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Unveiling Cesàro summability under neutrosophic 2-norms

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ABSTRACT: This paper introduces the concepts of Cesàro summability within the framework of neutrosophic 2-normed spaces (N2-NS). We establish that Cesàro summability does not necessarily imply ordinary convergence with regard to neutrosophic 2-norm, providing a concrete example to illustrate this distinction. In this connection, we prove necessary and sufficient condition for the sequences in N2-NS that their Cesàro summability guarantees ordinary convergence.

Key Words: Neutrosophic 2-normed linear space, t-norm, t-conorm $\mathscr{N}_2$-Cauchy sequence, Cesàro summability.

Contents

1. Introduction

Zadeh [15] is widely acknowledged as the pioneer who revolutionized classical set theory with the introduction of fuzzy set theory. This groundbreaking concept has become a cornerstone of modern mathematics, driving innovations and inspiring applications across a vast spectrum of scientific and engineering disciplines, from chaos control [2] to nonlinear dynamical systems [4], and beyond.

A significant milestone in the evolution of fuzzy set theory came with Atanassov's [1] introduction of intuitionistic fuzzy sets. By incorporating a non-membership function alongside the traditional membership function, intuitionistic fuzzy sets provide a more refined and comprehensive framework, extending the capabilities of fuzzy logic. This conceptual breakthrough has led to the emergence of transformative ideas in mathematical analysis, further broadening the scope of fuzzy set theory.

Expanding upon these foundations, Smarandache [11] introduced neutrosophic sets, a profound generalization of intuitionistic fuzzy sets. By integrating an indeterminacy function, neutrosophic sets represent each element through a triplet: truth-membership, indeterminacy-membership, and falsity-membership functions. This innovative framework offers an unprecedented level of granularity, allowing for a precise and nuanced characterization of elements based on their degrees of truth, uncertainty, and falsehood.

Building on these advancements, Kirişci and Şimşek [6] introduced the concept of neutrosophic normed linear spaces, laying the groundwork for the study of statistical convergence within this novel setting. Their work has ignited a surge of research into various forms of sequence convergence in neutrosophic spaces. For an in-depth exploration of this rapidly evolving field, see [7,8].

Parallel to these developments, the concept of 2-normed linear spaces was pioneered by Gähler [3], sparking widespread interest and extensive research in this area. Over time, this foundational idea has been refined and expanded by numerous mathematicians, shaping a dynamic and flourishing field of study.

Continuing this trajectory of innovation, Murtaza et al. [9] introduced the concept of neutrosophic 2-normed linear spaces in 2023-an advanced extension of neutrosophic normed spaces. Their work delves into statistical convergence and completeness within this enriched framework, further pushing the boundaries of mathematical analysis.

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Research on sequence convergence in neutrosophic 2-normed spaces is still in its early stages, with only modest progress made so far. However, preliminary studies reveal intriguing parallels in the behavior of sequence convergence within these spaces. Building on these insights, we introduce and explore the concept of Cesàro summability in neutrosophic 2-normed spaces (N2-NS). We establish that Cesàro summability in an N2-NS does not necessarily imply ordinary convergence, providing a concrete example to illustrate this distinction. Furthermore, we derive necessary and sufficient conditions under which the Cesàro summability of sequences in N2-NS ensures their ordinary convergence, thereby enriching the theoretical foundations of sequence analysis in this emerging framework.

2. Preliminaries

In this section, we present an overview of key definitions and terminology essential for describing our main results. Throughout the study, $\mathbb{N}$ will denote the set of all natural numbers.

Definition 2.1 *[10] A binary operation $\boxdot : \mathscr{J} \times \mathscr{J} \to \mathscr{J}$, where $\mathscr{J} = [0,1]$ is named to be a continuous t-norm if for each $\nu_1, \nu_2, \nu_3, \nu_4 \in \mathscr{J}$, the below conditions hold:*

1. *$\boxdot$ is associative and commutative;*
2. *$\boxdot$ is continuous;*
3. *$\nu_1 \boxdot 1 = \nu_1$ for all $\nu_1 \in \mathscr{J}$;*
4. *$\nu_1 \boxdot \nu_2 \le \nu_3 \boxdot \nu_4$ whenever $\nu_1 \le \nu_3$ and $\nu_2 \le \nu_4$.*

Definition 2.2 *[10] A binary operation $\oplus : \mathscr{J} \times \mathscr{J} \to \mathscr{J}$, where $\mathscr{J} = [0,1]$ is named to be a continuous t-conorm if for each $\nu_1, \nu_2, \nu_3, \nu_4 \in \mathscr{J}$, the below conditions hold:*

1. *$\oplus$ is associative and commutative;*
2. *$\oplus$ is continuous;*
3. *$\nu_1 \oplus 0 = \nu_1$ for all $\nu_1 \in \mathscr{J}$;*
4. *$\nu_1 \oplus \nu_2 \le \nu_3 \oplus \nu_4$ whenever $\nu_1 \le \nu_3$ and $\nu_2 \le \nu_4$.*

Example 2.1 [5] Some illustrations of $\boxdot$ and $\oplus$ are: $\tau_1 \boxdot \tau_2 = \min\{\tau_1, \tau_2\}$ and $\tau_1 \boxdot \tau_2 = \tau_1.\tau_2$. $\tau_1 \oplus \tau_2 = \max\{\tau_1, \tau_2\}$ and $\tau_1 \oplus \tau_2 = \tau_1 + \tau_2 - \tau_1.\tau_2$.

Lemma 2.1 *[12] If $\boxdot$ is a continuous t-norm, $\oplus$ is a continuous t-conorm, $\tau_i \in (0,1)$ and $1 \le i \le 7$, then the following statements hold:*

1. *If $\tau_1 > \tau_2$, there are $\tau_3, \tau_4 \in (0,1)$ such that $\tau_1 \boxdot \tau_3 \ge \tau_2$ and $\tau_1 \ge \tau_2 \oplus \tau_4$*
2. *If $\tau_5 \in (0,1)$, there are $\tau_6, \tau_7 \in (0,1)$ such that $\tau_6 \boxdot \tau_6 \ge \tau_5$ and $\tau_5 \ge \tau_7 \oplus \tau_7$.*

Definition 2.3 *[3] Let $\mathscr{Y}$ be a real vector space of dimension d, where $2 \le d < \infty$. A 2-norm on $\mathscr{Y}$ is a function $\|.,.\| : \mathscr{Y} \times \mathscr{Y} \to \mathbb{R}$ which satisfies the following conditions:*

1. *$\|\tau_1, \tau_2\| = 0$ if and only if τ_1 and τ_2 are linearly dependent in $\mathscr{Y}$;*
2. *$\|\tau_1, \tau_2\| = \|\tau_2, \tau_1\|$ for all τ_1, τ_2 in $\mathscr{Y}$;*
3. *$\|\kappa\tau_1, \tau_2\| = |\kappa| \, \|\tau_1, \tau_2\|$ for all κ in $\mathbb{R}$ and for all τ_1, τ_2 in $\mathscr{Y}$;*
4. *$\|\tau_1 + \tau_2, \tau_3\| \le \|\tau_1, \tau_3\| + \|\tau_2, \tau_3\|$ for all τ_1, τ_2, τ_3 in $\mathscr{Y}$.*

Example 2.2 [13] Let $\mathscr{Y} = \mathbb{R}^2$. Define $\|\cdot,\cdot\|$ on $\mathbb{R}^2$ by $\|u, v\| = |\tau_1\tau_4 - \tau_2\tau_3|$, where $u = (\tau_1, \tau_2), v = (\tau_3, \tau_4) \in \mathbb{R}^2$. Then, $(\mathscr{Y}, \|\cdot,\cdot\|)$ is a 2-normed space.

Definition 2.4 *[9] Let $\mathscr{W}$ be a vector space, $\mathscr{N}_2 = (\{(w,v), \Im(w,v), \Re(w,v), \wp(w,v)\} : (w,v) \in \mathscr{W} \times \mathscr{W})$ be a 2-norm space such that $\mathscr{N}_2 : \mathscr{W} \times \mathscr{W} \times \mathbb{R}^+ \to [0,1]$. If $\boxdot$ and $\oplus$ stand for continuous t-norm and t-conorm respectively, then four-tuple $(\mathscr{W}, \mathscr{N}_2, \boxdot, \oplus)$ is named to be neutrosophic 2-normed space (in short N2-NS) if for every $w, v, u \in \mathscr{W}$, $\zeta, \varepsilon > 0$ and $\kappa \neq 0$, the following conditions are gratified:*

1. $0 \leq \Im(w,v;\zeta) \leq 1$, $0 \leq \Re(w,v;\zeta) \leq 1$ *and* $0 \leq \wp(w,v;\zeta) \leq 1$;
2. $\Im(w,v;\zeta) + \Re(w,v;\zeta) + \wp(w,v;\zeta) \leq 3$;
3. $\Im(w,v;\zeta) = 1$ *iff* w, v *are linearly dependent;*
4. $\Im(\kappa w, v;\zeta) = \Im\left(w, v; \frac{\zeta}{|\kappa|}\right)$ *for each* $\kappa \neq 0$;
5. $\Im(w, v+u; \zeta+\varepsilon) \geq \Im(w,v;\zeta) \boxdot \Im(w,u;\varepsilon)$;
6. $\Im(w,v;\cdot) : (0,\infty) \to [0,1]$ *is a non decreasing continuous function;*
7. $\lim_{\zeta\to\infty} \Im(w,v;\zeta) = 1$;
8. $\Im(w,v;\zeta) = \Im(v,w;\zeta)$;
9. $\Re(w,v;\zeta) = 0$ *iff* w, v *are linearly dependent;*
10. $\Re(\kappa w, v;\zeta) = \Re\left(w, v; \frac{\zeta}{|\kappa|}\right)$ *for each* $\kappa \neq 0$;
11. $\Re(w, v+u; \zeta+\varepsilon) \leq \Re(w,v;\zeta) \oplus \Re(w,u;\varepsilon)$;
12. $\Re(w,v;\cdot) : (0,\infty) \to [0,1]$ *is a non increasing continuous function;*
13. $\lim_{\zeta\to\infty} \Re(w,v;\zeta) = 0$;
14. $\Re(w,v;\zeta) = \Re(v,w;\zeta)$;
15. $\wp(w,v;\zeta) = 0$ *iff* w, v *are linearly dependent;*
16. $\wp(\kappa w, v;\zeta) = \wp\left(w, v; \frac{\zeta}{|\kappa|}\right)$ *for each* $\kappa \neq 0$;
17. $\wp(w, v+u; \zeta+\varepsilon) \leq \wp(w,v;\zeta) \oplus \wp(w,u;\varepsilon)$;
18. $\wp(w,v;\cdot) : (0,\infty) \to [0,1]$ *is a non increasing continuous function;*
19. $\lim_{\zeta\to\infty} \wp(w,v;\zeta) = 0$;
20. $\wp(w,v;\zeta) = \wp(v,w;\zeta)$;
21. *If* $\zeta \leq 0$, $\Im(w,v;\zeta) = 0$, $\Re(w,v;\zeta) = 1$ *and* $\wp(w,v;\zeta) = 1$.

In the sequel, we denote $\mathscr{H}$ for neutrosophic 2-normed space instead of $(\mathscr{W}, \Im, \Re, \wp, \boxdot, \oplus)$. And, we denote $\mathscr{N}_2$ to mean neutrosophic 2-norm on $\mathscr{H}$. Throughout our discussions we use continuous t-norms $\nu_1 \boxdot \nu_2 = \min\{\nu_1, \nu_2\}$ and continuous t-conorms $\nu_1 \oplus \nu_2 = \max\{\nu_1, \nu_2\}$.

Definition 2.5 *[9] Let $\{w_k\}$ be a sequence in a N2-NS $\mathscr{H}$. Choose $\sigma \in (0,1)$ and $\zeta > 0$. Then, $\{w_k\}$ is named to be convergent in relation to $\mathscr{N}_2$ (in short $\mathscr{N}_2$ -convergent) if there exists a $k_0 \in \mathbb{N}$ and $\varrho \in \mathscr{W}$ such that $\Im(w_k - \varrho, v;\zeta) > 1-\sigma$, $\Re(w_k - \varrho, v;\zeta) < \sigma$ and $\wp(w_k - \varrho, v;\zeta) < \sigma$ for all $k \geq k_0$ and $v \in \mathscr{W}$. In this case, we write $\mathscr{N}_2 - \lim w_k = \varrho$ or $w_k \xrightarrow{\mathscr{N}_2} \varrho$ and ϱ is called $\mathscr{N}_2$ -limit of $\{w_k\}$.*

Definition 2.6 *[9] Let $\{w_k\}$ be a sequence in a N2-NS $\mathscr{H}$. Choose $\varepsilon \in (0,1)$ and $\eta > 0$. Then, $\{w_k\}$ is named to be Cauchy if there exists $m_0 \in \mathbb{N}$ such that $\Im(w_k - w_m, z;\zeta) > 1-\sigma$, $\Re(w_k - w_m, z;\zeta) < \sigma$ and $\wp(w_k - w_m, z;\zeta) < \sigma$ for all $k, m \geq m_0$ and $z \in \mathscr{H}$.*

Lemma 2.2 *[14] For any $\Psi(>0) \in \mathbb{R}$, let $\{\Psi\} = \Psi - [\Psi]$ and $\Psi_\upsilon = [\upsilon\Psi]$, $\upsilon \in \mathbb{N}$, where $[\cdot]$ stands for greatest integer function. Then, the following statements hold true:*

1. *For each $\upsilon \in \mathbb{N} \setminus \{0\}$, $\upsilon \geq \frac{1}{\{\Psi\}} \implies \Psi_\upsilon > \upsilon$ whenever $\Psi > 1$.*

2. *For each $\upsilon \in \mathbb{N} \setminus \{0\}$, $\Psi_\upsilon < \upsilon$ whenever $0 < \Psi < 1$.*

3. *For each $\upsilon \in \mathbb{N} \setminus \{0\}$, $\upsilon \geq \frac{(3\Psi-1)}{\Psi(\Psi-1)} \implies \frac{\Psi}{\Psi-1} < \frac{\Psi_\upsilon+1}{\Psi_\upsilon-\upsilon} < \frac{2\Psi}{\Psi-1}$, whenever $\Psi > 1$.*

4. *For each $\upsilon \in \mathbb{N} \setminus \{0\}$, $\upsilon > \frac{1}{\Psi} \implies 0 < \frac{\Psi_\upsilon+1}{\upsilon-\Psi_\upsilon} < \frac{2\Psi}{1-\Psi}$ whenever $0 < \Psi < 1$.*

3. Main Results

Throughout this section, we elegantly denote $\boxdot$ by min and $\oplus$ by max for clarity and consistency.

Definition 3.1 *A sequence $\{\ell_n\}$ in $\mathscr{H}$ is referred to as Cesàro summable to $\varrho \in \mathscr{H}$ if the sequence $\{\gamma_n\}$ of arithmetic means of $\{\ell_n\}$ is $\mathscr{N}_2$-convergent to ϱ, i.e., $\gamma_n = \frac{1}{n+1}\sum_{k=0}^{n}(\ell_k) \xrightarrow{\mathscr{N}_2} \varrho$.*

Theorem 3.1 *Consider a sequence $\{\ell_n\}$ in $\mathscr{H}$ such that $\ell_n \xrightarrow{\mathscr{N}_2} \varrho$. Then, $\{\ell_n\}$ is Cesàro summable to $\varrho \in \mathscr{H}$.*

Proof: Let $\ell_n \xrightarrow{\mathscr{N}_2} \varrho$. Then, for a specified $\zeta > 0$ and every $\sigma \in (0,1)$, $\upsilon \in \mathscr{W}$ there exists a $t_0 \in \mathbb{N}$ such that $\Im\left(\ell_n - \varrho, \upsilon; \frac{\zeta}{2}\right) > 1 - \sigma$, $\Re\left(\ell_n - \varrho, \upsilon; \frac{\zeta}{2}\right) < \sigma$ and $\wp\left(\ell_n - \varrho, \upsilon; \frac{\zeta}{2}\right) < \sigma$ for all $n \geq t_0$. So, in alignment with Definition 2.4, we arrive at

$$\lim_{n\to\infty} \Im\left(\sum_{k=0}^{t_0}(\ell_k - \varrho), \upsilon; \frac{(n+1)\zeta}{2}\right) = 1,$$
$$\lim_{n\to\infty} \Re\left(\sum_{k=0}^{t_0}(\ell_k - \varrho), \upsilon; \frac{(n+1)\zeta}{2}\right) = 0$$
$$\text{and} \lim_{n\to\infty} \wp\left(\sum_{k=0}^{t_0}(\ell_k - \varrho), \upsilon; \frac{(n+1)\zeta}{2}\right) = 0.$$

Therefore, there can be found $t_1 \in \mathbb{N}$ for which

$$\Im\left(\sum_{k=0}^{t_0}(\ell_k - \varrho), \upsilon; \frac{(n+1)\zeta}{2}\right) > 1 - \sigma,$$
$$\Re\left(\sum_{k=0}^{t_0}(\ell_k - \varrho), \upsilon; \frac{(n+1)\zeta}{2}\right) < \sigma$$
$$\text{and } \wp\left(\sum_{k=0}^{t_0}(\ell_k - \varrho), \upsilon; \frac{(n+1)\zeta}{2}\right) < \sigma$$

for all $n \geq t_1$. Let $t = \max\{t_0, t_1\}$. Then, for all $n \geq t$, we conclude that

$$\begin{aligned}
&\Im\left(\frac{1}{n+1}\sum_{k=0}^{n}\ell_k - \varrho, v; \zeta\right)\\
&= \Im\left(\sum_{k=0}^{n}(\ell_k - \varrho), v; (n+1)\zeta\right)\\
&\geq \min\left\{\Im\left(\sum_{k=0}^{t_0}(\ell_k - \varrho), v; \frac{(n+1)\zeta}{2}\right), \Im\left(\sum_{k=t_0+1}^{n}(\ell_k - \varrho), v; \frac{(n+1)\zeta}{2}\right)\right\}\\
&\geq \min\left\{\Im\left(\sum_{k=0}^{t_0}(\ell_k - \varrho), v; \frac{(n+1)\zeta}{2}\right), \Im\left(\sum_{k=t_0+1}^{n}(\ell_k - \varrho), v; \frac{(n-t_0)\zeta}{2}\right)\right\}\\
&\geq \min\left\{\Im\left(\sum_{k=0}^{t_0}(\ell_k - \varrho), v; \frac{(n+1)\zeta}{2}\right), \Im\left(\ell_{t_0+1} - \varrho, v; \frac{\zeta}{2}\right), \ldots, \Im\left(\ell_n - \varrho, v; \frac{\zeta}{2}\right)\right\}\\
&> 1 - \sigma,
\end{aligned}$$

$$\begin{aligned}
&\Re\left(\frac{1}{n+1}\sum_{k=0}^{n}\ell_k - \varrho, v; \zeta\right)\\
&= \Re\left(\sum_{k=0}^{n}(\ell_k - \varrho), v; (n+1)\zeta\right)\\
&\leq \max\left\{\Re\left(\sum_{k=0}^{t_0}(\ell_k - \varrho), v; \frac{(n+1)\zeta}{2}\right), \Re\left(\sum_{k=t_0+1}^{n}(\ell_k - \varrho), v; \frac{(n+1)\zeta}{2}\right)\right\}\\
&\leq \max\left\{\Re\left(\sum_{k=0}^{t_0}(\ell_k - \varrho), v; \frac{(n+1)\zeta}{2}\right), \Re\left(\sum_{k=t_0+1}^{n}(\ell_k - \varrho), v; \frac{(n-t_0)\zeta}{2}\right)\right\}\\
&\leq \max\left\{\Re\left(\sum_{k=0}^{t_0}(\ell_k - \varrho), v; \frac{(n+1)\zeta}{2}\right), \Re\left(\ell_{t_0+1} - \varrho, v; \frac{\zeta}{2}\right), \ldots, \Re\left(\ell_n - \varrho, v; \frac{\zeta}{2}\right)\right\}\\
&< \sigma.
\end{aligned}$$

Likewise, we obtain

$$\begin{aligned}
&\wp\left(\frac{1}{n+1}\sum_{k=0}^{n}\ell_k - \varrho, v; \zeta\right)\\
&\leq \max\left\{\wp\left(\sum_{k=0}^{t_0}(\ell_k - \varrho), v; \frac{(n+1)\zeta}{2}\right), \wp\left(\ell_{t_0+1} - \varrho, v; \frac{\zeta}{2}\right), \ldots, \wp\left(\ell_n - \varrho, v; \frac{\zeta}{2}\right)\right\}\\
&< \sigma.
\end{aligned}$$

Hence, $\{\ell_n\}$ is Cesàro summable to $\varrho \in \mathscr{H}$. It ends the proof. □

The converse of Theorem 3.1 does not hold, as demonstrated by the following example.

Example 3.1 Let $\mathscr{W} = \mathbb{R}^2$ and $(\mathbb{R}^2, \|, \cdot, \|)$ be a 2-normed space with 2-norm defined as in Example 2.2. We take t-norm and t-conorm as $\nu_1 \boxdot \nu_2 = \nu_1\nu_2$ and $\nu_1 \oplus \nu_2 = \min\{\nu_1 + \nu_2, 1\}$ for $\nu_1, \nu_2 \in [0, 1]$. Now choose $\sigma \in (0, 1)$ and $w, v \in \mathscr{W}, \zeta > 0$ with $\zeta > \|w, v\|$. Now we consider $\Im(w, v; \zeta) = \frac{\zeta}{\zeta + \|w,v\|}$, $\Re(w, v; \zeta) = \frac{\|w,v\|}{\zeta + \|w,v\|}$, $\wp(w, v; \zeta) = \frac{\|w,v\|}{\zeta}$. Then $\mathscr{N}_2 = (\Im, \Re, \wp)$ is a neutrosophic 2-norm on $\mathscr{W}$ and the four tuple

$(\mathscr{W}, \mathscr{N}_2, \boxdot, \ominus)$ becomes a neutrosophic 2-normed space. Now we define a sequence $\{\ell_n\} \in \mathscr{W}$ by $\ell_n = \big((-1)^{n+1}, 0\big)$. Let $\Theta = (0,0) \in \mathbb{R}^2$. Then for every nonzero $z = (z_1, z_2) \in \mathscr{W}$ we have

$$\begin{aligned}\lim_{n\to\infty} \Im(\gamma_{2n}, z; \zeta) &= \lim_{n\to\infty} \frac{\zeta}{\zeta + \|\gamma_{2n}, z\|} \\ &= \lim_{n\to\infty} \frac{\zeta}{\zeta + z_2 \cdot \frac{-1}{2n+1}} \\ &= \frac{\zeta}{\zeta + z_2 \times 0} \\ &= 1,\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} \Re(\gamma_{2n}, z; \zeta) &= \lim_{n\to\infty} \frac{\|\gamma_{2n}, z\|}{\zeta + \|\gamma_{2n}, z\|} \\ &= \lim_{n\to\infty} \frac{z_2 \cdot \frac{-1}{2n+1}}{\zeta + z_2 \cdot \frac{-1}{2n+1}} \\ &= \frac{z_2 \times 0}{\zeta + z_2 \times 0} \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} \wp(\gamma_{2n}, z; \zeta) &= \lim_{n\to\infty} \frac{\|\gamma_{2n}, z\|}{\zeta} \\ &= \lim_{n\to\infty} \frac{z_2 \cdot \frac{-1}{2n+1}}{\zeta} \\ &= \frac{z_2 \times 0}{\zeta} \\ &= 0.\end{aligned}$$

Therefore $\gamma_{2n} \xrightarrow{\mathscr{N}_2} \Theta$. In the similar way, we can arrive at $\gamma_{2n+1} \xrightarrow{\mathscr{N}_2} \Theta$. Hence $\{\ell_n\}$ is Cesàro summable to Θ. Again, we observe

$$\begin{aligned}\lim_{n\to\infty} \Im(\ell_{2n}, z; \zeta) &= \lim_{n\to\infty} \frac{\zeta}{\zeta + \|\ell_{2n} - (-1,0), z\|} \\ &= \lim_{n\to\infty} \frac{\zeta}{\zeta + \|0, z\|} \;\; (\ell_{2n} = (-1,0)) \\ &= \frac{\zeta}{\zeta + z_1 \cdot 0 - 0 \cdot z_2} \\ &= 1,\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} \Re(\ell_{2n}, z; \zeta) &= \lim_{n\to\infty} \frac{\|\ell_{2n} - (-1,0), z\|}{\zeta + \|\ell_{2n} - (-1,0), z\|} \\ &= \lim_{n\to\infty} \frac{\|0, z\|}{\zeta + \|0, z\|} \;\; (\ell_{2n} = (-1,0)) \\ &= \frac{z_1 \cdot 0 - 0 \cdot z_2}{\zeta + z_1 \cdot 0 - 0 \cdot z_2} \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} \wp(\ell_{2n}, z; \zeta) &= \lim_{n\to\infty} \frac{\|\ell_{2n} - (-1,0), z\|}{\zeta} \\ &= \lim_{n\to\infty} \frac{\|0, z\|}{\zeta} \; (\ell_{2n} = (-1,0)) \\ &= \frac{z_1 \cdot 0 - 0 \cdot z_2}{\zeta} \\ &= 0.\end{aligned}$$

Therefore $\ell_{2n} \xrightarrow{\mathscr{N}_2} (-1,0)$. In the similar way, we can show that $\ell_{2n+1} \xrightarrow{\mathscr{N}_2} (1,0)$. This yields that $\{\ell_n\}$ is not convergent with regard to $\mathscr{N}_2$.

It is natural to arise the question that under which condition the converse of the Theorem 3.1 holds good. The answer is shown by the ensuing theorem.

Theorem 3.2 *Consider $\{\ell_n\}$ to be a sequence in $\mathscr{H}$ such that it is Cesàro summable to $\varrho \in \mathscr{W}$. Then $\ell_n \xrightarrow{\mathscr{N}_2} \varrho$ if and only if for every $\zeta > 0$ and nonzero $z \in \mathscr{W}$ the conditions listed below are met:*

$$\sup_{\Psi>1}\left[\liminf_{n\to\infty} \Im\left(\frac{1}{\Psi_n - n}\sum_{k=n+1}^{\Psi_n}(\ell_k - \ell_n), z; \zeta\right)\right] = 1 \tag{3.1}$$

$$\inf_{\Psi>1}\left[\limsup_{n\to\infty} \Re\left(\frac{1}{\Psi_n - n}\sum_{k=n+1}^{\Psi_n}(\ell_k - \ell_n), z; \zeta\right)\right] = 0 \tag{3.2}$$

$$\inf_{\Psi>1}\left[\limsup_{n\to\infty} \wp\left(\frac{1}{\Psi_n - n}\sum_{k=n+1}^{\Psi_n}(\ell_k - \ell_n), z; \zeta\right)\right] = 0 \tag{3.3}$$

Proof: Consider $\{\ell_n\}$ to be a sequence in $\mathscr{H}$ such that it is Cesàro summable to $\varrho \in \mathscr{W}$ and suppose that $\ell_n \xrightarrow{\mathscr{N}_2} \varrho$. Our analysis reveals that Equation (3.1), (3.2) and (3.3) are met. By Lemma 2.2, we observe

$$\ell_n - \gamma_n = \frac{\Psi_n + 1}{\Psi_n - n}(\gamma_{\Psi_n} - \gamma_n) - \frac{1}{\Psi_n - n}\sum_{k=n+1}^{\Psi_n}(\ell_k - \ell_n). \tag{3.4}$$

Again, by Lemma 2.2, for $n \geq \frac{3\Psi-1}{\Psi(\Psi-1)}$, we find, for $\zeta > 0$ and nonzero $z \in \mathscr{W}$,

$$\Im\left(\frac{\Psi_n+1}{\Psi_n-n}(\gamma_{\Psi_n}-\gamma_n), z; \zeta\right) = \Im\left((\gamma_{\Psi_n}-\gamma_n), z; \frac{\zeta}{\frac{\Psi_n+1}{\Psi_n-n}}\right) \geq \Im\left((\gamma_{\Psi_n}-\gamma_n), z; \frac{\zeta}{\frac{2\Psi}{\Psi_n-1}}\right) \tag{3.5}$$

$$\Re\left(\frac{\Psi_n+1}{\Psi_n-n}(\gamma_{\Psi_n}-\gamma_n), z; \zeta\right) = \Re\left((\gamma_{\Psi_n}-\gamma_n), z; \frac{\zeta}{\frac{\Psi_n+1}{\Psi_n-n}}\right) \leq \Re\left((\gamma_{\Psi_n}-\gamma_n), z; \frac{\zeta}{\frac{2\Psi}{\Psi_n-1}}\right) \tag{3.6}$$

$$\wp\left(\frac{\Psi_n+1}{\Psi_n-n}(\gamma_{\Psi_n}-\gamma_n), z; \zeta\right) = \wp\left((\gamma_{\Psi_n}-\gamma_n), z; \frac{\zeta}{\frac{\Psi_n+1}{\Psi_n-n}}\right) \leq \wp\left((\gamma_{\Psi_n}-\gamma_n), z; \frac{\zeta}{\frac{2\Psi}{\Psi_n-1}}\right). \tag{3.7}$$

$\{\gamma_n\}$ being Cauchy with regard to $\mathscr{N}_2$, we arrive at

$$\lim_{n\to\infty} \Im\left(\frac{\Psi_n+1}{\Psi_n-n}(\gamma_{\Psi_n}-\gamma_n), z; \zeta\right) = 1,$$

$$\lim_{n\to\infty} \Re\left(\frac{\Psi_n+1}{\Psi_n-n}(\gamma_{\Psi_n}-\gamma_n), z; \zeta\right) = 0,$$

$$\lim_{n\to\infty} \wp\left(\frac{\Psi_n+1}{\Psi_n-n}(\gamma_{\Psi_n}-\gamma_n), z; \zeta\right) = 0.$$

Hence, $\lim_{n\to\infty} \frac{\Psi_n+1}{\Psi_n-n}(\gamma_{\Psi_n}-\gamma_n)=0$. By the Equation 3.4, for nonzero $z\in\mathscr{W}$, We can have

$$\lim_{n\to\infty}\Im\left(\frac{1}{\Psi_n-n}\sum_{k=n+1}^{\Psi_n}(\ell_k-\ell_n),z;\zeta\right)=1,$$
$$\lim_{n\to\infty}\Re\left(\frac{1}{\Psi_n-n}\sum_{k=n+1}^{\Psi_n}(\ell_k-\ell_n),z;\zeta\right)=0,$$
$$\lim_{n\to\infty}\wp\left(\frac{1}{\Psi_n-n}\sum_{k=n+1}^{\Psi_n}(\ell_k-\ell_n),z;\zeta\right)=0.$$

This yields that the Equation (3.1), (3.2 and 3.3 are satisfied.

Conversely, let the equations (3.1), (3.2) and (3.3) holds good. Let $\zeta>0$. Then for specified $\sigma\in(0,1)$ there can be found $\Psi>1$ and $i_0\in\mathbb{N}$ for which

$$\lim_{n\to\infty}\Im\left(\frac{1}{\Psi_n-n}\sum_{k=n+1}^{\Psi_n}(\ell_k-\ell_n),z;\frac{\zeta}{3}\right)>1-\sigma,$$
$$\lim_{n\to\infty}\Re\left(\frac{1}{\Psi_n-n}\sum_{k=n+1}^{\Psi_n}(\ell_k-\ell_n),z;\frac{\zeta}{3}\right)<\sigma,$$
$$\lim_{n\to\infty}\wp\left(\frac{1}{\Psi_n-n}\sum_{k=n+1}^{\Psi_n}(\ell_k-\ell_n),z;\frac{\zeta}{3}\right)<\sigma,$$

$\forall n\geq i_0$ and for every nonzero $z\in\mathscr{W}$. Again, we can have $i_1\in\mathbb{N}$ such that for all $n\geq i_1$,

$$\begin{cases}\Im\left(\gamma_n-\varrho,z;\frac{\zeta}{3}\right)>1-\sigma\\ \Re\left(\gamma_n-\varrho,z;\frac{\zeta}{3}\right)<\sigma\\ \wp\left(\gamma_n-\varrho,z;\frac{\zeta}{3}\right)<\sigma.\end{cases}$$

As, $\lim_{n\to\infty}\frac{\Psi_n+1}{\Psi_n-n}(\gamma_{\Psi_n}-\gamma_n)=0$ there exists $i_2\in\mathbb{N}$ in a way that

$$\lim_{n\to\infty}\Im\left(\frac{\Psi_n+1}{\Psi_n-n}(\gamma_{\Psi_n}-\gamma_n),z;\frac{\zeta}{3}\right)>1-\sigma,$$
$$\lim_{n\to\infty}\Re\left(\frac{\Psi_n+1}{\Psi_n-n}(\gamma_{\Psi_n}-\gamma_n),z;\frac{\zeta}{3}\right)<\sigma,$$
$$\lim_{n\to\infty}\wp\left(\frac{\Psi_n+1}{\Psi_n-n}(\gamma_{\Psi_n}-\gamma_n),z;\frac{\zeta}{3}\right)<\sigma$$

are satisfied. Now, we observe for $n>\max\{i_0,i_1,i_2\}$

$$\begin{aligned}&\Im(\ell_n-\varrho,z;\zeta)\\ &\geq\min\left\{\Im\left(\frac{1}{\Psi_n-n}\sum_{k=n+1}^{\Psi_n}(\ell_k-\ell_n),z;\frac{\zeta}{3}\right),\Im\left(\gamma_n-\varrho,z;\frac{\zeta}{3}\right),\Im\left(\frac{\Psi_n+1}{\Psi_n-n}(\gamma_{\Psi_n}-\gamma_n),z;\frac{\zeta}{3}\right)\right\}\\ &>1-\sigma,\end{aligned}$$

$$
\begin{aligned}
&\Re(\ell_n - \varrho, z; \zeta) \\
&\leq \max\left\{\Re\left(\frac{1}{\Psi_n - n}\sum_{k=n+1}^{\Psi_n}(\ell_k - \ell_n), z; \frac{\zeta}{3}\right), \Re\left(\gamma_n - \varrho, z; \frac{\zeta}{3}\right), \Re\left(\frac{\Psi_n + 1}{\Psi_n - n}\left(\gamma_{\Psi_n} - \gamma_n\right), z; \frac{\zeta}{3}\right)\right\} \\
&< \sigma,
\end{aligned}
$$

and

$$
\begin{aligned}
&\wp(\ell_n - \varrho, z; \zeta) \\
&\leq \max\left\{\wp\left(\frac{1}{\Psi_n - n}\sum_{k=n+1}^{\Psi_n}(\ell_k - \ell_n), z; \frac{\zeta}{3}\right), \wp\left(\gamma_n - \varrho, z; \frac{\zeta}{3}\right), \wp\left(\frac{\Psi_n + 1}{\Psi_n - n}\left(\gamma_{\Psi_n} - \gamma_n\right), z; \frac{\zeta}{3}\right)\right\} \\
&< \sigma.
\end{aligned}
$$

Hence $\ell_n \xrightarrow{\mathscr{N}_2} \varrho$. This ends the proof. □

Conclusion and future scope

In this work, we introduce the concept of Cesàro summability within an N2NS-one of the most comprehensive mathematical frameworks that seamlessly integrates some algebraic properties. Our results not only broaden the scope of Cesàro summability but also extend some important existing theorems, under neutrosophic 2-norms, offering new insights into the field. A natural direction for future research is the exploration of Tauberian theorems in this setting as well as its statistical version which promises to further enrich the theoretical foundations of summability.

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