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Some Fixed Point Results For Nonexpansive and G-Nonexpansive Mappings In Hilbert Spaces

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ABSTRACT: This paper is concerned with the convergence of a three-step Picard iteration scheme introduced by Javid et al. for nonexpansive and $G-$ nonexpansive mappings in Hilbert spaces endowed with binary relation under some assumptions. After that same results are obtained by replacing binary relation with directed graph. Main results are justified with some numerical examples. Here, an application of fixed point theory is discussed to obtain solution of system of equations and fastness of a three-step Picard iteration scheme with other well-known iteration schemes.

Key Words: Hilbert space, fixed point, G-nonexpansive mappings, binary relation, Fejer monotone.

Contents

1. Introduction

Fixed point theory has great implications in the filed of analysis. Fixed point theory is used to find solutions of different mathematical problems like integral equations, differential equations, optimization problems, convex minimization problems, image recovery, signal processing etc. (refer [6,7,22]).

Several mathematicians studied fixed point results over different spaces as metric space, Banach space, Reflexive space, Hilbert space and many more. One of the most important and fruitful result in metric space was given by Banach [4] called "Banach Contraction Principle". This principle was generalized and its several variants were studied by mathematicians over different spaces.

Let X be a Hilbert space and let K be a non-empty subset of X. A point $x \in X$ is called a fixed point of a mapping $T : X \to X$ if $T(x) = x$. Through-out the literature $F(T)$ denotes the set of fixed points of T, i.e., $F(T) = \{x \in X : Tx = x\}$. Note that a mapping $T : K \to K$ is called

(i) Lipschitz, if $||Tx - Ty|| \leq L||x - y||$, for all $x, y \in K$, where $L > 0$.

(ii) Contraction, if $||Tx - Ty|| \leq L||x - y||$, for all $x, y \in K$, where $0 < L < 1$.

(iii) Nonexpansive, if $||Tx - Ty|| \leq ||x - y||$, for all $x, y \in K$ and $L = 1$.

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Note that the Banach contraction principle is no longer true for nonexpansive mappings. The study of fixed points of mappings with certain contraction condition attract many researcher, but nonexpansive mapping has also an important role in the fixed point theory. In-fact various researchers investigated the theory of nonexpansive mappings for establishing the existence of fixed points (refer [9,18,23]). The study of nonexpansive mappings were basically motivated by Browder's [5] work on relationship between monotone operators and nonexpansive mappings and the significance of the geometric properties of the norm for the existence of fixed point for nonexpansive mapping given by Kirk [16].

In few years, the use of graph theory and the fixed point theory has increased rapidly. The study of fixed point theory along with graph theory was started by Echenique [10] in 2005. After that Kirk and Espinola [11] established some fixed point results in graph theory. Alfuraidan [2] studied some fixed point results for nonexpansive type mappings in hyperbolic metric space along with directed graph. Tiammee [27] proved Browder's fixed point theorem for nonexpansive type mappings in Hilbert space along with directed graph.

All the authors used some iterative techniques to obtain theoretical results. In fact Banach [4] used Picard iteration [21] to get fixed point of contraction mapping. The Picard iteration is as follows:
Let X be a non-empty set and $T : X \to X$. Let $\{x_k\}$ be a sequence in X with initial point $x_0 \in X$ such that

$$x_{k+1} = Tx_k = T^k x_0.$$

But this iteration was not enough to assure some kind of convergence in the case of more general class of nonexpansive mappings. In this direction, Agarwal [1], Ishikawa [12], Krasnoselkii [17], Mann [19], Noor [20] introduced iteration schemes to study behaviour of fixed point of nonexpansive type of mappings.

Recently, Javid Ali et al. [3] introduced three-step Picard iteration scheme (one can call new Picard iteration) in the framework of Banach space as follows:
Let X be a Banach space and T a self-mapping on X. The sequence $\{x_k\}$ with initial guess $x_0 \in X$ is defined in the following manner:

$$\begin{cases} x_{k+1} = Ty_k, \\ y_k = Tz_k, \\ z_k = Tx_k, \end{cases} \tag{1.1}$$

In [3], authors proved the convergence and stability of (1.1) in Banach spaces for Zamfirescu operator and proved that the iteration scheme (1.1) is faster than the iteration scheme given by Agarwal [1], Ishikawa [12], Picard-Mann hybrid iteration [14], Mann [19], Noor [20], Picard [21], normal $S-$ iteration [24], $SP-$ iteration [26], etc.

Vetro [30] established some fixed point results for single-valued and multi-valued nonexpansive mapping and proved convergence of Picard iteration in metric space endowed with binary relation. Inspired by work of Vetro [30], here the aim is to study convergence of iteration scheme (1.1) for nonexpansive mappings and $G-$ nonexpansive mappings in the framework of Hilbert spaces under some suitable conditions.

2. Preliminaries

This section proceeds with some necessary concepts and include some useful results.

Definition 2.1 *[11] A graph is an ordered pair (V, E), where V is a set and E is a binary relation on V, $E \subseteq V \times V$. Elements of E are called edges. Given a graph $G = (V, E)$, a path of G is a sequence $a_0 a_1 ... a_{n-1} ...$ with $(a_i, a_{i+1}) \in E$ for each $i = 0, 1, 2, ...$ A graph is connected if there is a finite path joining any two of its vertices.*

Definition 2.2 *[29] A directed graph also called a digraph is a graph in which the edges have a direction.*

Definition 2.3 *[29] A digraph is weakly connected if when considering it as an undirected graph, it is connected, i.e., for every pair of distinct vertices u and v there exists an undirected path from u to v.*

Definition 2.4 *[13] A mapping $T : X \to X$ is called $G-$ continuous if for $x \in X$ and $\{x_k\}$ be a sequence in X such that $x_k \to x$ and $(x_k, x_{k+1}) \in E(G)$ for $k \in \mathbb{N}$, $Tx_k \to Tx$.*

Definition 2.5 *[27] Let $G = (V(G), E(G))$ be a directed graph. A graph G is called transitive if for any $x, y, z \in V(G)$ such that $(x, y) \in E(G)$ and $(y, z) \in E(G)$, then $(x, z) \in E(G)$.*

Definition 2.6 *[2] Let K be a non-empty subset of a Hilbert space X. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = K$ and $E(G)$ contains all the loops, i.e., $(x, x) \in E(G)$ for any $x \in V(G)$. A mapping $T : K \to K$ is called an edge-preserving mapping (or $G-$ edge preserving mapping) if*

$$for\ all\ x, y \in K, (x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G).$$

Definition 2.7 *[27] Let K be a non-empty convex subset of a Hilbert space X, $G = (V(G), E(G))$ be a graph such that $V(G) = K$ and $T : K \to K$. Then T is said to be $G-$ nonexpansive if the following conditions hold:*

(i) T is an edge-preserving;

(ii) $||Tx - Ty|| \leq ||x - y||$, whenever $(x, y) \in E(G)$ for any $x, y \in K$.

Definition 2.8 *[15] Let K be a non-empty subset of a Hilbert space X. A sequence $\{x_k\}$ in X is said to be Fejer monotone with respect to subset K, if*

$$||x_{k+1} - p|| \leq ||x_k - p||,$$

for all $p \in K$, $k \geq 1$.

Proposition 2.1 *[15] Let K be a non-empty subset of a Hilbert space X. Suppose that $\{x_k\}$ is Fejer monotone sequence with respect to K. Then the followings are hold:*

(a) Sequence $\{x_k\}$ is bounded.

(b) For every $x \in K$, $\{||x_k - x||\}$ converges.

Lemma 2.1 *[25] Let X be a Hilbert space, and $\{\alpha_k\}$ be a sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. Suppose that $\{x_k\}$ and $\{y_k\}$ are in X such that $\limsup_{k\to\infty} ||x_k|| \leq c$, $\limsup_{k\to\infty} ||y_k|| \leq c$, and $\limsup_{k\to\infty} ||\alpha_k x_k + (1 - \alpha_k)y_k|| = c$ for some $c \geq 0$. Then $\lim_{k\to\infty} ||x_k - y_k|| = 0$.*

3. Main results

Lemma 3.1 *If $\{x_k\}$ is a non-increasing sequence of non-negative real numbers, then the sequence $\{\frac{x_k+x_{k+1}}{x_k+x_{k+1}+1}\}$ is also non-increasing.*

Proof: Since $\{x_k\}$ is a non-increasing sequence, we have

$$x_{k+1} \leq x_k,\ x_{k+2} \leq x_{k+1}, ...,\ x_{k+n} \leq x_{k+n-1}, ...$$

Therefore

$$\begin{aligned} x_{k+1} + x_{k+2} &\leq x_k + x_{k+1} \\ \Rightarrow x_{k+1} + x_{k+2} + 1 &\leq x_k + x_{k+1} + 1 \\ \Rightarrow \frac{x_{k+1} + x_{k+2}}{x_{k+1} + x_{k+2} + 1} &\leq \frac{x_k + x_{k+1}}{x_k + x_{k+1} + 1}. \end{aligned}$$

It conclude that $\{\frac{x_k+x_{k+1}}{x_k+x_{k+1}+1}\}$ is also non-increasing. □

Corollary 3.1 *Let X be a Hilbert space and $T : X \to X$ be a nonexpansive mapping. Let $\{x_k\}$ be a sequence in X generated by iteration scheme (1.1) with initial point $x_0 \in X$. Then the sequence $\{\frac{||x_k - x_{k-1}|| + ||x_{k+1} - x_k||}{||x_k - x_{k-1}|| + ||x_{k+1} - x_k|| + 1}\}$ is non-increasing.*

Proof: Let $\{x_k\}$ be a sequence in X defined by (1.1) with initial point $x_0 \in X$. By using nonexpansiveness of T, we have

$$\begin{aligned} ||x_{k+1} - x_k|| &= ||Ty_k - Ty_{k-1}|| \\ &\leq ||y_k - y_{k-1}|| \\ &= ||Tz_k - Tz_{k-1}|| \\ &\leq ||z_k - z_{k-1}|| \\ &= ||Tx_k - Tx_{k-1}|| \\ &\leq ||x_k - x_{k-1}||. \end{aligned}$$

This implies that $\{||x_{k+1} - x_k||\}$ is non-increasing sequence, hence from Lemma 3.1, the sequence $\{\frac{||x_k - x_{k-1}|| + ||x_{k+1} - x_k||}{||x_k - x_{k-1}|| + ||x_{k+1} - x_k|| + 1}\}$ is non-increasing. □

Theorem 3.1 *Let X be a Hilbert space and w be a binary relation on X, i.e., $w \subseteq X \times X$ and $T : X \to X$ be a nonexpansive mapping such that*

$$||Tx - Ty|| \leq \left(\frac{||x - Ty|| + ||y - Tx||}{||x - Tx|| + ||y - Ty|| + 1} + k \right) ||x - y||, \tag{3.1}$$

for all $(x, y) \in w$, $k \in [0, 1)$. Suppose that

(i) If $\{x_k\}$ is a sequence in X such that $(x_{k-1}, x_k) \in w$ for all $k \in \mathbb{N}$ and $x_k \to p \in X$ as $k \to \infty$, then $(x_{k-1}, p) \in w$ for all $k \in \mathbb{N}$.

(ii) $F(T)$ is well ordered with respect to w, i.e., for $x, y \in X$, either $(x, y) \in w$ or $(y, x) \in w$.

If there exists $x_0 \in X$ such that $(x_0, Tx_0) \in w$ and

$$\left(\frac{||x_0 - Tx_0|| + ||Tx_0 - T^2x_0||}{||x_0 - Tx_0|| + ||Tx_0 - T^2x_0|| + 1} + k \right) < 1, \tag{3.2}$$

then

(a) T has at-least one fixed point $p \in X$.

(b) sequence $\{x_k\}$ defined by (1.1) with initial point $x_0 \in X$ converges to a fixed point of T.

Proof: Let $x_0 \in X$ such that $(x_0, Tx_0) \in w$. Let $\{x_k\}$ be a sequence defined by (1.1) with initial point $x_0 \in X$. If $x_{k-1} = x_k$ for some $k \in \mathbb{N}$, then we obtain existence of fixed point of T. Suppose that $x_{k-1} \neq x_k$ for all $k \in \mathbb{N}$. By using (1.1), and (3.1), we have

$$\begin{aligned} ||x_{k+1} - x_k|| &= ||Tx_k - Tx_{k-1}|| \\ &\leq \left(\frac{||x_k - Tx_{k-1}|| + ||x_{k-1} - Tx_k||}{||x_k - Tx_k|| + ||x_{k-1} - Tx_{k-1}|| + 1} + k \right) ||x_k - x_{k-1}|| \\ &= \delta ||x_k - x_{k-1}|| \\ &\leq ||x_k - x_{k-1}||. \end{aligned}$$

Where $\delta = \left(\frac{||x_k - Tx_{k-1}|| + ||x_{k-1} - Tx_k||}{||x_k - Tx_k|| + ||x_{k-1} - Tx_{k-1}|| + 1} + k\right) < 1$. It conclude that $\{x_k\}$ is a Cauchy sequence in X. Since X is Hilbert space, there exists $p \in X$ such that $x_k \to p$ as $k \to \infty$. We claim that $p = Tp$. By assumption (i), $(x_k, p) \in M$. By using (1.1) and (3.1), we have

$$\begin{aligned} ||x_{k+1} - Tp|| &= ||Tx_k - Tp|| \\ &\leq \left(\frac{||x_k - Tp|| + ||p - Tx_k||}{||x_k - Tx_k|| + ||p - Tp|| + 1} + k\right) ||x_k - p|| \\ &\leq \left(\frac{||x_k - Tp|| + ||p - x_{k+1}||}{||x_k - Tx_k|| + ||p - Tp|| + 1} + k\right) ||x_k - p|| \\ &\to 0 \ as \ k \to \infty. \end{aligned}$$

By uniqueness of limit, $p = Tp$. Thus (a) and (b) holds. □

Theorem 3.2 *Let X be a Hilbert space and w be a binary relation on X, i.e., $w \subseteq X \times X$ and $T : X \to X$ be a nonexpansive mapping such that*

$$||Tx - Ty|| \leq \left(\frac{||x - Ty|| + ||y - Tx||}{||x - Tx|| + ||y - Ty|| + 1} + k\right) ||x - y|| + L||y - Tx||, \tag{3.3}$$

for all $(x, y) \in w$, $k \in [0, 1)$ and L is non-negative real number. Suppose that

(i) If $\{x_k\}$ is a sequence in X such that $(x_{k-1}, x_k) \in w$ for all $k \in \mathbb{N}$ and $x_k \to p \in X$ as $k \to \infty$, then $(x_{k-1}, p) \in w$ for all $k \in \mathbb{N}$.

(ii) $F(T)$ is well ordered with respect to w.

If there exists $x_0 \in X$ such that $(x_0, Tx_0) \in w$ and (3.2) holds, then

(a) T has at-least one fixed point $p \in X$.

(b) sequence $\{x_k\}$ defined by (1.1) with initial point $x_0 \in X$ converges to a fixed point of T.

Proof: Let $x_0 \in X$ such that $(x_0, Tx_0) \in w$. Let $\{x_k\}$ be a sequence defined by (1.1) with initial point $x_0 \in X$. If $x_{k-1} = x_k$ for some $k \in \mathbb{N}$, then we obtain existence of fixed point of T. Suppose that $x_{k-1} \neq x_k$ for all $k \in \mathbb{N}$. Now By using (3.3) and (1.1), we have

$$\begin{aligned} ||x_{k+1} - x_k|| &= ||Tx_k - Tx_{k-1}|| \\ &\leq \left(\frac{||x_k - Tx_{k-1}|| + ||x_{k-1} - Tx_k||}{||x_k - Tx_k|| + ||x_{k-1} - Tx_{k-1}|| + 1} + k\right) ||x_k - x_{k-1}|| + L||x_k - Tx_{k-1}|| \\ &= \delta||x_k - x_{k-1}||| \\ &\leq ||x_k - x_{k-1}||. \end{aligned}$$

Where $\delta = \left(\frac{||x_k - Tx_{k-1}|| + ||x_{k-1} - Tx_k||}{||x_k - Tx_k|| + ||x_{k-1} - Tx_{k-1}|| + 1} + k\right) < 1$. It conclude that $\{x_k\}$ is Cauchy sequence in X. Since X is Hilbert space, there exists $p \in X$ such that $x_k \to p$. We claim that $p = Tp$. By assumption (i), $(x_k, p) \in w$. By using (3.3), we have

$$\begin{aligned} ||x_{k+1} - Tp|| &= ||Tx_k - Tp|| \\ &\leq \left(\frac{||x_k - Tp|| + ||p - Tx_k||}{||x_k - Tx_k|| + ||p - Tp|| + 1} + k\right) ||x_k - p|| + L||p - Tx_k|| \\ &\leq \left(\frac{||x_k - Tp|| + ||p - x_{k+1}||}{||x_k - Tx_k|| + ||p - Tp|| + 1} + k\right) ||x_k - p|| + L||p - x_{k+1}|| \\ &\to 0 \ as \ k \to \infty. \end{aligned}$$

By uniqueness of limit, $p = Tp$. Thus (a) and (b) holds. □

Corollary 3.2 *Let X be a Hilbert space and $T : X \to X$ be a nonexpansive mapping such that all the conditions of the Theorem 3.1 and 3.2 satisfied. Let*

$$M_1 = \{T : ||Tx - Ty|| \leq \left(\frac{||x - Ty|| + ||y - Tx||}{||x - Tx|| + ||y - Ty|| + 1} + k \right) ||x - y||\},$$

and

$$M_2 = \{T : ||Tx - Ty|| \leq \left(\frac{||x - Ty|| + ||y - Tx||}{||x - Tx|| + ||y - Ty|| + 1} + k \right) ||x - y|| + L||y - Tx||\}.$$

Let $F = M_1 \cup M_2$ and $T \in F$, then T has a fixed point in X.

By putting $w = X \times X$ in Theorems 3.1 and 3.2, the following results are obtained.

Theorem 3.3 *Let X be a Hilbert space and $T : X \to X$ be a nonexpansive mapping such that (3.1) is satisfied for all $(x, y) \in X \times X$, $k \in [0, 1)$ and $x_0 \in X$ such that (3.2) satisfied. Then*

(a) T has at-least one fixed point $p \in X$.

(b) sequence $\{x_k\}$ defined by (1.1) with initial point $x_0 \in X$ converges to a fixed point of T.

Proof: All the conditions of Theorem 3.1 are satisfied with $w = X \times X$ and hence Theorem 3.3 follows from Theorem 3.1. □

Theorem 3.4 *Let X be a Hilbert space and $T : X \to X$ be a nonexpansive mapping such that (3.3) is satisfied for all $(x, y) \in X \times X$, $k \in [0, 1)$ and $x_0 \in X$ such that (3.2) satisfied. Then*

(a) T has at-least one fixed point $p \in X$.

(b) sequence $\{x_k\}$ defined by (1.1) with initial point $x_0 \in X$ converges to a fixed point of T.

Example 3.1 *Let $X = \mathbb{R}$ be a Hilbert space and $T : X \to X$ defined by*

$$Tx = \begin{cases} \frac{1}{3} + \frac{x}{3}, & x \in [0, 1], \\ 1 + \frac{x}{3}, & x \in [2, \infty). \end{cases}$$

First, we show that T is nonexpansive mapping. Consider the following cases:

Case I: when $x, y \in [0, 1]$. Then

$$\begin{aligned} ||Tx - Ty|| &= \frac{1}{3}||x - y|| \\ &\leq ||x - y||. \end{aligned}$$

Case II: when $x, y, \in [2, \infty)$. Then

$$\begin{aligned} ||Tx - Ty|| &= \frac{1}{3}||x - y|| \\ &\leq ||x - y||. \end{aligned}$$

Case III: when $x \in [0, 1]$, $y \in [2, \infty)$. Then

$$||Tx - Ty|| \leq \frac{2}{3} + \frac{1}{3}||x - y||.$$

Clearly $||Tx - Ty|| \leq ||x - y||$ for $x \in [0, 1]$, $y \in [2, \infty)$.

Next, to prove that T satisfies (3.1), for this, let $x \in [0,1]$ and $y \in [2,\infty)$. Then

$$\begin{aligned}\frac{||x-Ty||+||y-Tx||}{||x-Tx||+||y-Ty||+1} &= \frac{||3x-3-y||+||3y-x-1||}{||2x-1||+||2y-3||+3} \\ &\geq 1.\end{aligned}$$

Now choose $x_0 = \frac{1}{4}$ and $k = \frac{2}{3}$, we have

$$\begin{aligned}\frac{||x_0-Tx_0||+||Tx_0-T^2x_0||}{||x_0-Tx_0||+||Tx_0-T^2x_0||+1}+k &= \frac{25}{33} \\ &< 1.\end{aligned}$$

Since all the hypothesis of the Theorem 3.3 are satisfied, hence T has at-least one fixed point in X.

Example 3.2 *Let $X = \mathbb{R}$ be a Hilbert space and $T : X \to X$ defined by*

$$Tx = x.$$

Clearly T is nonexpansive mapping. Also for $x, y \in X$, with $x \neq y$,

$$\begin{aligned}||Tx-Ty|| &= ||x-y|| \\ &\leq \left(3||x-y||+\frac{1}{3}\right)||x-y||+\frac{1}{3}||x-y||.\end{aligned}$$

And

$$\frac{||x_0-Tx_0||+||Tx_0-T^2x_0||}{||x_0-Tx_0||+||Tx_0-T^2x_0||+1}+k < 1.$$

Hence all the hypothesis of the Theorem 3.4 are satisfied for $k = \frac{1}{3}$, and $L = \frac{1}{3}$, therefore T has at-least one fixed point in X.

4. Fixed point results for $G-$ nonexpansive mappings

In this section, the convergence of (1.1) is proved for $G-$ nonexpansive mapping by replacing binary relation with a directed graph.

Lemma 4.1 *Let X be a Hilbert space and $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = X$ and $E(G)$ contains all the loops. Let $T : V(G) \to V(G)$ be a $G-$ nonexpansive mapping. Fix $x_0 \in V(G)$ such that $(x_0, Tx_0) \in E(G)$. Let $\{x_k\}$ be a sequence in $V(G)$ defined by (1.1). Then we have the following:*

(a) $(T^kx_0, T^{k+1}x_0) \in E(G)$ for any $k \geq 1$.

(b) $(x_k, x_{k+1}) \in E(G)$ for any $k \geq 0$.

(c) $(x_k, Tx_k) \in E(G)$ for any $k \geq 1$.

(d) $(x_{k+1}, Tx_k) \in E(G)$ for any $k \geq 0$.

Proof:

(a) By assumption $(x_0, Tx_0) \in E(G)$ and T is an edge-preserving mapping, hence $(Tx_0, T^2x_0) \in E(G)$. Again by edge-preserving of T, $(T^2x_0, T^3x_0) \in E(G)$. Continuing this process, $(T^kx_0, T^{k+1}x_0) \in E(G)$ for any $k \geq 1$.

(b) By assumption $(x_0, Tx_0) \in E(G)$, i.e., $(x_0, z_0) \in E(G)$ and T is an edge-preserving mapping, so $(Tx_0, Tz_0) \in E(G)$, i.e. $(z_0, y_0) \in E(G)$. Again using edge-preserving of T, $(Tz_0, Ty_0) \in E(G)$, i.e., $(y_0, x_1) \in E(G)$. Since $(x_0, y_0) \in E(G)$, $(y_0, x_1) \in E(G)$, by transitivity of $E(G)$, $(x_0, x_1) \in E(G)$. By using edge-preserving of T, $(Tx_0, Tx_1) \in E(G)$, i.e., $(z_0, z_1) \in E(G)$, $(Tz_0, Tz_1) \in E(G)$, $(y_0, y_1) \in E(G)$, $(Ty_0, Ty_1) \in E(G)$, $(x_1, x_2) \in E(G)$. Continuing this process, $(x_k, x_{k+1}) \in E(G)$ for any $k \geq 0$.

(c) To show that $(x_k, Tx_k) \in E(G)$ for any $k \geq 1$, proceeds by value of k. By assumption $(x_0, Tx_0) \in E(G)$, hence result is true for $k = 0$. Now suppose that $(x_k, Tx_k) \in E(G)$ for $k \geq 1$. Since $(x_k, x_{k+1}) \in E(G)$ by (b), $(x_k, Tx_k) \in E(G)$, $(x_{k+1}, Tx_k) \in E(G)$ as G is transitive. Since $(x_k, x_{k+1}) \in E(G)$, $(Tx_k, Tx_{k+1}) \in E(G)$ due to edge-preserving of T. Again, $(x_{k+1}, Tx_k) \in E(G)$, and $(Tx_k, Tx_{k+1}) \in E(G)$, which gives that $(x_{k+1}, Tx_{k+1}) \in E(G)$.

(d) By part (c), $(x_{k+1}, Tx_k) \in E(G)$ for any $k \geq 0$.

□

Lemma 4.2 *Let X be a Hilbert space and $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = X$ and $E(G)$ contains all the loops. Let $T : V(G) \to V(G)$ be a $G-$ nonexpansive mapping. Fix $x_0 \in V(G)$ such that $(x_0, Tx_0) \in E(G)$. Let $\{x_k\}$ be a sequence in $V(G)$ defined by (1.1). Let $F(T) \neq \emptyset$ with $r \in F(T)$ such that $(x_0, r), (r, x_0) \in E(G)$, then we have following:*

(a) (x_k, r) *and* (r, x_k) *are in* $E(G)$ *for* $k \geq 1$.

(b) $\lim_{k\to\infty} ||x_k - r||$ *exists.*

(c) $\lim_{k\to\infty} ||Tx_k - x_k|| = 0$.

Proof:

(a) Proceeds on the value of k. Since T is an edge-preserving and $(x_0, r) \in E(G)$, $(Tx_0, r) \in E(G)$, i.e., $(z_0, r) \in E(G)$. Again by edge-preserving of T, $(Tz_0, r) \in E(G)$, i.e., $(y_0, r) \in E(G)$. Now $(Ty_0, r) \in E(G)$, $(x_1, r) \in E(G)$. Continuing this process, $(x_k, r) \in E(G)$. Using similar argument, one can prove that $(r, x_k) \in E(G)$ for $k \geq 1$.

(b) Let $r \in F(T)$. By (a), $(x_k, r) \in E(G)$ for $k \geq 1$. Note that

$$\begin{aligned} ||x_{k+1} - r|| &= ||Tx_k - Tr|| \\ &\leq ||x_k - r||. \end{aligned}$$

It follows that $\{x_k\}$ is Fejer monotone with respect to $r \in F(T)$. Hence by Proposition 2.1, $\lim_{k\to\infty} ||x_k - r||$ exists.

(c) As $\lim_{k\to\infty} ||x_k - r||$ exists, so suppose that $\lim_{k\to\infty} ||x_k - r|| = w$.
If $w = 0$, then by using nonexpansiveness of T,

$$\begin{aligned} ||Tx_k - x_k|| &\leq ||Tx_k - r|| + ||r - x_k|| \\ &\leq ||x_k - r|| + ||r - x_k||. \end{aligned}$$

Therefore, the result follows.
Suppose that $w > 0$. As $\lim_{k\to\infty} ||x_k - r|| = w$, it follows that $\limsup_{k\to\infty} ||x_k - r|| \leq w$. Also $||Tx_k - r|| \leq ||x_k - r||$, this implies that $\limsup_{k\to\infty} ||Tx_k - r|| \leq w$. Now for any sequence $\{\alpha_k\}$ in $[\delta, 1 - \delta]$, for $\delta \in (0, 1)$,

$$\begin{aligned} \limsup_{k\to\infty} ||\alpha_k(x_k - r) + (1 - \alpha_k)(Tx_k - r)|| \leq \alpha_k \limsup_{k\to\infty} ||x_k - r|| \\ + (1 - \alpha_k) \limsup_{k\to\infty} ||x_k - r||. \end{aligned}$$

Which gives
$$\limsup_{k\to\infty} ||\alpha_k(x_k - r) + (1-\alpha_k)(Tx_k - r)|| \le w.$$
Hence, by Lemma 2.1, $\lim_{k\to\infty} ||(x_k - r) - (Tx_k - r)|| = 0$. Therefore $\lim_{k\to\infty} ||x_k - Tx_k|| = 0$.

□

Theorem 4.1 *Let X be a Hilbert space and $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = X$ and $E(G)$ contains all the loops. Let $\{x_k\}$ be a sequence in $V(G)$ defined by (1.1) and $T : V(G) \to V(G)$ be a $G-$ nonexpansive mapping such that it satisfies (3.1) for all $(x, y) \in E(G)$, $k \in [0, 1)$. Suppose that*

(i) If $\{x_k\}$ is a sequence in $V(G)$ such that $x_k \to x$ as $k \to \infty$, and $(x_k, x_{k+1}) \in E(G)$ for all $k \ge 0$, then there is a subsequence $\{x_{k_n}\}$ with $(x_{k_n}, x) \in E(G)$ for all $k \ge 0$.

(ii) If there exists $x_0 \in V(G)$ such that $(x_0, Tx_0) \in E(G)$ and (3.2) satisfied, then

(a) T has at-least one fixed point $p \in V(G)$.

(b) sequence $\{x_k\}$ defined by (1.1) with initial point $x_0 \in V(G)$ converges to a fixed point of T.

Proof: For $x_0 \in V(G)$, $(x_0, Tx_0) \in E(G)$. Let $\{x_k\}$ be a sequence in $V(G)$ defined by (1.1). If $x_{k-1} = x_k$ for some $k \in \mathbb{N}$, then the fixed point of T exists. Suppose that $x_{k-1} \ne x_k$ for all $k \in \mathbb{N}$. Now by using Lemma 4.2 and (3.1),

$$\begin{aligned}
||x_{k+1} - x_k|| &= ||Tx_k - Tx_{k-1}|| \\
&\le \left(\frac{||x_k - Tx_{k-1}|| + ||x_{k-1} - Tx_k||}{||x_k - Tx_k|| + ||x_{k-1} - Tx_{k-1}|| + 1} + k \right) ||x_k - x_{k-1}|| \\
&= \left(\frac{||x_k - x_k|| + ||x_{k-1} - x_{k+1}||}{||x_k - Tx_k|| + ||x_{k-1} - Tx_{k-1}|| + 1} + k \right) ||Tx_{k-1} - x_{k-1}|| \\
&\to 0 \ as \ k \to \infty.
\end{aligned}$$

It conclude that $\{x_k\}$ is a Cauchy sequence in $V(G) = X$. Hence there exists $p \in X$ such that $x_k \to p$ as $k \to \infty$. So by assumption (i), there is a subsequence $\{x_{k_n}\}$ with $(x_{k_n}, p) \in E(G)$ for all $k \ge 0$. To prove that $p = Tp$.

$$\begin{aligned}
||x_{k_{n+1}} - Tp|| &= ||Tx_{k_n} - Tp|| \\
&\le \left(\frac{||x_{k_n} - Tp|| + ||p - Tx_{k_n}||}{||x_{k_n} - Tx_{k_n}|| + ||p - Tp|| + 1} + k \right) ||x_{k_n} - p|| \\
&\to 0 \ as \ k \to \infty.
\end{aligned}$$

By uniqueness of limit, $p = Tp$. Thus (a) and (b) holds. □

Theorem 4.2 *Let X be a Hilbert space and $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = X$ and $E(G)$ contains all the loops. Let $\{x_k\}$ be a sequence in $V(G)$ defined by (1.1) and $T : V(G) \to V(G)$ be a $G-$ nonexpansive mapping such that it satisfies (3.3) for all $(x, y) \in E(G)$, $k \in [0, 1)$. Suppose that*

(i) If $\{x_k\}$ is a sequence in $V(G)$ such that $x_k \to x$ as $k \to \infty$, and $(x_k, x_{k+1}) \in E(G)$ for all $k \ge 0$, then there is a subsequence $\{x_{k_n}\}$ with $(x_{k_n}, x) \in E(G)$ for all $k \ge 0$.

(ii) If there exists $x_0 \in V(G)$ such that $(x_0, Tx_0) \in E(G)$ and (3.2) is satisfied, then

(a) T has at-least one fixed point $p \in V(G)$.

(b) sequence $\{x_k\}$ defined by (1.1) with initial point $x_0 \in V(G)$ converges to a fixed point of T.

Proof: Note that for $x_{k-1} = x_k$ for some $k \in \mathbb{N}$, the result is true. For $x_{k-1} \neq x_k$ for all $k \in \mathbb{N}$,

$$\begin{aligned} ||x_{k+1} - x_k|| &= ||Tx_k - Tx_{k-1}|| \\ &\leq \left(\frac{||x_k - Tx_{k-1}|| + ||x_{k-1} - Tx_k||}{||x_k - Tx_k|| + ||x_{k-1} - Tx_{k-1}|| + 1} + k \right) ||x_k - x_{k-1}|| + L||x_k - Tx_{k-1}|| \\ &= \left(\frac{||Tx_{k-1} - Tx_{k-1}|| + ||x_{k-1} - x_{k+1}||}{||x_k - Tx_k|| + ||x_{k-1} - Tx_{k-1}|| + 1} + k \right) ||x_k - x_{k-1}|| + L||x_k - Tx_{k-1}|| \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

It conclude that $\{x_k\}$ is a Cauchy sequence in $V(G) = X$. Hence there exists $p \in X$ such that $x_k \to p$ as $\to \infty$. So by assumption, there is a subsequence $\{x_{k_n}\}$ with $(x_{k_n}, p) \in E(G)$ for all $k \geq 0$. Now

$$\begin{aligned} ||x_{k_{n+1}} - Tp|| &= ||Tx_{k_n} - Tp|| \\ &\leq \left(\frac{||x_{k_n} - Tp|| + ||p - Tx_{k_n}||}{||x_{k_n} - Tx_{k_n}|| + ||p - Tp|| + 1} + k \right) ||x_{k_n} - p|| + L||p - Tx_{k_n}|| \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

By uniqueness of limit, $p = Tp$. Thus (a) and (b) holds. □

Theorem 4.3 *Let X be a Hilbert space and $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = X$ and $E(G)$ contains all the loops. Let $\{x_k\}$ be a sequence in $V(G)$ defined by (1.1) and $T : V(G) \to V(G)$ be a $G-$ nonexpansive mapping such that it satisfies (3.1). Let $X_T = \{x_k \in X : (x_k, Tx_k) \in E(G)\}$ for all $k \geq 0$. Then the following statements hold:*

(a) For any $x_k \in X_T$ for $k \geq 0$, $T_{[x_k]_G}$ has a fixed point, where $T_{[x_k]_G} = \{y \in X :$ there is path between x_k and $y\}$.

(b) If $x_k \in X$ with $(x_k, z) \in E(G)$, where $z \in F(T)$, then $\{T^k x_k\}$ converges to z for $k \geq 1$.

(c) If G is weakly connected, then T has a fixed point in G.

(d) If $F(T) \subseteq E(G)$, then T has a fixed point.

(e) $F(T) \neq \emptyset$ if and only if $X_T \neq \emptyset$.

Proof:

(a) Let $x_k \in X_T$ for $k \geq 0$. From Lemma 4.1, $(x_k, Tx_k) \in E(G)$ for $k \geq 0$. Now for $x_k \in X_T$ there is $z_k \in F(T)$ such that $z_k = Tx_k$. From Theorem 4.1, $\{x_k\}$ is a Cauchy sequence in X, hence there exists $z \in X$ such that $x_k \to z$. Now

$$\begin{aligned} ||z_k - z|| &= ||Tx_k - x_k|| + ||x_k - z|| \\ &\to \ 0 \text{ as } k \to \infty. \end{aligned}$$

This implies that $z_k \to z$. Since from Lemma 4.2, $(x_k, z) \in E(G)$ for all $k \geq 0$, $(x_0, x_1, ..., x_k, z)$ is a path in G, so $z \in T_{[x_k]_G}$.

(b) By part (a), $Tx_k \to z \in F(T)$. Since T is $G-$ nonexpansive, it is $G-$ continuous, therefore $T(Tx_k) \to Tz \Rightarrow T^2 x_k \to z$. Continuing this process, $\{T^k x_k\}$ converges to $z \in F(T)$ for $k \geq 1$.

(c) Since from Lemma 4.1, $(x_k, Tx_k) \in E(G)$ for $k \geq 1$, with assumption that $(x_0, Tx_0) \in E(G)$, therefore $X_T \neq \emptyset$. Also G is weakly connected, $[x_k]_G = X$. By part (a), T has a fixed point in G.

(d) Let $F(T) \subseteq E(G)$. This implies that for any $z \in F(T)$, we have $(x_k, z) \in E(G)$ (see Lemma 4.2), so $X_T = X$, so by (c), T has a fixed point.

(e) Let $X_T \neq \emptyset$. Let $x_k \in X$ with $(x_k, Tx_k) \in E(G)$. From Lemma 4.1, $(x_k, x_{k+1}) \in E(G)$ and from Lemma 2.1, $x_k \to z$ for $z \in F(T) \Rightarrow F(T) \neq \emptyset$. Conversely, suppose that $F(T) \neq \emptyset$. Let $z \in F(T)$, them by Lemma 4.2, $(x_k, z) \in E(G) \Rightarrow X_T \neq \emptyset$.

□

Example 4.1 *Let $X = \mathbb{R}$ be a Hilbert space. Let $G = (V(G), E(G))$ be a graph in X such that $V(G) = X$ and for $x, y \in V(G)$, $(x, y) \in E(G)$ such that $||x - y|| \leq 1$. Let $T : X \to X$ is mapping defined by*

$$Tx = \begin{cases} \frac{1}{2} + \frac{x}{2}, & x \in [0,1], \\ 1 + \frac{x}{2}, & x \in [2, \infty). \end{cases}$$

Clearly T is edge-preserving and nonexpansive mapping, i.e., it is $G-$ nonexpansive mapping. Next to prove that T satisfies (3.1), for this, let $x \in [0, 1)$ and $y \in (1, \infty)$. Then

$$\begin{aligned} \frac{||x - Ty|| + ||y - Tx||}{||x - Tx|| + ||y - Ty|| + 1} &= \frac{||2x - 2 - y|| + ||2y - 1 - x||}{||x - 1|| + ||y - 1|| + 1} \\ &\geq 1. \end{aligned}$$

Clearly

$$||Tx - Ty|| \leq \left(\frac{||x - Ty|| + ||y - Tx||}{||x - Tx|| + ||y - Ty|| + 1} + k \right) ||x - y||$$

Now choose $x_0 = \frac{1}{4}$ and $k = \frac{2}{5}$,

$$\frac{||x_0 - Tx_0|| + ||Tx_0 - T^2x_0||}{||x_0 - Tx_0|| + ||Tx_0 - T^2x_0|| + 1} + k < 1.$$

Since all the conditions of the Theorem 4.1 are satisfied, hence T has at-least one fixed point in X.

Example 4.2 *Let $X = \mathbb{R}$ be a Hilbert space and $T : X \to X$ defined by*

$$Tx = x.$$

Let $G = (V(G), E(G))$ be a graph such that $V(G) = X$ and for $x, y \in V(G)$, $(x, y) \in E(G)$ such that $||x - y|| \leq 1$. Clearly T is $G-$ nonexpansive mapping. Also for $x, y \in X$, with $x \neq y$,

$$\begin{aligned} ||Tx - Ty|| &= ||x - y|| \\ &\leq \left(2||x - y|| + \frac{1}{2} \right) ||x - y|| + \frac{1}{2}||x - y||. \end{aligned}$$

And

$$\frac{||x_0 - Tx_0|| + ||Tx_0 - T^2x_0||}{||x_0 - Tx_0|| + ||Tx_0 - T^2x_0|| + 1} + k < 1.$$

Hence all the hypothesis of the Theorem 4.2 are satisfied for $k = \frac{1}{2}$ and $L = \frac{1}{2}$, therefore T has at-least one fixed point in X.

Now, with the help of following example, the fastness of iteration scheme (1.1) with some well-known iteration schemes are shown here:

Example 4.3 *Let $X = \mathbb{R}$ with norm $||x - y|| = |x - y|$. Define $T : X \to X$ by*

$$Tx = \frac{x}{7},$$

for any $x \in X$.

Clearly T is nonexpansive mapping and 0 is a fixed point of T.

5. Tables and Figures

Table 1: Strong convergence of (1.1), Picard, Mann [19], Ishikawa [12], Noor [20] for the mapping F in Example 4.3.

Iteration	New Picard	Picard	Mann	Ishikawa	Noor
0	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000
1	0.00145773	0.07142857	-0.35714286	-0.60204082	-0.54956268
2	0.00000425	0.01020408	0.10204082	0.33788005	0.29040302
3	0.00000001	0.00145773	-0.01457726	-0.12182070	-0.10094021
4	0.00000000	0.00020825	0.00104123	0.03200901	0.02610872
5	0.00000000	0.00002975	-0.00002975	-0.00655858	-0.00537154
6	0.00000000	0.00000425	0.00000000	0.00109310	0.00091657
7	0.00000000	0.00000061	-0.00000000	-0.00015251	-0.00013349
8	0.00000000	0.00000009	0.00000000	0.00001819	0.000001695
9	0.00000000	0.00000001	0.00000000	-0.00000188	-0.00000191
10	0.00000000	0.00000000	0.00000000	0.00000017	0.00000019
11	0.00000000	0.00000000	0.00000000	-0.00000001	-0.00000002
12	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000

1.

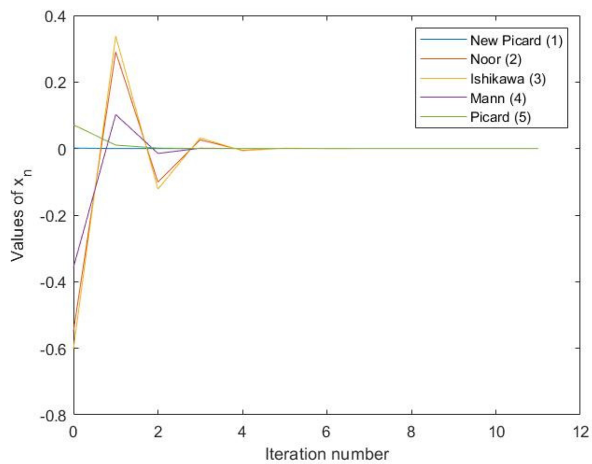

Figure 1: Behaviors of (1.1) (cyan), Picard iteration (green), Mann iteration (magenta), Ishikawa iteration (yellow), Noor iteration (carrot orange)

Table 2: Strong convergence of (1.1), $S-$ iteration [1], Picard Hybrid (PH) iteration [14], CR iteration [8], Thakur iteration [28] for the mapping F in Example 4.3.

Iteration	New Picard	CR	Picard Hybrid(PH)	$S-$ iteration	Thakur
0	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000
1	0.00145773	0.12390671	-0.5102041	-0.01020408	-0.02478134
2	0.00000425	0.00469617	0.00208247	-0.00020825	0.00046962
3	0.00000001	0.00005020	-0.00004250	-0.12182070	-0.00000502
4	0.00000000	0.00000017	0.00000043	-0.00000708	0.00000003
5	0.00000000	0.00000000	0.00000000	-0.00000029	0.00000000
6	0.00000000	0.00000000	0.00000000	-0.00000001	0.00000000
7	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
8	0.00000000	0.00000000	0.00000000	0.00000000	0.000001695
9	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
10	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
11	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
12	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000

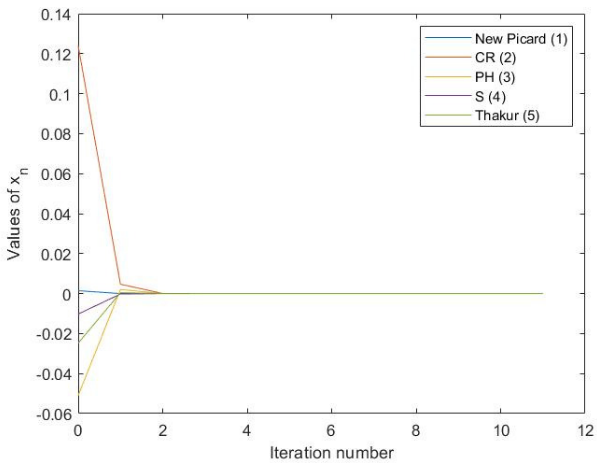

Figure 2: Behaviors of (1.1) (cyan), $CR-$ iteration (carrot orange), Picard Hybrid iteration (PH) (yellow), $S-$ iteration (magenta), Thakur iteration (green)

6. Application of fixed point theory in solving system of equations

Let $X = \mathbb{R}^n$ with the norm $||.||$ such that for $x, y \in X$,

$$||x - y|| = \max |x_j - y_j|. \tag{6.1}$$

Let $T : X \to X$ defined by

$$Tx = Cx + b, \tag{6.2}$$

where $C = [c_{jk}]$ be a $n \times n$ matrix, b is the fixed vector of X. Equation (6.2) can be written in component form as

$$Tx_j = \sum_{k=1}^{n} c_{jk} x_{jk} + \beta_j, \tag{6.3}$$

$b = (\beta_j)$, $j = 1, 2, ..., n$. Finding solution of system of equation (6.3) is equivalent to finding fixed points of T.

Theorem 6.1 *Let $X = \mathbb{R}^n$ with the norm $||.||$ such that for $x, y \in X$,*

$$||x - y|| = \max |x_j - y_j| \tag{6.4}$$

and $T : X \to X$ defined by (6.2) with the assumption that $|C| \leq 1$. Let $\{x_k\}$ be a sequence in X defined by (1.1) with initial point $x_0 \in X$. Then T has a fixed point, which is solution of the system of equation (6.3).

Proof: Since

$$\begin{aligned} ||Tx - Ty|| &= |C| ||x - y|| \\ &\leq ||x - y||. \end{aligned}$$

This implies that T is nonexpansive mapping. Now

$$\begin{aligned} \frac{||x - Ty|| + ||y - Tx||}{||x - Tx|| + ||y - Ty|| + 1} &= \frac{|1 - c| ||x + y|| + 2b}{|1 - c| ||x + y|| + 2b + 1} \\ &\leq \frac{2||x + y|| + 2b}{2||x + y|| + 2b + 1}. \end{aligned}$$

And

$$\left(\frac{||x - Ty|| + ||y - Tx||}{||x - Tx|| + ||y - Ty|| + 1} + k \right) ||x - y|| \leq \left(\left(\frac{2||x + y|| + 2b}{2||x + y|| + 2b + 1} \right) + k \right) ||x - y||.$$

Clearly

$$\begin{aligned} ||Tx - Ty|| &\leq ||x - y|| \\ &< \left(\left(\frac{2||x + y|| + 2b}{2||x + y|| + 2b + 1} \right) + k \right) ||x - y|| \\ &= \left(\frac{||x - Ty|| + ||y - Tx||}{||x - Tx|| + ||y - Ty|| + 1} + k \right) ||x - y||. \end{aligned}$$

Also

$$\begin{aligned} \frac{||x_0 - Tx_0|| + ||Tx_0 - T^2x_0||}{||x_0 - Tx_0|| + ||Tx_0 - T^2x_0|| + 1} &\leq \frac{4||x_0|| + 2b}{4||x_0|| + 2b + 1} \\ &< 1. \end{aligned}$$

For $x_0 = \frac{1}{2}$ and for any $b \in X$. Therefore $\frac{||x_0 - Tx_0|| + ||Tx_0 - T^2x_0||}{||x_0 - Tx_0|| + ||Tx_0 - T^2x_0|| + 1} + k < 1$. Hence from Theorem 3.3, T has existence of fixed point and the sequence $\{x_k\}$ generated by (1.1) converges to this fixed point and this fixed point will be solution of the Problem (6.3). □

7. Conclusion

Here some convergence results are obtained for a three-step Picard iteration scheme for nonexpansive mapping in Hilbert spaces endowed with a binary relation under some suitable conditions. After that the strong convergence of the same iteration scheme for $G-$ nonexpansive mapping is proved here. To justify main results, some examples and an application of fixed point theory is discussed here.

8. Conflict of Interest

The author declares that there is no conflict of interest.

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