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Rental Cost minimization for FSSP with equipotential machines

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ABSTRACT: Scheduling, the method that deals with arranging the jobs on machines in industries or preparing a sequence of programs to be run on computers or in setting a time table and in many other such situations, is a powerful tool to handle these conditions easily. Usually, in industries for doing specific tasks, there are multiple machines instead of a single one. But all the machines are not having the same operational cost. So, jobs are assigned to these machines in a sequence so that the unit operational cost can be reduced to minimum along with reducing the total elapsed time. The present paper gives a brief description of a situation where the first stage of jobs is completed on equipotential machines and for the second stage the machine is taken on rent. So, the objective considered in this paper is to arrange the jobs in such a way that the rental cost can be minimized along with the total elapsed time.

Key Words: Utilization time; Rental cost; Earliest time

Contents

1. Introduction

Scheduling of jobs is day to day problem faced by industries. Flow shop scheduling problem(FSSP) is the scheduling problem that occurs frequently in manufacturing industry. To schedule a job on a machine so that the time taken to complete all the jobs can be minimized is the classical situation that is to be handled in the industries on a regular basis. When the machines are rented, one of the major issues to be taken care of by the managers is to reduce the rental cost to its minimum. Preparing a schedule of jobs in such a way that the output is maximized in minimum time and with minimum cost is the foundation of the scheduling theory. Along with the basic needs of scheduling, there are many other points of comparison that are to be taken care of while preparing a schedule. To complete the job before due date, to minimize the transportation cost, or to take minimum time while setting up machines are some different aspects that are considered as and when needed while scheduling.
In 1954 Johnsons[10] gave an algorithm for the first time to solve such problems. He considered two-stage as well as three-stage scheduling problems in his papers. Ignall et al.[9] worked on the model to develop an algorithm called branch and bound (B&B) method that can solve the two stage problems in a systematic way very easily. Lomniciki[11] solved three stage problems with the help of B&B method.

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Mokotoff[17][18] studied the problems related to parallel machines. Maggu et al.[13][12] worked with the different aspects related to flow shop scheduling problems like transportation cost, equivalent jobs, job block criteria etc. Different heuristic and meta heuristic methods were also developed by different mathematicians to solve the problems under different environments. Aggoune[1], Ho et al.[8], Gemmill[2] devised different heuristic methods. Rental cost also play an important role while studying the cost in the industries as many a times machines are hired on rent. Singh T. P. et al.[19] worked on the concept of rental cost and devise the algorithm to find the solution of such problems. Gupta[7][3] further researched on flow shop scheduling theory under different circumstances and using different parameters. The research by Gupta was further extended by Goel to solve the two stage scheduling problem [6, 15] containing equipotential machines at first stage as well as on both the stages. Further the research continued to solve the problems of three stage[4] with the equipotential machines. Malhotra et al. [14] gave a comparison beween branch and bound method and meta-heuristic methods. Mansouri et al. [16] are also working on parallel machine fss problems. Present paper is the work on rental cost and using the concepts used in the paper by Goel et al.[5].

2. Practical Situation

The model of equipotential machines, which involves using identical machines for initial stages of production followed by a rented machine for subsequent specialized tasks, has diverse applications across various industries. In manufacturing and production lines, this model is used where multiple identical machines handle the same processing tasks, and a rented machine is employed for finishing or quality control. Similarly, in food processing and the pharmaceutical industry, it facilitates the use of uniform machines for primary tasks, with rented equipment used for packaging or labeling. The textile and electronics industries benefit from this model by employing identical machines for initial production phases and renting specialized equipment for dyeing, finishing, or testing. In the automotive sector, it supports high-throughput assembly with rented machines for painting or final inspection. Additionally, in printing and publishing, this approach allows for efficient printing operations with rented equipment for binding or laminating. This model is particularly valuable in settings where the cost-effectiveness of purchasing specialized equipment is outweighed by the benefits of rental flexibility.

3. Problem Formulation

The problem considered in this paper is two stage FSSP where in first stage there are equipotential machines to carry out the jobs and the jobs are scheduled on these machines in such a way that the utilization cost of the machines can be minimised. For the second stage there is one rental machine for the jobs. The rent is to be calculated according to the total utilization time of the machine. So to save the rent the idle time of the machine is to be reduced. The concept of earliest time is also used. Some assumptions are also to be taken into consideration

- All equipotential machines of first stage are available at time zero and have different utilization cost.
- All jobs are to processed on the machine of stage one first and then on the machine of stage second.
- Machine is to be rented when needed and returned back as soon as the task is complete
- Jobs are not required to be processed on all the equipotential machines of stage one.

4. Mathematical Model

The problem considered in this paper consist of k equipotential machines of type M and a single machine of type N. The number of jobs to be executed on the machine M first and then on machine N is p. The time taken by job i {i=1, 2, 4, ..., p} to be processed, on machine M is κ_i and on machine N is η_i. Per unit cost of processing job i on machine $M_j\{j = 1, 2, 3, ..., k\}$ is ζij. Time available on machine M_j is α_j. The mathematical representation of the data is given in Table 1.

Table 1: Mathematical Representation of the Data

Job / Machine	M_1	M_2	$\cdots$	M_k	Processing Time on M (κ_i)	Processing Time on N (η_i)
1	ζ_{11}	ζ_{12}	$\cdots$	ζ_{1k}	κ_1	η_1
2	ζ_{21}	ζ_{22}	$\cdots$	ζ_{2k}	κ_2	η_2
3	ζ_{31}	ζ_{32}	$\cdots$	ζ_{3k}	κ_3	η_3
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
p	ζ_{p1}	ζ_{p2}	$\cdots$	ζ_{pk}	κ_p	η_p
Weight	α_1	α_2	$\cdots$	α_k		

5. Proposed Methodology

Step 1: Optimized Processing Time using MODI Method The objective is to find the optimized processing time for machines M_j such that the total cost is minimized.

For applying the Modified Distribution Method (MODI), the following condition must be satisfied:

$$\sum_{j=1}^{k} \alpha_j = \sum_{i=1}^{p} \kappa_i$$

Step 2: Calculation of l_t for Job Sequencing

To determine the job that should be scheduled first, compute the value of l_t for each job $t = 1, 2, \ldots, p$ using the following formula:

$$l_t = \max \left\{ \sum_{i=1}^{p} \eta_i + \max_{1 \le j \le k} \zeta_{tj}, \quad \max_{1 \le j \le k} \left(\sum_{i=1}^{p} \zeta_{ij} \right) + \min_{k \ne t} \eta_k \right\}$$

Step 3 : Repeat step 2 by considering the job t is processed. Continue the process till the optimal sequence is achieved.

Step 4 : Prepare the table representing the in-out times of jobs on the machines $M_1, M_2, ..., M_k$ and N.

Step 5 : Calculate the Earliest time by subtracting the sum of the processing times of all jobs on machine N from the total elapsed time i.e. the time when the machine N ends processing the last job in the in-out table.

Step 6 : By deducting the beginning time of the first task on machine N from the end time of the final job on the same machine, the Utilization Time of the machine can be determined.

Step 7 : Once the new in-out table has been created, machine N's utilisation time can be deter- mined and will undoubtedly be lower than the previous utilisation time discovered in step 6 if the starting time of the first job on machine N is less than the Earliest time. Job 1's starting time can then be assigned as the Earliest time. The rental cost will therefore reduce if the Utilisation time does too.

Rental Cost = Utilazation Time * Rent

6. Numerical Problem

Illustration 1 : Here is an example that shows the application of the model in which 3 machines of type A are taken into consideration and 5 jobs are to be completed on these machines. The table containing the data in the format of Table 1 is given in Table 2.

Step 1: $\sum_{i=1}^{p} \kappa_i = 96 = \sum_{j=1}^{k} \alpha_j$

The condition for the application of the MODI method is satisfied. Hence, after applying the method, the table that gives the processing times of the jobs on all the machines is presented in Table 3.

Table 2: Numerical Example 1

Job	**A1**	**A2**	**Proc. Time on A**	**Proc. Time on B**	**A**	**B**
1	5	2	4	20	7	
2	6	1	3	25	3	
3	2	8	9	23	6	
4	3	8	7	18	4	
5	4	3	2	10	2	
Total	20	22	25	96	22	

Table 3: Processing Times of Jobs on Machine A and B

Job	**A1**	**A2**	**A3**	**Proc. Time on B**
1	–	3	17	7
2	–	25	–	3
3	23	–	–	6
4	9	–	9	4
5	–	–	10	2
	32	28	36	22

Step 2: Calculate the lower bounds:

$$l_1 = \max\{39, 38\} = 39$$
$$l_2 = \max\{47, 38\} = 47$$
$$l_3 = \max\{45, 38\} = 45$$
$$l_4 = \max\{31, 38\} = 38$$
$$l_5 = \max\{32, 39\} = 39$$

The job with the minimum lower bound value is job 4 (with $l_4 = 38$). Therefore, **job 4 is to be processed first in the sequence**.

Step 3: To find the optimal sequence, calculate:

$$l_{41} = \max\{48, 38\} = 48$$
$$l_{42} = \max\{47, 38\} = 47$$
$$l_{43} = \max\{54, 38\} = 54$$
$$l_{45} = \max\{41, 39\} = 41$$

Among these, the minimum is $l_{45} = 41$, so the job to be processed next after job 4 is **job 5**.

Continuing similarly, we calculate:

$$l_{451} = \max\{58, 39\} = 58$$
$$l_{452} = \max\{47, 42\} = 47$$
$$l_{453} = \max\{54, 39\} = 54$$

For further branching of l_{452}:

$$l_{4521} = \max\{58, 42\} = 58$$
$$l_{4523} = \max\{54, 43\} = 54$$

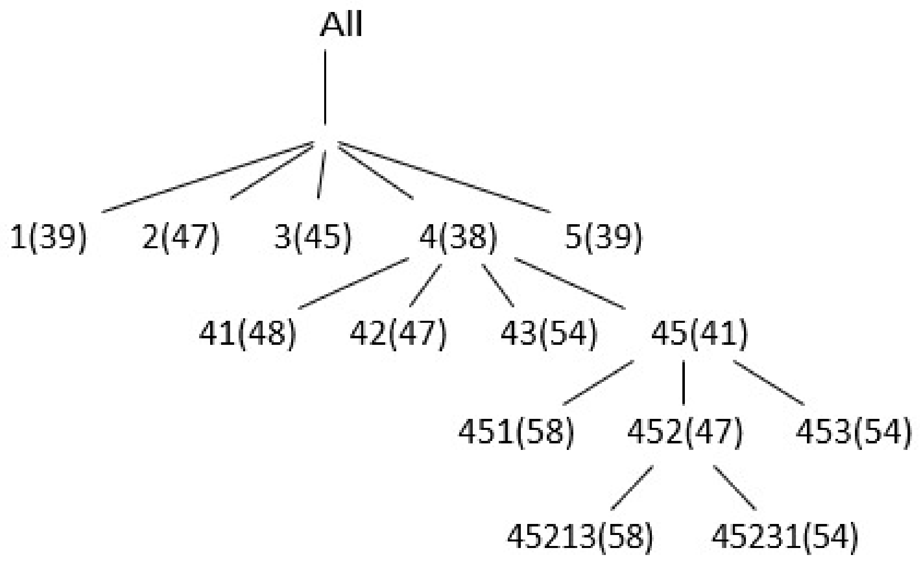

Figure 1: Tree representing the stages of branch and bound method

Hence, the optimal sequence of jobs turns out to be as follows:

$$\boxed{4 \rightarrow 5 \rightarrow 2 \rightarrow 3 \rightarrow 1}$$

This branching process is illustrated in Figure 1.

Step 4 : The in-out table is given in Table 4.

Table 4: In-Out Table according to Step 4

Job	A1	A2	A3	B
4	0–9	–	0–9	9–13
5	–	–	9–19	19–21
2	–	0–25	–	25–28
3	9–32	–	–	32–38
1	–	25–28	19–36	38–45

Step 5 : Earliest Time = 45-22 = 23 hours

Step 6 : Utilization Time = 45 - 9 = 36 h

Step 7 : The first job starts on machine B at the 9th hour although the earliest time is 23 Hrs. So, the first job can be assigned to machine B at 23 Hrs, then the new in-out table is represented as Table 5.

Now, usage time = 45 - 23 = 22 h. Tables 4 and Table 5 now make it abundantly evident that, while machine B's utilization time is 36 hours when using the B&B technique, it is only 22 hours when utilizing the suggested method- ology. If the rental cost of machine B is Rs. 4320/- per hour, then the cost can be decreased by Rs. 60,480/-.

Illustration 2 : The problems in which number of jobs is small or the number of equipotential machines is less can be calculated manually, but in the actual scenario the number of machines or the number of jobs is not so small that it can be interpreted manually. Sometimes, it may take hours of calculations to solve the problems. So to solve the calculations the mathematical programming software MATLAB is used. In the illustration MATLAB 2014a is used to find the hiring time of machine B and to plot the

Table 5: Final In-Out Table

Job	A1	A2	A3	B
4	0–9	–	0–9	23–27
5	–	–	9–19	27–29
2	–	0–25	–	29–32
3	9–32	–	–	32–38
1	–	25–28	19–36	38–45

Gantt Chart. Table 6 shows the data that was taken into consideration, and Table 7 shows the outcomes of using the suggested procedure in MATLAB.

Table 6: Numerical Example 2

Job	A1	A2	A3	A4	A5	Proc. A	Proc. B
1	99	96	95	98	97	50	2
2	70	65	60	65	65	38	5
3	90	80	48	70	62	30	8
4	99	95	87	95	99	20	6
5	65	70	48	67	60	44	3
6	85	88	75	82	80	46	4
7	75	70	71	73	76	55	5
8	70	51	93	94	92	10	1
9	85	84	80	86	90	29	1
10	92	91	93	80	89	36	7
	80	63	93	52	70		

Table 7: In-Out Table

Job	A1	A2	A3	A4	A5	B (B&B)	B (Present Algo)
10	–	–	–	0–36	–	36–43	54–61
1	0–50	–	–	–	–	50–52	61–63
2	–	–	–	36–50	0–24	52–57	63–68
3	–	–	0–30	–	–	57–65	68–76
4	–	–	30–49	50–51	–	65–71	76–82
8	–	0–10	–	–	–	71–72	82–83
7	50–51	10–63	–	51–52	–	72–77	83–88
6	–	–	–	–	24–70	77–81	88–92
9	51–80	–	–	–	–	81–82	92–93
5	–	–	49–93	–	–	93–96	93–96

The gantt chart given by Figure 2 gives the times of the jobs on machine B. Blue bars gives the time obtained using the BB algorithm while the red tiles shows the times obtained using the present algorithm. In earlier case the utilization time of machine B is from 36th hour to 96th hour i.e. in total 60 hours while on the other hand the utilization time is 42 hours. The machine B is to be hired at 54th hour to minimize the rental cost. This difference in time can be viewed through the starting point of first blue tile and the starting point of first red tile in the graph.

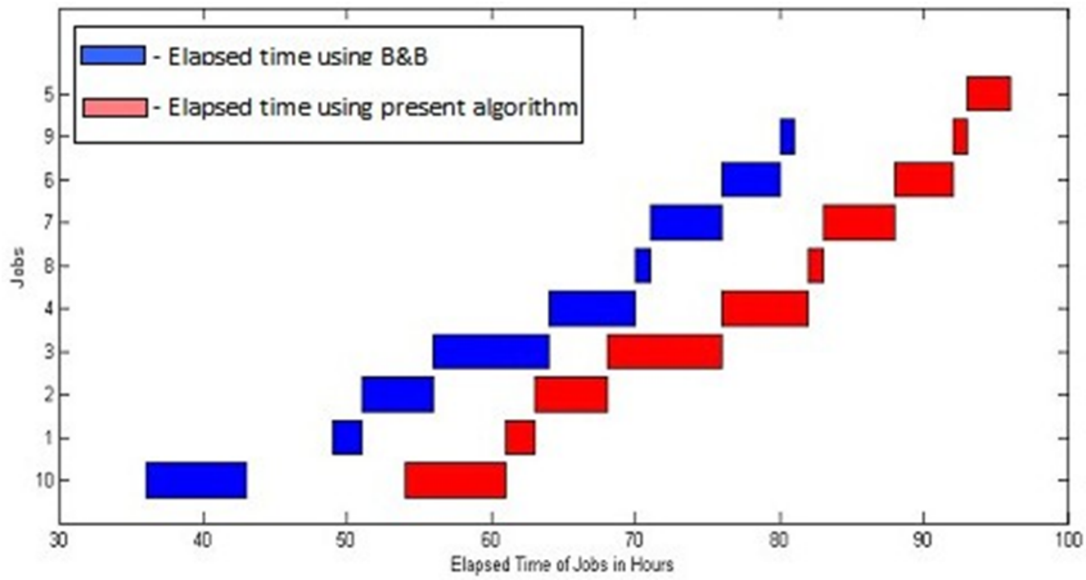

Figure:2 Comparison of Elapsed Time using B&B and Present Algorithm

7. Computational Results

The MATLAB code is rigorously tested with various configurations, involving different numbers of jobs and machines, to ensure its accuracy and reliability. The tests demonstrated the code's capability to handle diverse scheduling scenarios and produce optimal or near-optimal solutions. This comprehensive testing validates the robustness of the MATLAB implementation and its applicability to real-world industrial scheduling problems. Table 8 lists the outcomes that were thus obtained. The table lists the number of jobs and the rent obtained with the different number of machines using b&b method and the present modification in b&b method.The results obtained using the current algorithm and the b&b approach are shown differently in Figure 3 and Figure 4.

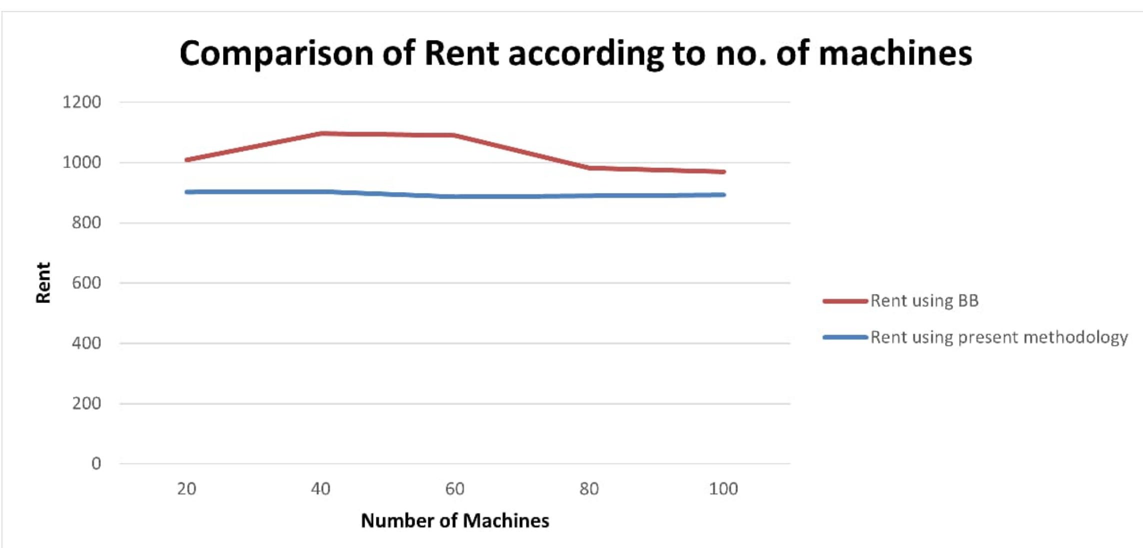

Figure 3: Comparison Chart 1

Table 8: Comparison of Rent using B&B and Present Algorithm

No. of Jobs	No. of Machines	Rent using B&B	Rent using Present Algorithm
20	20	306	273
20	40	313	291
20	60	339	298
20	80	352	311
20	100	341	303
40	20	633	608
40	40	591	585
40	60	589	581
40	80	657	638
40	100	603	593
60	20	962	902
60	40	928	917
60	60	1054	909
60	80	854	845
60	100	1201	926
80	20	1370	1211
80	40	1201	1193
80	60	1394	1122
80	80	1401	1190
80	100	1203	1189
100	20	1768	1515
100	40	2447	1531
100	60	2072	1518
100	80	1649	1466
100	100	1499	1455

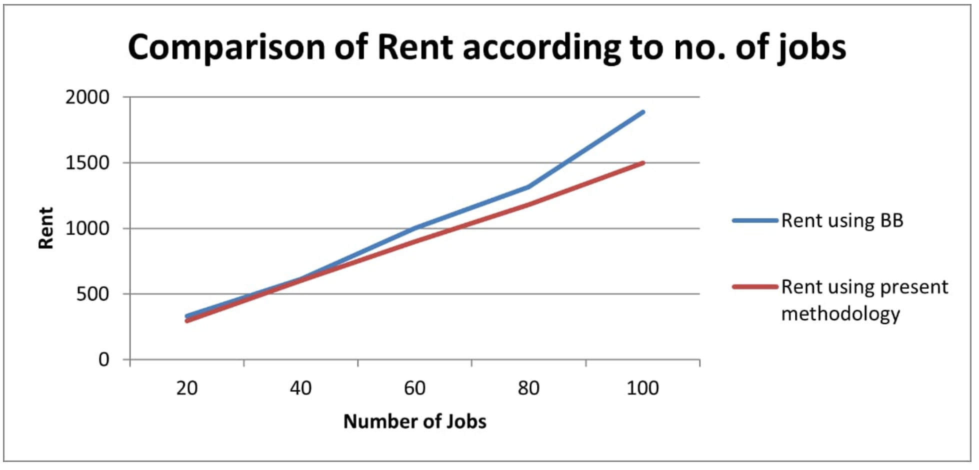

Figure 4: Comparison Chart 1

8. Conclusion

In conclusion, this research paper presents a novel approach to solving FSSP by extending the traditional B&B method. The proposed methodology, which integrates equipotential machines at the first stage with a rented machine at the second stage, demonstrates significant improvements over the conventional B&B method. By introducing modifications to the B&B approach, the new methodology effectively addresses the complexities associated with the two-stage scheduling problem, yielding better quality solutions with enhanced computational efficiency. The empirical results, obtained through extensive testing with various job and machine configurations, validate the effectiveness of the proposed methodology in optimizing flow shop schedules. This advancement not only provides a more robust and flexible solution framework but also contributes valuable insights into the application of scheduling theory in real-world scenarios involving rented resources.

9. Future Work

For future work, it is proposed to extend the investigation of the proposed methodology by testing its performance against various heuristic methods. This will provide a comparative analysis to assess the relative advantages of the new approach in different contexts. Additionally, the methodology will be evaluated with a focus on diverse performance measures, including transportation costs, weight considerations, and applications in fuzzy environments. By incorporating these factors, the research aims to explore the robustness and adaptability of the proposed method in addressing complex and realistic scheduling scenarios. This comprehensive evaluation will enhance the understanding of the methodology's practical applicability and effectiveness, paving the way for further refinements and broader implementations in the field of flow shop scheduling.

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