Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–6.
ISSN-0037-8712
doi:10.5269/bspm.76550

Some Remarks on Pseudocompactness

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ABSTRACT: In this paper, we introduced the notion of d_G-pseudocompactness of topological spaces with respect to some metric space (G, d) and we gave characterization for a topological space to be d_G-pseudocompact with respect to some metric group (G, d). We investigated the relationship of d_G-pseudocompactness with other types of compactness and found some conditions under which the notion of d_G-pseudocompactness becomes equivalent to pseudocompactness.

Key Words: d_G-pseudocompact, μ_G-space, G-regular, $G^\star$-regular, $G^{\star\star}$-regular.

Contents

1. Introduction

A topological space X is pseudocompact if $f(X)$ is a bounded subset of $\mathbb{R}$ for all continuous real valued function f on X. X is feebly compact if every locally finite family of non-empty open subsets of X is finite. Both pseudocompactness and feebly compactness have been studied explicitly in [3,4]. It is shown that a Tychonoff space is pseudocompact if and only if it is feebly compact [1]. In this paper we introduced the notion of d_G-pseudocompactness of topological spaces with respect to some metric space/metric group (G, d). Our work includes several results on d_G-pseudocompactness. Our main result is to test when the notion of d_G-pseudocompactness becomes equivalent to pseudocompactness. In addition, we found the relationship of d_G-pseudocompactness with other types of compactness.

In notation and terminology, we follow [2] if not stated otherwise. Throughout the paper, (X, τ), or simply X, denotes a topological space and (Y, d)(resp. (G, d)) denotes a metric space (resp. metric group) with metric d, unless stated otherwise. All metric spaces/metric groups are assumed to be unbounded. By a 'space', we always mean a 'topological space'. The symbol $\mathbb{R}$ denotes the space of all real numbers with the usual topology, $\mathbb{N}$ denotes the space of all natural numbers with the discrete topology and $\omega = \mathbb{N} \cup \{0\}$. $C(X, Y)$ denotes the set of all continuous functions from space X to space Y. For a subset A of (X, τ), $\bar{A}$ denotes the closure of A in (X, τ) and (A, τ_A) denotes the subspace of (X, τ) with subspace topology τ_A on A. For $A \subseteq Y \subseteq X, cl_Y(A)$ denotes the closure of A in subspace Y of X. The letter e denotes the identity of metric group G.

Definition 1.1 *Let (Y, d) be a metric space. A subset A of Y is bounded if* $\sup\{d(x, y) : x, y \in A\}$ *is finite.*

It is clear that a subset of a bounded set is bounded and finite union of bounded sets is again bounded. Also, the closure of a bounded set is bounded as $\sup\{d(x, y) : x, y \in A\} = \sup\{d(x, y) : x, y \in \bar{A}\}$.

Theorem 1.2 *In a metric space (Y, d), a set $A \subseteq Y$ is bounded if and only if* $\sup\{d(x, y_0) : x \in A\}$ *is finite, where y_0 is some fixed element of Y.*

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2010 *Mathematics Subject Classification:* Primary 54D30, Secondary 54C05, 54E35, 54H11.
Submitted April 26, 2025. Published September 23, 2025

Proof: Let $A \subseteq Y$ be bounded and $y_0 \in Y$ be fixed. Then by Definition 1.1, $\sup\{d(x,y) : x, y \in A\}$ is finite, say k. Let $x \in A$ be arbitrary. By Triangle's inequality, we have $d(x, y_0) \leq d(x, x_0) + d(x_0, y_0)$ for some $x_0 \in A$. This gives that $d(x, y_0) \leq d(x_0, y_0) + k$. Thus, $\sup\{d(x, y_0) : x \in A\}$ is finite. Conversely, let us suppose that $\sup\{d(x, y_0) : x \in A\}$ is finite, say k_1 and let $x, y \in A$. Then by Triangle's inequality, we have $d(x,y) \leq d(x, y_0) + d(y_0, y) \leq 2k_1$. Thus, $\sup\{d(x,y) : x, y \in A\}$ is finite and hence, A is a bounded subset of Y.

□

Corollary 1.3 *In a metric group (G,d), a set $A \subseteq G$ is bounded if and only if $\sup\{d(x,e) : x \in A\}$ is finite, where e is the identity of G.*

Definition 1.4 *Let (X,τ) be a topological space and (Y,d) be a metric space. X is said to be d_Y-pseudocompact if for each continuous function $f : (X,\tau) \to (Y,d)$, $f(X)$ is a bounded subset of Y.*

Definition 1.5 *A subset A of a space (X,τ) is said to be d_Y-pseudocompact if for each continuous function $f : (A,\tau_A) \to (Y,d), f(A)$ is a bounded subset of (Y,d), where (Y,d) is a metric space.*

Theorem 1.6 *Let (X,τ) be a topological space and (Y,d) be a metric space. Then the following statements hold:*

(i) *Closure of a d_Y-pseudocompact set is d_Y-pseudocompact.*

(ii) *Finite union of d_Y-pseudocompact sets is d_Y-pseudocompact.*

(iii) *Every compact subset of X is d_Y-pseudocompact.*

Theorem 1.7 *Let $f : (X,\tau) \to (Z,\sigma)$ be a continuous function between spaces X and Z. If X is d_Y-pseudocompact, then $f(X)$ is d_Y-pseudocompact, where (Y,d) is a metric space.*

Definition 1.8 *Let (X,τ) be a space. A Family $\alpha \subseteq P(X)$ is centered if $\cap\beta \neq \emptyset$ for any finite $\beta \subseteq \alpha$.*

Theorem 1.9 *If for every centered family $\{U_n : n \in \omega\}$ of open subsets of X, the intersection $\bigcap \left\{\bar{U}_n : n \in \omega\right\} \neq \emptyset$ then the space X is d_G-pseudocompact.*

Proof: Let us suppose that X is not d_G-pseudocompact. Then there exists an unbounded G-valued continuous function on X, say f. For $n \geq 1$, consider $A_n = \{g \in G : d(g,e) > n\}$. Clearly $A_n = \phi^{-1}(n,\infty)$, where $\phi = d \circ \psi \circ \xi$, $\xi : G \to G \times \{e\}$ defined by $\xi(g) = (g,e)$ and ψ is the inclusion map. Since all the maps d, ψ and ξ are continuous, therefore the map ϕ is also continuous. It is observed that $< A_n >$ is a decreasing sequence of non-empty open subsets of G. Also $\bar{A}_{n+1} = \{g \in G : d(g,e) \geq n+1\} \subseteq \{g \in G : d(g,e) > n\} = A_n$. For each $n \in \mathbb{N}$, let $U_n = f^{-1}(A_n)$. Since $< U_n >$ is a decreasing sequence, so it is centered and $\bar{U}_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$. This implies that $\bigcap \left\{\bar{U}_n : n \in \omega\right\} = \bigcap \{U_n : n \in \omega\} = \emptyset$. A contradiction. □

Theorem 1.10 *If Y is a dense d_G-pseudocompact subspace of a topological space X, then X is d_G-pseudocompact.*

Proof: Let f be a G-valued continuous map on X. Then $f(X) = f(\bar{Y}) \subseteq \overline{f(Y)}$. Since Y is d_G-pseudocompact, $f(Y)$ is a bounded subset of G. Since closure of a bounded set is bounded, $f(X)$ is a bounded subset of G. Thus, X is d_G-pseudocompact. □

Definition 1.11 *A topological space X is μ_G-space if every countable subset of X is discrete and every G-valued continuous function on each countable subset of X can be extended continuously to the whole of X.*

Theorem 1.12 *Let X be a μ_G-space. If X is d_G-pseudocompact, then every subset of X is finite. In particular, X is finite.*

Proof: Let us suppose, if possible, that there exists a countable infinite subset of X, say A. Let $A = \{a_n : n \in \mathbb{N}\}$. Define a function $f : A \to G$ by $f(a_n) = g_n$ such that $d(g_n, e) > n$. Since X is μ_G-space, A is a discrete subset of X. So f is a continuous function on A and can be extended continuously to X. Let $F : X \to G$ be a continuous extension of f. Clearly F is not a bounded function. A contradiction to the given hypothesis that X is d_G-pseudocompact. Thus, every subset of X is finite. □

2. Main Results

Our Main question is,

(a). Given a metric group (G, d), for which family of topological spaces the notion of d_G-pseudocompactness is equivalent to pseudocompactness?

(b). Given a topological space X, for which family of metric groups the notion of d_G-pseudocompactness is equivalent to pseudocompactness?

For this, firstly we recall some definitions and then we shall give characterization for a topological space to be d_G-pseudocompact.

Definition 2.1 *[5] Let G be a non-trivial topological group with identity element e. A topological space X is called*

(a). *G-regular if for each closed set $F \subseteq X$ and every point $x \in X\backslash F$, there exist $f \in C(X, G)$ and a point $g \in G\backslash\{e\}$ such that $f(x) = g$ and $f(F) \subseteq \{e\}$.*

(b). *$G^\star$-regular if there exists a point $g \in G\backslash\{e\}$ such that for every closed set $F \subseteq X$ and each point $x \in X\backslash F$, there exists $f \in C(X, G)$ such that $f(x) = g$ and $f(F) \subseteq \{e\}$.*

(c). *$G^{\star\star}$-regular provided that, whenever F is a closed subset of $X, x \in X\backslash F$ and $g \in G\backslash\{e\}$, there exists $f \in C(X, G)$ such that $f(x) = g$ and $f(F) \subseteq \{e\}$.*

It is clear that X is $G^{\star\star}$-regular $\Longrightarrow$ X is $G^\star$-regular $\Longrightarrow$ X is G-regular.

Theorem 2.2 *Let G be a topological group containing atleast three elements. Then arbitrary product of $G^{\star\star}$-regular spaces is $G^{\star\star}$-regular.*

Proof: Let $\{X_\alpha : \alpha \in I\}$ be a family of $G^{\star\star}$-regular spaces, where I is any arbitrary index set. Let $X = \prod_{\alpha\in I} X_\alpha$. Let $b \in X$ and F be a closed subset of X such that $b \in X\backslash F$. Then there exists a basic open set $U \subseteq X$ such that $b \in U \subseteq X\backslash F$, where $U = \prod_{\alpha\in I} U_\alpha$ such that $U_\alpha = X_\alpha$ for all except finitely many indices. Let $U_\alpha \neq X_\alpha$ for $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$. Let $g \in G\backslash\{e\}$. Then g can be written as $g = g_{\alpha_1} g_{\alpha_2} \ldots g_{\alpha_n}$, where $g_{\alpha_i} \in G\backslash\{e\}$ for all $i = 1, 2, \ldots, n$. Since $b_{\alpha_i} \in U_{\alpha_i}$ and $g_{\alpha_i} \in G\backslash\{e\}$ for all $i = 1, 2, \ldots, n$, there exists $f_i \in C(X, G)$ such that $f_i(b_{\alpha_i}) = g_{\alpha_i}$ and $f_i(X_{\alpha_i}\backslash U_{\alpha_i}) = \{e\}$. Now define $h_i : X \to G$ by $h_i(x) = f_i(\pi_{\alpha_i}(x))$, where $\pi_{\alpha_i} : X \to X_{\alpha_i}$ is a projection on X_{α_i}. Clearly each h_i is continuous and $h_i(x) = e$ for all $x \in X\backslash\pi^{-1}_{\alpha_i}(U_{\alpha_i})$. Now the function $f : X \to G$ defined by $f(x) = h_1(x)h_2(x)\ldots h_n(x)$ is a continuous function such that $f(b) = g$ and $f(X\backslash F) = \{e\}$. Hence, X is $G^{\star\star}$-regular. □

Definition 2.3 *A family $\sigma = \{A_i : i \in I\}$ of subsets of a space X is locally finite if for each $x \in X$ there exists a neighborhood of x which intersects with finitely many members of σ.*

Definition 2.4 *A family $\sigma = \{A_i : i \in I\}$ of subsets of a space X is star finite if every member of σ intersects with finitely many members of σ.*

It is obvious that every star finite open covering is locally finite.

Theorem 2.5 *Let (G,d) be a metric space. If X is feebly compact, then X is d_G-pseudocompact.*

Proof: Let f be a G-valued continuous function on X and $h \in G$ be fixed. Consider $A_0 = \{g \in G : 0 \leq d(g,h) < 1\}$ and for $n \geq 1, A_n = \{g \in G : n-1 < d(g,h) < n+1\}$. Clearly each A_n is open in G and $G = \bigcup_{n \in \omega} A_n$. This implies that $X = \bigcup_{n \in \omega} f^{-1}(A_n)$. Since f is continuous, $\{f^{-1}(A_n) : n \in \omega\}$ is an open cover of X and each $f^{-1}(A_n)$ does not meet $f^{-1}(A_i)$ for $i \neq n-1, n, n+1$. Therefore, the covering $\{f^{-1}(A_n) : n \in \omega\}$ becomes star finite. Since X is feebly compact, every locally finite family of non-empty open subsets of X is finite. Therefore, every star finite open cover of X has a finite subcover. Let $X = \bigcup_{i=1}^{m} f^{-1}(A_{n_i})$ for some $m \in \mathbb{N}$. Then $f(X) \subseteq \bigcup_{i=1}^{m} A_{n_i}$. Let $M = \text{Max}\{n_1, n_2, \ldots, n_m\}$. Then $d(g,h) < M+1$ for all $g \in f(X)$. Thus, $f(X)$ is a bounded subset of G and hence, X is d_G-pseudocompact. □

Corollary 2.6 *Every countably compact space is d_G-pseudocompact.*

Proof: The proof follows as every countably compact space is feebly compact. □

Corollary 2.7 *Let X be a Tychonoff space. If X is pseudocompact, then X is d_G-pseudocompact.*

Proof: In Tychonoff spaces, the pseudocompactness is equivalent to feebly compactness [1]. Therefore, the proof follows by Theorem 2.5. □

The following theorems give an answer to our main question.

Theorem 2.8 *For a $G^{\star\star}$-regular space X, X is d_G-pseudocompact if and only if X is feebly compact.*

Proof: Let X be d_G-pseudocompact but not feebly compact. Then there exist a locally finite family $\sigma = \{A_i : i \in I\}$ of non-empty open subsets of X which is not finite. Let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be a countable subfamily of σ. To each A_n, associate a point $a_n \in A_n$. Since X is G-regular, there exists $f_1 \in C(X,G)$ and a point $b_1 \in G \backslash \{e\}$ such that $f_1(a_1) = b_1$ and $f(X \backslash A_1) = \{e\}$. Also G is an unbounded metric group, for each $n \geq 2$, so we can find $b_n \in G$ such that $d(e, b_{n+1}) \geq d(e, b_n) + 1$ for all $n \in \mathbb{N}$. Since X is $G^{\star\star}$-regular, there exists $f_n \in C(X,G)$ such that $f_n(a_n) = b_n$ and $f_n(X \backslash A_n) = \{e\}$ for all $n \geq 2$. Since $\mathcal{A}$ is locally finite, $A_x = \{A_n : x \in A_n\}$ is finite for every $x \in X$. Define $f : X \to G$ by $f(x) = \prod_{x \in A_n} f_n(x)$ if $A_x \neq \emptyset$ and $f(x) = e$ otherwise. Clearly f is well defined. To see that f is continuous, let $x \in X$ and W be an open neighborhood of $f(x)$. Let O be an open neighborhood of x. If O does not intersects with any A_n, then $f(O) = \{e\} \subseteq W$ and therefore, f is continuous on x. Otherwise, $\{n : A_n \cap O \neq \emptyset\}$ is a non-empty finite set. Let $O \cap A_{n_i} \neq \emptyset$ for $i \in \{1, 2, \ldots, k\}$. Then $f(x) = f_{n_1}(x) f_{n_2}(x) \ldots f_{n_k}(x)$. Since G is a topological group, there exists open neighborhoods W_{n_i} of $f_{n_i}(x)$ such that $\prod_{i=1}^{k} W_{n_i} \subseteq W$. Since each $f_{n_i}(x)$ is continuous at x, there exists open neighborhoods U_{n_i} of x such that $f_{n_i}(U_{n_i}) \subseteq W_{n_i}$ for all $i = 1, 2, \ldots, k$. Let $U = O \cap U_{n_1} \cap \ldots \cap U_{n_k}$. Then $f(U) \subseteq W$ as for $y \in U, f(y) = \prod_{i=1}^{k} f_{n_i}(y) \in \prod_{i=1}^{k} W_{n_i} \subseteq W$. Thus, f is a G-valued unbounded continuous function on X. A contradiction. Converse part follows by Theorem 2.5. □

Theorem 2.9 *For a $G^{\star\star}$-regular space X, X is d_G-pseudocompact if and only if X is pseudocompact.*

Proof: In Tychonoff spaces, the pseudocompactness is equivalent to feebly compactness [1]. Therefore, the proof follows by Theorem 2.8. □

Proposition 2.10 *[5]*

(i) If topological group G is pathwise connected, then X is $G^{\star\star}$-regular.

(ii) If X is zero-dimensional, then X is $G^{\star\star}$-regular.

Thus, we have the following:

(i) For a family of zero-dimensional topological spaces, the notion of d_G-pseudocompactness is equivalent to pseudocompactness for any metric group (G, d).

(ii) For a family of pathwise connected metric group, the notion of d_G-pseudocomp actness is equivalent to pseudocompactness for any topological space X.

(iii) For a family of countable regular topological spaces, the notion of d_G-pseudocompactness is equivalent to pseudocompactness for any metric group (G, d) because every countable regular space is zero-dimensional.

Theorem 2.11 *For a $G^{\star\star}$-regular space X, the following statements are equivalent:*

(a) X is d_G-pseudocompact.

(b) For every decreasing family $\{U_n : n \in \omega\}$ of non-empty open subsets of X the intersection $\bigcap \bar{U}_n \neq \emptyset$.

(c) For every countable family $\{V_n : n \in \omega\}$ of open subsets of X which has the finite intersection property, the intersection $\bigcap \bar{V}_n \neq \emptyset$.

Proof: The proof follows by Theorem 3.10.23 [2] and Theorem 2.9.

□

Theorem 2.12 *Let U be an open subset of a $G^{\star\star}$-regular d_G-pseudocompact space X. Then $\bar{U}$ is d_G-pseudocompact.*

Proof: Let X be a $G^{\star\star}$-regular d_G-pseudocompact space and U be an open subset of X. Let $\sigma = \{U_n : n \in \omega\}$ be a decreasing family of non-empty open subsets of $\bar{U}$. Then $\sigma_U = \{U_n \cap U : n \in \omega\}$ is also a decreasing family of non-empty open subsets of X. Since X is d_G-pseudocompact, $\bigcap \{\overline{U_n \cap U} : n \in \omega\} \neq \emptyset$ (By Theorem 2.11). Thus, $\bigcap \{cl_{\bar{U}}(U_n) : n \in \omega\} \neq \emptyset$. Hence, $\bar{U}$ is d_G-pseudocompact. □

Theorem 2.13 *Let X and Y be $G^{\star\star}$-regular spaces such that X is d_G-pseudocompact and Y is d_G-pseudocompact k-space. Then $X \times Y$ is d_G-pseudocompact.*

Proof: By Theorem 2.2, $X \times Y$ is $G^{\star\star}$-regular as G is an unbounded metric group. So the proof follows by Theorem 3.10.26 [2]. □

Corollary 2.14 *Let X and Y be $G^{\star\star}$-regular spaces such that X is d_G-pseudocompact and Y is compact. Then $X \times Y$ is d_G-pseudocompact.*

Proof: We know that every compact space is d_G-pseudocompact (Theorem 1.6) and every compact space is k-space [2]. So, the proof follows by Theorem 2.13. □

Corollary 2.15 *Let X and Y be $G^{\star\star}$-regular spaces such that X is d_G-pseudocompact and Y is a d_G-pseudocompact sequential space. Then $X \times Y$ is d_G-pseudocompact.*

Proof: Since every sequential space is k-space [2], the proof follows by Theorem 2.13. □

Theorem 2.16 *Let X and Y be $G^{\star\star}$-regular spaces such that X is d_G-pseudocompact and Y is sequentially compact. Then $X \times Y$ is d_G-pseudocompact.*

Proof: Let $f : X \times Y \to G$ be an unbounded cotinuous function. Then there exist points $z_i = (x_i, y_i) \in X \times Y$ such that $d(f(z_i), e) \geq i$ for each $i \in \mathbb{N}$. Let $< y_{n_k} >$ be a subsequence of $< y_n >$ that converge to a point $y \in Y$. Then the subspace $F = \{y, y_{n_1}, y_{n_2}, \ldots\}$ is compact and hence, $X \times F$ is d_G-pseudocompact by Corollary 2.14. But the function $f \mid X \times F : X \times F \to G$ is not bounded. Hence, $X \times Y$ is d_G-pseudocompact. □

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