ and

$$\Lambda_0=\min\left\{1-\Lambda_\Phi\sum_{j=1}^{4}\mathcal{Y}_{H_j},1-\Lambda_\Phi\sum_{j=1}^{4}\mathcal{Z}_{H_j},1-\Lambda_\Phi\sum_{j=1}^{4}\mathcal{Q}_{H_j},1-\Lambda_\Phi\sum_{j=1}^{4}\mathcal{W}_{H_j}\right\}.$$

Theorem 3.2 *Assume that $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Then system (3.1) has at least one solution.*

Proof: Initially, we provide the complete continuity of $\Phi:\Omega\to\Omega$. Since $H_i,(i=1,2,3,4)$ is continuous, Φ is likewise continuous. Let $\mathfrak{B}\subseteq\Omega$ be bounded set. Hence, there exist constants $\mathsf{C}_{H_i}>0$, such that $\max_{t\in[0,\mathsf{T}]}|H_i(t,\mathcal{A}(t),\mathcal{P}(t),E(t),\mathcal{I}(t))|\leq\mathsf{C}_{H_i}$, for all $(\mathcal{A},\mathcal{P},E,\mathcal{I})\in\mathfrak{B}$. Consequently, for any $(\mathcal{A},\mathcal{P},E,\mathcal{I})\in\mathfrak{B}$, we have

$$\begin{aligned}\|\Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I})\|\leq&|\mathcal{A}_0|+\frac{(1-\alpha)M_1}{\mathcal{GHF}(\alpha)}\max_{t\in[0,\mathsf{T}]}|H_1(t,\mathcal{A}(t),\mathcal{P}(t),E(t),\mathcal{I}(t))|\\&+\frac{\alpha M_1}{\mathcal{GHF}(\alpha)\Gamma(\beta)}\max_{t\in[0,\mathsf{T}]}\int_0^t(t-s)^{\beta-1}\frac{w(s)}{w(t)}|H_1\left(s,\mathcal{A}(s),\mathcal{P}(s),E(s),\mathcal{I}(s)\right)|\mathrm{d}s,\end{aligned}$$

which implies that

$$\|\Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I})\|\leq|\mathcal{A}_0|+\Lambda_\Phi\mathsf{C}_{H_1}\tag{3.11}$$

Similarly, we get

$$\begin{aligned}&\|\Phi_2(\mathcal{A},\mathcal{P},E,\mathcal{I})\|\leq|\mathcal{P}_0|+\Lambda_\Phi\mathsf{C}_{H_2},\\&\|\Phi_3(\mathcal{A},\mathcal{P},E,\mathcal{I})\|\leq|E_0|+\Lambda_\Phi\mathsf{C}_{H_3},\\&\|\Phi_4(\mathcal{A},\mathcal{P},E,\mathcal{I})\|\leq|\mathcal{I}_0|+\Lambda_\Phi\mathsf{C}_{H_4}.\end{aligned}\tag{3.12}$$

This proves the uniform boundedness of Φ.

Secondly, we demonstrate the equicontinuity of Φ. Consider $0<t_2<t_1<\mathsf{T}$, we have

$$\begin{aligned}|\ \Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I})(t_1)-\Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I})(t_2)\ |\leq&\left|\frac{(1-\alpha)M_1}{\mathcal{GHF}(\alpha)}(t_1-t_2)\right|\bigg|H_1\left(t_1,\mathcal{A}(t_1),\mathcal{P}(t_1),E(t_1),\mathcal{I}(t_1)\right)\\&-H_1\left(t_2,\mathcal{A}(t_2),\mathcal{P}(t_2),E(t_2),\mathcal{I}(t_2)\right)\bigg|\\&+\left|\frac{\alpha M_1\mathsf{C}_{H_1}}{\mathcal{GHF}(\alpha)\Gamma(\beta+1)}\left(t_1^\beta-t_2^\beta\right)\right|.\end{aligned}$$

Then

$$\lim_{t_1 \longrightarrow t_2} \mid \Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I})\,(t_1) - \Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I})\,(t_2) \mid = 0.$$

Analogously, we obtain

$$\begin{aligned}
&\lim_{t_1 \longrightarrow t_2} \mid \Phi_2(\mathcal{A},\mathcal{P},E,\mathcal{I})\,(t_1) - \Phi_2(\mathcal{A},\mathcal{P},E,\mathcal{I})\,(t_2) \mid = 0,\\
&\lim_{t_1 \longrightarrow t_2} \mid \Phi_3(\mathcal{A},\mathcal{P},E,\mathcal{I})\,(t_1) - \Phi_3(\mathcal{A},\mathcal{P},E,\mathcal{I})\,(t_2) \mid = 0,\\
&\lim_{t_1 \longrightarrow t_2} \mid \Phi_4(\mathcal{A},\mathcal{P},E,\mathcal{I})\,(t_1) - \Phi_4(\mathcal{A},\mathcal{P},E,\mathcal{I})\,(t_2) \mid = 0.
\end{aligned}$$

Thus, $\Phi(\mathcal{A},\mathcal{P},E,\mathcal{I})$ is equicontinuous. It follows from Arzela-Ascoli theorem that Φ is relatively compact, which confirms that Φ completely continuous.
It remains to show that $\mathfrak{B}(\Phi) = \{(\mathcal{A},\mathcal{P},E,\mathcal{I}) \in \Omega : (\mathcal{A},\mathcal{P},E,\mathcal{I}) = \lambda\Phi(\mathcal{A},\mathcal{P},E,\mathcal{I}), \lambda \in [0,1]\}$ is bounded. Let $(\mathcal{A},\mathcal{P},E,\mathcal{I}) \in \mathfrak{B}(\Phi)$. Then $(\mathcal{A},\mathcal{P},E,\mathcal{I}) = \lambda\Phi(\mathcal{A},\mathcal{P},E,\mathcal{I})$. For $t \in [0,\mathsf{T}]$, we have $\mathcal{A}(t) = \lambda\Phi_1(\mathcal{A},\mathcal{P},E,\mathcal{I})(t), \mathcal{P}(t) = \lambda\Phi_2(\mathcal{A},\mathcal{P},E,\mathcal{I})(t), E(t) = \lambda\Phi_3(\mathcal{A},\mathcal{P},E,\mathcal{I})(t)$ and $\mathcal{I}(t) = \lambda\Phi_4(\mathcal{A},\mathcal{P},E,\mathcal{I})(t)$. Hence,

$$\begin{aligned}
|\mathcal{A}(t)| \le |\mathcal{A}_0| + \left[\frac{(1-\alpha)M_1}{\mathcal{GHF}(\alpha)} + \frac{\alpha M_1 \mathsf{T}^\beta}{\mathcal{GHF}(\alpha)\Gamma(\beta+1)}\right] \Big(& \mathcal{X}_{H_1}(t) + \mathcal{Y}_{H_1}(t)|\mathcal{A}(t)| \\
& + \mathcal{Z}_{H_1}(t)|\mathcal{P}(t)| + \mathcal{Q}_{H_1}(t)|E(t)| + \mathcal{W}_{H_1}(t)|\mathcal{I}(t)|\Big),
\end{aligned}$$

which leads to

$$\|\mathcal{A}\| \le |\mathcal{A}_0| + \Lambda_\Phi \left(\mathcal{X}^*_{H_1} + \mathcal{Y}^*_{H_1}\|\mathcal{A}\| + \mathcal{Z}^*_{H_1}\|\mathcal{P}\| + \mathcal{Q}^*_{H_1}\|E\| + \mathcal{W}^*_{H_1}\|\mathcal{I}\|\right). \tag{3.13}$$

Similarly, we have

$$\begin{aligned}
\|\mathcal{P}\| &\le |\mathcal{P}_0| + \Lambda_\Phi \left(\mathcal{X}^*_{H_2} + \mathcal{Y}^*_{H_2}\|\mathcal{A}\| + \mathcal{Z}^*_{H_2}\|\mathcal{P}\| + \mathcal{Q}^*_{H_2}\|E\| + \mathcal{W}^*_{H_2}\|\mathcal{I}\|\right),\\
\|E\| &\le |E_0| + \Lambda_\Phi \left(\mathcal{X}^*_{H_3} + \mathcal{Y}^*_{H_3}\|\mathcal{A}\| + \mathcal{Z}^*_{H_3}\|\mathcal{P}\| + \mathcal{Q}^*_{H_3}\|E\| + \mathcal{W}^*_{H_3}\|\mathcal{I}\|\right),\\
\|\mathcal{I}\| &\le |\mathcal{I}_0| + \Lambda_\Phi \left(\mathcal{X}^*_{H_4} + \mathcal{Y}^*_{H_4}\|\mathcal{A}\| + \mathcal{Z}^*_{H_4}\|\mathcal{P}\| + \mathcal{Q}^*_{H_4}\|E\| + \mathcal{W}^*_{H_4}\|\mathcal{I}\|\right).
\end{aligned} \tag{3.14}$$

According to (3.13) and (3.14), we deduce that

$$\begin{aligned}
\|\mathcal{A}\| + \|\mathcal{P}\| + \|E\| + \|\mathcal{I}\| \le\; & |\mathcal{A}_0| + |\mathcal{P}_0| + |E_0| + |\mathcal{I}_0| + \Lambda_\Phi\left(\mathcal{X}^*_{H_1} + \mathcal{X}^*_{H_2} + \mathcal{X}^*_{H_3} + \mathcal{X}^*_{H_4}\right)\\
& + \Lambda_\Phi\left(\mathcal{Y}^*_{H_1} + \mathcal{Y}^*_{H_2} + \mathcal{Y}^*_{H_3} + \mathcal{Y}^*_{H_4}\right)\|\mathcal{A}\|\\
& + \Lambda_\Phi\left(\mathcal{Z}^*_{H_1} + \mathcal{Z}^*_{H_2} + \mathcal{Z}^*_{H_3} + \mathcal{Z}^*_{H_4}\right)\|\mathcal{P}\|\\
& + \Lambda_\Phi\left(\mathcal{Q}^*_{H_1} + \mathcal{Q}^*_{H_2} + \mathcal{Q}^*_{H_3} + \mathcal{Q}^*_{H_4}\right)\|E\|\\
& + \Lambda_\Phi\left(\mathcal{W}^*_{H_1} + \mathcal{W}^*_{H_2} + \mathcal{W}^*_{H_3} + \mathcal{W}^*_{H_4}\right)\|\mathcal{I}\|.
\end{aligned}$$

Therefore,

$$\|\mathcal{A},\mathcal{P},E,\mathcal{I}\| \le \frac{|\mathcal{A}_0| + |\mathcal{P}_0| + |E_0| + |\mathcal{I}_0| + \Lambda_\Phi\left(\mathcal{X}^*_{H_1} + \mathcal{X}^*_{H_2} + \mathcal{X}^*_{H_3} + \mathcal{X}^*_{H_4}\right)}{\Lambda_0}.$$

Consequently, $\mathfrak{B}(\Phi)$ is bounded. It follows from the alternative Leray-Schauder Theorem 2.2 that Φ has at least one fixed point, which guarantees the existence of at least one solution of (3.1). □

4. Ulam-Hyers stability

In this section, we investigate the Ulam-Hyers stability of system (3.1).
Let $\varepsilon > 0$ and consider the following inequality:

$$|{}^{FFR}D^{\alpha,\beta,\beta}_{0,t,w} Y(t) - H(t, Y(t))| \le \varepsilon. \tag{4.1}$$

We recall that system (3.1) is Ulam-Hyers stable if there exists a real number $V_H > 0$ such that for all $\varepsilon > 0$ and for each solution $Y \in \Omega$ of inequality (4.1), there exists a solution $\overline{Y} \in \Omega$ of (3.1) satisfying

$$|Y(t) - \overline{Y}(t)| \leq \varepsilon V_H, \text{ for all } t \in [0, \mathsf{T}].$$

The concept of Ulam-Hyers stability differs from that of asymptotic stability, which focuses on the convergence of the solution to the equilibrium state.

Let $\varepsilon > 0$ and $Y \in \Omega$ be a function satisfying (4.1). Then there exists a function $\chi(t)$ such that

(i) $|\chi(t)| \leq \varepsilon$,

(ii) ${}^{FRR}D^{\alpha,\beta,\beta}_{0,t,w} Y(t) = H(t, Y(t)) + \chi(t)$.

Obviously, we have the following result.

Lemma 4.1 *The solution of the following perturbed system*

$$\begin{cases} {}^{FRR}D^{\alpha,\beta,\beta}_{0,t,w} Y(t) = H(t, Y(t)) + \chi(t) \\ Y(0) = Y_0, \end{cases}$$

satisfies the following inequality:

$$\Bigg| Y(t) - \Bigg(\frac{w(0)Y(0)}{w(t)} + \frac{1-\alpha}{\mathcal{GHF}(\alpha)} \frac{\partial g(\eta, t)}{\partial t} H(t, Y(t)) + \frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \frac{\partial g(\eta, s)}{\partial s} w(s) H(s, Y(s)) ds \Bigg) \Bigg| \leq \Lambda_\Phi \varepsilon.$$

Theorem 4.1 *Assume that ($\mathcal{H}_1$) and ($\mathcal{H}_2$) hold. If $\Lambda_\Phi \sum_{i=1}^{4} \aleph_{H_i} < 1$, then system (3.1) is Ulam–Hyers stable.*

Proof: It follows from Theorem 3.1 that system (3.1) has a unique solution $\overline{Y} \in \Omega$. According to Lemma 4.1, we have

$$\begin{aligned}
\big| Y(t) - \overline{Y}(t) \big| &= \Bigg| Y(t) - \Bigg\{ \frac{\overline{Y}(0)w(0)}{w(t)} + \frac{\partial g(\eta, t)}{\partial t} H\big(t, \overline{Y}(t)\big) \frac{1-\alpha}{\mathcal{GHF}(\alpha)} \\
&\quad + \frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \frac{\partial g(\eta, s)}{\partial s} w(s) H\big(s, \overline{Y}(s)\big) \mathrm{d}s \Bigg\} \Bigg| \\
&\leq \Bigg| Y(t) - \Bigg\{ \frac{Y(0)w(0)}{w(t)} + \frac{\partial g(\eta, t)}{\partial t} H\big(t, Y(t)\big) \frac{1-\alpha}{\mathcal{GHF}(\alpha)} \\
&\quad + \frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \frac{\partial g(\eta, s)}{\partial s} w(s) H\big(s, Y(s)\big) \mathrm{d}s \Bigg\} \\
&\quad + \Bigg\{ \frac{Y(0)w(0)}{w(t)} + \frac{\partial g(\eta, t)}{\partial t} H\big(t, Y(t)\big) \frac{1-\alpha}{\mathcal{GHF}(\alpha)} \\
&\quad + \frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \frac{\partial g(\eta, s)}{\partial s} w(s) H\big(s, Y(s)\big) \mathrm{d}s \Bigg\} \\
&\quad - \Bigg\{ \frac{\overline{Y}(0)w(0)}{w(t)} + \frac{\partial g(\eta, t)}{\partial t} H\big(t, \overline{Y}(t)\big) \frac{1-\alpha}{\mathcal{GHF}(\alpha)} \\
&\quad + \frac{\alpha}{w(t)\mathcal{GHF}(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \frac{\partial g(\eta, s)}{\partial s} w(s) H\big(s, \overline{Y}(s)\big) \mathrm{d}s \Bigg\} \Bigg|.
\end{aligned}$$

Hence,

$$
\begin{aligned}
\|Y-\overline{Y}\| &\leq \Lambda_{\Phi}\varepsilon + \frac{(1-\alpha)M_1}{\mathcal{GHF}(\alpha)}\sum_{i=1}^{4}\aleph_{H_i}\|Y-\overline{Y}\| \\
&\quad + \frac{\alpha M_1 \mathsf{T}^{\beta}}{\mathcal{GHF}(\alpha)\Gamma(\beta+1)}\sum_{i=1}^{4}\aleph_{H_i}\|Y-\overline{Y}\| \\
&= \Lambda_{\Phi}\varepsilon + \Lambda_{\Phi}\sum_{i=1}^{4}\aleph_{H_i}\|Y-\overline{Y}\|.
\end{aligned}
$$

Therefore, we obtain

$$
\|Y-\overline{Y}\| \leq \varepsilon V_H, \quad \textit{with} \;\; V_H = \frac{\Lambda_{\Phi}}{1-\Lambda_{\Phi}\sum\limits_{i=1}^{4}\aleph_{H_i}}.
$$

This demonstrates that the system (3.1) is Ulam-Hyers stable. □

5. Numerical simulations

This section presents numerical simulations that use fractional orders over time to investigate the dynamic complexity of the forest biomass system's compartments. Let $t_n = n\triangle t$, where the discretization time step is $\triangle t$ and $n \in \mathbb{N}$. The numerical method [30] allows us to get

$$
\begin{cases}
\mathcal{A}(t_n) = \dfrac{\mathcal{A}(0)w(0)}{w(t_n)} + \dfrac{1-\alpha}{\mathcal{GHF}(\alpha)}\dfrac{\partial g(\eta,t_n)}{\partial t_n}H_1(t_n, Y(t_n)) \\
\qquad + \dfrac{\alpha(\triangle t)^{\beta}}{\mathcal{GHF}(\alpha)\Gamma(\beta+1)w(t_n)}\displaystyle\sum_{k=0}^{n} w(t_k)\dfrac{\partial g(\eta,t_k)}{\partial t_k}H_1(t_k, Y(t_k))\mathcal{R}^{\beta}_{n,k}, \\
\mathcal{P}(t_n) = \dfrac{\mathcal{P}(0)w(0)}{w(t_n)} + \dfrac{1-\alpha}{\mathcal{GHF}(\alpha)}\dfrac{\partial g(\eta,t_n)}{\partial t_n}H_2(t_n, Y(t_n)) \\
\qquad + \dfrac{\alpha(\triangle t)^{\beta}}{\mathcal{GHF}(\alpha)\Gamma(\beta+1)w(t_n)}\displaystyle\sum_{k=0}^{n} w(t_k)\dfrac{\partial g(\eta,t_k)}{\partial t_k}H_2(t_k, Y(t_k))\mathcal{R}^{\beta}_{n,k}, \\
E(t_n) = \dfrac{E(0)w(0)}{w(t_n)} + \dfrac{1-\alpha}{\mathcal{GHF}(\alpha)}\dfrac{\partial g(\eta,t_n)}{\partial t_n}H_3(t_n, Y(t_n)) \\
\qquad + \dfrac{\alpha(\triangle t)^{\beta}}{\mathcal{GHF}(\alpha)\Gamma(\beta+1)w(t_n)}\displaystyle\sum_{k=0}^{n} w(t_k)\dfrac{\partial g(\eta,t_k)}{\partial t_k}H_3(t_k, Y(t_k))\mathcal{R}^{\beta}_{n,k}, \\
\mathcal{I}(t_n) = \dfrac{\mathcal{I}(0)w(0)}{w(t_n)} + \dfrac{1-\alpha}{\mathcal{GHF}(\alpha)}\dfrac{\partial g(\eta,t_n)}{\partial t_n}H_4(t_n, Y(t_n)) \\
\qquad + \dfrac{\alpha(\triangle t)^{\beta}}{\mathcal{GHF}(\alpha)\Gamma(\beta+1)w(t_n)}\displaystyle\sum_{k=0}^{n} w(t_k)\dfrac{\partial g(\eta,t_k)}{\partial t_k}H_4(t_k, Y(t_k))\mathcal{R}^{\beta}_{n,k},
\end{cases}
\tag{5.1}
$$

where $\mathcal{R}^{\beta}_{n,k} = (n-k+1)^{\beta} - (n-k)^{\beta}$ and $\mathcal{GHF}(\alpha)$ is the normalization function defined by:

$$
\mathcal{GHF}(\alpha) = 1-\alpha+\frac{\alpha}{\Gamma(\alpha)}.
$$

Additionally, it is assumed that $g(t,\eta) = \frac{t^{2-\eta}}{2-\eta}$. The values of the others parameters are given from [26] (see, Table 1).

Table 1: Parameter values.

Parameters	Description	Value
λ	Constant input rate of nutrients	4
ν	Rate of nutrient consumption by forestry biomass	0.7
η_1	Rate of nutrient depletion	0.3
r	Adjusted intrinsic growth rate of forestry biomass	0.2
r_0	Constant	0.01
K	Forest biomass carrying capacity	70
a	Forest biomass harvesting rate	0.2
δ	New plantation rate	0.1
η_2	Forest biomass illegal logging	0.03
b	Forest biomass unit fixed price	0.35
τ	Forest biomass per unit tax	0.25
c	Fixed cost per unit of effort expanded to harvest forestry biomass	0.08
d	Constant	0.03
μ	Maximum growth due to alternative industries	0.1
η_3	Reduction rate of industries	0.3

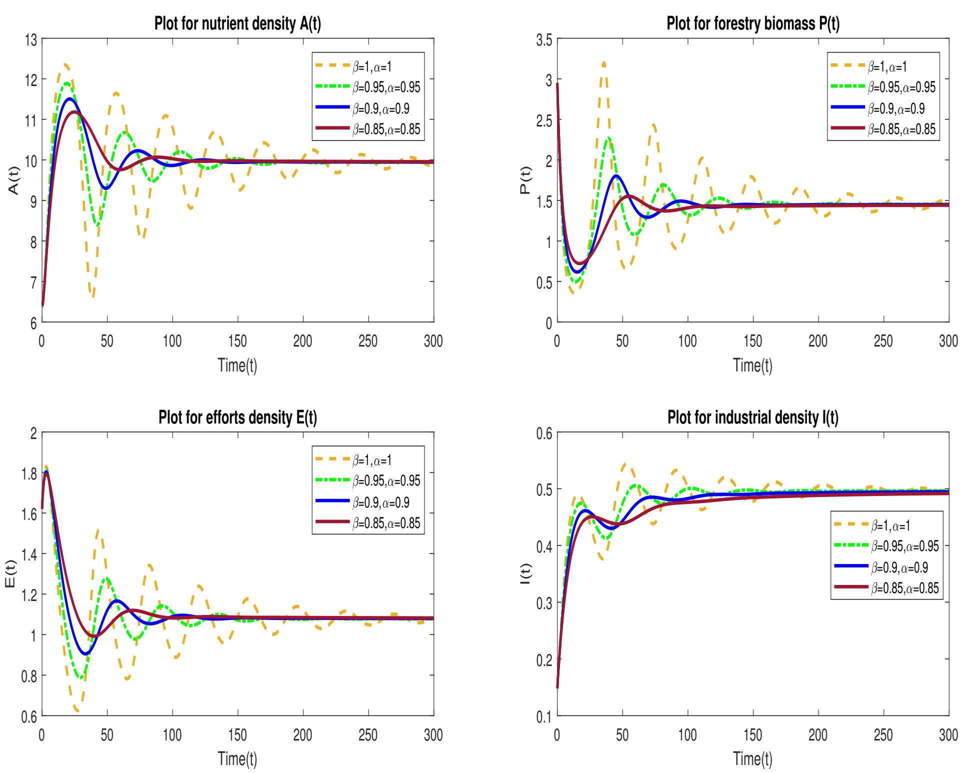

Figure 1: Solutions of the system (3.1) with $\eta = 1$ and varying α and β.

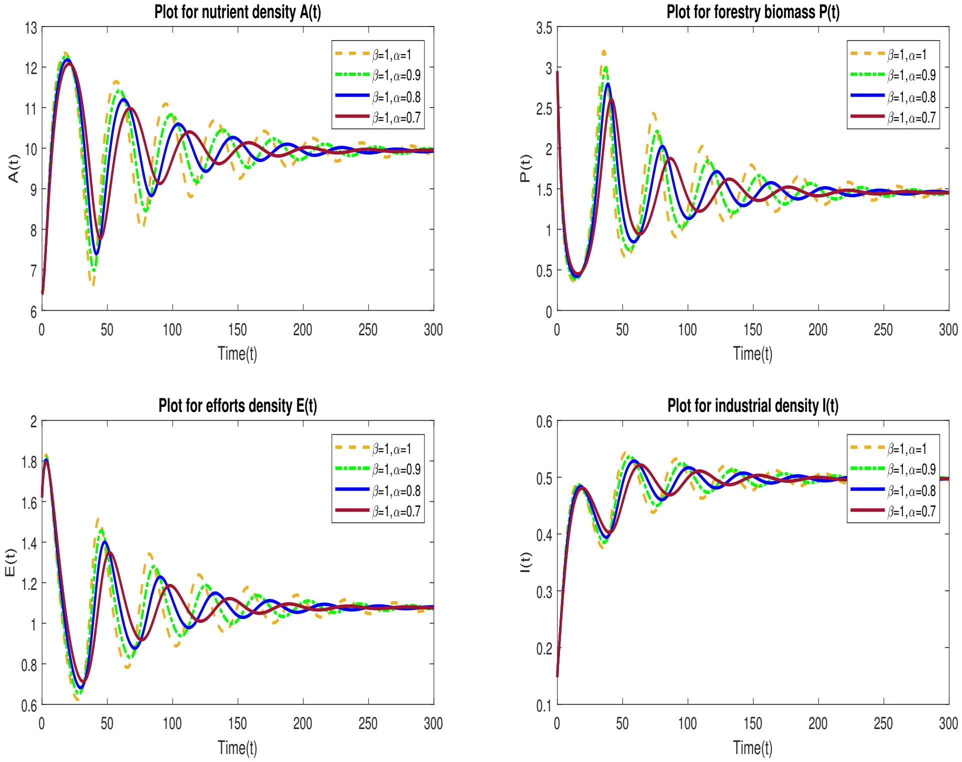

Figure 2: Solutions of the system (3.1) with $\eta = 1$, $\beta = 1$ and different values of α.

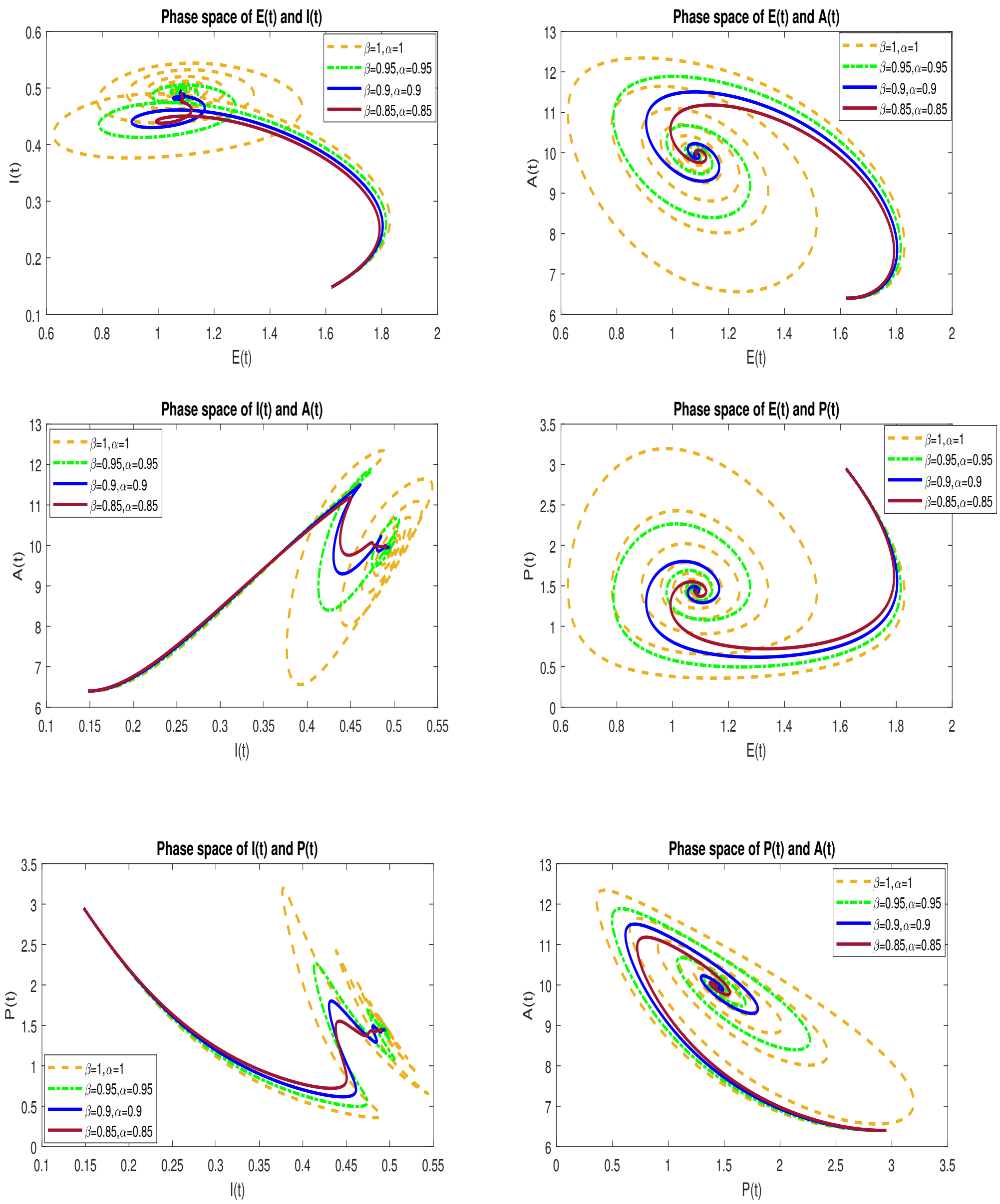

Figure 3: Dynamical behavior of the system (3.1) with $\eta = 1$ and varying α and β.

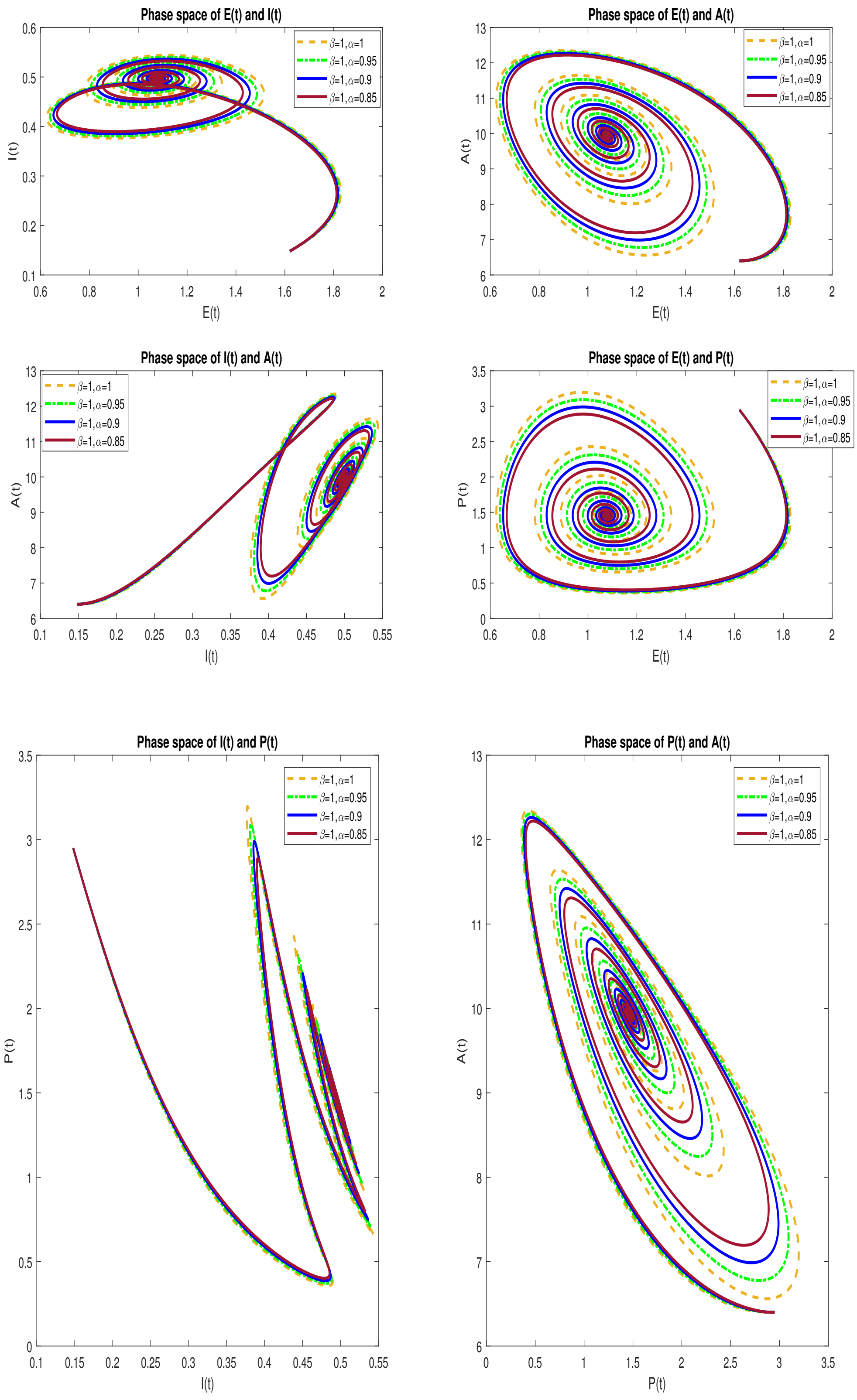

Figure 4: Dynamical behavior of the system (3.1) with $\eta = 1$, $\beta = 1$ and different values of α.

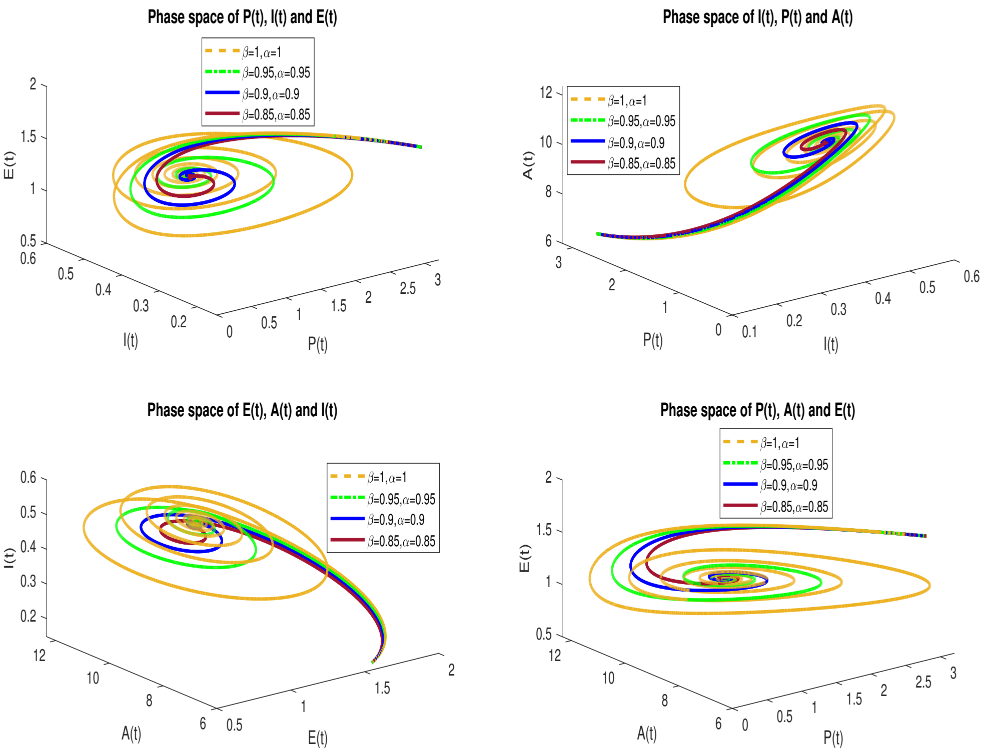

Figure 5: Phase space of the system (3.1) with $\eta = 1$, $\beta = 1$ and different values of α.

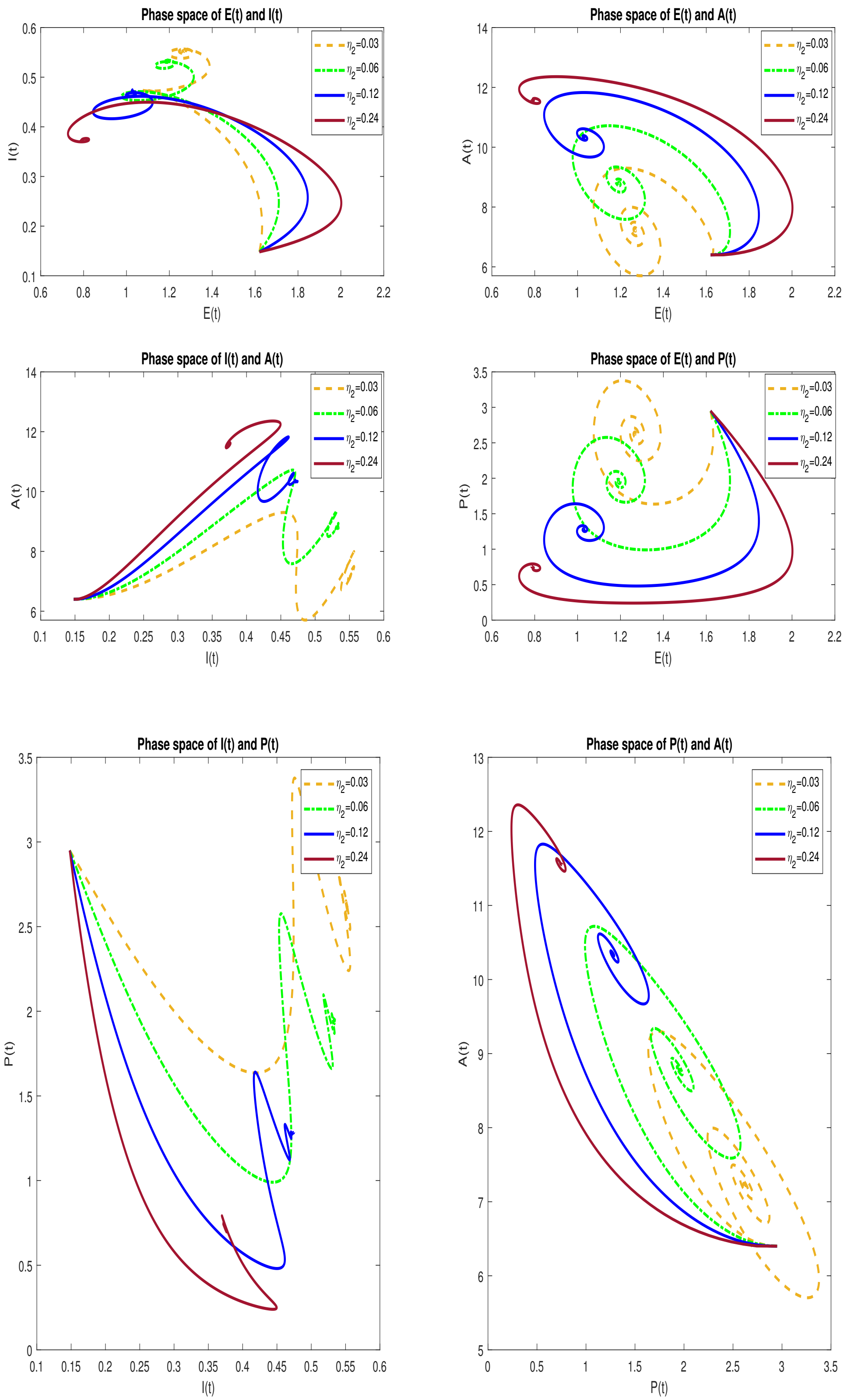

Figure 6: Dynamical behavior of the system (3.1) with $\eta = 1$ and different values of η_2.

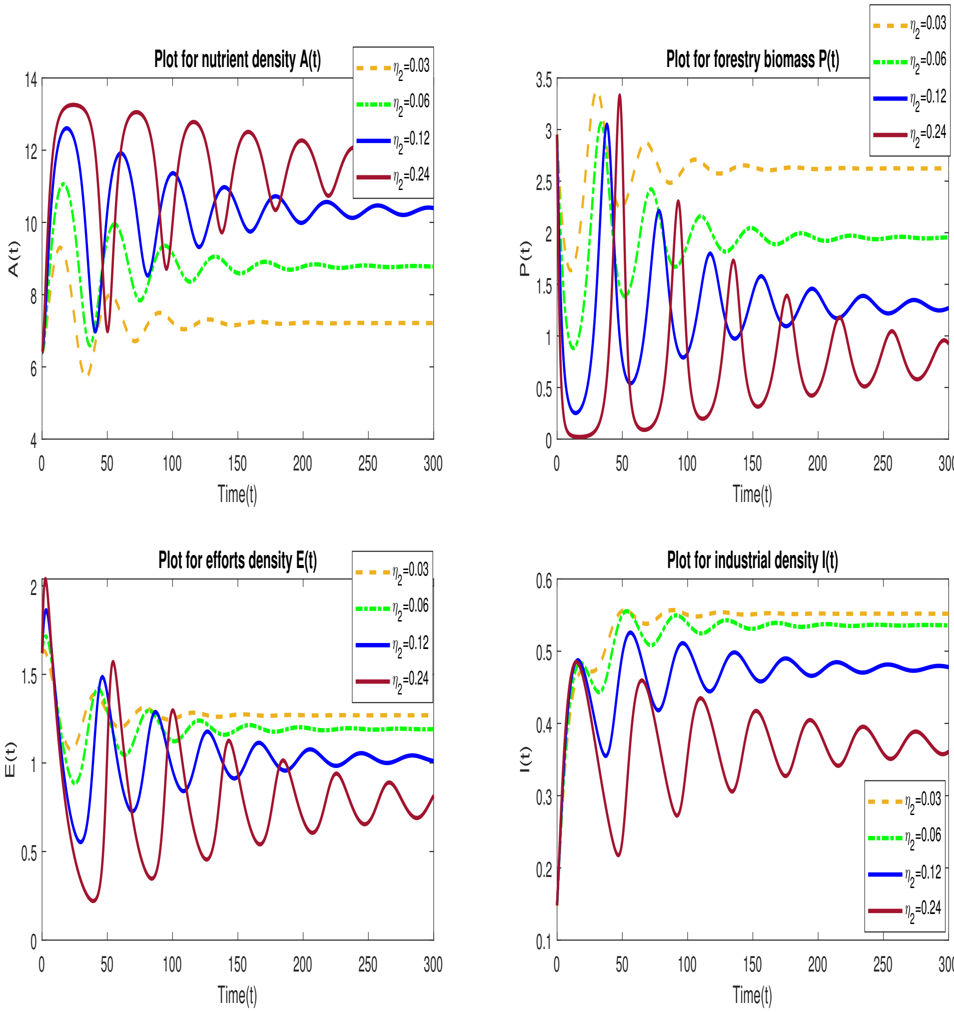

Figure 7: Solutions of the system (3.1) with $\eta = 1$ and different values of η_2.

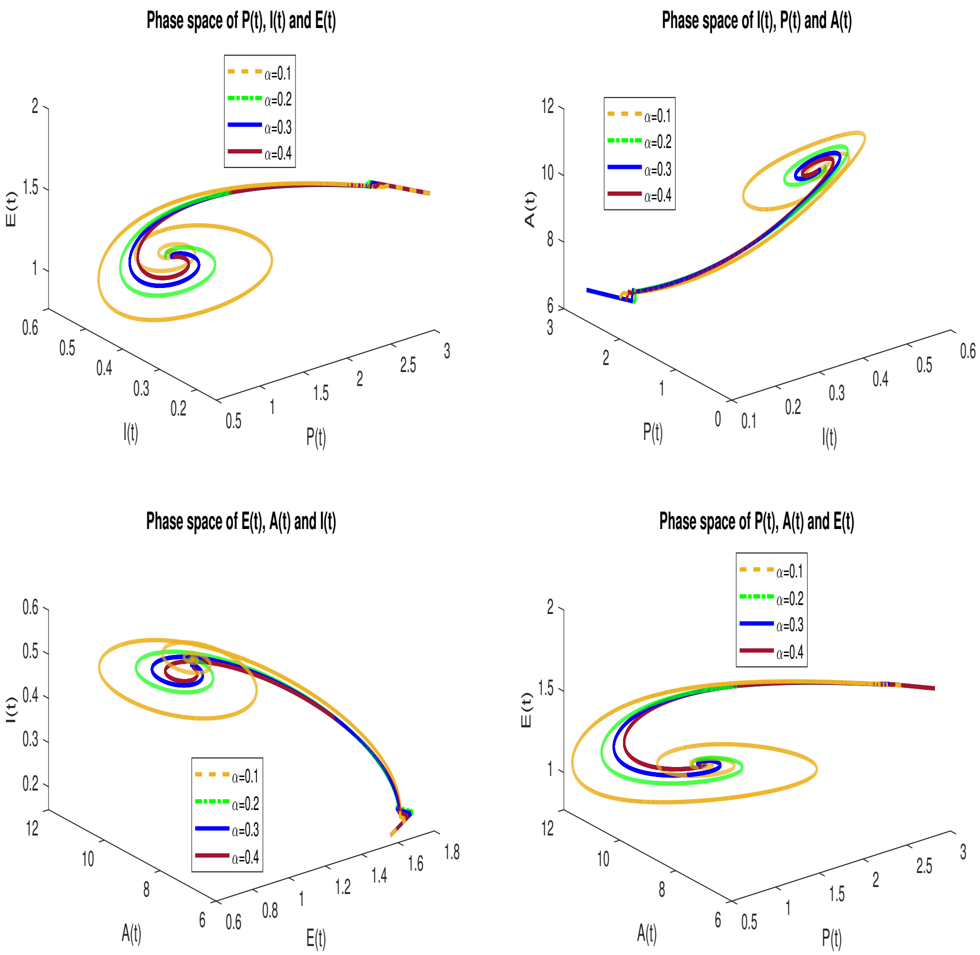

Figure 8: Phase space of the system (3.1) with $\eta = 1$ and different values of α.

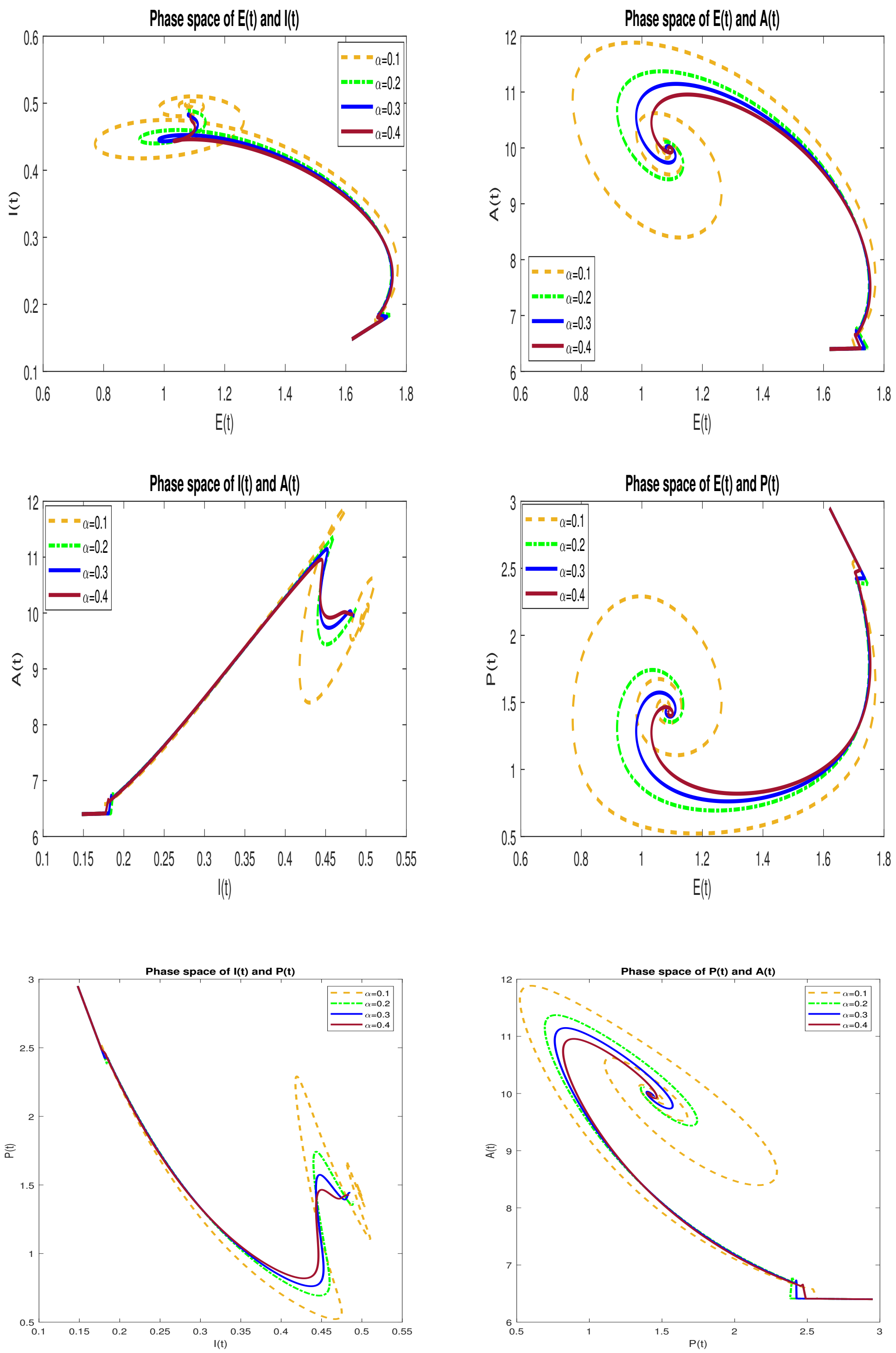

Figure 9: Dynamical behavior of the system (3.1) with $\eta = 1$ and different values of α.

First, Figure 1 illustrates the time series plots of the forestry biomass model (3.1) at various fractional orders and fractal dimensions using the fractal-fractional Hattaf derivative. The nutrient density fluctuates at different rates for combinations of α and β before stabilizing at an equilibrium point during the simulation. In contrast, the forestry biomass shows an initial sharp decline, followed by an increase, and then a gradual decrease across different fractional orders and dimensions. A similar pattern is observed in the effort intensity, which fluctuates along various pathways depending on the fractional orders. The industrial density, however, increases rapidly before stabilizing over time. Additionally, when the fractal dimension is fixed at $\beta = 1$, Figure 2 analyzes the behavior of all state variables in the proposed model, demonstrating that the model quickly achieves stability when the fractional order α is varied, compared to the integer-order model.

Second, Figure 3 depicts the two-dimensional dynamic behavior of the forestry biomass model (3.1) across different fractional orders and fractal dimensions using the fractal-fractional Hattaf derivative. Figure 4 shows the two-dimensional dynamic behavior of the same model with varying fractional orders α and a fixed β of 1. Figure 5 presents the three-dimensional dynamic behavior of the forestry biomass model (3.1) for different fractional orders and fractal dimensions using the fractal-fractional Hattaf derivative. Figure 6 depicts the 2D dynamic behavior of the forestry biomass model (3.1) under various rates of illegal logging, with values of $\eta_2 = 0.03$, 0.06, 0.12, and 0.24. Figure 7 shows the time series plots for these illegal logging rates, revealing that an increase in illegal logging leads to a decline in forest biomass.

Finally, Figure 8 illustrates the three-dimensional dynamic behavior of the forestry biomass model (3.1) for different fractional orders of α, while Figure 9 depicts the two-dimensional dynamic behavior of the same model across various fractional orders of α.

6. Conclusion

In this work, we have proposed a fractal-fractional model to describe the dynamics of a forest biomass model under the Hattaf fractal-fractional operator. The conditions for existence and uniqueness have been successfully established using fixed-point theory and the Ulam-Hyers conditions have been rigorously investigated. Furthermore, the impact of illegal logging on biomass utilization and conservation, along with the effects of taxation and industrial development on biomass sustainability, have been studied. The results show that even minor changes in the order of derivatives and system parameters can significantly affect the graphical representation of the system's behavior, suggesting that the fractal-fractional technique is well-suited to explain the complex dynamic configurations under consideration. The findings also reveal that forest biomass can be preserved more efficiently and economically through government-imposed taxes. Moreover, in the absence of industrialization, forest biomass densities remain high. However, with industrialization, biomass levels stabilize at lower densities, underscoring the detrimental impact of industrial activities on forest ecosystems. This reduction in biomass not only affects wildlife but also contributes to global warming. In future work, we plan to use other types of fractional operators such as [31,32] in order to model the complex dynamics of forestry biomass and investigate the impact of illegal logging and industrial effort on this forestry biomass.

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