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Geonardo numbers

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ABSTRACT: Inspired by the term Gibonacci numbers, which was coined by A. T. Benjamin and J. J. Quinn as shorthand for generalized Fibonacci numbers, Geonardo numbers are considered. More concretely, for each $a \in \mathbb{N}_0$, the study of the sequence of generalized Leonardo numbers associated with a, introduced in an earlier work, is continued and new properties of these numbers are studied: parity; forms of Binet's formula; growth of consecutive Geonardo numbers plus $a \in \mathbb{N}_0$; generating functions – ordinary, exponential, Poisson; identities – sum-binomial, Catalan, Cassini, d'Ocagne, Melham. In addition, some unknown equalities and inequalities related to Leonardo numbers are previously established.

Key Words: Leonardo number, Geonardo number, identity, inequality.

Contents

1. Introduction

Throughout the years, sequences of integers, and also of polynomials as in [13], have attracted a lot of attention. One of the most known is the sequence of Fibonacci numbers, but many more and their generalizations have been studied, such as the sequences of: Leonardo numbers, [5]; Padovan numbers, [14]; Perrin numbers, [14]; telephone numbers, [2]; Pell numbers, [12].

The sequence of Leonardo numbers is entry A001595 of the On-Line Encyclopedia of Integer Sequences, [10], and was studied by Catarino and Borges in [5], Alp and Koçer in [1], Beites and Catarino in [3]. Interestingly enough, it is part of the smoothsort algorithm of the renowned computer scientist Dijkstra, [7].

In [5], Catarino and Borges established Binet's formula, the ordinary generating function and some identities for the sequence of Leonardo numbers, namely involving expressions for sums and products of Leonardo numbers. Later on, they defined and studied diverse properties of the sequence of incomplete Leonardo numbers in [6].

Alp and Koçer continued the research [5] on Leonardo numbers; in [1], using Binet's formula for Leonardo numbers, the cited authors obtained new identities. Beites and Catarino, in [3], defined and studied the Leonardo quaternions sequence; in addition to the results on this sequence, known results on Leonardo numbers were extended.

Moreira, França and Beites defined the sequence of Geonardo numbers associated with $a \in \mathbb{N}_0$ – a sequence of generalized Leonardo numbers – in [8]. In this reference, using Linear Algebra, they established a Binet-type formula for these numbers. Later on, Santos, Costa and Catarino used the term Gersenne sequence, [9] – a new sequence of generalized Mersenne numbers.

The present article has three sections. Section 2 is devoted to brief preliminaries on Leonardo numbers and Geonardo numbers. In section 3, unknown equalities and inequalities related to Leonardo numbers are established. The study of the sequence of Geonardo numbers asociated with $a \in \mathbb{N}_0$ is continued in section 4, and new properties of these numbers are studied.

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2. Preliminaries

Throughout the manuscript, $\{F_n\}_{n=0}^{\infty}$ denotes the well-known sequence of Fibonacci numbers defined by

$$F_0 = 0, F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2}, n \in \mathbb{N}_0 \text{ such that } n \geq 2,$$

[5]. These numbers were generalized by Benjamin and Quinn to Gibonacci numbers, [4]. Inspired by this term, Geonardo numbers were introduced by Moreira, França and Beites in [8].

Let $a \in \mathbb{N}_0$, fixed. We consider the sequence $\{Ge_n\}_{n=0}^{\infty}$ of Geonardo numbers associated with a, defined by

$$Ge_0 = Ge_1 = a \text{ and } Ge_n = Ge_{n-1} + Ge_{n-2} + a, n \in \mathbb{N}_0 \text{ such that } n \geq 2,$$

[8]. The particular case $a = 1$ leads to the sequence $\{Le_n\}_{n=0}^{\infty}$ of Leonardo numbers. These numbers can be extended to negative subscripts, as in [1] and [3], defining

$$Le_{-n} = (-1)^n (Le_{n-2} + 1) - 1, n \in \mathbb{N}.$$

3. Leonardo numbers, revisited

In the present section, equalities and inequalities related to Leonardo numbers are presented.

Theorem 3.1 *Let $\phi = \frac{1+\sqrt{5}}{2}$. For $n \in \mathbb{N}_0$, $2\phi^{n-1} - 1 \leq Le_n < 2^{n+2} - 1$.*

Proof: Notice that $2\phi^{-1} - 1 \leq 1 = Le_0$. We now use strong mathematical induction on $n \in \mathbb{N}$. As $2\phi^0 - 1 = 1 = Le_1$, the base case is valid. Concerning the induction step, fix k and assume that the first inequality holds for $j \leq k$. As $\phi^2 = 1 + \phi$, then we get

$$Le_{k+1} = Le_k + Le_{k-1} + 1 \geq 2\phi^{k-1} + 2\phi^{k-2} - 1 = 2\phi^{k-2}(1 + \phi) - 1 = 2\phi^k - 1.$$

Observe that $Le_0 = 1 < 3 = 2^2 - 1$. Let us proceed, once again, by strong mathematical induction on $n \in \mathbb{N}$. The base case holds since $Le_1 = 1 < 7 = 2^3 - 1$. For the induction step, fix k and assume that the second inequality holds for $j \leq k$. Then we obtain

$$Le_{k+1} = Le_k + Le_{k-1} + 1 < 2^{k+2} + 2^{k+1} - 1 < 2^{k+2} + 2^{k+2} - 1 < 2^{k+3} - 1.$$

□

In the subsequent theorem, the asymptotic form of Binet's formula for Leonardo numbers is deduced.

Theorem 3.2 *Let $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. For $n \in \mathbb{N}$, $Le_n \approx 2\frac{\phi^{n+2}}{\phi - \psi} - 1$.*

Proof: From [4, p. 137], the asymptotic form of Binet's formula for Fibonacci numbers is $F_n \approx \frac{\phi^{n+1}}{\sqrt{5}}$. Hence,

$$2F_{n+1} - 1 \approx 2\frac{\phi^{n+2}}{\sqrt{5}} - 1.$$

By [5, Proposition 2.2], $Le_n = 2F_{n+1} - 1$, and we arrive at the statement. □

A simplification of Binet's formula for Leonardo numbers is presented in the following theorem.

Theorem 3.3 *Let $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. For $n \in \mathbb{N}$, Le_n is the integer closest to $2\frac{\phi^{n+1}}{\phi - \psi} - 1$.*

Proof: We want to prove that $\left| Le_n - (2\frac{\phi^{n+1}}{\phi - \psi} - 1) \right| < \frac{1}{2}$, that is, by Binet's formula $Le_n = 2\frac{\phi^{n+1} - \psi^{n+1}}{\phi - \psi} - 1$ for Leonardo numbers in [5, Proposition 2.4], that $\left| -2\frac{\psi^{n+1}}{\phi - \psi} \right| < \frac{1}{2}$, where $\psi = -\frac{1}{\phi}$. The last inequality is equivalent to

$$(\phi - \psi)\phi^{n+1} > 4.$$

It is straightforward that the inequality is valid when $n = 1$. In addition, assuming that it holds for n, we get $(\phi-\psi)\phi^{n+2} = (\phi-\psi)\phi^{n+1}\phi > 4\phi > 4$. Thus, by induction on n, $(\phi-\psi)\phi^{n+1} > 4$ for all $n \in \mathbb{N}$. $\square$

Next result implies that consecutive Leonardo numbers plus 1 essentially grow by a factor of Φ.

Theorem 3.4 *Let* $\phi = \frac{1+\sqrt{5}}{2}$. *For* $n, m \in \mathbb{N}_0$, $\lim_{n\to\infty} \frac{Le_{n+m}+1}{Le_n+1} = \phi^m$.

Proof: From Binet's formula for Leonardo numbers in [5, Proposition 2.4] and Binet's formula $F_n = \frac{\phi^n-\psi^n}{\phi-\psi}$ for Fibonacci numbers in [5, equality (2)], we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{Le_{n+m}+1}{Le_n+1} &= \lim_{n\to\infty} \frac{2\frac{\phi^{n+m+1}-\psi^{n+m+1}}{\phi-\psi} - 1 + 1}{2\frac{\phi^{n+1}-\psi^{n+1}}{\phi-\psi} - 1 + 1} \\
&= \lim_{n\to\infty} \frac{\frac{\phi^{n+m+1}-\psi^{n+m+1}}{\phi-\psi}}{\frac{\phi^{n+1}-\psi^{n+1}}{\phi-\psi}} \\
&= \lim_{n\to\infty} \frac{F_{n+1+m}}{F_{n+1}},
\end{aligned}$$

which, by [4, Corollary 31], is equal to ϕ^m. $\square$

4. Geonardo numbers, visited

In the present section, the new sequence of Geonardo numbers associated with a is studied.

Theorem 4.1 *For* $n \in \mathbb{N}_0$, $Ge_n = aLe_n$.

Proof: For $n = 0$, $Ge_0 = a = a \times 1 = aLe_0$. We now use strong mathematical induction on $n \in \mathbb{N}$. As $Ge_1 = a = a \times 1 = aLe_1$ then the base case holds. Concerning the induction step, fix k and assume that the result holds for $j \leq k$. Then we have

$$Ge_{k+1} = Ge_k + Ge_{k-1} + a = aLe_k + aLe_{k-1} + a = a(Le_k + Le_{k-1} + 1) = aLe_{k+1}.$$

$\square$

Corollary 4.2 *For* $n \in \mathbb{N}_0$, *if* a *is odd (respectively, even) then* $Ge_n = aLe_n$ *is odd (respectively, even).*

Proof: Taking into account Theorem 4.1, the result is a consequence of [5, Lemma 2.1] which states that, for $n \in \mathbb{N}_0$, Le_n is an odd number. $\square$

Theorem 4.3 *For* $n \in \mathbb{N}_0$, $2a\phi^{n-1} - a \leq Ge_n < 2^{n+2}a - a$.

Proof: By Theorem 4.1 and Theorem 3.1, we get

$$a(2\phi^{n-1} - 1) \leq Ge_n = aLe_n < a(2^{n+2} - 1).$$

$\square$

In the subsequent theorem, Binet's formula for Geonardo numbers is deduced.

Theorem 4.4 *Let* $\phi = \frac{1+\sqrt{5}}{2}$ *and* $\psi = \frac{1-\sqrt{5}}{2}$. *For* $n \in \mathbb{N}_0$,

$$Ge_n = 2a\frac{\phi^{n+1} - \psi^{n+1}}{\phi - \psi} - a.$$

Proof: By Theorem 4.1 and Binet's formula for Leonardo numbers in [5, Proposition 2.4], we obtain $Ge_n = aLe_n = a\left(2\frac{\Phi^{n+1}-\Psi^{n+1}}{\Phi-\Psi} - 1\right)$. □

The asymptotic form of Binet's formula for Geonardo numbers is established in the next result.

Theorem 4.5 *Let* $\phi = \frac{1+\sqrt{5}}{2}$ *and* $\psi = \frac{1-\sqrt{5}}{2}$. *For* $n \in \mathbb{N}$, $Ge_n \approx 2a\frac{\Phi^{n+2}}{\Phi-\Psi} - a$.

Proof: From Theorem 4.1 and the asymptotic form of Binet's formula for Leonardo numbers in Theorem 3.2, we have $Ge_n = aLe_n \approx a\left(2\frac{\Phi^{n+2}}{\Phi-\Psi} - 1\right)$. □

The following theorem implies that consecutive Geonardo numbers plus a essentially grow by a factor of ϕ.

Theorem 4.6 *Let* $\phi = \frac{1+\sqrt{5}}{2}$. *For* $n, m \in \mathbb{N}_0$, $\lim_{n\to\infty} \frac{Ge_{n+m} + a}{Ge_n + a} = \phi^m$.

Proof: From Theorem 4.1 and Theorem 3.4, we get

$$\lim_{n\to\infty} \frac{Ge_{n+m} + a}{Ge_n + a} = \lim_{n\to\infty} \frac{Le_{n+m} + 1}{Le_n + 1} = \phi^m.$$

□

The ordinary generating function for Geonardo numbers is deduced in the subsequent result.

Theorem 4.7 *The ordinary generating function of* $\{Ge_n\}_{n=0}^{\infty}$ *is given by*

$$G(t) = \frac{a - at + at^2}{1 - 2t + t^3}.$$

Proof: From Theorem 4.1 and the ordinary generating function of Leonardo numbers in [5, Proposition 5.1], we get $Ge_n = aLe_n = a\dfrac{1 - t + t^2}{1 - 2t + t^3}$. □

The exponential generating function for Geonardo numbers is presented in the next result, and, as a consequence, the Poisson generating function.

Theorem 4.8 *Let* $\phi = \frac{1+\sqrt{5}}{2}$ *and* $\psi = \frac{1-\sqrt{5}}{2}$. *The exponential generating function of* $\{Ge_n\}_{n=0}^{\infty}$ *is given by*

$$G_E(t) = 2a\frac{\phi e^{\phi t} - \psi e^{\psi t}}{\phi - \psi} - ae^t.$$

Proof: Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then, by Binet's formula for Geonardo numbers in Theorem 4.4, we have

$$\begin{aligned} G_E(t) &= \sum_{n=0}^{\infty} Ge_n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(2a\frac{\phi^{n+1} - \psi^{n+1}}{\phi - \psi} - a\right)\frac{t^n}{n!} \\ &= \frac{2a}{\phi - \psi}\left(\phi \sum_{n=0}^{\infty} \frac{(\phi t)^n}{n!} - \psi \sum_{n=0}^{\infty} \frac{(\psi t)^n}{n!}\right) - a\sum_{n=0}^{\infty} \frac{t^n}{n!} \\ &= \frac{2a}{\phi - \psi}\left(\phi e^{\phi t} - \psi e^{\psi t}\right) - ae^t, \end{aligned}$$

which leads to the intended expression. □

Corollary 4.9 *Let* $\phi = \frac{1+\sqrt{5}}{2}$ *and* $\psi = \frac{1-\sqrt{5}}{2}$. *The Poisson generating function of* $\{Ge_n\}_{n=0}^{\infty}$ *is given by*

$$Ge_P(t) = 2a\frac{\phi e^{\phi t} - \psi e^{\psi t}}{(\phi - \psi)e^t} - a. \tag{4.1}$$

Proof: As $Ge_P(t) = e^{-t}Ge_E(t)$, then, invoking Theorem 4.8, the expression for $Ge_P(t)$ follows. □

The subsequent theorem presents a binomial-sum identity for Geonardo numbers.

Theorem 4.10 *For* $n \in \mathbb{N}_0$,

$$\sum_{i=0}^{n} Ge_{k+i} = Ge_{n+k+2} - Ge_{k+1} - (n+1)a.$$

Proof: For $n = 0$, the left-hand side and the right-hand side of the equality are equal, respectively, to Ge_k and $Ge_{k+2} - Ge_{k+1} - a$, and the result follows. We now proceed by induction on $n \in \mathbb{N}$. The base case holds since we get $Ge_k + Ge_{k+1}$ and $Ge_{k+3} - Ge_{k+1} - 2a$, both equal to $Ge_{k+2} - a$. As far as the induction step, we have

$$\begin{aligned}
\sum_{i=0}^{n+1} Ge_{k+i} &= \sum_{i=0}^{n} Ge_{k+i} + Ge_{k+n+1} \\
&= Ge_{n+k+2} - Ge_{k+1} - (n+1)a + Ge_{k+n+1} \\
&= Ge_{n+k+3} - Ge_{k+1} - (n+2)a.
\end{aligned}$$

□

Catalan's identity for Geonardo numbers is deduced in the following result, and, as a consequence, Cassini's identity is obtained.

Theorem 4.11 *For* $n, r \in \mathbb{N}$ *such that* $n > r$,

$$Ge_n^2 - Ge_{n-r}Ge_{n+r} = aGe_{n-r} + aGe_{n+r} - 2aGe_n - (-1)^{n-r}(Ge_{r-1} + a)^2.$$

Proof: From Theorem 4.1 and Catalan's identity for Leonardo numbers in [5, Proposition 4.1], we obtain

$$\frac{1}{a^2}(Ge_n^2 - Ge_{n-r}Ge_{n+r}) = \frac{1}{a}Ge_{n-r} + \frac{1}{a}Ge_{n+r} - \frac{2}{a}Ge_n - (-1)^{n-r}\left(\frac{1}{a}Ge_{r-1} + 1\right)^2.$$

The multiplication of both sides of the equality by a^2 leads to the result. □

Corollary 4.12 *For* $n \in \mathbb{N}$ *such that* $n \geq 2$,

$$Ge_n^2 - Ge_{n-1}Ge_{n+1} = aGe_{n-1} + aGe_{n+1} - 2aGe_n - 4(-1)^{n-1}a^2.$$

Proof: Take $r = 1$ in Theorem 4.11. □

Next result exhibits d'Ocagne's identity for Geonardo numbers.

Theorem 4.13 *For* $m, n \in \mathbb{N}$ *such that* $m > n$,

$$Ge_mGe_{n+1} - Ge_{m+1}Ge_n = 2a(-1)^{n+1}(Ge_{m-n-1} + a) + aGe_{m-1} - aGe_{n-1}.$$

Proof: From Theorem 4.1 and d'Ocagne's identity for Leonardo numbers in [5, Proposition 4.3], we get

$$\begin{array}{rcl}\frac{1}{a^2}(Ge_mGe_{n+1}-Ge_{m+1}Ge_n) & = & \frac{1}{a^2}\left[2a(-1)^{n+1}(Ge_{m-n-1}+a)\right.\\ & & \left.+aGe_{m-1}-aGe_{n-1}\right]\end{array}$$

The multiplication of both sides of the equality by a^2 leads to the result. □

Melham's identity for Geonardo numbers is established in the following theorem.

Theorem 4.14 *For* $n \in \mathbb{N}$,

$$\begin{array}{rcl}Ge_{n+1}Ge_{n+2}Ge_{n+6}-Ge_{n+3}^3 & = & -(Ge_n+a)^3+(Ge_n+a)(Ge_{n-1}+a)^2\\ & & +(Ge_n+a)^2(Ge_{n-1}+a)-14a(Ge_n+a)^2\\ & & -5a(Ge_{n-1}+a)^2\\ & & -17a(Ge_n+a)(Ge_{n-1}+a)+7a^2Ge_n\\ & & +4a^2Ge_{n-1}+11a^3.\end{array}$$

Proof: The sequence $\{Le_n\}_{n=0}^{\infty}$ of Leonardo numbers is a particular case of the generalized Leonardo sequence $\{W_n\}_{n=0}^{\infty}$ in [11], with $W_0 = W_1 = 1$ and $W_2 = 3$. From [5, Proposition 2.2], which states that $Le_n = 2F_{n+1} - 1$, Melham's identity for Leonardo numbers in [11, p. 321] can be expressed by

$$\begin{array}{rcl}Le_{n+1}Le_{n+2}Le_{n+6}-Le_{n+3}^3 & = & -(Le_n+1)^3+(Le_n+1)(Le_{n-1}+1)^2\\ & & +(Le_n+1)^2(Le_{n-1}+1)-14(Le_n+1)^2\\ & & -5(Le_{n-1}+1)^2-17(Le_n+1)(Le_{n-1}+1)\\ & & +7Le_n+4Le_{n-1}+11\end{array}$$

Hence, by Theorem 4.1, we get $\frac{1}{a^3}(Ge_{n+1}Ge_{n+2}Ge_{n+6}-Ge_{n+3}^3)$ equal to

$$\begin{array}{l}\frac{1}{a^3}\left[-(Ge_n+a)^3+(Ge_n+a)(Ge_{n-1}+a)^2+(Ge_n+a)^2(Ge_{n-1}+a)\right.\\ -14a(Ge_n+a)^2-5a(Ge_{n-1}+a)^2-17a(Ge_n+a)(Ge_{n-1}+a)\\ \left.+7a^2Ge_n+4a^2Ge_{n-1}+11a^3\right].\end{array}$$

The result follows from multiplying both sides of the equality by a^3. □

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