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Solutions of a Markoff type equation in the Jacobsthal–Lucas numbers

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ABSTRACT: Let $\{j_n\}_{n\geq 0}$ be the sequence of the Jacobsthal–Lucas numbers, given by the relation $j_0 = 2$, $j_1 = 1$, $j_n = j_{n-1} + 2j_{n-2}$ for all $n \geq 2$. In this paper, we study the solutions (X, Y, Z) of the following equation that is so called the Jin-Schmidt equation:

$$AX^2 + BY^2 + CZ^2 = DXYZ + 1,$$

where $X = j_i$, $Y = j_j$ and $Z = j_k$ with $i, j, k \geq 1$. As this equation has infinitely many integer solutions, that are connected to the lower part of the approximated spectrum for quaternions, to the approximated constants for complex numbers on certain circles, and to the Diophantine approximation of irrational numbers, we show that this equation has a finite number of such special solutions. This result gives a deep insight to the mentioned connections of the solutions in the case of Jacobsthal–Lucas numbers.

Key Words: Diophantine equations, Jin-Schmidt equation, Linear recurrences, Jacobsthal–Lucas numbers.

Contents

1. Introduction

The study of Diophantine equations has been the main interest for many mathematicians for its applications in many fields of sciences. In fact, one of the most known studied Diophantine equations is called the Markoff equation that has the form

$$X^2 + Y^2 + Z^2 = 3XYZ, \tag{1.1}$$

where $X, Y, Z \in \mathbb{N}$ with $X \leq Y \leq Z$. This equation was firstly studied by the scientist Markoff in 1879-1880 [9,8], who studied the solutions of the equation in positive integers and found that it has an infinite number of solutions generated by

$$(X, Y, Z) \in \{(1, 1, 1), (X, Z, 3XZ - Y), (Y, Z, 3YZ - X)\}.$$

Note that the solutions are known by the Markoff triples. Many generalizations of the Markoff equation have been studied, and one of these generalizations was studied by Hurwitz [1] (called by the Markoff-Hurwitz) in 1907 and has the form

$$X_1 + X_2^2 + ... + X_n^2 = AX_1X_2...X_n$$

such that A is a positive integer and $n \geq 3$. Another generalization was studied by Rosenberger [10] in 1979, which has the form

$$AX^2 + BY^2 + BZ^2 = DXYZ, \tag{1.2}$$

where A, B, C and D are a positive integers such that $A, B, C\backslash D$, and $gcd(A, B) = gcd(A, C) = gcd(C, D) = 1$. Rosenberger proved that when

$$(A, B, C, D) \in \{(1, 2, 3, 6), (1, 1, 1, 3), (1, 1, 1, 1), (1, 1, 2, 2), (1, 1, 5, 5), (1, 1, 2, 4)\},$$

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then equation (1.2) has infinitely many solutions. Also, Jin and Schmidt [6] proposed another generalization of the Markoff equation in 2001, that has the form

$$AX^2 + BY^2 + CZ^2 = DXYZ + 1, \tag{1.3}$$

where X, Y and Z are positive integers with $gcd(A, B, C) = 1$, and A, B, $C \setminus D$. Jin and Schmidt stated that the equation (1.3) has solutions only when

$$(A, B, C, D) \in T = \{(7, 2, 14, 14), (2, 2, 3, 6), (6, 10, 15, 30), (2, 1, 2, 2), (5, 1, 5, 5), (3, 1, 6, 6)\}.$$

Note that we call equation (1.3) by the Jin-Schmidt equation. Another well known concept in number theory is a linear recurrence sequence, that's given by the relation:

$$M_{n+d} = b_1 M_{n+d-1} + b_2 M_{n+d-2} + ... + b_d M_n \quad \text{for} \quad n \geq 0 \tag{1.4}$$

with $b_1, b_2, ..., b_d \in \mathbb{C}$, $b_d \neq 0$ and the sequence $\{M_n\}$ has the order d. If $d = 2$, then the sequence is called a binary linear recurrence sequence such as the Fibonacci sequence $\{F_n\}$ that is defined by

$$F_n = F_{n-1} + F_{n-2}, \tag{1.5}$$

where $n \geq 0$ with $F_0 = 0$, $F_1 = 1$ and $F_2 = 1$. The Binet's formula of $\{F_n\}$ is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.6}$$

where $(\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ and $\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$ for $n \geq 1$. Also, $\beta = \frac{-1}{\alpha}$, and α is called the golden ratio. The sequence Jacobsthal–Lucas numbers $\{j_n\}$ is also a binary linear recurrence sequence defined by

$$j_n = j_{n-1} + 2j_{n-2}, \tag{1.7}$$

where $n \geq 2$ with $j_0 = 2$, $j_1 = 1$. The Binet's formula of $\{j_n\}$ is defined as follows:

$$j_n = \alpha_1^n + \beta_1^n \qquad \forall n \geq 0, \tag{1.8}$$

where $\alpha_1^n - 1 \leq j_n \leq \alpha_1^n + 1$ for $n \geq 1$ and $(\alpha_1, \beta_1) = (2, -1)$.

If $d = 3$, then the sequence (1.4) is called a ternary recurrence sequence, as the Tribonacci sequence $\{T_n\}$, that is represented as follows:

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad \text{for} \quad n \geq 3, \quad \text{where} \quad T_0 = 0, T_1 = 1, T_2 = 1.$$

An interesting study of the Markoff equation (1.1) was defined by Luca and Srinivasan [7] in 2018 in which they studied the solutions of the Markoff equation in the Fibonacci sequence, namely where $X = F_i$, $Y = F_j$ and $Z = F_k$ with $i \leq j \leq k$. Tengely [13] in 2020 found the solutions of equation (1.2), where $X = F_i$, $Y = F_j$ and $Z = F_k$ with $i, j, k \geq 1$. In 2020, Hashim and Tengely [5] conducted a significant study by examining the solutions of the Jin-Schmidt equation (1.3) within the Fibonacci sequence (1.5), namely if $X = F_i, Y = F_j$ and $Z = F_k$ such that $i, j, k \geq 2$. In 2023, Hashim [3] determined the solutions (X, Y, Z) of the Markoff equation, where $(X, Y, Z) = (T_i, T_j, T_k)$ with $i, j, k \geq 2$.

In this paper, we also study the solutions of the Jin-Schmidt equation (1.3) in the Jacobsthal–Lucas numbers sequence $\{j_n\}$, namely where $(X, Y, Z) = (j_i, j_j, j_k)$ with $i, j, k \geq 1$ In other words, we investigate the solutions (j_i, j_j, j_k) of the following equation:

$$7j_i^2 + 2j_j^2 + 14j_k^2 = 14j_i j_j j_k + 1, \tag{1.9}$$

$$2j_i^2 + 2j_j^2 + 3j_k^2 = 6j_i j_j j_k + 1, \tag{1.10}$$

$$6j_i^2 + 10j_j^2 + 15j_k^2 = 30j_i j_j j_k + 1, \tag{1.11}$$

$$2j_i^2 + j_j^2 + 2j_k^2 = 2j_i j_j j_k + 1, \tag{1.12}$$

$$5j_i^2 + j_j^2 + 5j_k^2 = 5j_i j_j j_k + 1, \tag{1.13}$$

$$3j_i^2 + j_j^2 + 6j_k^2 = 6j_i j_j j_k + 1, \tag{1.14}$$

where $i, j, k \geq 1$. Mostly, we are interested in examining whether or not the Jin-Shcmdit equation still has infinitely many solutions where the unknowns are terms in the sequence of Jacobsthal–Lucas numbers. One of the important applications that encourages us to study the Jin-Schmidt equation (1.3) is its connection to the lower part of the approximated spectrum for quaternions, as well as its connection to the approximated constants for complex numbers on the circle $\{z \in \mathbb{C} \,|\, |z| = \frac{1}{\sqrt{2}}\}$. Furthermore, the solutions of the Markoff equation and its generalizations are connected to the study of Diophantine approximation. For more details about their applications, see e.g [11]. Also, for more results connecting Diophantine equations connected to linear recurrence sequences, see e.g. [2] and [4].

2. Main approach

Here, we give an approach for solving the Jin-Schmidt equation (1.3) where the unknowns are terms in the Jacobsthal–Lucas numbers sequence given in (1.7). Namely, we get all the solutions (X, Y, Z) that satisfy the equation

$$AX^2 + BY^2 + CZ^2 = DXYZ + 1, \tag{2.1}$$

where $(X, Y, Z) = (j_i, j_j, j_k)$ with $i, j, k \geq 1$. Our main approach is based on solving the Jin-Schmidt equation for each $(A, B, C, D) \in T$ (i.e. the equations (1.9)–(1.14)) by considering all the possible cases $X \leq Y \leq Z$, $X \leq Z \leq Y$, $Y \leq X \leq Z$, $Y \leq Z \leq X$, $Z \leq X \leq Y$, $Z \leq Y \leq X$. So equation (1.3) becomes

$$aj_i^2 + bj_j^2 + cj_k^2 = dj_ij_jj_k + 1, \tag{2.2}$$

where $(a, b, c, d) \in \mathbb{S}$, and $\mathbb{S}$ is the set of all the tuples obtained from permutations of A, B, C and the fixing D. For example, if we permute the coefficients of the equations (2.2), the following distinct equations are obtained:

$$aj_i^2 + bj_j^2 + cj_k^2 = dj_ij_jj_k + 1, \tag{2.3}$$
$$aj_i^2 + cj_j^2 + bj_k^2 = dj_ij_jj_k + 1, \tag{2.4}$$
$$bj_i^2 + aj_j^2 + cj_k^2 = dj_ij_jj_k + 1, \tag{2.5}$$
$$bj_i^2 + cj_j^2 + aj_k^2 = dj_ij_jj_k + 1, \tag{2.6}$$
$$cj_i^2 + bj_j^2 + aj_k^2 = dj_ij_jj_k + 1, \tag{2.7}$$
$$cj_i^2 + aj_j^2 + bj_k^2 = dj_ij_jj_k + 1, \tag{2.8}$$

where $i, j, k \geq 1$ and $i \leq j \leq k$. In fact, in order to solve equation (2.1) at (A, B, C, D) completely we solve one of the equations (2.3)-(2.8) under the condition $i \leq j \leq k$, and the solutions of equation (2.1) at the intended (A, B, C, D) can be obtained by permuting components the of the obtained solutions of the distinct equations. Next, we present a summary for the idea of solving equation (2.2) with $1 = i \leq j \leq k$ or $1 = j_i \leq j_j \leq j_k$:

(i) First, we determine an upper bound for i to equation (2.2) by rewriting it in the form:

$$aj_i^2 + bj_j^2 + cj_k^2 - (dj_ij_jj_k + 1) = 0. \tag{2.9}$$

By dividing equation (2.9) by j_k, we get that

$$cj_k - dj_ij_j = \frac{-aj_i^2 - bj_j^2}{j_k} + \frac{1}{j_k}. \tag{2.10}$$

From equation (1.8), we obtain that

$$c\alpha_1^k - d\alpha_1^{i+j} = \frac{-aj_i^2 - bj_j^2}{j_k} + \frac{1}{j_k} - c\beta_1^k + d(\alpha_1^i\beta_1^j + \alpha_1^j\beta_1^i + \beta_1^{i+j}). \tag{2.11}$$

By the condition of $1 \leq i \leq j \leq k$ (or $1 \leq j_i \leq j_j \leq j_k$) with substituting the following facts in the terms of the absolute values of both sides of equation (2.11):

$$\begin{aligned} &\left|\frac{-aj_i^2 - bj_j^2}{j_k}\right| = \left|\frac{aj_i^2 + bj_j^2}{j_k}\right| \\ &\leq (a+b)j_j^2 j_k^{-1} \leq (a+b)j_j \quad \text{by Binet's formula (1.8)} \\ &\leq (a+b)(\alpha_1^j + \beta_1^j) \quad \text{since } \alpha_1 > \beta_1 \\ &< (a+b)(\alpha_1^j + \alpha_1^j) = 2(a+b)\alpha_1^j, \end{aligned} \tag{2.12}$$

$$\left|\frac{1}{j_k}\right| \leq 1 < \alpha_1^j, \tag{2.13}$$

$$\left|-c\beta_1^k\right| = \left|c\beta_1^k\right| < c\alpha_1^j \quad \text{since} \quad \alpha_1 = 2,\ \beta_1 = -1, \tag{2.14}$$

$$\begin{aligned} &\left|d(\alpha_1^i\beta_1^j + \alpha_1^j\beta_1^i + \beta_1^{i+j})\right| = \left|\pm d(\alpha_1^i + \alpha_1^j + 1)\right| \\ &< \left|d(\alpha_1^j + \alpha_1^j + \alpha_1^j)\right| = 3d\alpha_1^j, \end{aligned} \tag{2.15}$$

we obtain the following inequality:

$$\left|c\alpha_1^k - d\alpha_1^{i+j}\right| < (1 + 2(a+b) + c + 3d)\alpha_1^j. \tag{2.16}$$

Dividing both sides of the inequality (2.16) by $c\alpha_1^{i+j}$ implies that

$$\left|\alpha_1^{k-i-j} - \frac{d}{c}\right| < \frac{w}{\alpha_1^i}, \tag{2.17}$$

such that $w = \frac{1}{c}(1 + 2(a+b) + c + 3d)$. Suppose that

$$M = \min_{n\in\mathbb{Z}} \left|\alpha_1^n - \frac{d}{c}\right| > 0, \tag{2.18}$$

so inequality (2.17) becomes

$$\alpha_1^i < \frac{w}{M}. \tag{2.19}$$

By taking the natural logarithm for both sides of the inequality (2.19), we get the upper bound for i as follows:

$$i \leq \left\lfloor \frac{\ln(\frac{w}{M})}{\ln(\alpha_1)} \right\rfloor = l, \tag{2.20}$$

where $l \in \mathbb{N}$.

(ii) We use the inequality (2.17) to determine an upper bound of $k - j$. From the set $\mathbb{S}$, we find $d/c \in \{1, 2, 3, 5, 6, 7\}$. So, we get that $a/c, b/c \leq 7$, $1/c \leq 1$, and $d/c \leq 7$. Therefore,

$$w = \frac{1}{c}(1 + 2(a+b) + c + 3d) = (\frac{1}{c} + \frac{2(a+b)}{c} + \frac{c}{c} + \frac{3d}{c}) \leq 51.$$

Substituting w and d/c into the inequality (2.17) implies that

$$\left|\alpha_1^{k-i-j} - \frac{d}{c}\right| < \frac{51}{\alpha_1 = 2} = 25.5 \quad \text{as} \quad i \geq 1, \tag{2.21}$$

or

$$\left|\alpha_1^{k-i-j}\right| < 25.5 + \left|\frac{d}{c}\right| < 25.5 + 7 < 32.5. \tag{2.22}$$

Taking the natural logarithm for both sides of the inequality (2.22) leads to

$$k - j < i + \frac{\ln(32.5)}{\ln(\alpha_1 = 2)} < l + 6.$$

As $i \le l$, we obtain that

$$k \le j + l + 5. \tag{2.23}$$

(iii) We find the value of M in (2.18) to get the upper bounds values for i and k. Since

$$0 < M = \min_{n \in \mathbb{Z}} \left| \alpha_1^n - \frac{d}{c} \right|, \quad \text{where} \quad d/c \in \{1, 2, 3, 5, 6, 7\}.$$

We take different values for n to find the value of M as follows:

– First, we substitute the value $n = 0$ into inequality (2.18), we get that $M = \min_{n=0} \left|1 - \frac{d}{c}\right| \ge 0$. If $\frac{d}{c} = 1$, then $M = \min_{n=0} \left|1 - \frac{d}{c}\right| = 0$, but this is a contradiction with $M > 0$. But, if $\frac{d}{c} \ge 2$, then $M = \min_{n=0} \left|\alpha^0 - \frac{d}{c}\right| \ge 1$.

– If $n = 1$, then $M = \min_{n=1} \left|2 - \frac{d}{c}\right|$. If $\frac{d}{c} = 2$, thus $M = 0$, but this is not possible since $M > 0$. But, if $\frac{d}{c} \ne 2$, then $M = \min_{n=1} \left|2 - \frac{d}{c}\right| \ge 1$ as $d/c \in \{1, 2, 3, 5, 6, 7\}$.

– After that, we choose $n = 2$, then $\alpha^2 = 4$. Therefore, $M = \min_{n=2} \left|\alpha_1^n - \frac{d}{c}\right| \ge 1$ as $d/c \in \{1, 2, 3, 5, 6, 7\}$.

– Now, if $n \ge 3$, we obtain that $\alpha_1^n \ge \alpha_1^3 = 8$. So, $M = \min_{n \ge 3} \left|\alpha_1^n - \frac{d}{c}\right| \ge \left|8 - \frac{d}{c}\right| \ge 1$ as $d/c \in \{1, 2, 3, 5, 6, 7\}$.

– Finally, in case of $n \le -1$, then $\alpha_1^n \le \alpha_1^{-1} = \frac{1}{2}$. Thus, $M = \min_{n \le -1} \left|\alpha_1^n - \frac{d}{c}\right| \ge \frac{1}{2}$ with $d/c \in \{1, 2, 3, 5, 6, 7\}$.

Therefore, $M = \min_{n \in \mathbb{Z}} \left|\alpha_1^n - \frac{d}{c}\right| \ge \frac{1}{2}$. Inequality (2.20) becomes

$$i \le l = \left\lfloor \frac{\ln(\frac{51}{0.5})}{\ln(2)} \right\rfloor < 7 \quad \text{or} \quad i \le 6. \tag{2.24}$$

Also, from inequalities (2.23) and (2.24) we get that

$$k \le j + 6 + 5 \quad \text{or} \quad k \le j + 11, \tag{2.25}$$

where $k \ge j \ge i$.

(iv) Since $i \le 6$, we have that $i \in \{1, 2, 3, 4, 5, 6\}$. Next, we reduce the values of i by using the Sagemath program with the algorithm `solve_ Diophantine()` [12]. Namely, we determine the values of i for which the equation

$$aj_i^2 + by^2 + cz^2 - dj_i yz - 1 = 0 \tag{2.26}$$

is solvable in y and z with $i \le 6$. So, we remove the values of i with which equation (2.26) is not solvable.

(v) Finally, we substitute the values of i, which we obtained from (iv) and the values of k obtained from the inequality (2.25) with $j \ge 1$ in the equation (2.2) to set an equation with respect to j, which can be solved using the formula of the Jacobsthal–Lucas sequence given in (1.7). From every obtained solution (j_i, j_j, j_k) of (2.2), we acquire the corresponding solutions $(X, Y, Z) = (j_i, j_j, j_k)$ of the equation (2.1) by comparing the positions of the components (a, b, c, d) and (A, B, C, D), respectively.

Theorem 2.1 *If $(X, Y, Z) = (j_i, j_j, j_k)$ is a solution of equation (1.3) where $(A, B, C, D) \in T$, the following table shows the complete set of its solutions:*

Eq.	*(A,B,C,D)*	$\{(j_i, j_j, j_k)\}$
(1.9)	$(7, 2, 14, 14)$	$\{(1, 5, 1)\}$
(1.10)	$(2, 2, 3, 6)$	$\{(1, 1, 1)\}$
(1.11)	$(6, 10, 15, 30)$	$\{(1, 1, 1), (31, 1, 7)\}$
(1.12)	$(2, 1, 2, 2)$	$\{\}$
(1.13)	$(5, 1, 5, 5)$	$\{\}$
(1.14)	$(3, 1, 6, 6)$	$\{\}$

Proof: [Proof of Theorem 2.1] By the arguments listed in the Main approach section, we get the solutions shown in the above table as follows:

☐ `First Case` . If $(A, B, C, D) = (7, 2, 14, 14)$. To find all the solutions, we must study all the distinct equations obtained by permuting the coefficients of equation (1.9). This distinct equations are given as follows:

$$7j_i^2 + 2j_j^2 + 14j_k^2 = 14j_ij_jj_k + 1, \tag{2.27}$$
$$2j_i^2 + 7j_j^2 + 14j_k^2 = 14j_ij_jj_k + 1, \tag{2.28}$$
$$7j_i^2 + 14j_j^2 + 2j_k^2 = 14j_ij_jj_k + 1, \tag{2.29}$$
$$14j_i^2 + 7j_j^2 + 2j_k^2 = 14j_ij_jj_k + 1, \tag{2.30}$$
$$14j_i^2 + 2j_j^2 + 7j_k^2 = 14j_ij_jj_k + 1, \tag{2.31}$$
$$2j_i^2 + 14j_j^2 + 7j_k^2 = 14j_ij_jj_k + 1, \tag{2.32}$$

where $1 \leq i \leq j \leq k$. From inequalities (2.24) and (2.25) in argument (iii), we have that $i \leq 6$ and $1 \leq k \leq j + 11$ for equations (2.27)-(2.32). Now, we investigate the solutions to equation (2.27). We first follow the argument given in (iv) for eliminating the values of i to get that $i \in \{1, 4\}$ such that

$$7j_i^2 + 2y^2 + 14z^2 - 14j_iyz - 1 = 0$$

is solvable in y and z. If $i = 1$, we get that

$$2j_j^2 + 14j_k^2 - 14j_jj_k + 6 = 0,$$

which leads to

$$j_j^2 + 3 = 7j_k(j_j - j_k), \tag{2.33}$$

where $1 \leq j \leq k \leq j+11$. We can show that equation (2.33) is not satisfied for all $1 = j \leq k \leq j+11$. In fact, equation (2.33) can be written in the form

$$j_j^2 + 3 = 7j_k(j_j - j_k),$$

which is satisfied only if $j_j - j_k > 0$ (or $j_j > j_k$) for all $1 = j \leq k \leq j + 11$, since the left-hand side of equation (2.33) is positive for all $j \geq 1$. This is impossible as we assume that $k \geq j$ i.e. $j_k \geq j_j$. Therefore, equation (2.27) has no solution with $i = 1$.
Now, if $i = 4$, we obtain that

$$2j_j^2 + 14j_k^2 - 14(17)j_jj_k + 2022 = 0,$$

which can be written in the form

$$2j_j^2 + 2022 = 14j_k(j_j - j_k). \tag{2.34}$$

Similarly, the latter equation is satisfied only if $j_j > j_k$, which contradicts the assumption of $j_j \leq j_k$. Again, equation (2.27) is not solvable in the case of $i = 4$.

Also, we check the solutions of equation (2.28). We follow the argument given in (iv) to eliminate the not needed values of i. We obtain that

$$2j_i^2+7y^2+14z^2-14j_iyz-1=0$$

is solvable only with $i=2$. Substituting the value of $i=2$ in equation (2.28), we get

$$7j_j^2+14j_k^2-14(5)j_jj_k+49=0,$$

or

$$j_j^2+7=2j_k(5j_j-j_k). \tag{2.35}$$

Note that the left-hand side of equation (2.35) is positive for all $j\geq 2$. So, we obtain that $5j_j-j_k>0$ for all $j\leq k\leq j+2$ because if $k=j+1$, we have that (by the relation (1.7)) $5j_j-j_{j-1}=5j_j-(j_j+2j_{j-1})=4j_j-2j_{j-1}>0$, since $j_j>j_{j-1}$ for $j\geq 2$. Similarly, for $k=j+2$ we have $5j_j-j_{j+2}=2j_j-2j_{j-1}>0$ for all $j\geq 2$. But, this is a contradiction if $j+3\leq k\leq j+11$ with $2\leq i\leq j\leq k$ because if $j+3\leq k\leq j+11$ we have $5j_j-j_k=5j_j-(j_{k-1}+2j_{k-2})<0$ for all $k\geq j+3$ with $j\geq 2$. Therefore, we get that equation (2.28) is possibly solvable if $j\leq k\leq j+2$ and surely unsolvable if $j+3\leq k\leq j+11$ since $2\leq i\leq j\leq k$. Now, we consider equation (2.28) in case of $k\in\{j,j+1,j+2\}$.

- If $k=j$. We replace the values of $k=j$ in equation (2.35), we get that

$$j_j^2=1,$$

and this is no possible as $j\geq 2$. Hence, equation (2.35) has no solution with $k=j\geq 2$.

- If $k=j+1$. By substituting the value of $k=j+1$ in equation (2.35), we have that

$$j_j^2+2j_{j+1}^2-10j_jj_{j+1}+7=0, \tag{2.36}$$

where $2=i\leq j\leq k$. We show that the latter equation is not satisfied for all $j\geq 2$ by proving its left-hand side is always negative. In the case of $j=2$, we get that $(5)^2+2(7)^2-10(5)(7)+7=-220$. If $j=3$, than $(7)^2+2(17)^2-10(7)(17)+7=-556$. Next, we consider the case with $j\geq 4$. From (1.7), we have that $j_{j+1}=j_j+2j_{j-1}$ which we substitute in the left hand side of equation (2.36) to get that

$$\begin{aligned}
j_j^2+2j_{j+1}^2-10j_jj_{j+1}+7 &= j_j^2+2(j_j+2j_{j-1})^2-10j_j(j_j+2j_{j-1})+7\\
&= j_j^2+2j_j^2+8j_{j-1}j_1^2+8j_{j-1}^2-10j_j^2-20jj_{j-1}+7\\
&= 8j_{j-1}^2-7j_j^2-12jj_{j-1}+7\\
&< 8j_{j-1}^2-7j_{j-1}^2-12j_{j-1}^2+7 \quad \text{as} \quad -j_{j-1}>-j_j \quad \forall j\geq 4\\
&= -11j_{j-1}^2+7<0 \quad \text{ for all } \quad j_j\geq 4.
\end{aligned}$$

Similarly, we can show that the equation (2.28) has no solution when $k=j+2$ with $2=i\leq j\leq k$. We conclude that equation (2.28) has no solution in the case of $1\leq i\leq j\leq k$.

After that, we study the solutions of equation (2.29). By argument (iv), we obtain that

$$7j_i^2+14y^2+2z^2-14j_iyz-1=0$$

has a solution only at $i\in\{1,4\}$. If $i=1$, then $j_i=j_1=1$. We replace $j_i=j_1=1$ in equation (2.29). This leads to

$$14j_j^2+2j_k^2-14j_jj_k+6=0$$

We can write it as follows

$$7j_j^2+3=j_k(7j_j-j_k), \tag{2.37}$$

where $1\le j\le k$. We can show that equation (2.37) is possibly satisfied if $j\le k\le j+2$ (except when $k=j+2$ and $j=1$). In fact, equation (2.37) can be written in the form $7j_j^2+3=j_k(7j_j-j_k)$. Note that the latter equation is possibly satisfied only if $7j_j-j_k>0$ (or $7j_j>j_k$) for all $1\le j\le k\le j+2$ (except when $k=j+2$ and $j=1$), since the left hand side of equation (2.37) is positive for all $j\ge 1$. But, this is impossible if $j+3\le k\le j+11$ because if $j+3\le k\le j+11$ we have $7j_j-j_k=7j_j-(j_{k-1}+2j_{k-2})<0$ for all $k\ge j+3$ with $j\ge 2$. Therefore, we find that equation (2.29) is possibly solvable only when $j\le k\le j+2$.

– If $k=j$, We replace $k=j$ in equation (2.37), we get that

$$j_j^2=-3,$$

and this is impossible for all $j\ge 1$. Therefore, equation (2.37) has no solution with $k=j\ge 1$.

– If $k=j+1$, by substituting the value of $k=j+1$ in equation (2.37), we obtain that

$$14j_j^2+2j_{j+1}^2-14j_jj_{j+1}+6=0, \tag{2.38}$$

where $j\ge 1$. We note that the equation (2.38) is satisfied only when $j=1$. Otherwise, it is not possible. We prove this as follows. If $j=1$, we obtain that

$$14(1)^2+2(5)^2-14(1)(5)+6=0.$$

Therefore, we get that $(j_i,j_j,j_k)=(j_1,j_1,j_2)=(1,1,5)$ is a solution in equation (2.29). We find that equation (2.38) is not satisfied for all $j\ge 2$ by proving its left-hand side is always negative with $j\ge 2$. This can be shown by substituting the relation (1.7) in equation (2.38) several times. Namely,

$$\begin{aligned}
14j_j^2+2j_{j+1}^2-14j_jj_{j+1}+6 &= 14j_j^2+2(j_j+2j_{j-1})^2-14j_j(j_j+2j_{j-1})+6\\
&= 14j_j^2+2j_j+8j_jj_{j-1}+8j_{j-1}^2-14j_j^2-28j_{j-1}j_j+6\\
&= 2j_j^2+8j_{j-1}^2-20j_{j-1}j_j+6\\
&= 2(j_{j-1}+2j_{j-2})^2+8j_{j-1}^2-20j_{j-1}(j_{j-1}+2j_{j-2})+6\\
&= 2j_{j-1}^2+8j_{j-1}j_{j-2}+8j_{j-2}^2+8j_{j-1}^2-20j_{j-1}^2-40j_{j-1}j_{j-2}+6\\
&= 8j_{j-1}j_{j-2}+8j_{j-2}^2-10j_{j-1}^2-40j_{j-1}j_{j-2}+6\\
&< -32j_{j-1}j_{j-2}+8j_{j-2}j_{j-1}-10j_{j-1}^2-+6 \quad \text{as} \quad j_{j-1}>j_{j-2}\\
&= -10j_{j-1}^2-24j_{j-1}j_{j-2}+6<0 \quad \text{for all} \quad j\ge 2.
\end{aligned}$$

Similarly, we notice that equation (2.37) does not have any solution if $k=j+2$ with $j\ge 1$. Therefore, we get that equation (2.29) has no solution when $1=i\le j\le k$.

Next, we deal with equation (2.29) in the case of $i=4$, which leads to $j_i=j_4=17$. By substituting $j_i=17$ in equation (2.29), we get that

$$14j_j^2+2j_k^2-238j_jj_k+2022=0.$$

We can rewrite it as follows

$$14j_j^2+2022=2j_k(119j_j-j_k), \tag{2.39}$$

where $4\le j\le k$. We can show that equation (2.39) is possibly satisfied if $j\le k\le j+6$. In fact, equation (2.39) can be written in the form $14j_j^2+2022=2j_k(119j_j-j_k)$, which is possibly satisfied only if $119j_j - j_k > 0$ (or $119j_j > j_k$) for all $j \leq k \leq j+6$, since the left-hand side of equation (2.37) is positive for all $j \geq 4$. But, it's not satisfied with $j+7 \leq k \leq j+11$ because if $j+7 \leq k \leq j+11$ we have $119j_j - j_k = 119j_j - (j_{k-1} + 2j_{k-2}) < 0$ for all $k \geq j+6$ with $j \geq 4$. Therefore, we obtain that equation (2.29) is possibly solvable only when $j \leq k \leq j+6$.

– Substituting the value of $k = j$ in equation (2.39) leads to

$$-222j_j^2 + 2022 = 0, \tag{2.40}$$

which does not hold for all $j \geq 4$ since $j_j \geq j_4 = 17$, that gives $-222j_j^2 + 2022 \leq -222(17)^2 + 2022 = -62136$. This leads to equation (2.29) having no solution when $k = j$ with $i = 4$.

– If $k = j+1$, we obtain that

$$14j_j^2 + 2j_{j+1}^2 - 238j_j j_{j+1} + 2022 = 0. \tag{2.41}$$

Next, we prove the left-hand side of equation (2.41) is always negative to show that equation (2.41) is not satisfied for all $j \geq 4$. If $j = 4$, we obtain that $14(17)^2 + 2(31)^2 - 238(17)(31) + 2022 = -117436$. Next, we consider $j \geq 5$. By relation (1.7), we have that $j_{j+1} = j_j + 2j_{j-1}$. By substituting it in the left-hand side of equation (2.41), we get that

$$\begin{aligned}
14j_j^2 + 2j_{j+1}^2 - 238j_j j_{j+1} + 2022 &= 14j_j^2 + 2(j_j + 2j_{j-1})^2 - 238j_j(j_j + 2j_{j-1}) + 2022 \\
&= 14j_j^2 + 2j_j + 8j_j j_{j-1} + 8j_{j-1}^2 - 238j_j^2 - 476j_{j-1}j_j + 2022 \\
&= 8j_{j-1}^2 - 222j_j^2 - 238j_{j-1}j_j + 6 \\
&< 8j_j^2 - 222j_j^2 - 468j_{j-1}j_j + 2022 \quad \text{as} \quad j_j > j_{j-1} \\
&= -214j_j^2 - 468j_{j-1}j_j + 2022 < 0 \quad \text{for all} \quad j \geq 5.
\end{aligned}$$

Thus, equation (2.41) is not satisfied if $k = j+1$ with $i = 4$. Therefore, the equation (2.29) is not a solution if $k = j+1$ with $i = 4$. Similarly, we find that equation (2.29) does not have any solution when $k = j+2$, $k = j+3$, $k = j+4$, $k = j+5$ and $k = j+6$ with $i = 4$.

Now, we study the solution of equation (2.30). We follow the argument given in (iv) to eliminate the values of i. We obtain the equation

$$7j_i^2 + 2y^2 + 14z^2 - 14j_i yz - 1 = 0$$

is solvable only with $i \in \{1, 2\}$. If $i = 1$, then $j_i = j_1 = 1$. Substituting $j_i = j_1 = 1$ in equation (2.30), we obtain that

$$7j_j^2 + 2j_k^2 - 14j_j j_k + 13 = 0,$$

which can be written in the form

$$7j_j^2 + 13 = 2j_k(7j_j - j_k), \tag{2.42}$$

where $j \geq 1$. We note that equation (2.42) is possibly satisfied if $j \leq k \leq j+2$ (except when $k = j+2$ and $j = 1$). We show that the latter equation is possibly satisfied only if $7j_j - j_k > 0$ (or $7j_j > j_k$) for all $j \leq k \leq j+2$ (except when $k = j+1$ and $j = 1$), since the left-hand side of equation (2.42) is positive for all $j \geq 1$. But, it's not satisfied with $j+3 \leq k \leq j+11$ because if $j+3 \leq k \leq j+11$ we have the right hand side of equation (2.42) can be written in the form $7j_j - j_k = 7j_j - (j_{k-1} + 2j_{k-2}) < 0$ for all $k \geq j+3$ with $j \geq 1$. Hence, we get that equation (2.30) is possibly solvable if $j \leq k \leq j+2$. But, equation (2.30) is unsolvable if $j+3 \leq k \leq j+11$. Now, we study the solutions of equation (2.42) with $k \in \{j, j+1, j+2\}$.

– If $k = j$, we get from equation (2.42) that

$$-5j_j^2 + 13 = 0, \tag{2.43}$$

which does not lead to a solution as $j \geq 1$. Therefore, the equation (2.30) is not satisfied for all $k = j \geq 1$.

– If $k = j + 1$, this leads to

$$7j_j^2 + 2j_{j+1}^2 - 14j_j j_{j+1} + 13 = 0, \tag{2.44}$$

where $j \geq 1$. We can show that equation (2.44) is satisfied only with $j = 1$, and this is not possible for all $j \geq 2$. In the following, we show the left-hand side of equation (2.44) is always negative for all $j \geq 2$. However, if $j = 1$, we get that $7(1)^2 + 2(5)^2 - 14(1)(5) + 13 = 0$. So, the equation (2.30) has the solution $(j_i, j_j, j_k) = (j_1, j_1, j_2) = (1, 1, 5)$. Now, we deal with $j \geq 2$. From (1.7), we have that $j_{j+1} = j_j + 2j_{j-1}$. Then, equation (2.44) becomes as

$$\begin{aligned}
7j_j^2 + 2j_{j+1}^2 - 14j_j j_{j+1} + 13 &= 7j_j^2 + 2(j_j + 2j_{j-1})^2 - 14j_j(j_j + 2j_{j-1}) + 13 \\
&= 7j_j^2 + 2j_j + 8j_j j_{j-1} + 8j_{j-1}^2 - 14j_j^2 - 28j_{j-1}j_j + 13 \\
&= 8j_{j-1}^2 - 5j_j^2 - 20j_{j-1}j_j + 13 \\
&< 8j_{j-1}^2 - 5j_{j-1}^2 - 20j_{j-1}j_{j-1} + 13 \quad \text{as} \quad -j_j < -j_{j-1} \\
&= -17j_{j-1}^2 + 13 < 0 \quad \forall j \geq 2.
\end{aligned}$$

Therefore, equation (2.30) is not satisfied in the case of $k = j + 1$ with $j \geq 1$. As shown above, we note that the equation (2.30) has no solution when $k = j + 2$ and $i = 1$.

If $i = 2$, then $j_i = j_2 = 5$. Substituting $j_2 = 5$ in equation (2.30), we get that

$$7j_j^2 + 2j_k^2 - 70j_j j_k + 349 = 0, \tag{2.45}$$

where $2 \leq j \leq k$. We will show that equation (2.45) is possibly satisfied for all $j \leq k \leq j+5$. Since equation (2.45) can be written in the form

$$7j_j^2 + 349 = 2j_k(35j_j - j_k), \tag{2.46}$$

we note that equation (2.46) is possibly satisfied only if $2 \leq j \leq k \leq j+4$ and $35j_j - j_k > 0$ and the left-hand side of equation (2.45) is positive for all $j \geq 2$. But, this is impossible if $j+5 \leq k \leq j+11$ because if $j+5 \leq k \leq j+11$ we have $35j_j - j_k = 35j_j - (j_{k-1} + 2j_{k-2}) < 0$ for all $k \geq j+5$ with $j \geq 2$. Therefore, equation (2.30) is possibly solvable if $j \leq k \leq j+4$. But, this is not possible to be solved if $j+5 \leq k \leq j+11$ (except when $k = j+5$ and $j = 2$). Now, we study the solution of equation (2.45) with $j \leq k \leq j+5$.

– If $k = j$, we get (from equation (2.45) or (2.46)) that

$$61j_j^2 = 349, \tag{2.47}$$

where $j \geq 2$. We note that this is not possible for all values of j. Therefore, equation (2.47) has no solution with $k = j \geq 2$.

– If $k = j + 1$, substituting $k = j + 1$ in equation (2.45) we obtain that

$$7j_j^2 + 2j_{j+1}^2 - 70j_j j_{j+1} + 349 = 0, \tag{2.48}$$

where $j \geq 2$. We find that equation (2.48) is not satisfied for all $j \geq 2$ because the left-hand side of equation (2.48) is always negative. This can be proven as follows. If $j = 2$, we get that

$7(5)^2 + 2(7)^2 - 70(5)(7) + 349 = -1828$. Next, we study equation (2.48) with $j \geq 3$. By (1.7), we have that $j_{j+1} = j_j + 2j_{j-1}$ that we substitute in equation (2.48), we get that

$$\begin{aligned} 7j_j^2 + 2j_{j+1}^2 - 70j_jj_{j+1} + 349 &= 7j_j^2 + 2(j_j + 2j_{j-1})^2 - 70j_j(j_j + 2j_{j-1}) + 349 \\ &= 7j_j^2 + 2j_j + 8j_jj_{j-1} + 8j_{j-1}^2 - 70j_j^2 - 140j_{j-1}j_j + 349 \\ &= 8j_{j-1}^2 - 61j_j^2 - 132j_{j-1}j_j + 349 \\ &< 8j_j^2 - 61j_j^2 - 132j_{j-1}j_j + 349 \quad \text{as} \quad j_j > j_{j-1} \quad \forall j \geq 3 \\ &= -53j_j^2 - 132j_{j-1}j_j + 349 < 0 \quad \text{for all} \quad j \geq 3. \end{aligned}$$

Therefore, we obtain that equation (2.30) has no solution in the case of $k = j + 1$ and $j \geq 2$. Also, we can find that equation (2.30) has no solution when $k = j + 2$, $k = j + 3$, $k = j + 4$ with $i = 2$ and $j \geq 2$. Also. if $k = j + 5$ and $j = 2$, then equation (2.48) can be as follows. $7(5)^2 + 2(127)^2 - 70(5)(127) + 349 = -11668$. Therefore, equation (2.30) has no solution in the case of $k = j + 5$ and $j = 2$.

We next the investigate the solutions of equation (2.31). By argument (iv), we get that equation

$$14j_i^2 + 2y^2 + 7z^2 - 14j_iyz - 1 = 0$$

is solvable only with $i \in \{1, 2\}$. If $i = 1$, we get that

$$2j_j^2 + 7j_k^2 - 14j_jj_k + 13 = 0,$$

or

$$2j_j^2 + 13 = 7j_k(2j_j - j_k), \tag{2.49}$$

where $k \geq j \geq 1$. We get equation (2.49) is possibly satisfied only if $j \leq k \leq j + 1$. Since we find that the left-hand side of equation (2.49) is positive. So, we obtain that $(2j_j - j_k) > 0$ for all $j \leq k \leq j + 1$. Therefore, equation (2.31) is possibly solvable only if $j \leq k \leq j + 1$ (except when $k = j + 1$ and $j = 1$). But, equation (2.31) is unsolvable if $j + 2 \leq k \leq j + 11$ because if $j + 2 \leq k \leq j + 11$ we have $2j_j - j_k = 2j_j - (j_{k-1} + 2j_{k-2}) < 0$ for all $k \geq j + 2$ with $j \geq 1$. Now, we study the solutions of equation (2.31) at $k \in \{j, j + 1\}$.

– If $k = j$, we get that

$$5j_j^2 = 13, \tag{2.50}$$

where $j \geq 1$. We obtain that equation (2.50) is not satisfied for all $j \geq 1$. So, equation (2.31) is not satisfied if $k = j \geq 1$.

– If $k = j + 1$, we obtain that

$$2j_j^2 + 7j_{j+1}^2 - 14j_jj_{j+1} + 13 = 0, \tag{2.51}$$

where $1 = i \leq j \leq k$. We show that the latter equation is not satisfied for all $j \geq 1$ by proving its left hand said is not satisfied for all $j \geq 1$. In case of $j = 1$, we get that $2(1)^2+7(5)^2-14(1)(5)+13 = 120$. Suppose that $j = 2$, we obtain that $2(5)^2 + 7(7)^2 - 14(5)(7) + 13 = -84$. Assume that $j = 3$, we have that $2(7)^2+7(17)^2-14(7)(17)+13 = 468$. In the case of $j = 4$, then $2(17)^2+7(31)^2-14(17)(31)+13 = -60$. If $j = 5$, then $2(31)^2 + 7(65)^2 - 14(31)(65) + 13 = 3300$. Now, we deal with $j \geq 6$. This can be shown by substituting the relation (1.7) in equation (2.51) several times. Namely,

$$\begin{aligned}
2j_j^2+7j_{j+1}^2-14j_jj_{j+1}+13 &= 2j_j^2+7(j_j+2j_{j-1})^2-14j_j(j_j+2j_{j-1})+13\\
&= 2j_j^2+7j_j^2+28j_jj_{j-1}+28j_{j-1}^2-14j_j^2-28j_jj_{j-1}+367\\
&= 28j_{j-1}^2-5j_j^2+13\\
&= 28j_{j-1}^2-5(j_{j-1}+2j_{j-2})^2+13\\
&= 23j_{j-1}^2-20j_{j-1}j_{j-2}-20j_{j-2}^2+13\\
&= 23(j_{j-2}+2j_{j-3})^2-20(j_{j-2}+2j_{j-3})j_{j-2}-20j_{j-2}^2+13\\
&= 92j_{j-3}^2-17j_{j-2}^2+52j_{j-2}j_{j-3}+13\\
&= 92j_{j-3}^2-17(j_{j-3}+2j_{j-4})^2+52(j_{j-3}+2j_{j-4})j_{j-3}+13\\
&< -68j_{j-4}^2+127j_{j-4}^2+36j_{j-4}+13 \quad \text{as} \quad j_{j-3}>j_{j-4} \quad \forall j\geq 6\\
&= 95j_{j-4}^2+13>0 \quad \text{for all} \quad j\geq 6.
\end{aligned}$$

Then equation (2.31) is not satisfy if $k=j+1$ with $j\geq i=1$.

If $i=2$, this leads to $j_j=j_2=5$. Substituting $j_2=5$ in equation (2.31), we get that

$$2j_j^2+7j_k^2-70j_jj_k+349=0,$$

or

$$2j_j^2+349=7j_k(10j_j-j_k), \tag{2.52}$$

where $k\geq j\geq 2$. We show that equation (2.52) is possibly satisfied only at $j\leq k\leq j+3$ since the left-hand side of equation (2.52) is positive for all $j\geq 2$ and $(10j_j-j_k)>0$ is satisfied only at $j\leq k\leq j+3$ for all $j\geq 2$. Therefore, equation (2.31) is possibly solvable if $j\leq k\leq j+3$. But, it's not satisfied with $j+4\leq k\leq j+11$ because if $j+4\leq k\leq j+11$ we have $10j_j-j_k=10j_j-(j_{k-1}+2j_{k-2})<0$ for all $k\geq j+4$ with $j\geq 1$. So, we study the solutions of equation (2.31) by studying the solution of (2.52) when $j\leq k\leq j+3$.

- If $k=j$, we get that

$$61j_j^2=349, \tag{2.53}$$

which does not lead to a solution as $j\geq 2$. Therefore, equation (2.31) having no solution when $k=j$ with $i=2$.

- If $k=j+1$, we obtain that

$$2j_j^2+7j_{j+1}^2-70j_jj_{j+1}+349=0, \tag{2.54}$$

where $j\geq 2$. We note that equation (2.54) is not satisfied for all $2=i\leq j$ by proving its left-hand side always negative. Now, if $j=2$, we get that $2(5)^2+7(7)^2-70(5)(7)+349=-1708$. Next, we consider $j\geq 3$. By (1.7), we have that $j_{j+1}=j_j+2j_{j-1}$. Then, equation (2.54) becomes as

$$\begin{aligned}
2j_j^2+7j_{j+1}^2-70j_jj_{j+1}+349 &= 2j_j^2+7(j_j+2j_{j-1})^2-70j_j(j_j+2j_{j-1})+349\\
&= 2j_j^2+7j_j+28j_jj_{j-1}+28j_{j-1}^2-70j_j^2-140j_{j-1}j_j+349\\
&= 28j_{j-1}^2-61j_j^2-112j_{j-1}j_j+349\\
&< 28j_j^2-61j_j^2-112j_{j-1}j_j+349 \quad \text{as} \quad j_j>j_{j-1} \quad \forall j\geq 3\\
&= -33j_j^2-112j_{j-1}j_j+349<0 \quad \text{for all} \quad j\geq 3.
\end{aligned}$$

Therefore, equation (2.31) having no solutions in the Jacobsthal–Lucas numbers when $k = j+1$ with $2 = i \geq j$. In the same way, we find that the equation (2.31) does not have any solutions when $k = j+2$ and $k = j+3$ with $j \geq i = 2$.

Finally, we study the solution of equation (2.32). We follow the argument given in (iv) to eliminate the values of i. We get that

$$2j_i^2 + 14y^2 + 7z^2 - 14j_iyz - 1 = 0$$

is solvable only with $i = 2$, which leads to

$$14j_j^2 + 7j_k^2 - 70j_jj_k + 49 = 0,$$

that can be written as follows

$$14j_j^2 + 49 = 7j_k(10j_j - j_k), \tag{2.55}$$

where $2 \leq j \leq k$. We note that equation (2.55) is possibly satisfied only if $(10j_j - j_k) > 0$. Since the left-hand side is positive integers for all $j \geq 2$. So, equation (2.55) might be solvable if $2 \leq j \leq k \leq j+3$. But, it's not satisfied with $j+4 \leq k \leq j+11$ because if $j+4 \leq k \leq j+11$ we have $10j_j - j_k = 10j_j - (j_{k-1} + 2j_{k-2}) < 0$ for all $k \geq j+4$ with $j \geq 2$. This leads to equation (2.32) is possibly solvable only if $j \leq k \leq j+3$. But, it's not satisfied with $j+4 \leq k \leq j+11$.

– Now, if $k = j$, we substitute the value of $k = j$ into the equation (2.55) we get that

$$49j_j^2 = 7, \tag{2.56}$$

we note that equation (2.56) is not satisfied for all $j \geq i = 2$. Then, equation (2.32) has no solution at $k = j \geq 2$.

– If $k = j+1$, we substitute the values of $k = j+1$ into the equation (2.55) to obtain that

$$14j_j^2 + 7j_{j+1}^2 - 70j_jj_{j+1} + 49 = 0, \tag{2.57}$$

where $j \geq 2$. In the following, we prove that the equation (2.57) is not satisfied for all $j \geq 2$ by showing the left-hand side of equation (2.57) is always negative. This can be proven as follows. If $j = 2$, we obtain that $14(5)^2 + 7(7)^2 - 70(5)(7) + 49 = -1708$. If $j = 3$, then $14(7)^2 + 7(17)^2 - 70(7)(17) + 49 = -5572$. After that, we study equation (2.57) with $j \geq 4$. This can be shown by substituting the relation (1.7) in equation (2.57) several times. Namely,

$$\begin{aligned}
14j_j^2 + 7j_{j+1}^2 - 70j_jj_{j+1} + 49 &= 14j_j^2 + 7(j_j + 2j_{j-1})^2 - 70j_j(j_j + 2j_{j-1}) + 49 \\
&= 14j_j^2 + 7j_j^2 + 28j_{j-1}j_1^2 + 28j_{j-1}^2 - 70j_j^2 - 140jj_{j-1} + 49 \\
&= 28j_{j-1}^2 - 49j_j^2 - 42jj_{j-1} + 49 \\
&= 28j_{j-1}^2 - 49(j_{j-1} + 2j_{j-2})^2 - 42(j_{j-1} + 2j_{j-2})j_{j-1} + 49 \\
&= -63j_{j-1}^2 - 280j_{j-1}j_{j-2} - 196j_{j-1}^2 + 49 < 0 \quad \text{for all} \quad j_j \geq 4.
\end{aligned}$$

Then, equation (2.28) has no solution if $k = j+1$ for all $j \geq 4$. Similarly, we obtain that the equation (2.28) does not have a solution if $k = j+2$, $k = j+3$, and $k = j+4$ for all $j \geq 4$.

Therefore, through the above studies to equations (2.27)-(2.32), we obtain only one solution to equation (1.9) with $j = 1$, $j = 1$, and $k = 2$. Hence, we get the solution $(j_i, j_j, j_k) = (1, 1, 5)$. By permuting the component of $(j_i, j_j, j_k) = (1, 1, 5)$, we get the solutions of equation (1.9) given:

$$(j_i, j_j, j_k) = (1, 5, 1)$$

☐ `Case 2`. If $(A,B,C,D)=(2,2,3,6)$. By permuting the coefficients of the equation (1.10), we get the equations:

$$2j_i^2+2j_j^2+3j_k^2=6j_ij_jj_k+1, \tag{2.58}$$

$$2j_i^2+3j_j^2+2j_k^2=6j_ij_jj_k+1, \tag{2.59}$$

$$3j_i^2+2j_j^2+2j_k^2=6j_ij_jj_k+1, \tag{2.60}$$

where $1\le i\le j\le k$. From argument (iii), we have that $i\le 6$ and $j\le k\le j+11$. We now investigate the solutions of equation (2.58). We next follow the argument given in (iv) to eliminate the values of i. We get that

$$2j_i^2+2y^2+3z^2-6j_iyz-1=0$$

is solvable only with $i\in\{1,2,3,4\}$.

If $i=1$, we get that

$$2j_j^2+3j_k^2-6j_jj_k+1=0$$

becomes as follows:

$$2j_j^2+1=3j_k(2j_j-j_k), \tag{2.61}$$

where $1\le j\le k$. We obtain that equation (2.61) is possibly satisfied for all $j\le k\le j+1$. Since equation (2.61) can be written in the form $2j_j^2+1=3j_k(2j_j-j_k)$, we find that equation (2.61) is possibly satisfied only if $(2j_j-j_k)>0$ for all $j\le k\le j+1$ (except when $k=j+1$ and $j=1$) as the left-hand side of equation (2.61) is possible for all $j\ge 1$. But, it's not satisfied with $j+2\le k\le j+11$ because if $j+2\le k\le j+11$ we have $2j_j-j_k=2j_j-(j_{k-1}+2j_{k-2})<0$ for all $k\ge j+2$ with $j\ge 1$. Therefore, equation (2.58) might be solvable if $j\le k\le j+1$. But, this is not possible if $j+2\le k\le j+11$. Now, we consider equation (2.61) in the case of $k\in\{j,j+1\}$.

– If $k=j$, we get that

$$j_j^2=1, \tag{2.62}$$

where $j\ge 1$. Therefore, equation (2.62) is satisfied with $i=1$. So, equation (2.58) has the solution $(j_1,j_1,j_1)=(1,1,1)$. But, in the case of $j\ge 2$, we note that the left-hand side of equation (2.62) is not satisfied for all $k=j\ge 2$ because ${j_j}^2-1=(j_{j-1}+2j_{j-2})^2=j_{j-1}^2+4j_{j-1}j_{j-2}+4j_{j-2}^2-1>0$ for all $j\ge 1$. Therefore, equation (2.58) has no solution when $k=j\ge 2$.

– If $k=j+1$, we obtain that

$$2j_j^2+3j_{j+1}^2-6j_jj_{j+1}+1=0, \tag{2.63}$$

where $j\ge 1$. We show the left-hand side equation (2.63) is not satisfied for all $j\ge 1$. This can be proven as follows. Let $j=1$, we obtain that

$$2j_j^2+3j_{j+1}^2-6j_jj_{j+1}+1=2+3(7)^2-6(5)+1=48.$$

Now, we consider equation (2.63) if $j\ge 2$. By (1.7), we have that $j_{j+1}=j_j+2j_{j-1}$ which is substituted in the left-hand side of equation (2.63), to have that

$$\begin{aligned}2j_j^2+3j_{j+1}^2-6j_jj_{j+1}+1&=2j_j^2+3(j_j+2j_{j-1})^2-6j_j(j_j+2j_{j-1})+1\\&=2j_j^2+3j_j+12j_jj_{j-1}+12j_{j-1}^2-6j_j^2-12j_{j-1}j_j+1\\&=12j_{j-1}^2-j_j^2+1.\\&>12j_{j-1}^2-j_{j-1}^2+1\quad\text{as}\quad -j_j<-j_{j-1}\quad\forall j\ge 3\\&=11j_{j-1}^2+1>0\quad\text{for all}\quad j\ge 3.\end{aligned}$$

So, equation (2.58) has no solution with $k = j + 1$ and $j \geq 1$. Similarly, we find that equation (2.58) has no solution when $i \in \{2, 3, 4\}$ and $2 \leq j \leq k \leq j + 1$. Therefore, we omit the details of computations.

Next, we study the solutions of equation (2.59). We follow the argument given in (iv) to eliminate the not needed values of i. We get that

$$2j_i^2 + 3y^2 + 2z^2 - 6j_i yz - 1 = 0$$

is solvable only with $i \in \{1, 2, 3, 4\}$. If $i = 1$, then $j_i = j_1 = 1$. Substituting $j_i = j_1 = 1$ in equation (2.59), we obtain that

$$3j_j^2 + 2j_k^2 - 6j_j j_k + 1 = 0,$$

that leads to

$$3j_j^2 + 1 = 2j_k(3j_j - j_k), \tag{2.64}$$

where $1 \leq j \leq k$. We obtain that equation (2.64) is possibly satisfied only if $j \leq k \leq j + 1$, as the left-hand side of equation (2.64) is positive for all $j \geq 2$ and $(3j_j - j_k) > 0$ when $j \leq k \leq j + 1$ (except when $k = j + 1$ and $j = 1$). But, it's not satisfied with $j + 2 \leq k \leq j + 11$ because if $j + 2 \leq k \leq j + 11$ we have $3j_j - j_k = 3j_j - (j_{k-1} + 2j_{k-2}) < 0$ for all $k \geq j + 2$ with $j \geq 1$. Therefore, we obtain that equation (2.59) is possibly solvable only when $j \leq k \leq j + 1$. But, when $j+2 \leq k \leq j+11$, equation (2.59) is unsolvable. Now, we study equation (2.64) when $k \in \{j, j+1\}$.

- If $k = j$, we get that

$$j_j^2 = 1, \tag{2.65}$$

which gives that $j = 1$. Then equation (2.59) has a solution if $(j_i, j_j, j_k) = (j_1, j_1, j_1) = (1, 1, 1)$. But, we note that equation (2.65) is not satisfied for all $j \geq 2$ because ${j_j}^2 - 1 = (j_{j-1} + 2j_{j-2})^2 = j_{j-1}^2 + 4j_{j-1}j_{j-2} + 4j_{j-2}^2 - 1 > 0$ for all $j \geq 1$. Therefore, equation (2.59) has no solution for all $k = j \geq 2$.

- If $k = j + 1$, this leads to

$$3j_j^2 + 2j_{j+1}^2 - 6j_j j_{j+1} + 1 = 0, \tag{2.66}$$

where $1 \leq j \leq k$. We get that equation (2.66) is not satisfied for all $j \geq 1$. This can be explained as follows. If $j = 1$, we get that $3(1)^2 + 2(5)^2 - 6(1)(5) + 1 = 24$. In the case of $j = 2$, this leads to $3(5)^2 + 2(7)^2 - 6(5)(7) + 1 = -36$. For $j = 3$, we get that $3(7)^2 + 2(17)^2 - 6(7)(17) + 1 = 12$. Assuming that $j = 4$, we obtain that $3(17)^2 + 2(31)^2 - 6(17)(31) + 1 = -372$. If $j = 5$, we get that $3(31)^2 + 2(65)^2 - 6(31)(65) + 1 = -756$. Now, we consider equation (2.66) with $j \geq 6$. This can be studied by substituting the relation (1.7) in equation (2.66) several times. Namely,

$$\begin{aligned}
3j_j^2 + 2j_{j+1}^2 - 6j_j j_{j+1} + 1 &= 3j_j^2 + 2(j_j + 2j_{j-1})^2 - 6j_j(j_j + 2j_{j-1}) + 1 \\
&= 3j_j^2 + 2j_j + 8j_j j_{j-1} + 8j_{j-1}^2 - 6j_j^2 - 12j_{j-1}j_j + 1 \\
&= 8j_{j-1}^2 - j_j^2 - 4j_{j-1}j_j + 1 \\
&= 8j_{j-1}^2 - (j_{j-1} + 2j_{j-2})^2 - 4j_{j-1}(j_{j-1} + 2j_{j-2}) + 1 \\
&= 3j_{j-1}^2 - 12j_{j-1}j_{j-2} - 4j_{j-2}^2 + 1 \\
&= 3(j_{j-2} + 2j_{j-3})^2 - 12(j_{j-2} + 2j_{j-3})j_{j-2} - 4j_{j-2}^2 + 1 \\
&< 12j_{j-3}^2 - 13j_{j-2}^2 - 12j_{j-2}j_{j-3} + 1 \quad \text{as} \quad j_{j-3} < j_{j-2} \quad \forall j \geq 6 \\
&= -j_{j-2}^2 - 12j_{j-2}j_{j-3} + 1 < 0 \quad \forall j \geq 6.
\end{aligned}$$

Therefore, equation (2.59) has no solution if $k = j+1$ for all $j \geq 1$. In the same way, we find that equation (2.59) does not have any solution when $i \in \{2, 3, 4\}$.

Now, we study the solutions of equation (2.60). We follow the argument given in (iv) to eliminate the not needed values of i. We get that

$$3j_i^2 + 2y^2 + 2z^2 - 6j_i yz - 1 = 0$$

is solvable only with $i \in \{1, 6\}$.

– If $i = 1$, we obtain that

$$2j_j^2 + 2j_k^2 - 6j_j j_k + 2 = 0, \tag{2.67}$$

where $2 = i \leq j \leq k$. We can show that equation (2.67) is possibly satisfied if $j \leq k \leq j+1$. In fact, equation (2.67) can be written in the form

$$j_j^2 + 1 = j_k(3j_j - j_k). \tag{2.68}$$

where $k \geq j \geq 2$. Note that the latter equation is possibly satisfied only if $(3j_j - j_k) > 0$ (or $3j_j > j_k$) for all $j \leq k \leq j+1$, since the left-hand side of equation (2.68) is positive for all $j \geq 1$. But, it's not satisfied with $j+2 \leq k \leq j+11$ because if $j+2 \leq k \leq j+11$ we have $3j_j - j_k = 3j_j - (j_{k-1} + 2j_{k-2}) < 0$ for all $k \geq j+2$ with $j \geq 2$. Therefore, equation (2.60) is possibly solvable at $j \leq k \leq j+1$ as $j \leq k$. But, this is impossible when $j+2 \leq k \leq j+11$.

– If $k = j$. From equation (2.68), we get that

$$j_j^2 = 1. \tag{2.69}$$

where $1 = i \leq j \leq k$. We obtain that (2.69) is satisfied only if $j = 1$. But, one can easily show that equation (2.69) has no solution with $k = j \geq 2$. Therefore, the equation (2.60) has a solution only if $i = j = k = 1$.

– If $k = j+1$, replacing in equation (2.68), we obtain that

$$2j_j^2 + 2j_{j+1}^2 - 6j_j j_{j+1} + 2 = 0, \tag{2.70}$$

where $1 = i \leq j \leq k$. We find that equation (2.70) is not satisfied for all $j \geq 1$. This can be proven as follows. If $j = 1$, we obtain that $2(1)^2 + 2(5)^2 - 6(1)(5) + 2 = 24$. If $j = 2$, we get that $2(5)^2 + 2(7)^2 - 6(5)(7) + 2 = -60$. Also, we note that equation (2.70) is not satisfied for all $j \geq 3$ by proving its left-hand side is always negative. This can be shown by substituting the relation (1.7) in equation (2.70) several times. Namely,

$$\begin{aligned}
2j_j^2 + 2j_{j+1}^2 - 6j_j j_{j+1} + 2 &= 2j_j^2 + 2(j_j + 2j_{j-1})^2 - 6j_j(j_j + 2j_{j-1}) + 2 \\
&= 2j_j^2 + 2j_j + 8j_j j_{j-1} + 8j_{j-1}^2 - 6j_j^2 - 12j_{j-1}j_j + 2 \\
&= 8j_{j-1}^2 - 2j_j^2 - 4j_{j-1}j_j + 2 \\
&= 8j_{j-1}^2 - 2(j_{j-1} + 2j_{j-2})^2 - 4j_{j-1}(j_{j-1} + 2j_{j-2}) + 2 \\
&= 2j_{j-1}^2 - 8j_{j-2}^2 - 16j_{j-1}j_{j-2} + 2 \\
&= 2(j_{j-2} + 2j_{j-3})^2 - 8j_{j-2}^2 - 16j_{j-1}j_{j-2} + 2 \\
&= 8j_{j-2}j_{j-3} + 8j_{j-3}^2 - 6j_{j-2}^2 - 16j_{j-1}j_{j-2} + 2 \\
&< 8j_{j-2}^2 + 8j_{j-2}^2 - 6j_{j-2}^2 - 16j_{j-2}^2 + 2 \\
&= -6j_{j-2}^2 + 13 < 0 \quad \text{as,} \quad -j_{j-2} > -j_{j-1}, \quad j_{j-2} > j_{j-3} \quad \forall j \geq 3.
\end{aligned}$$

Therefore, equation (2.60) is not satisfied if $k = j + 1$ with $1 = i \leq j \leq k$. Also, in a similar way we notice that equation (2.60) has no solution when $i = 6$. Hence, we leave out the details of the proof.

Based on what was studied above, we notice that equations (2.58)-(2.60) have only one solution that is $(j_i, j_j, j_k) = (j_1, j_1, j_1) = (1, 1, 1)$. Therefore, equation (1.10) has a single solution given by

$$(j_i, j_j, j_k) = (j_1, j_1, j_1) = (1, 1, 1).$$

☐ `Case 3` . If $(A, B, C, D) = (6, 10, 15, 30)$. By permuting the coefficients of equation (1.11), we obtain the distinct equations as follows:

$$6j_i^2 + 10j_j^2 + 15j_k^2 = 30j_ij_jj_k + 1, \tag{2.71}$$
$$10j_i^2 + 6j_j^2 + 15j_k^2 = 30j_ij_jj_k + 1, \tag{2.72}$$
$$6j_i^2 + 15j_j^2 + 10j_k^2 = 30j_ij_jj_k + 1, \tag{2.73}$$
$$15j_i^2 + 6j_j^2 + 10j_k^2 = 30j_ij_jj_k + 1, \tag{2.74}$$
$$10j_i^2 + 15j_j^2 + 6j_k^2 = 30j_ij_jj_k + 1, \tag{2.75}$$
$$15j_i^2 + 10j_j^2 + 6j_k^2 = 30j_ij_jj_k + 1, \tag{2.76}$$

where $1 \leq i \leq j \leq k$. From argument (iii), we obtain that $1 \leq i \leq 6$ and $j \leq k \leq j + 11$ for all equation (2.71)-(2.76). We first study the solutions of equation (2.71). We follow the argument given in (iv) to eliminate the not needed values of i. We get that

$$6j_i^2 + 10y^2 + 15z^2 - 30j_iyz - 1 = 0$$

is solvable only with $i \in \{1, 5\}$. If $i = 1$, then $j_i = j_1 = 1$. Substituting $j_i = j_1 = 1$ in equation (2.71), we obtain that

$$10j_j^2 + 15j_k^2 - 30j_jj_k + 5 = 0, \tag{2.77}$$

where $1 = i \leq j \leq k$. We can show that equation (2.77) is possibly satisfied only if $1 = j \leq k \leq j+1$. In fact, equation (2.77) can be written in the form

$$2j_j^2 + 1 = 3j_k(2j_j - j_k), \tag{2.78}$$

where $1 \leq j \leq k$. Note that the latter equation is possibly satisfied only if $2j_j - j_k > 0$ (or $2j_j > j_k$) for all $j \leq k \leq j + 1$ (except when $k = j + 1$ and $j = 1$). Since the left-hand side of equation (2.78) is positive for all $j \geq 1$. But, it's not satisfied with $j + 2 \leq k \leq j + 11$ because if $j + 2 \leq k \leq j + 11$ we have $2j_j - j_k = 2j_j - (j_{k-1} + 2j_{k-2}) < 0$ for all $k \geq j + 2$ with $j \geq 1$. So, we get that if $j \leq k \leq j + 1$, the equation (2.71) is possibly solvable. But, this is impossible if $j + 2 \leq k \leq j + 11$. Now, we study equation (2.78) at $k \in \{j, j + 1\}$.

– If $k = j$, we get that

$$j_j^2 = 1. \tag{2.79}$$

where $j \geq 1$. We get that equation (2.79) is satisfied only if $j = 1$. But, one can easily show that equation (2.79) is not satisfied when $j_j \geq 2$. Then, equation (2.71) has a solution only if $1 = i = j = k$.

– If $k = j + 1$, we obtain that

$$2j_j^2 + 3j_{j+1}^2 - 6j_jj_{j+1} + 1 = 0, \tag{2.80}$$

where $j \geq 1$. We can show that the equation (2.80) has no solution for all $j \geq 1$. This can be proven as follows. In the case of $j = 1$, we get the left hand side of equation (2.80) as $2(1)^2 + 3(5)^2 - 6(1)(5) + 1 = 8$. Suppose that $j = 2$, we obtain that $2(5)^2 + 3(7)^2 - 6(5)(7) + 1 = -12$. If $j = 3$, this leads to $2(7)^2 + 3(17)^2 - 6(7)(17) + 1 = 252$. Next, we study the equation (2.80) for all $j \geq 4$. This can be shown by substituting the relation (1.7) in equation (2.80) several times. Namely,

$$\begin{aligned}
2j_j^2 + 3j_{j+1}^2 - 6j_j j_{j+1} + 1 &= 2j_j^2 + 3(j_j + 2j_{j-1})^2 - 6j_j(j_j + 2j_{j-1}) + 1 \\
&= 2j_j^2 + 3j_j + 12j_j j_{j-1} + 12j_{j-1}^2 - 6j_j^2 - 12j_{j-1}j_j + 1 \\
&= 12j_{j-1}^2 - j_j^2 + 1 \\
&= 12j_{j-1}^2 - (j_{j-1} + 2j_{j-2})^2 + 1 \\
&= 11j_{j-1}^2 - 4j_{j-1}j_{j-2} - 4j_{j-2}^2 + 1 \\
&= 11(j_{j-2} + 2j_{j-3})^2 - 4(j_{j-2} + 2j_{j-3})j_{j-2} - 4j_{j-2}^2 + 1 \\
&= 44j_{j-3}^2 + 3j_{j-2}^2 + 36j_{j-2}j_{j-3} + 1 > 0 \quad \forall j \geq 4.
\end{aligned}$$

Therefore, we have that equation (2.80) is not satisfied if $k = j + 1$ with $j \geq 1$. This implies that the equation (2.71) has no solution with $k = j + 1$ and $j \geq 1$.

Now, if $i = 5$. In the same way, we get that equation (2.71) is possibly solvable if $j \leq k \leq j + 5$ with $5 = i \leq j$. After checking the solutions of the equation (2.71), we find that the equation does not have a solution in all cases.

We now study the solutions of equation (2.72). We next follow the argument given in (iv) to eliminate the not needed values of i. We get that

$$10j_i^2 + 6y^2 + 15z^2 - 30j_i yz - 1 = 0$$

is solvable only with $i \in \{1, 3\}$. If $i = 1$, we get that

$$2j_j^2 + 5j_k^2 - 10j_j j_k + 3 = 0,$$

which leads to

$$j_j^2 + 3 = 5j_k(2j_j - j_k), \tag{2.81}$$

where $1 = i \leq j \leq k$. We can show that the equation (2.81) is possibly satisfied only with $1 \leq j \leq k \leq j+1$, since the left-hand side of equation (2.81) is positive for $j \geq 1$ and $(2j_j - j_k) > 0$ when $1 \leq j \leq k \leq j + 1$ (except when $k = j + 1$ and $j = 1$). But, it's not satisfied with $j + 2 \leq k \leq j + 11$ because if $j + 2 \leq k \leq j + 11$ we have $2j_j - j_k = 2j_j - (j_{k-1} + 2j_{k-2}) < 0$ for all $k \geq j + 2$ with $j \geq 1$. Therefore, equation (2.72) is possibly solvable if $j \leq k \leq j + 1$. But, equation (2.72) unsolvable when $j + 2 \leq k \leq j + 11$. As done earlier, we find that equation (2.72) has no solution if $j \leq k \leq j + 1$.

Now, if $i = 3$. Substituting $i = 3$ in equation (2.72), we get that

$$6j_j^2 + 15j_k^2 - 210j_j j_k + 489 = 0,$$

or

$$2j_j^2 + 163 = 14j_k(5j_j - j_k), \tag{2.82}$$

where $3 = i \leq j \leq k$. We note the left-hand side of equation (2.82) is positive for all $j \geq 3$. So, $(5j_j - j_k) > 0$ is satisfied only if $j \leq k \leq j + 2$. But, it's not satisfied with $j + 3 \leq k \leq j + 11$ because if $j + 3 \leq k \leq j + 11$ we have $5j_j - j_k = 5j_j - (j_{k-1} + 2j_{k-2}) < 0$ for all $k \geq j + 3$ with $j \geq 3$. Therefore, we get that equation (2.72) is possibly solvable when $j \leq k \leq j + 2$. But, this is impossible in case $j + 3 \leq k \leq j + 11$ with $j \geq 3$.

- If $k = j$. Substituting it in equation (2.82), we get that

$$63j_j^2 = 163, \tag{2.83}$$

where $j \geq 3$. Clearly, equation (2.83) has no solution for all $j \geq 3$. Therefore, equation (2.72) has no solution for all $j \geq 3$.

– If $k = j+1$, replacing $k = j+1$ in equation (2.82), we obtain that

$$2j_j^2 + 5j_{j+1}^2 - 70j_jj_{j+1} + 163 = 0, \tag{2.84}$$

where $j \geq 3$. We note that the equation (2.84) is not satisfied for all $j \geq 3$ by proving its left-hand side is always negative. In the case of $j = 3$, we get that $2(7)^2 + 5(17)^2 - 70(7)(17) + 163 = -6624$. If $j = 4$, we obtain that $2(17)^2 + 5(31)^2 - 70(17)(31) + 163 = -31344$. Next, we study equation (2.84) with $j \geq 5$. By (1.7), we have that $j_{j+1} = j_j + 2j_{j-1}$ which we substitute in the left hand side of equation (2.84) to get that

$$\begin{aligned}
&2j_j^2 + 5j_{j+1}^2 - 70j_jj_{j+1} + 163 = 2j_j^2 + 5(j_j + 2j_{j-1})^2 - 70j_j(j_j + 2j_{j-1}) + 163\\
&= 2j_j^2 + 7j_j + 20j_jj_{j-1} + 20j_{j-1}^2 - 70j_j^2 - 140j_{j-1}j_j + 163\\
&= 20j_{j-1}^2 - 63j_j^2 - 120j_{j-1}j_j + 163\\
&< 20j_j^2 - 63j_j^2 - 120j_{j-1}j_j + 163 \quad \text{as} \quad j_j > j_{j-1} \quad \forall j \geq 5\\
&= -43j_j^2 - 120j_{j-1}j_j + 163 < 0 \quad \text{for all} \quad j \geq 5.
\end{aligned}$$

Therefore, equation (2.72) having no solutions in the Jacobsthal–Lucas numbers when $k = j+1$ with $3 = i \leq j$. Similarly, we find that the equation (2.72) does not have a solution when $k = j+2$ with $j \geq 3$.

Now, we investigate the solutions of equation (2.73). We follow the argument given in (iv) to eliminate the not needed values of i. We obtain that the equation

$$6j_i^2 + 15y^2 + 10z^2 - 30j_iyz - 1 = 0$$

is solvable only with $i \in \{1, 5\}$. If $i = 1$, then $j_i = j_1 = 1$. Substitute $j_i = j_1 = 1$ in equation (2.73), we obtain that

$$15j_j^2 + 10j_k^2 - 30j_jj_k + 5 = 0,$$

which leads to

$$3j_j^2 + 1 = 2j_k(3j_j - j_k), \tag{2.85}$$

where $1 \leq i \leq j \leq k$. We can show that equation (2.85) is possibly satisfied only if $j \leq k \leq j+1$. In fact, equation (2.85) can be written in the form $3j_j^2 + 1 = 2j_k(3j_j - j_k)$. Note that the latter equation is satisfied only if $3j_j - j_k > 0$ (or $3j_j > j_k$) for all $j \leq k \leq j+1$, since the left hand side of equation (2.85) is positive for all $j \geq 1$. Then, equation (2.73) is possibly solvable if $j \leq k \leq j+1$. But, it's not satisfied with $j+2 \leq k \leq j+11$ because if $j+2 \leq k \leq j+11$ we have $3j_j - j_k = 3j_j - (j_{k-1} + 2j_{k-2}) < 0$ for all $k \geq j+2$ with $j \geq 1$. As done earlier, we can easily show that equation (2.73) does not have any solution with $k \in \{j, j+1\}$ and $j \geq i = 1$. Hence, we obtain that equation (2.73) does not have any solution at $i = 1$. Also, following the same method, we find that equation (2.73) has no solution when $i = 5$.

We investigate the solutions of equation (2.74). By argument (iv), we have that

$$15j_i^2 + 6y^2 + 10z^2 - 30j_iyz - 1 = 0$$

can be solved with respect to y and z only with $i \in \{1, 3\}$. If $i = 1$, we get that

$$6j_j^2 + 10j_k^2 - 30j_j j_k + 14 = 0,$$

or

$$3j_j^2 + 7 = 5j_k(3j_j - j_k). \tag{2.86}$$

where $1 \leq i \leq j \leq k$. We note that equation (2.86) is possibly satisfied only if $j \leq k \leq j+1$. In fact, equation (2.86) can be written in the form $3j_j^2 + 7 = 5j_k(3j_j - j_k)$. Note that the latter equation is satisfied only if $3j_j - j_k > 0$ (or $3j_j > j_k$) for all $1 = j \leq k \leq j+1$, since the left hand side of equation (2.86) is positive for all $j \geq 1$. But, it's not satisfied with $j+2 \leq k \leq j+11$ because if $j+2 \leq k \leq j+11$ we have $3j_j - j_k = 3j_j - (j_{k-1} + 2j_{k-2}) < 0$ for all $k \geq j+2$ with $j \geq 1$. Therefore, equation (2.74) is solvable if $j \leq k \leq j+1$. But, this is impossible if $j+2 \leq k \leq j+11$ with $j \geq 1$. By studying equation (2.86) with $k \in \{j, j+1\}$ and $j \geq 1$ as done in the pervious cases, we get that equation (2.86) has no solution. Therefore, we find that equation (2.74) has no solution when $i = 1$. In a similar way, one can show that the equation has no solutions with $i = 3$.

Now, we study the solutions of equation (2.75). By argument (iv), we have that

$$10j_i^2 + 15y^2 + 6z^2 - 30j_i yz - 1 = 0$$

can be solved with respect to y and z only with $i \in \{1, 3\}$. If $i = 1$, we get that

$$15j_j^2 + 6j_k^2 - 30j_j j_k + 9 = 0,$$

which leads to

$$5j_j^2 + 3 = 2j_k(5j_j - j_k), \tag{2.87}$$

where $1 \leq i \leq j \leq k$. Note that the latter equation is possibly satisfied only if $(5j_j - j_k) > 0$ (or $5j_j > j_k$) for all $j \leq k \leq j+2$ since the left-hand side of equation (2.87) is positive for all $j \geq 1$. But, it's not satisfied with $j+3 \leq k \leq j+11$ with which we have $5j_j - j_k = 5j_j - (j_{k-1} + 2j_{k-2}) < 0$ for all $k \geq j+3$ with $j \geq 1$. Therefore, equation (2.74) is possibly solvable if $j \leq k \leq j+2$. But, this is impossible if $j+3 \leq k \leq j+11$ and $1 \leq j \leq k$. Now, we consider equation (2.87) with $j \leq k \leq j+2$.

– If $k = j$, we get that

$$j_j^2 = 1, \tag{2.88}$$

where $j \geq 1$. We obtain that equation (2.88) is satisfied only $j = 1$, and it is clear that equation (2.88) has no solutions with $k = j \geq 2$. Therefore, the equation (2.75) has a solution only if $i = j = k = 1$.

– If $k = j+1$, we obtain that

$$5j_j^2 + 2j_{j+1}^2 - 10j_j j_{j+1} + 3 = 0, \tag{2.89}$$

where $j \geq 1$. We find that equation (2.89) is not satisfied for all $j \geq 1$. We can explain this as follows. If $j = 1$, we obtain from the left hand side of equation (2.89) that $5(1)^2 + 2(5)^2 - 10(1)(5) + 3 = 8$. For $j = 2$, we obtain that $5(5)^2 + 2(7)^2 - 10(5)(7) + 3 = -124$. In the case of $j = 3$, we get that $5(7)^2 + (17)^2 - 10(7)(17) + 3 = -349$. Now, we consider equation (2.89) for all $j \geq 4$. From equation

(1.7) we have that $j_{j+1} = j_j + 2j_{j-1}$ in which we substitute in the left hand side of equation (2.89) to get that

$$\begin{aligned} 5j_j^2 + 2j_{j+1}^2 - 10j_jj_{j+1} + 3 &= 5j_j^2 + 2(j_j + 2j_{j-1})^2 - 10j_j(j_j + 2j_{j-1}) + 3 \\ &= 5j_j^2 + 2j_j + 8j_jj_{j-1} + 8j_{j-1}^2 - 10j_j^2 - 20j_{j-1}j_j + 3 \\ &= 8j_{j-1}^2 - 3j_j^2 - 12j_jj_{j-1} + 3 \\ &< 8j_j^2 - 3j_j^2 - 12j_jj_{j-1} + 3 \quad \text{as} \quad j_{j-1} < j_j \quad \forall j \geq 4 \\ &= -3j_j^2 - 4j_{j-2}^2 + 3 < 0 \quad \text{for all} \quad j \geq 4, \end{aligned}$$

which leads to the no satisfaction of equation (2.89). So, the equation (2.75) has no solution if $k = j + 1$ and $j \geq 1$.

– If $k = j + 2$, this is leads to

$$5j_j^2 + 2j_{j+2}^2 - 10j_jj_{j+2} + 3 = 0, \tag{2.90}$$

where $j \geq 1$. We note that the equation (2.90) is satisfied only when $j = 3$. Otherwise, we find that it is not satisfied. This can be explained as follows. Assume that $j = 1$, we get that $5(1)^2+2(7)^2-10(1)(7)+3 = 36$. In the case of $j = 2$, we obtain that $5(5)^2+2(17)^2-10(5)(17)+3 = -144$. If $j = 3$, we have that $5(7)^2 + 2(31)^2 - 10(7)(31) + 3 = 0$. Therefore, the equation (2.75) has a solution that is given by $(i, j, k) = (1, 3, 5)$. After that, we get the left-hand side of equation (2.90) is always negative for all $j \geq 4$ which can be shown as follows. from equation (1.7), we obtain that $j_{j+2} = 3j_j + 2j_{j-1}$. Substituting $j_{j+2} = 3j_j + 2j_{j-1}$ in equation (2.90) implies that

$$\begin{aligned} 5j_j^2 + 2j_{j+2}^2 - 10j_jj_{j+2} + 3 &= 5j_j^2 + 2(3j_j + 2j_{j-1})^2 - 10j_j(3j_j + 2j_{j-1}) + 3 \\ &= 5j_j^2 + 18j_j + 24j_jj_{j-1} + 8j_{j-1}^2 - 30j_j^2 - 20j_{j-1}j_j + 3 \\ &= 8j_{j-1}^2 - 17j_j^2 + 4j_{j-1}j_j + 3 \\ &< 8j_{j-1}^2 - 17j_jj_{j-1} + 4j_jj_{j-1} + 3 \quad \text{as} \quad -j_{j-1} > -j_j \quad \forall j \geq 4 \\ &= 8j_{j-1}^2 - 13j_jj_{j-1} + 3 \\ &< 8j_{j-1}^2 - 13j_{j-1}^2 + 3 \quad \text{as} \quad -j_{j-1} > -j_j \\ &= -5j_{j-1}^2 + 3 < 0 \quad \text{for all} \quad j \geq 4. \end{aligned}$$

So, the equation (2.75) has no solutions with $k = j + 2$ and $j \geq 4$. In the same way, we obtain that the equation (2.75) does not have a solution when $i = 3$.

Finally, we investigate the solutions of equation (2.76). By argument (iv), we obtain

$$15j_i^2 + 10y^2 + 6z^2 - 6j_iyz - 1 = 0,$$

is solvable only with $i \in \{1, 3\}$. By argument (v) and in a similar way done in the above cases, we obtain that the equation (2.76) does not have any solution when $i \in \{1, 3\}$.

Finally, we find that the only equations, a mong (2.71)-(2.76) that have solutions, are listed in the below table.

Eq.No.	(i, j, k)	(j_i, j_j, j_k)
(2.71)	$(1, 1, 1)$	$(1, 1, 1)$
(2.75)	$(1, 1, 1)$	$(1, 1, 1)$
(2.76)	$(1, 3, 7)$	$(1, 7, 31)$

By permuting the components of the solutions (j_i, j_j, j_k) with which it satisfies equation (1.11), we get the solutions of equation (1.11) are as follows:

$$(j_i, j_j, j_k) \in \{(1, 1, 1), (31, 1, 7)\}$$

Note that the solutions of equations (1.12)-(1.14) are studied similarly. Therefore, in the remaining cases, we only give a summary for the details of components.

☐ `case 4` . If $(A, B, C, D) = (2, 1, 2, 2)$. By permuting the coefficients of equation (1.12), we obtain the different equations as follows:

$$2j_i^2 + j_j^2 + 2j_k^2 = 2j_ij_jj_k + 1, \tag{2.91}$$
$$j_i^2 + 2j_j^2 + 2j_k^2 = 2j_ij_jj_k + 1, \tag{2.92}$$
$$2j_i^2 + 2j_j^2 + j_k^2 = 2j_ij_jj_k + 1, \tag{2.93}$$

where $1 \leq i \leq j \leq k$. From (iii), we have that $1 \leq i \leq 6$ and $j \leq k \leq j + 11$ for equations (2.91)-(2.93).

Now, we study the solutions to equations (2.91) and (2.93). Using the SageMath program, as mentioned in argument (iv), we notice that no values for i exist to satisfy the equations. Therefore, the equations (2.91) and (2.93) have no solutions in the Jacobsthal–Lucas numbers.

We study the solutions of equation (2.92). In the same way, we obtain from argument (iv) that $i \in \{1, 2\}$, After substituting the values of i into the equation (2.92) and analyzing it as in the previous cases, we notice that the equation does not have any solution.

Therefore, equations (2.91), (2.92), and (2.93) do not have any solution in Jacobsthal– Lucas numbers.

☐ `case 5` . If $(A, B, C, D) = (5, 1, 5, 5)$. Using permutations of the coefficients of the equation (1.13) we get the equations:

$$5j_i^2 + j_j^2 + 5j_k^2 = 5j_ij_jj_k + 1, \tag{2.94}$$
$$j_i^2 + 5j_j^2 + 5j_k^2 = 5j_ij_jj_k + 1, \tag{2.95}$$
$$5j_i^2 + 5j_j^2 + j_k^2 = 5j_ij_jj_k + 1, \tag{2.96}$$

where $1 \leq i \leq j \leq k$. From argument (iii), we get that $i \leq 6$, and $1 \leq k \leq j + 11$ for equations (2.94)-(2.96). Now, we study the solutions of equations (2.94)-(2.96). Using (iv), we obtain that $i \in \{1, 3\}$ for the equations (2.94) and (2.95) and $i = 1$ for the equation (2.96). By argument (v), we get the equations (2.94)-(2.96) do not have any solutions. So, equation (1.13) does not have any solution in the Jacobsthal–Lucas numbers.

☐ `case 6` . If $(A, B, C, D) = (3, 1, 6, 6)$. In the same way as in the previous cases, from equation (1.14) we get that

$$3j_i^2 + j_j^2 + 6j_k^2 = 6j_ij_jj_k + 1, \tag{2.97}$$
$$j_i^2 + 3j_j^2 + 6j_k^2 = 6j_ij_jj_k + 1, \tag{2.98}$$
$$j_i^2 + 6j_j^2 + 3j_k^2 = 6j_ij_jj_k + 1, \tag{2.99}$$
$$6j_i^2 + j_j^2 + 3j_k^2 = 6j_ij_jj_k + 1, \tag{2.100}$$
$$6j_i^2 + 3j_j^2 + j_k^2 = 6j_ij_jj_k + 1, \tag{2.101}$$
$$3j_i^2 + 6j_j^2 + j_k^2 = 6j_ij_jj_k + 1, \tag{2.102}$$

where $1 \leq i \leq j \leq k$. From argument (iii), we get that $i \leq 6$, and $1 \leq k \leq j + 11$ for the equations (2.97)-(2.102).

Firstly, we study the solutions of equations (2.97)-(2.97) in the same approach used with the studied cases. Using the SageMath program, as mentioned in (iv) we obtain that $i \in \{1, 3, 4\}$ for equations (2.97) and (2.102), $i = 1$ for equations (2.98) and (2.99), and $i \in \{1, 2\}$ for equations (2.100) and (2.101). Also, by argument (v) we find that the equations (2.97)-(2.102) do not have any solutions. Hence, the equation (1.14) does not have any solution in the Jacobsthal–Lucas numbers. Therefore, Theorem 2.1 is completely proved.

3. Conclusion

The Markoff type equation, Jin-Schmidt equation, has infinitely many integer solutions, and these solutions are deeply connected to the lower part of the approximated spectrum for quaternions and to the approximated constants for complex numbers on the circles $\{z \in \mathbb{C} \mid |z| = \frac{1}{\sqrt{2}}\}$. Also, the Markoff equation is connected to Diophantine approximation of irrational numbers, more precisely there is one to one correspondence between the solutions of such equations and the approximation of certain irrational numbers. The paper's findings indicate that it has a finite number of solutions within the sequence of Jacobsthal–Lucas numbers. This outcome will provide a deep insight into the study of Diophantine approximations, as the solutions to the Markoff equation and its generalizations are linked to Diophantine approximations. □

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Conflict of interest

The authors declare that there is no conflict of interest.

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