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An Efficient Numerical Approach for Fractional Heat Equations with Nonlocal Memory Terms

Sami Baroudi*, Ali El Mfadel, Abderrazak Kassidi and M'hamed Elomari

ABSTRACT: This paper presents a numerical method for solving partial integro-differential equations with weakly singular kernels, using a tempered φ-Caputo fractional derivative of order $\alpha \in (0,1)$. We apply a second-order time discretization and use a tempered fractional integral operator along with piecewise linear interpolation to handle the singularity in the kernel. The stability of the method is analyzed using Von Neumann stability analysis. Finally, numerical examples are provided to demonstrate the effectiveness of the approach.

Key Words: Tempered $\varphi-$fractional integral, tempered $\varphi-$Caputo fractional derivative, integro-differential equations, finite difference scheme.

Contents

1. Introduction

The prevalence of integro-differential equations has surged in applied mathematics, reflecting increased efforts to model real-world phenomena [6,11,25]. This trend is particularly evident in the heightened attention devoted to partial integro-differential equations within the realm of applied mathematics. Fractional calculus techniques have been increasingly employed over the past few decades to model processes across diverse fields such as computer science, physics, neuroscience, biology, medicine, and engineering [5,23,34,36]. The inherent challenges in addressing these problems, both numerically and analytically, stem from various factors, including the multitude of variables, nonlinearity, nonlocal events, and multidimensionality. Analytically solving partial integro-differential equations, especially those involving fractional orders, proves challenging. Consequently, numerical approximation techniques are employed to obtain solutions, as demonstrated in [28,31,32].

The equation (1.1) finds application in modeling physical phenomena related to linear viscoelastic mechanics [6,9,30], heat flow in materials with memory [17,27], and other related phenomena. The integral term in (1.1) represents the viscoelasticity component of the equation. Over the last thirty years, fractional differential equations have gained significance and popularity, driven by their demonstrated utility across seemingly disparate domains of science and engineering. Recent research underscores that differential equations of fractional order serve as valuable tools for simulating various physical phenomena [20,24,44].

Presently, there exists a multitude of research studies focusing on integro-differential equations. In [8], C. Chen et al. utilize the Galerkin finite element method to approximate errors in the numerical solution of a parabolic integro-differential equation with a memory term featuring a weakly singular kernel. Tang,

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in [35], introduces and establishes the stability and convergence of the product trapezoidal approach for solving partial integro-differential equations containing a weakly singular kernel. In [9], M. Dehghan et al. employ finite difference schemes and the product trapezoidal numerical integration rule to develop various computational approaches for solving such equations. Addressing a fourth-order partial integro-differential equation with a weakly singular kernel, Yang et al. in [38] employ the Crank-Nicolson scheme and a quasi-wavelets numerical approach. Efficient numerical methods for solving one and two-dimensional multi-term time-fractional diffusion wave equations are proposed by J. Ren et al. in [39], using the compact difference technique for spatial discretization. Investigating the finite element method, Bu in [7] tackles the solution of multi-term fractional advection diffusion equations. In [37], Wang et al. develop a numerical approach based on the operational matrix of fractional order integration with second Chebyshev wavelets to solve fractional integro-differential equations. Alquran, in [4], presents an analytical series solution for a set of Caputo fractional integro-differential equations. Qiao et al. in [29] introduce a novel numerical method combining high-order orthogonal spline collocation in space and the alternate direction implicit method in time to solve a two-dimensional multi-term time fractional integro-differential equation.

Exploring chaos in a fractional predator-prey-pathogen model using the Atangana-Baleanu fractional operator, Ghanbari et al. in [16] employ three different schemes: the linear scheme, the quadratic scheme, and the quadratic-linear scheme.

Several authors have studied various forms of fractional derivatives, including Riemann-Liouville [19], Caputo [1], Hilfer [18], Erdélyi-Kober [22], and Hadamard [2], to investigate the existence of solutions to fractional differential equations. Almeida [3] introduces the tempered φ-Caputo derivative, offering a more general approach to studying fractional differential equations. Different choices of the function φ lead to well-known fractional derivatives, such as Caputo, Caputo-Hadamard, or Caputo-Erdelyi-Kober, which are kernel-dependent. This approach proves suitable for diverse applications, allowing some control over the modeling of the phenomenon of interest. Further information on the existence and uniqueness of solutions to fractional differential equations using φ-Caputo type fractional derivatives and fixed point theorems is available in the articles [3,14,13,15,12,46,43,42,45] and their references.

The numerical solution to a time-fractional diffusion heat problem with weakly singular integro-partial characteristics is obtained through the application of the finite difference method outlined in the following,

$$ {}^{C}\mathcal{D}_{0^+}^{\alpha,\lambda;\varphi} x(z,t) = \int_0^t e^{-\lambda(\varphi(t)-\varphi(s))}(\varphi(t)-\varphi(s))^{-1/2}\varphi'(s)\frac{\partial^2 x}{\partial z^2}(z,s)ds, \tag{1.1} $$

here, with α belonging to the interval (0,1), ${}^{C}\mathcal{D}_{0^+}^{\alpha,\lambda;\varphi}$ represents the tempered φ-Caputo fractional derivative of order α concerning the variable t. The initial condition complements Equation (1.1),

$$ x(z,0) = f(z)\ ,\ z \in [a,b] \tag{1.2} $$

and the boundary conditions:

$$ x(a,t) = f_1(t)\ ,\ x(b,t) = f_2(t)\ ,\ t \in [0,T], \tag{1.3} $$

The functions $f(z)$, $f_1(t)$, and $f_2(t)$ are explicitly defined, with T representing the final time.

The remaining sections of this paper are organized as follows. In the second section, we present fundamental tools related to $\varphi-$fractional integrals and tempered $\varphi-$Caputo fractional derivatives, which will be utilized in subsequent discussions. Section 3 outlines the finite-difference method employed to approach the solution for the system (1.1), incorporating piecewise linear interpolation. The stability analysis of the constructed discrete scheme is the central focus of Section 4. In Section 5, practical examples are provided as applications, and the paper concludes with a summary in Section 6..

2. Preliminary

This section introduces novel concepts related to tempered fractional integrals and derivatives of a function f with respect to another function φ. For a more in-depth exploration, readers are referred to [3,21]. Essential findings from this study will be highlighted in the subsequent discussion.

Definition 2.1 *[21] Suppose that the real function x is piecewise continuous on $(0,T)$ and $x \in \mathrm{L}([0,T])$. Let $\alpha \in (0,\infty)$ and $\lambda \in [0,\infty)$. Then, the tempered Riemann-Liouville fractional integral of order α is defined as*

$$\mathcal{I}_{0^+}^{\alpha,\lambda}x(t) = e^{-\lambda t}\mathcal{I}_{0^+}^{\alpha}\left(e^{\lambda t}x(t)\right) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-\lambda(t-s)}x(s)ds,$$

where $\mathcal{I}_{0^+}^{\alpha}$ is the Riemann-Liouville fractional integral of order $\alpha \in (0,1)$.

Definition 2.2 *[21] Let $n-1 < \alpha \leq n \in \mathbb{N}$ and $\lambda \in [0,\infty)$. Then, the tempered Caputo fractional derivative of order α is defined as*

$${}^{C}\mathcal{D}_{0^+}^{\alpha,\lambda}x(t) = e^{-\lambda t}\,{}^{C}\mathcal{D}_{0^+}^{\alpha}\left(e^{\lambda t}x(t)\right) = \frac{e^{-\lambda t}}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\frac{d^n}{ds^n}\left(e^{\lambda s}x(s)\right)ds,$$

where ${}^{C}\mathcal{D}_{0^+}^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$.

Definition 2.3 *[21,26] Let $\alpha,\ \lambda \in [0,\infty)$ and φ be a positive, monotone and increasing function on $[0,T]$ such that $\varphi' \neq 0$ on $[0,T]$. Then, the tempered φ- Riemann-Liouville fractional integral of $x \in C[0,T]$ is defined by*

$$\begin{aligned}\mathcal{I}_{0^+}^{\alpha,\lambda;\varphi}x(t) &= e^{-\lambda(\varphi(t)-\varphi(0))}\mathcal{I}_{0^+}^{\alpha;\varphi}\left(e^{\lambda(\varphi(t)-\varphi(0))}x(t)\right)\\ &= \frac{1}{\Gamma(\alpha)}\int_0^t \varphi'(s)\Big(\varphi(t)-\varphi(s)\Big)^{\alpha-1}e^{-\lambda(\varphi(t)-\varphi(s))}x(s)ds,\end{aligned} \tag{2.1}$$

Definition 2.4 *[21,26] Let $\lambda \in [0,\infty)$ and $\varphi \in C^n[0,T]$ be a non-decreasing function such that $\varphi' \neq 0$ on $[0,T]$. Then, the tempered φ-Caputo fractional derivative of a function $u \in C^n[0,T]$ of order $\alpha \in (n-1,\ n), n \in \mathbb{N}$ is defined by*

$$\begin{aligned}{}^{C}\mathcal{D}_{0^+}^{\alpha,\lambda;\varphi}x(t) &= e^{-\lambda(\varphi(t)-\varphi(0))}\,{}^{C}\mathcal{D}_{0^+}^{\alpha;\varphi}\left(e^{\lambda(\varphi(t)-\varphi(0))}x(t)\right)\\ &= e^{-\lambda(\varphi(t)-\varphi(0))}\mathcal{I}_{0^+}^{n-\alpha;\varphi}\left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n\Big(e^{\lambda(\varphi(t)-\varphi(0))}x(t)\Big)\end{aligned} \tag{2.2}$$

Theorem 2.5 *[21,26] Let $\alpha,\beta \in [0,\infty)$ such that $m-1 < \beta < m \leq n-1 < \alpha \leq n, m, n \in \mathbb{N}$. Then,*

$${}^{C}\mathcal{D}_{0^+}^{\beta,\lambda;\varphi}\mathcal{I}_{0^+}^{\alpha,\lambda;\varphi}x(t) = \mathcal{I}_{0^+}^{\alpha-\beta,\lambda;\varphi}x(t),\ \ t \in [0,T].$$

Definition 2.6 *[21,26] Let $n-1 < \alpha \leq n \in \mathbb{N}^*, \lambda \in [0,\infty)$ and $\varphi \in C^n[0,T]$ be a non-decreasing function such that $\varphi' \neq 0$ on $[0,T]$. Then, for $x \in C^n[0,T]$ we have,*

1. ${}^{C}\mathcal{D}_{0^+}^{\alpha,\lambda;\varphi}\mathrm{I}_{0^+}^{\alpha,\lambda;\varphi}x(t) = x(t)$,

2. $\mathcal{I}_{0^+}^{\alpha,\lambda;\varphi}\,{}^{C}\mathcal{D}_{0^+}^{\alpha,\lambda;\varphi}x(t) = x(t) - e^{-\lambda(\varphi(t)-\varphi(0))}\sum_{k=0}^{n-1}\frac{\big(\varphi(t)-\varphi(0)\big)^k}{\Gamma(k+1)}\Big[\mathcal{D}_{\varphi}^{[k],\lambda}x(t)\Big]_{t=0}$,

   where $\mathcal{D}_{\varphi}^{[k],\lambda}x(t) = \left(\frac{1}{\varphi(t)}\frac{d}{dt}\right)^k\Big(e^{\lambda(\varphi(t)-\varphi(0))}x(t)\Big)$,

3. *Let $\alpha_1, \alpha_2 \in (0,\infty)$ and $\lambda \in [0,\infty)$. Then, for $x \in C[0,T]$,*

$$\mathcal{I}_{0^+}^{\alpha_1,\lambda;\varphi}\mathcal{I}_{0^+}^{\alpha_2,\lambda;\varphi}x(t) = \mathcal{I}_{0^+}^{\alpha_1+\alpha_2,\lambda;\varphi}x(t),\ \ t \in [0,T].$$

Definition 2.7 *[21,26] Let $0 < \alpha < 1, \lambda \in [0,\infty),\ u \in C^1[0,T], \varphi \in C^1[0,T]$ be a non-decreasing function such that $\varphi' \neq 0$ on $[0,T]$ and $F \in C([0,T]\times\mathbb{R},\mathbb{R})$. Then, the folowing problem,*

$${}^{C}\mathcal{D}_{0^+}^{\alpha,\lambda;\varphi}x(t) = F\big(t,x(t)\big),\ \ t \in [0,T],\ \ x(0) = x_0 \in \mathbb{R}$$

is equivalent to the fractional integral equation

$$x(t) = e^{-\lambda(\varphi(t)-\varphi(0))}x_0 + \mathcal{I}_{0^+}^{\alpha,\lambda;\varphi}F\big(t,x(t)\big),\ \ t \in [0,T].$$

3. The finite-difference method

Consider a regular grid $\{z_i = ih\}_{i=0}^{M}$ and $\{t_j = jk\}_{j=0}^{L}$, where $h = \frac{b-a}{M}$ and $k = \frac{T}{L}$. Let $x_{i,j} \simeq x(z_i,\ t_j)$ represent an approximate solution for a given equation over $i = 0, \cdots, M$ and $j = 0, \cdots, L$. By substituting $z = z_i$ into equation (1.1) and subsequently applying the central finite difference formula to obtain the second-order derivative with respect to the variable t

$$\frac{\partial^2 x}{\partial z^2}(z_i, t) \simeq \frac{x(z_{i+1}, t) - 2x(z_i, t) + x(z_{i-1}, t)}{h^2}. \tag{3.1}$$

After substituting (3.1) into equation (1.1) and applying the operator $\mathcal{I}_{0^+}^{\alpha,\lambda;\varphi}$, we obtain

$${}^{C}\mathcal{D}_{0^+}^{\alpha,\lambda;\varphi} x(z_i, t) \simeq \frac{\Gamma(1/2)}{h^2}\Big[\mathcal{I}_t^{\frac{1}{2},\lambda;\varphi} x(z_{i+1},\ t) - 2\mathcal{I}_t^{\frac{1}{2},\lambda;\varphi} x(z_i,\ t) + \mathcal{I}_t^{\frac{1}{2},\lambda;\varphi} x(z_{i-1},\ t)\Big]. \tag{3.2}$$

Now, consider both sides of Eq. (3.2) and employ the operator $\mathcal{I}_{0^+}^{\alpha,\lambda;\varphi}$. Utilizing properties (b) and (c) outlined in Theorem 2.6, we derive

$$x(z_i,\ t) \simeq e^{-\lambda(\varphi(t)-\varphi(0))} f(z_i) + \frac{\Gamma(1/2)}{h^2}\Big[\mathcal{I}_{0^+}^{\alpha+\frac{1}{2},\lambda;\varphi} x(z_{i+1}, t) - 2\mathcal{I}_{0^+}^{\alpha+\frac{1}{2},\lambda;\varphi} x(z_i, t) + \mathcal{I}_{0^+}^{\alpha+\frac{1}{2},\lambda;\varphi} x(z_{i-1}, t)\Big]. \tag{3.3}$$

By employing the substitution $t = t_{n+1}$, where $n = 0, 1, \cdots, (L-1)$, the integrals of the fractional order can be computed as follows. To achieve this, we use the integration rule [10], taking into consideration

$$\int_0^t \Big(\varphi(t) - \varphi(s)\Big)^{\alpha-1} \psi(s)\varphi'(s)ds \simeq \int_0^{t_{n+1}} \varphi'(s)\Big(\varphi(t_{n+1}) - \varphi(s)\Big)^{\alpha-1} \tilde{\psi}(s)ds, \tag{3.4}$$

where $\tilde{\psi}(s)$ denotes the linear interpolation of $\psi(s)$,

$$\tilde{\psi}(s) = \frac{t_{j+1} - s}{k}\psi(t_j) + \frac{s - t_j}{k}\psi(t_{j+1}), \tag{3.5}$$

where $t_j \le s \le t_{j+1}$ for $j = 0, 1, \dots, (L-1)$.
The integral to the right of equation (3.4) is defined as $\mathrm{I}(\tilde{\psi},\ t_{n+1})$, such that

$$\mathrm{I}(\tilde{\psi},\ t_{n+1}) := \int_0^{t_{n+1}} \varphi'(s)\Big(\varphi(t_{n+1}) - \varphi(s)\Big)^{\alpha-1} \tilde{\psi}(s)ds, \tag{3.6}$$

by using a linear interpolation (3.5) in (3.6) and some calculus, we get

$$\begin{aligned}
\mathrm{I}(\tilde{\psi},\ t_{n+1}) &= \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \varphi'(s)\Big(\varphi(t_{n+1}) - \varphi(s)\Big)^{\alpha-1} \tilde{\psi}(s)ds \\
&= \sum_{j=0}^{n} \psi(t_j) \int_{t_j}^{t_{j+1}} \varphi'(s)\Big(\varphi(t_{n+1}) - \varphi(s)\Big)^{\alpha-1} \frac{t_{j+1} - s}{k} ds \\
&\quad + \sum_{j=0}^{n} \psi(t_{j+1}) \int_{t_j}^{t_{j+1}} \varphi'(s)\Big(\varphi(t_{n+1}) - \varphi(s)\Big)^{\alpha-1} \frac{s - t_j}{k} ds \\
&= \frac{k^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n+1} e_j \tilde{\psi}(t_j).
\end{aligned}$$

Where

$$e_j = \begin{cases} n^{\alpha+1} - (n-\alpha)(n\ +1)^\alpha & \textit{if } j = 0 \\ (n-j)^{\alpha+1} + (n\ +2-j)^{\alpha+1} - 2(n\ +1-j)^{\alpha+1} & \textit{if } j \in [1, n] \\ 1 & \textit{if } j = n+1. \end{cases} \tag{3.7}$$

Therefore, we can provide an estimate of the fractional order integral using

$$\mathcal{I}_{0^+}^{\alpha,\lambda;\varphi} x(z_{i+1},\ t_{n+1}) \simeq \frac{k^{\alpha} e^{-\lambda\varphi(t_{n+1})}}{\Gamma(\alpha+2)} \sum_{j=0}^{n+1} \tilde{e}_j x(z_{i+1}, \tilde{t}_j), \tag{3.8}$$

where $\tilde{e}_j = e^{\lambda\varphi(t_j)} e_j$.

In a manner akin to this, one can provide approximations for the integrals $\mathcal{I}_{0^+}^{\alpha\frac{1}{2},\lambda;\varphi} x(z_i,\ t_{n+1})$,

$$\mathcal{I}_{0^+}^{\alpha+\frac{1}{2},\lambda;\varphi} x(z_s,\ t_{n+1}) \simeq \frac{k^{\alpha+\frac{1}{2}} e^{-\lambda\varphi(t_{n+1})}}{\Gamma(\alpha+\frac{5}{2})} \sum_{j=0}^{n+1} \tilde{c}_j x(z_s, \tilde{t}_j),\ s = i-1, i, i+1, \tag{3.9}$$

in which the coefficients $\tilde{c}_j$ are defined by,

$$\tilde{c}_j = e^{\lambda\varphi(t_j)} \begin{cases} n^{\alpha+\frac{3}{2}} - \ (n-\frac{1}{2}-\alpha)(n+1)^{\alpha+\frac{1}{2}}, & \text{if } j=0, \\ (n-j)^{\frac{3}{2}+\alpha} + (n+2-j)^{\alpha+\frac{3}{2}} - 2(n+1-j)^{\alpha+\frac{3}{2}}\ , & \text{if } j \in [1,n] \\ 1, & \text{if } j = n+1. \end{cases}$$

Inserting (3.8) – (3.9) into equation (3.3) for $i = 1, 2, \cdots, M-1$ and $n = 0, 1, \cdots, N-1$ we obtain

$$\begin{aligned} x_{i,n+1} \simeq \mu_i + \chi \sum_{j=0}^{n} \tilde{c}_j \tilde{x}_{i+1,j} + \chi e^{\lambda\varphi(t_{n+1})} \tilde{x}_{i+1,n+1}, \\ - 2\chi \sum_{j=0}^{n} \tilde{c}_{j,n+1} \tilde{x}_{i,j} - 2\chi e^{\lambda\varphi(t_{n+1})} \tilde{x}_{i,n+1} \\ + \chi \sum_{j=0}^{n} \tilde{c}_j \tilde{x}_{i-1,j} + \chi e^{\lambda\varphi(t_{n+1})} \tilde{x}_{i-1,n+1}, \end{aligned} \tag{3.10}$$

where $\chi := \dfrac{\Gamma(\frac{1}{2})}{h^2} \dfrac{k^{\alpha+\frac{1}{2}} e^{-\gamma\varphi(t_{n+1})}}{\Gamma(\alpha+\frac{5}{2})}, \quad \tilde{x}_{i,j} = x(z_i, \tilde{t}_j)$

and $\mu_i = e^{-\lambda\left(\varphi(t_{n+1})-\varphi(0)\right)} f_i$

by introducing

$$\tau := e^{\lambda\varphi(t_{n+1})}\chi, \quad \zeta_j := \chi\tilde{c}_j, \quad j = 0, 1, \cdots, n. \tag{3.11}$$

we can compactly write the Eq. (3.10) as,

$$-\tau x_{i-1,n+1} + (1+2\tau)x_{i,n+1} - \tau x_{i+1,n+1} \simeq \mu_i + \sum_{j=0}^{n} \zeta_j(\tilde{x}_{i-1,j} - 2\tilde{x}_{i,j} + \tilde{x}_{i+1,j}), \tag{3.12}$$

where $i \in [1, M-1] and\ n \in [0, L-1]$.

For every $n \in [0, N-1]$, equation (3.12) is a system of $M-1$ equations and unknowns $x_{i,n+1}$, which takes the form of the following matrix

$$\mathcal{A}X_{n+1} + E_{n+1} \simeq P + \sum_{j=0}^{n} (R_j \tilde{X}_j + Y_j), \quad n \in [0, N-1], \tag{3.13}$$

where

$$\mathcal{A} = \begin{pmatrix} 1+2\tau & -\tau & & & 0 \\ -\tau & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\tau \\ 0 & & & -\tau & 1+2\tau \end{pmatrix}, \qquad R_j = \zeta_j \begin{pmatrix} -2 & 1 & & & 0 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & -2 \end{pmatrix},$$

$$Y_j = \zeta_j \begin{pmatrix} x_{0,j} \\ 0 \\ \vdots \\ 0 \\ x_{M,j} \end{pmatrix}, \quad P = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \vdots \\ \mu_{M-1} \end{pmatrix}, \quad \tilde{X}_j = \begin{pmatrix} \tilde{x}_{1,j} \\ \tilde{x}_{2,j} \\ \vdots \\ \vdots \\ \tilde{x}_{M-1,j} \end{pmatrix} \quad \textit{and} \quad E_{n+1} = -\tau \begin{pmatrix} x_{0,n+1} \\ 0 \\ \vdots \\ 0 \\ x_{M,n+1} \end{pmatrix}.$$

The matrix $\mathcal{A}$ possesses tridiagonal, symmetric, positive, and strictly diagonally dominant properties. Given the invertibility of matrix $\mathcal{A}$, the system (3.13) can be explicitly represented in terms of X_{n+1} for $n \in [0, N-1]$.

$$X_{n+1} \simeq \mathcal{A}^{-1}\big(P + \sum_{j=0}^{n}(R_j\tilde{X}_j + Y_j) - E_{n+1}\big). \tag{3.14}$$

4. Stability result

In this section, we elucidate the application of a method, akin to the renowned Von Neumann (or Fourier) approach, for the rapid and effective assessment of the stability of fractional numerical schemes, specifically applied to the tempered φ-Caputo fractional derivative scheme (3.12). Throughout this section, we make the assumption that $\mu_i = 0$.

Theorem 4.1 *If the following condition* $\sum_{j=0}^{n} \zeta_j(-1)^{j-n} < \tau + \frac{1}{4}$ *holds, then the fractional numerical system* (3.12) *is stable, whene* ζ_j *and* τ *defined in* (3.11).

Proof 4.2 *Let's assume that the solution is expressed as* $x_{m,n} = e^{i\rho m h}\vartheta_n$*, where* ρ *represents a real spatial wave number. When adding this expression to* (3.12)*, we obtain*

$$\vartheta_{n+1}(-\tau e^{-i\rho h} + (1+2\tau) - \tau e^{i\rho h}) = \sum_{j=0}^{n} \zeta_j \vartheta_j (e^{-i\rho h} - 2 + e^{i\rho h}),$$

After simplifying and applying the familiar Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$*, we obtain*

$$\vartheta_{n+1}\left(1 + 4\tau \sin^2\left(\frac{\rho h}{2}\right)\right) = -4\sin^2\left(\frac{\rho h}{2}\right)\sum_{j=0}^{n}\zeta_j\vartheta_j. \tag{4.1}$$

In the von Neumann method, the stability analysis is conducted through the utilization of the amplification factor ν*, defined as*

$$\vartheta_{n+1} = \nu\vartheta_n, \tag{4.2}$$

For the time being, let's assume that $\nu \equiv \nu(\rho)$ *is not subject to changes over time. As a result,* (4.1) *presents a self-contained equation for the amplification factor of the subdiffusion mode*

$$\nu\left(1 + 4\tau\sin^2\left(\frac{\rho h}{2}\right)\right) = -4\sin^2\left(\frac{\rho h}{2}\right)\sum_{j=0}^{n}\zeta_j\nu^{j-n}, \tag{4.3}$$

If $|\nu| > 1$ *for a certain* ρ*, as per* (4.2)*, the temporal factor of the solution diverges to infinity, rendering the mode unstable. Conversely, when* $|\nu| \le 1$*, the mode remains stable, i.e.*

$$-1 \le \frac{-4\sin^2(\frac{\rho h}{2})}{1 + 4\tau\sin^2(\frac{\rho h}{2})}\sum_{j=0}^{n}\zeta_j\nu^{j-n} \le 1.$$

Given the time-independent limit $\nu = -1$ *and* $1 + 4\tau\sin^2\left(\frac{\rho h}{2}\right) > 0$*, we get*

$$-1 - 4\tau\sin^2\left(\frac{\rho h}{2}\right) + 4\sin^2\left(\frac{\rho h}{2}\right)\sum_{j=0}^{n}\zeta_j(-1)^{j-n} \le 0 \tag{4.4}$$

Therefore, the final expression gives

$$\sin^2\left(\frac{\rho h}{2}\right)\sum_{j=0}^{n}\zeta_j(-1)^{j-n}\le\tau\sin^2\left(\frac{\rho h}{2}\right)+\frac{1}{4}\le\tau+\frac{1}{4},$$

Given that $0\le\sin^2\left(\frac{\rho h}{2}\right)\le 1$, *we need to establish a constrained growth rate to ensure and thereby conclude the proof of the Theorem.*

5. Numerical Example

To assess the effectiveness of the proposed method in solving the problem (1.1)-(1.3), we examine a test example and conduct numerical simulations using Matlab.

Example: Let's consider the following problem

$$^{C}\mathcal{D}_{0^+}^{\alpha,\lambda;\varphi}x(z,t)=\Gamma(1/2)\mathcal{I}_{0^+}^{\frac{1}{2},\lambda;\varphi}\frac{\partial^2 x}{\partial z^2}(z,t), \tag{5.1}$$

where $x(z,0)=\sin(\pi z)$ for $0\le z\le 1$, $x(0,t)=x(1,t)=0$ and $0\le t\le T$.
For $g(z,t)=0$, $\alpha=1$, $\lambda=0$ and $\varphi(t)=t$ the exact solution of (5.1) is given by:

$$x(z,t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^n c^{n-k}\pi^{2n+\frac{k}{2}}t^{n+\frac{k}{2}}}{\Gamma\left(\frac{k}{2}+n+1\right)}\sin(\pi z)\ . \tag{5.2}$$

Table 1 present the approximate solution for $(z=0.1,0.2,\dots,0.9)$, $t=T$ and $\alpha=1$ where $M,L=10$.

z	Numerical solution
0.1	0.009955451375736
0.2	0.018936393807118
0.3	0.026063710075020
0.4	0.030639728804273
0.5	0.032216517398567
0.6	0.030639728804278
0.7	0.026063710075016
0.8	0.018936393807117
0.9	0.009955451375739

Table 1: The approximate solution for $z=0.1,0.2,\dots,0.9$, $t=T$ and $\alpha=1$

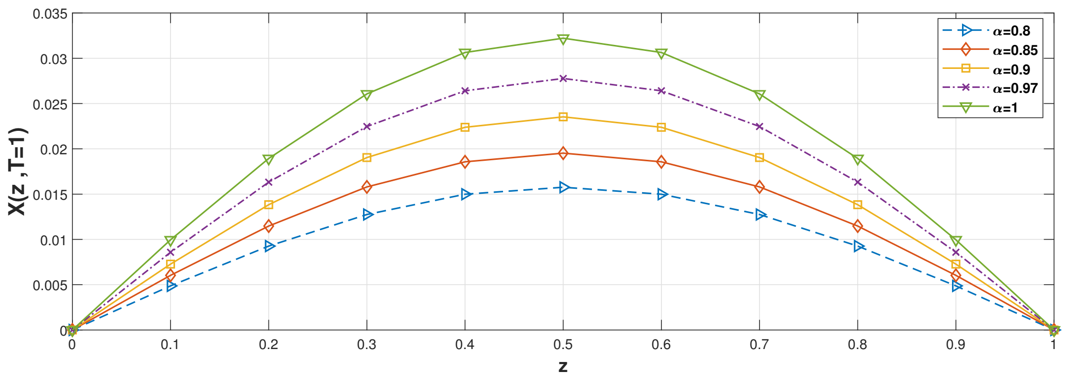

Figure 1: Numerical solutions for $0.8 \leq \alpha < 1$ and exact solution for $\alpha = 1$

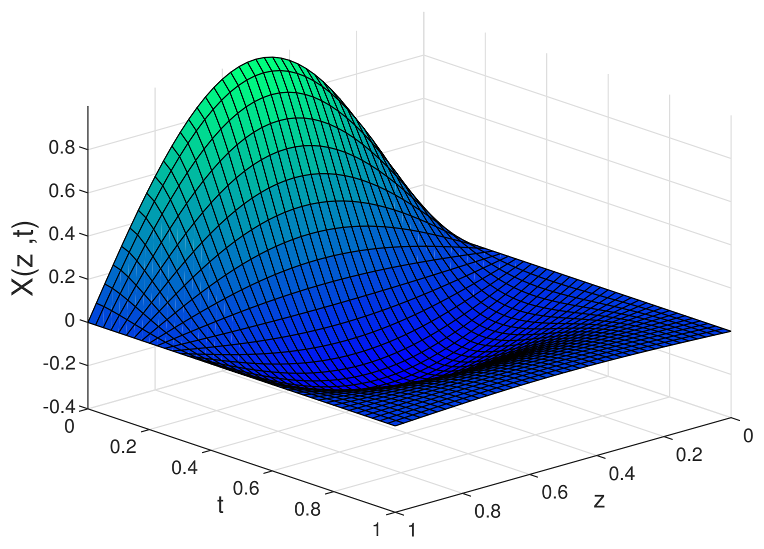

Figure 2: Graph of numerical solution calculated with M=L=40

The computed solutions for different values of α consistently approach the exact solution corresponding to $\alpha = 1$, as depicted in Figure 1. Given the absence of an exact solution for the problem with $0.8 \leq \alpha < 1$, $\lambda = 0$, $\varphi(t) = t$, it can be demonstrated that the approximated solutions remain acceptable in this scenario. The graph of $x(z,t)$ computed with $M = L = 40$ is presented in Figure 2.

6. Conclusion

In this investigation, we present a finite difference method for solving temporal fractional integro-differential equations with weakly singular kernels, wherein fractional derivatives are defined as tempered φ-Caputo fractional derivatives. The study includes error and stability analyses, and numerical example are employed to showcase the convergence of the finite difference scheme. An advantageous feature of the proposed scheme is its accuracy, which is not contingent on the specific fractional value α. Our interest in exploring the integration of other numerical methodologies to address complex problems involving high-dimensional fractional integro-differential equations, irregular regions, and nonlinear equations.

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Sami Baroudi,
Ali El Mfadel,
Abderrazak Kassidi,
M'hamed Elomari
Laboratory of Applied Mathematics and Scientific Computing,

Sultan Moulay Slimane University,
Morocco.
E-mail address: sami.baroudi@usms.ma
E-mail address: a.elmfadel@usms.ma
E-mail address: a.kassidi@usms.ma
E-mail address: m.elomari@usms.ma