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Fuzzy Structures to Quadruple Coincidence Points in Fréchet Spaces

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ABSTRACT: Within the framework of fuzzy Fréchet spaces (FFS), we have derived results on quadruple coincidence points for commuting mappings, notably without requiring a partially ordered set. Our findings are complemented by compelling examples and new perspectives that extend prior studies. Additionally, we explored an application focused on establishing unique solutions for the Lipschitzian quadruple system.

Key Words: Quadruple coincidence points, Fuzzy Fréchet spaces, Lipschitzian quadruple system, Commuting mappings.

Contents

1. Introduction

Fixed point theory, a vital field within functional analysis, finds extensive applications across both pure and applied mathematics. In the context of metric fixed points, Banach's contraction theorem stands as a fundamental result with broad applications [1]. Nonlinear analysis continues to benefit from fixed point theory as a foundational framework, providing critical tools for tackling differential and integral equations, solving problems in nonlinear and functional analysis, and addressing computational challenges in fields like computer science and engineering. The theory of fuzzy sets, initially proposed by Zadeh [2], introduced basic operations with wide applicability. Fuzzy sets have proven effective in modeling human involvement in human-computer interactions, thereby driving advances in data analysis, data mining, image processing, and interpretation. They also underpin intelligent systems –innovative conceptual frameworks that support human-centered approaches. The idea of fuzzy semi-norms, which was first put forth by katsaras in 1984 [3], has since played a significant role in this evolving field. Later, in 2007, Sadeqi and Solaty [4] expanded on this idea. In 2021, Ahmed Ghanawi and Al Nafie [5] further developed the concept by applying it to the formulation of fuzzy Fréchet spaces. Recently, there has been growing interest in investigating fixed-point theorems within ordered spaces, particularly under contractive conditions that apply to all points in the partial order (see [6]-[15]). The main aim of our work is to present new results on quadruple coincidence points for commuting mappings, without the necessity of a partially ordered set. To illustrate and support these theoretical concepts, we provided compelling examples and applications, demonstrating how they can be used to identify a unique solution in the Lipschitzian quadruple system.

2. Preliminaries

Definition (2.1). [16] A continuous t-norm is defined as a binary operation $* : [0,\ 1] \times [0,1] \to [0,1]$ if and only if

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1. $*$ is commutative, continuous, and associative,

2. $\forall\, m_1 \leq m_3$ and $m_2 \leq m_4$, where $m_1,\ m_2,\ m_3$, and $m_4 \in [0,1]$, $m_1 * 1 = m_1$ and $m_1 *\ m_2 \leq m_3 * m_4$.

$\forall\, m \in [0,1]$, the sequence $\{*^n m\}_{n=1}^{\infty}$ is defined inductively by $*^1 m = m,\ \ *^n m = \left(*^{n-1} m\right) * m$. A t-norm $*$ is classified as y-type if $\{*^n m\}_{n=1}^{\infty}$ is continuous at $m = 1$. This means that, for every $\varepsilon \in (0,1)$, there is $\delta \in (0,1)$ such that if $m \in (1-\delta, 1]$, then $*^n m > 1 - \varepsilon$ for all $n \in \mathbb{N}$.
$* = min$ is the most well-known continuous t-norm of the y-type and satisfies $\min(m_1,\ m_2) \geq m_1 m_2,\ \forall m_1 m_2 \in [0,1]$.
Definition (2.2). [4] A fuzzy set Υ in $Q \times \mathbb{R}$ is considered a fuzzy semi-norm on Q if the following conditions are met, where Q is a vector space over the field K: $\forall\ v,\ u \in Q$ and $\forall m_1,\ m_2 \in \mathbb{R}$

1. $\Upsilon(v,\ m_1) = 0,\ \forall m_1 \leq 0$,

2. $\Upsilon(\alpha v,\ m_1) = \Upsilon\left(v, \frac{m_1}{\alpha}\right),\ \forall m_1 > 0,\ \forall \alpha \in \frac{K}{\{0\}}$,

3. $\Upsilon(v + u, m_1, +m_2) \geq \Upsilon(v, m_1) * \Upsilon(u, m_2)$,

4. $\lim_{m_1 \to 0} \Upsilon(v,\ m_1) = 0,\ \lim_{m_1 \to \infty} \Upsilon(v,\ m_1) = 1$, and for all $v \in Q$ $\Upsilon(v,\ m_1)$ is non-decreasing with respect to m_1.

Definition (2.3). [4] If there is at least one $\Upsilon \in B$ and $m > 0$ such that $\Upsilon(v,\ m) \neq 1$ for every $v \neq 0$ in a family B of fuzzy semi-norms on a vector space Q, then the family B is said to be separating.
Definition (2.4). [5] A complete fuzzy topological vector space with fuzzy topology τ_B produced by a countable separating family of fuzzy semi-norms $B = \{\Upsilon_i\}_{i \in I}$, then is known as fuzzy Fréchet space (FFS).

The study in [5], explores the construction of fuzzy Fréchet spaces, as well as the notions of fuzzy convergence, fuzzy Cauchy sequences, and fuzzy continuity within these spaces.

3. Main results

Let us define Q to be FFS and $B = \{\Upsilon_i\}_{i \in I}$ be the family of fuzzy semi-norms such that B generates Q's fuzzy topology.
Definition (3.1). Let $\Phi : Q^4 \to Q$ and $S : Q \to Q$ are two mappings

1. We refer to Φ and S as commuting if $S\Phi_{uvwz} = \Phi_{s_u s_v s_w s_z},\ \forall\ u,\ v,\ w,\ z \in Q$.

2. We refer to $(u,\ v,\ w,\ z) \in Q^4$ is a quadruple coincidence point of Φ and S if $\Phi_{uvwz} = Su,\ \Phi_{vwzu} = Sv,\ \Phi_{wzuv} = Sw$, and $\Phi_{zuvw} = Sz$.

Theorem (3.1). Let $*$ is a t-norm of the y-type with $a * b \geq ab,\ \forall a,\ b \in [0,1]$. Let $\Phi : Q^4 \to Q$ and $S : Q \to Q$ are two mappings such that

1. $\Phi\left(Q^4\right) \subseteq S(Q)$,

2. S is a fuzzy continuous,

3. S and Φ are commuting,

4. $\forall\ u_1,\ \ v_1,\ \ w_1,\ \ z_1,\ \ u_2,\ v_2,\ \ w_2,\ \ z_2 \in Q$, and $\forall\ \Upsilon_i \in B$,

$$\Upsilon_i\left(\Phi_{u_1 v_1 w_1 z_1} - \Phi_{u_2 v_2 w_2 z_2},\ rm\right) \geq$$
$$\left(\Upsilon_i\left(S_{u_1} - S_{u_2}, m\right)\right)^{k_1} * \left(\Upsilon_i\left(S_{v_1} - S_{v_2}, m\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{w_1} - S_{w_2}, m\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{z_1} - S_{z_2}, m\right)\right)^{k_4}, \quad (3.1)$$

where $r \in (0,1)$, and $k_1,\ k_2,\ k_3,\ k_4 \in [0,1]$ such that $k_1 + k_2 + \ k_3 + \ k_4 \leq 1$.
Thus, the ensuing deductions are valid.

1. A unique u exists where $u = S_u = \Phi_{uuuu}$.

2. For the mappings S and Φ, there is at least a quadruple coincidence point; additionally, if $\Phi{=}u_0$, there is a constant on Q^4. This is only true if the mapping is its inverse is exists and satisfies $S^{-1}(u_0) = \{u_0\}$; then, we get $(u,\ u,\ u,\ u)$ is a unique quadruple coincidence point of S and Φ.

Proof. At first, in order to avoid the unknown quantity 0^0, we consider here $(\Upsilon_i(S_{u_1} - S_{u_2}, m))^0 = 1$, $\forall\ \Upsilon_i \in B,\ m > 0$, and $\forall\ u_1,\ \ u_2 \in Q$.
Now, the proof split into two cases.
Case I. If $\Phi{\subseteq}\mathrm{Q}$ is constant, then there is $u_0 \in Q$ such that $\forall\ u,\ v,\ w,\ z \in Q,\ \Phi_{uvwz} = u_0$. By commuting of S and Φ, It is possible to write $S_{u_0} = S\Phi_{uvwz} = \Phi_{s_u s_v s_w s_z} = u_0$. Thus, $u_0 = S_{u_0} = \Phi_{u_0 u_0 u_0 u_0}$ and $(u_0,\ u_0, u_0, u_0)$ is a quadruple coincidence point of S and Φ. However, suppose that $S^{-1}(u_0) = \{u_0\}$ and $(u,\ v,\ w,\ z) \in Q^4$ is a another quadruple coincidence point of S and Φ. Then, $S_u = \Phi_{uvwz} = u_0$, thus $u \in S^{-1}(u_0) = \{u_0\}$. analogously, we can write $u = v = w = z = u_0$; hence, $(u_0,\ u_0, u_0, u_0)$ is a unique quadruple coincidence point of S and Φ.
Case II. If $\Phi{\in}\mathrm{Q}$ is not constant; then, let $(k_1,\ k_2,\ k_3,\ k_4) \neq (0,\ 0,\ 0,\ 0)$. In this instance, $m \in [0, \infty)$, while n and j are regarded as non-negative integers. There are five steps in this case.
Step 1. Assume that the points $u_0,\ v_0, w_0$, and z_0 in Q are arbitrary. Since $\Phi\left(Q^4\right) {\subseteq} \mathrm{S}\left(\mathrm{Q}\right)$, we can decide $u_1,\ v_1, w_1,\ z_1 \in Q$ so as to $S_{u_1} = \Phi_{u_0 v_0 w_0 z_0},\ S_{v_1} = \Phi_{v_0 w_0 z_0 u_0},\ S_{w_1} = \Phi_{w_0 z_0 u_0 v_0}$ and $S_{z_1} = \Phi_{z_0 u_0 v_0 w_0}$. Once more, with $\Phi\left(Q^4\right) {\subseteq} \mathrm{S}\left(\mathrm{Q}\right)$, we can decide $u_2,\ v_2, w_2,\ z_2 \in Q$ so as to $S_{u_2} = \Phi_{u_1 v_1 w_1 z_1},\ S_{v_2} = \Phi_{v_1 w_1 z_1 u_1},\ S_{w_2} = \Phi_{w_1 z_1 u_1 v_1}$ and $S_{z_2} = \Phi_{z_1 u_1 v_1 w_1}$.
Under identical circumstances, we can create four sequences $\{u_n\},\ \{v_n\},\ \{w_n\}$, and $\{z_n\}$ in order for $n \geq 0,\ \ S_{u_{n+1}} = \Phi_{u_n v_n w_n z_n},\ \ S_{v_{n+1}} = \Phi_{v_n w_n z_n u_n},\ \ S_{w_{n+1}} = \Phi_{w_n z_n u_n v_n}$ and $S_{z_{n+1}} = \Phi_{z_n u_n v_n w_n}$.
Step 2. We will prove that $\{u_n\},\ \{v_n\},\ \{w_n\}$, and $\{z_n\}$ are Cauchy sequences. For $n \geq 0$ and $\forall m > 0$, let

$$\mathrm{H}_n(m) = \Upsilon_i\left(S_{u_n} - S_{u_{n+1}}, m\right) * \Upsilon_i\left(S_{v_n} - S_{v_{n+1}}, m\right) * \Upsilon_i\left(S_{w_n} - S_{w_{n+1}}, m\right) * \Upsilon_i\left(S_{z_n} - S_{z_{n+1}}, m\right)$$

$\forall\ \Upsilon_i \in B$. H_n is a non-decreasing mapping and $m - mr \leq r \leq \frac{r}{m}$, thus we obtain

$$\mathrm{H}_n(m - mr) \leq \mathrm{H}_n(m) \leq \mathrm{H}_n\left(\frac{m}{r}\right), \quad \forall m > 0 \quad textand \quad n \geq 0. \tag{3.2}$$

It is derived from (3.1) that, $\forall n \in N,\ \forall m \geq 0$, and $\forall \Upsilon_i \in B$,

$$\begin{aligned}
&\Upsilon_i\left(S_{u_n} - S_{u_{n+1}}, m\right) = \Upsilon_i\left(\Phi_{u_{n-1}v_{n-1}w_{n-1}z_{n-1}} - \Phi_{u_n v_n w_n z_n}, m\right) \geq \left(\Upsilon_i\left(S_{u_{n-1}} - S_{u_n}, \frac{m}{r}\right)\right)^{k_1} \\
&* \left(\Upsilon_i\left(S_{v_{n-1}} - S_{v_n}, \frac{m}{r}\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{w_{n-1}} - S_{w_n}, \frac{m}{r}\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{z_{n-1}} - S_{z_n}, \frac{m}{r}\right)\right)^{k_4}
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&\Upsilon_i\left(S_{v_n} - S_{v_{n+1}}, m\right) = \Upsilon_i\left(\Phi_{v_{n-1}w_{n-1}z_{n-1}u_{n-1}} - \Phi_{v_n w_n z_n u_n}, m\right) \geq \left(\Upsilon_i\left(S_{v_{n-1}} - S_{v_n}, \frac{m}{r}\right)\right)^{k_1} * \\
&\left(\Upsilon_i\left(S_{w_{n-1}} - S_{w_n}, \frac{m}{r}\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{z_{n-1}} - S_{z_n}, \frac{m}{r}\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{u_{n-1}} - S_{u_n}, \frac{m}{r}\right)\right)^{k_4}
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\Upsilon_i\left(S_{w_n} - S_{w_{n+1}}, m\right) = \Upsilon_i\left(\Phi_{w_{n-1}z_{n-1}u_{n-1}v_{n-1}} - \Phi_{w_n z_n u_n v_n}, m\right) \geq \left(\Upsilon_i\left(S_{w_{n-1}} - S_{w_n}, \frac{m}{r}\right)\right)^{k_1} \\
&* \left(\Upsilon_i\left(S_{z_{n-1}} - S_{z_n}, \frac{m}{r}\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{u_{n-1}} - S_{u_n}, \frac{m}{r}\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{v_{n-1}} - S_{v_n}, \frac{m}{r}\right)\right)^{k_4}
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\Upsilon_i\left(S_{z_n} - S_{z_{n+1}}, m\right) = \Upsilon_i\left(\Phi_{z_{n-1}u_{n-1}v_{n-1}w_{n-1}} - \Phi_{z_n u_n v_n w_n}, m\right) \geq \left(\Upsilon_i\left(S_{z_{n-1}} - S_{z_n}, \frac{m}{r}\right)\right)^{k_1} \\
&* \left(\Upsilon_i\left(S_{u_{n-1}} - S_{u_n}, \frac{m}{r}\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{v_{n-1}} - S_{v_n}, \frac{m}{r}\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{w_{n-1}} - S_{w_n}, \frac{m}{r}\right)\right)^{k_4}.
\end{aligned} \tag{3.6}$$

It is derived from (3.3)-(3.6) that,

$$\Upsilon_i\left(S_{u_n}-S_{u_{n+1}},m\right)\geq\left(\Upsilon_i\left(S_{u_{n-1}}-S_{u_n},\frac{m}{r}\right)\right)^{k_1}*\left(\Upsilon_i\left(S_{v_{n-1}}-S_{v_n},\frac{m}{r}\right)\right)^{k_2}*\left(\Upsilon_i\left(S_{w_{n-1}}-S_{w_n},\frac{m}{r}\right)\right)^{k_3}$$
$$*\left(\Upsilon_i\left(S_{z_{n-1}}-S_{z_n},\frac{m}{r}\right)\right)^{k_4}\geq\Upsilon_i\left(S_{u_{n-1}}-S_{u_n},\frac{m}{r}\right)*\Upsilon_i\left(S_{v_{n-1}}-S_{v_n},\frac{m}{r}\right)*\Upsilon_i\left(S_{w_{n-1}}-S_{w_n},\frac{m}{r}\right)*$$
$$\Upsilon_i\left(S_{z_{n-1}}-S_{z_n},\frac{m}{r}\right)=\mathrm{H}_{n-1}\left(\frac{m}{r}\right);$$

$$\Upsilon_i\left(S_{v_n}-S_{v_{n+1}},m\right)\geq\left(\Upsilon_i\left(S_{v_{n-1}}-S_{v_n},\frac{m}{r}\right)\right)^{k_1}*\left(\Upsilon_i\left(S_{w_{n-1}}-S_{w_n},\frac{m}{r}\right)\right)^{k_2}*\left(\Upsilon_i\left(S_{z_{n-1}}-S_{z_n},\frac{m}{r}\right)\right)^{k_3}$$
$$*\left(\Upsilon_i\left(S_{u_{n-1}}-S_{u_n},\frac{m}{r}\right)\right)^{k_4}\geq\Upsilon_i\left(S_{v_{n-1}}-S_{v_n},\frac{m}{r}\right)*\Upsilon_i\left(S_{w_{n-1}}-S_{w_n},\frac{m}{r}\right)*\Upsilon_i\left(S_{z_{n-1}}-S_{z_n},\frac{m}{r}\right)*$$
$$\Upsilon_i\left(S_{u_{n-1}}-S_{u_n},\frac{m}{r}\right)=\mathrm{H}_{n-1}\left(\frac{m}{r}\right);$$

$$\Upsilon_i\left(S_{w_n}-S_{w_{n+1}},m\right)\geq\left(\Upsilon_i\left(S_{w_{n-1}}-S_{w_n},\frac{m}{r}\right)\right)^{k_1}*\left(\Upsilon_i\left(S_{z_{n-1}}-S_{z_n},\frac{m}{r}\right)\right)^{k_2}*\left(\Upsilon_i\left(S_{u_{n-1}}-S_{u_n},\frac{m}{r}\right)\right)^{k_3}$$
$$*\left(\Upsilon_i\left(S_{v_{n-1}}-S_{v_n},\frac{m}{r}\right)\right)^{k_4}\geq\Upsilon_i\left(S_{w_{n-1}}-S_{w_n},\frac{m}{r}\right)*\Upsilon_i\left(S_{z_{n-1}}-S_{z_n},\frac{m}{r}\right)*\Upsilon_i\left(S_{u_{n-1}}-S_{u_n},\frac{m}{r}\right)$$
$$*\,\Upsilon_i\left(S_{v_{n-1}}-S_{v_n},\frac{m}{r}\right)=\mathrm{H}_{n-1}\left(\frac{m}{r}\right);$$

And

$$\Upsilon_i\left(S_{z_n}-S_{z_{n+1}},m\right)\geq\left(\Upsilon_i\left(S_{z_{n-1}}-S_{z_n},\frac{m}{r}\right)\right)^{k_1}*\left(\Upsilon_i\left(S_{u_{n-1}}-S_{u_n},\frac{m}{r}\right)\right)^{k_2}*\left(\Upsilon_i\left(S_{v_{n-1}}-S_{v_n},\frac{m}{r}\right)\right)^{k_3}*$$
$$\left(\Upsilon_i\left(S_{w_{n-1}}-S_{w_n},\frac{m}{r}\right)\right)^{k_4}\geq\Upsilon_i\left(S_{z_{n-1}}-S_{z_n},\frac{m}{r}\right)*\Upsilon_i\left(S_{u_{n-1}}-S_{u_n},\frac{m}{r}\right)*\Upsilon_i\left(S_{v_{n-1}}-S_{v_n},\frac{m}{r}\right)*$$
$$\Upsilon_i\left(S_{w_{n-1}}-S_{w_n},\frac{m}{r}\right)=\mathrm{H}_{n-1}\left(\frac{m}{r}\right);$$

This demonstrates that for every $m>0$ and every $n\geq 0$,

$$\begin{aligned}\Upsilon_i\left(S_{u_n}-S_{u_{n+1}},m\right)&\geq\mathrm{H}_{n-1}\left(\frac{m}{r}\right)\geq\mathrm{H}_{n-1}(m),\\ \Upsilon_i\left(S_{v_n}-S_{v_{n+1}},m\right)&\geq\mathrm{H}_{n-1}\left(\frac{m}{r}\right)\geq\mathrm{H}_{n-1}(m),\\ \Upsilon_i\left(S_{w_n}-S_{w_{n+1}},m\right)&\geq\mathrm{H}_{n-1}\left(\frac{m}{r}\right)\geq\mathrm{H}_{n-1}(m),\\ \Upsilon_i\left(S_{z_n}-S_{z_{n+1}},m\right)&\geq\mathrm{H}_{n-1}\left(\frac{m}{r}\right)\geq\mathrm{H}_{n-1}(m),\ \forall\Upsilon_i\in B.\end{aligned}\tag{3.7}$$

When we substitute $(m-mr)$ for m, we get the following result for all $m>0$ and all $n\geq 0$,

$$\Upsilon_i\left(S_{u_n}-S_{u_{n+1}},m-mr\right),\ \Upsilon_i\left(S_{v_n}-S_{v_{n+1}},m-mr\right),\tag{3.8}$$
$$\Upsilon_i\left(S_{w_n}-S_{w_{n+1}},m-mr\right),\ \Upsilon_i\left(S_{z_n}-S_{z_{n+1}},m-mr\right)$$
$$\geq\mathrm{H}_{n-1}(m-mr),\ \forall\Upsilon_i\in B.\tag{3.9}$$

Given that $*$ is commutative, we can infer using (3.3)-(3.6) that

$$\mathrm{H}_n(m)=\Upsilon_i\left(S_{u_n}-S_{u_{n+1}},m\right)*\ \Upsilon_i\left(S_{v_n}-S_{v_{n+1}},m\right)*\Upsilon_i\left(S_{w_n}-S_{w_{n+1}},m\right)*\Upsilon_i\left(S_{z_n}-S_{z_{n+1}},m\right)\geq$$
$$\left(\left(\Upsilon_i\left(S_{u_{n-1}}-S_{u_n},\frac{m}{r}\right)\right)^{k_1}*\left(\Upsilon_i\left(S_{v_{n-1}}-S_{v_n},\frac{m}{r}\right)\right)^{k_2}*\left(\Upsilon_i\left(S_{w_{n-1}}-S_{w_n},\frac{m}{r}\right)\right)^{k_3}*\left(\Upsilon_i\left(S_{z_{n-1}}-S_{z_n},\frac{m}{r}\right)\right)^{k_4}\right)$$
$$*\left(\left(\Upsilon_i\left(S_{u_{n-1}}-S_{u_n},\frac{m}{r}\right)\right)^{k_2}*\left(\Upsilon_i\left(S_{v_{n-1}}-S_{v_n},\frac{m}{r}\right)\right)^{k_3}*\left(\Upsilon_i\left(S_{w_{n-1}}-S_{w_n},\frac{m}{r}\right)\right)^{k_4}*\left(\Upsilon_i\left(S_{z_{n-1}}-S_{z_n},\frac{m}{r}\right)\right)^{k_1}\right)$$
$$*\left(\left(\Upsilon_i\left(S_{u_{n-1}}-S_{u_n},\frac{m}{r}\right)\right)^{k_3}*\left(\Upsilon_i\left(S_{v_{n-1}}-S_{v_n},\frac{m}{r}\right)\right)^{k_4}*\left(\Upsilon_i\left(S_{w_{n-1}}-S_{w_n},\frac{m}{r}\right)\right)^{k_1}*\left(\Upsilon_i\left(S_{z_{n-1}}-S_{z_n},\frac{m}{r}\right)\right)^{k_2}\right)$$
$$*\left(\left(\Upsilon_i\left(S_{u_{n-1}}-S_{u_n},\frac{m}{r}\right)\right)^{k_4}*\left(\Upsilon_i\left(S_{v_{n-1}}-S_{v_n},\frac{m}{r}\right)\right)^{k_1}*\left(\Upsilon_i\left(S_{w_{n-1}}-S_{w_n},\frac{m}{r}\right)\right)^{k_2}*\left(\Upsilon_i\left(S_{z_{n-1}}-S_{z_n},\frac{m}{r}\right)\right)^{k_3}\right).$$

Consequently,

$$
\begin{aligned}
&\mathrm{H}_n(m) \geq \\
&\left(\left(\Upsilon_i\left(S_{u_{n-1}}-S_{u_n}, \frac{m}{r}\right)\right)^{k_1} . \left(\Upsilon_i\left(S_{v_{n-1}}-S_{v_n}, \frac{m}{r}\right)\right)^{k_2} . \left(\Upsilon_i\left(S_{w_{n-1}}-S_{w_n}, \frac{m}{r}\right)\right)^{k_3} . \left(\Upsilon_i\left(S_{z_{n-1}}-S_{z_n}, \frac{m}{r}\right)\right)^{k_4}\right) \\
&* \left(\left(\Upsilon_i\left(S_{u_{n-1}}-S_{u_n}, \frac{m}{r}\right)\right)^{k_2} . \left(\Upsilon_i\left(S_{v_{n-1}}-S_{v_n}, \frac{m}{r}\right)\right)^{k_3} . \left(\Upsilon_i\left(S_{w_{n-1}}-S_{w_n}, \frac{m}{r}\right)\right)^{k_4} . \left(\Upsilon_i\left(S_{z_{n-1}}-S_{z_n}, \frac{m}{r}\right)\right)^{k_1}\right) * \\
&\left(\left(\Upsilon_i\left(S_{u_{n-1}}-S_{u_n}, \frac{m}{r}\right)\right)^{k_3} . \left(\Upsilon_i\left(S_{v_{n-1}}-S_{v_n}, \frac{m}{r}\right)\right)^{k_4} . \left(\Upsilon_i\left(S_{w_{n-1}}-S_{w_n}, \frac{m}{r}\right)\right)^{k_1} . \left(\Upsilon_i\left(S_{z_{n-1}}-S_{z_n}, \frac{m}{r}\right)\right)^{k_2}\right) \\
&* \left(\left(\Upsilon_i\left(S_{u_{n-1}}-S_{u_n}, \frac{m}{r}\right)\right)^{k_4} . \left(\Upsilon_i\left(S_{v_{n-1}}-S_{v_n}, \frac{m}{r}\right)\right)^{k_1} . \left(\Upsilon_i\left(S_{w_{n-1}}-S_{w_n}, \frac{m}{r}\right)\right)^{k_2} . \left(\Upsilon_i\left(S_{z_{n-1}}-S_{z_n}, \frac{m}{r}\right)\right)^{k_3}\right) \\
&= \left(\Upsilon_i\left(S_{u_{n-1}}-S_{u_n}, \frac{m}{r}\right)\right)^{k_1+k_2+k_3+k_4} * \left(\Upsilon_i\left(S_{v_{n-1}}-S_{v_n}, \frac{m}{r}\right)\right)^{k_1+k_2+k_3+k_4} * \left(\Upsilon_i\left(S_{w_{n-1}}-S_{w_n}, \frac{m}{r}\right)\right)^{k_1+k_2+k_3+k_4} \\
&* \left(\Upsilon_i\left(S_{z_{n-1}}-S_{z_n}, \frac{m}{r}\right)\right)^{k_1+k_2+k_3+k_4} \\
&\geq \Upsilon_i\left(S_{u_{n-1}}-S_{u_n}, \frac{m}{r}\right) * \Upsilon_i\left(S_{v_{n-1}}-S_{v_n}, \frac{m}{r}\right) * \Upsilon_i\left(S_{w_{n-1}}-S_{w_n}, \frac{m}{r}\right) * \Upsilon_i\left(S_{z_{n-1}}-S_{z_n}, \frac{m}{r}\right) = \mathrm{H}_{n-1}\left(\frac{m}{r}\right).
\end{aligned}
$$

From inequality (3.2), it is possible to write

$$
\mathrm{H}_n(m) \geq \mathrm{H}_{n-1}\left(\frac{m}{r}\right) \geq \mathrm{H}_{n-1}(m) \geq \mathrm{H}_{n-1}(m-mr), \ \forall m>0, \quad textand \ \ pn \geq 1. \tag{3.10}
$$

By carrying on in the same way, we have
$\mathrm{H}_n(m) \geq \mathrm{H}_{n-1}\left(\frac{m}{r}\right) \geq \mathrm{H}_{n-2}\left(\frac{m}{r^2}\right) \geq \cdots \geq \mathrm{H}_0\left(\frac{m}{r^n}\right), \forall m>0$, and $n \geq 1$,
It results in the discovery that for any $m>0$

$$
\lim_{n \to \infty} \mathrm{H}_n(m) \ \geq \lim_{n \to \infty} \mathrm{H}_0\left(\frac{m}{r^n}\right) = 1 .
$$

Then

$$
\lim_{n \longrightarrow \infty} H_n(m) = 1 \tag{3.11}
$$

By (3.7) and (3.10), we get

$$
\begin{aligned}
&\Upsilon_i\left(S_{u_n}-S_{u_{n+1}}, m\right), \ \Upsilon_i\left(S_{v_n}-S_{v_{n+1}}, m\right), \ \Upsilon_i\left(S_{w_n}-S_{w_{n+1}}, m\right), \ \Upsilon_i\left(S_{z_n}-S_{z_{n+1}}, m\right) \geq \mathrm{H}_n(m) \geq \\
&\mathrm{H}_{n-1}(m-mr), \ \forall \ \Upsilon_i \in B.
\end{aligned} \tag{3.12}
$$

Next, we shall demonstrate that $n, \ q \geq 1$ for every $m>0$ and $\forall \ \Upsilon_i \in B$,

$$
\Upsilon_i\left(S_{u_n}-S_{u_{n+q}}, m\right), \ \Upsilon_i\left(S_{v_n}-S_{v_{n+q}}, m\right), \ \Upsilon_i\left(S_{w_n}-S_{w_{n+q}}, m\right), \ \Upsilon_i\left(S_{z_n}-S_{z_{n+q}}, m\right) \geq *^q \mathrm{H}_{n-1}(m-mr). \tag{3.13}
$$

Using induction in $q \geq 1$, we may demonstrate this as follows: If $q=1$ for all $n \geq 1$ and all $m>0$ by (3.12) then inequality (3.13) holds. Assume that for every $n \geq 1$ and all $m>0$ for some q, (3.13) is true. We now establish the relationship for $q+1$. The induction assumption (3.1) dictates that

$$
\begin{aligned}
&\Upsilon_i\left(S_{u_{n+1}}-S_{u_{n+q+1}}, \mathrm{rm}\right) = \Upsilon_i\left(\Phi_{u_n v_n w_n z_n} - \Phi_{u_{n+q} v_{n+q} w_{n+q} z_{n+q}}, rm\right) \\
&\geq \left(\Upsilon_i\left(S_{u_n}-S_{u_{n+q}}, m\right)\right)^{k_1} * \left(\Upsilon_i\left(S_{v_n}-S_{v_{n+q}}, m\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{w_n}-S_{w_{n+q}}, m\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{z_n}-S_{z_{n+q}}, m\right)\right)^{k_4} \\
&\geq \left(*^q \mathrm{H}_{n-1}(m-mr)\ \right)^{k_1} * \left(*^q \mathrm{H}_{n-1}(m-mr)\right)^{k_2} * \left(*^q \mathrm{H}_{n-1}(m-mr)\right)^{k_3} * \left(*^q \mathrm{H}_{n-1}(m-mr)\right)^{k_4} \\
&\geq \left(*^q \mathrm{H}_{n-1}(m-mr)\ \right)^{k_1} . \left(*^q \mathrm{H}_{n-1}(m-mr)\right)^{k_2} . \left(*^q \mathrm{H}_{n-1}(m-mr)\right)^{k_3} . \left(*^q \mathrm{H}_{n-1}(m-mr)\right)^{k_4} \\
&= \left(*^q \mathrm{H}_{n-1}(m-mr)\ \right)^{k_1+k_2+k_3+k_4} \geq *^q \mathrm{H}_{n-1}(m-mr), \ \forall \ \Upsilon_i \in B.
\end{aligned}
$$

In the same way, we reach at

$$
\begin{aligned}
&\left(\Upsilon_i\left(S_{u_{n+1}}-S_{u_{n+q+1}}, rm\right), \ \Upsilon_i\left(S_{v_{n+1}}-S_{v_{n+q+1}}, rm\right), \ \Upsilon_i\left(S_{w_{n+1}}-S_{w_{n+q+1}}, rm\right), \ \Upsilon_i\left(S_{z_{n+1}}-S_{z_{n+q+1}}, rm\right)\right) \\
&\geq *^q \mathrm{H}_{n-1}(m-mr), \ \forall \ \Upsilon_i \in B.
\end{aligned}
$$

From (3) in definition (2.2), and (3.9), considering the assumption of induction, we obtain $\forall\ \Upsilon_i \in B$;

$$\begin{aligned}\Upsilon_i\left(S_{u_{n+1}}-S_{u_{n+q+1}},m\right)&=\Upsilon_i\left(S_{u_{n+1}}-S_{u_{n+q+1}},m-rm+rm\right)\\ &\geq \Upsilon_i\left(S_{u_n}-S_{u_{n+1}},m-rm\right)*\Upsilon_i\left(S_{u_{n+1}}-S_{u_{n+q+1}},rm\right)\\ &\geq \mathrm{H}_{n-1}\left(m-mr\right)*\left(*^q\mathrm{H}_{n-1}\left(m-mr\right)\right)=*^q\mathrm{H}_{n-1}\left(m-mr\right).\end{aligned}$$

Furthermore, the same outcome remains true if we take into $\Upsilon_i\left(S_{v_{n+1}}-S_{v_{n+q+1}},m\right),\ \Upsilon_i\left(S_{w_{n+1}}-S_{w_{n+q+1}},m\right),\ \Upsilon_i\left(S_{z_{n+1}}-S_{z_{n+q+1}},m\right)$. This implies that (3.13) is valid.

We can demonstrate that $\{S_{u_n}\}$ is Cauchy as a result. Suppose that $m>0$ and $\epsilon\in(0,\ 1)$ are given. Since $*$ of the y-type and from (3.11), so $\exists n_0\in N$ such that

$$\mathrm{H}_n\left(m-mr\right)>1-\delta,$$

$\forall n\geq n_0$, and $\delta\in(0,1)$. Thus, by (3.13), we get

$$\Upsilon_i\left(S_{u_n}-S_{u_{n+q}},m\right),\ \Upsilon_i\left(S_{v_n}-S_{v_{n+q}},m\right),\ \Upsilon_i\left(S_{w_n}-S_{w_{n+q}},m\right),\ \Upsilon_i\left(S_{z_n}-S_{z_{n+q}},m\right)>1-\epsilon,$$

$\forall n\geq n_0,\ q\geq 1$, and $\forall\ \Upsilon_i\in B$. Accordingly, $\{S_{u_n}\}$ is a Cauchy sequence. Likewise, the sequences $\{S_{v_n}\}$, $\{S_{w_n}\}$, and $\{S_{z_n}\}$ are also Cauchy sequences.

Step 3. Establishing a quadruple coincidence point for Φ and since Q is complete, then there are $u,\ v,\ w,\ z\in Q$ such that
$\lim_{n\to\infty}S_{u_n}=u,\quad \lim_{n\to\infty}S_{v_n}=v,\quad \lim_{n\to\infty}S_{w_n}=w,\quad$ and $\lim_{n\to\infty}S_{z_n}=z$.
Because S is fuzzy continuous, it follows that
$\lim_{n\to\infty}SS_{u_n}=Su,\quad \lim_{n\to\infty}SS_{v_n}=Sv,\quad \lim_{n\to\infty}SS_{w_n}=Sw,\quad$ and $\lim_{n\to\infty}SS_{z_n}=Sz$.
Since Φ and S are commuting, this results in

$$SS_{u_{n+1}}=S\Phi_{u_nv_nw_nz_n}=\Phi_{S_{u_n}S_{v_n}S_{w_n}S_{z_n}}$$

From inequality (3.1), $\forall\ \Upsilon_i\in B$ we have

$$\begin{aligned}&\Upsilon_i\left(SS_{u_{n+1}}-\Phi_{uvwz},rm\right)=\Upsilon_i\left(\Phi_{S_{u_n}S_{v_n}S_{w_n}S_{z_n}}-\Phi_{uvwz},rm\right)\\ &\geq\left(\Upsilon_i\left(SS_{u_n}-S_u,m\right)\right)^{k_1}*\left(\Upsilon_i\left(SS_{v_n}-S_v,m\right)\right)^{k_2}*\left(\Upsilon_i\left(SS_{w_n}-S_w,m\right)\right)^{k_3}*\left(\Upsilon_i\left(SS_{z_n}-S_z,m\right)\right)^{k_4}\\ &\geq\Upsilon_i\left(SS_{u_n}-S_u,m\right)*\Upsilon_i\left(SS_{v_n}-S_v,m\right)*\Upsilon_i\left(SS_{w_n}-S_w,m\right)*\Upsilon_i\left(SS_{z_n}-S_z,m\right).\end{aligned}\tag{3.14}$$

When $n\to\infty$, we obtain that

$$\lim_{n\to\infty}SS_{u_{n+1}}=\Phi_{uvwz}=S_u.$$

Likewise, we conclude that $\Phi_{vwzu}=S_v,\ \Phi_{wzuv}=S_w,\ \Phi_{zuvw}=S_z$. this demonstrates that $(u,\ v,\ w,\ z)\in Q$ is a quadruple coincidence point of S and Φ.

$$\Phi_{uvwz}=S_u,\ \Phi_{vwzu}=S_v,\ \Phi_{wzuv}=S_w,\ \Phi_{zuvw}=S_z.\tag{3.15}$$

Step 4. Demonstrating that $\Phi_{uvwz}=u,\ \Phi_{vwzu}=v,\ \Phi_{wzuv}=w,\ \Phi_{zuvw}=z$: From inequality (3.1), $\forall\ \Upsilon_i\in B$ we obtain

$$\begin{aligned}&\Upsilon_i\left(S_u-S_{v_{n+1}},rm\right)=\Upsilon_i\left(\Phi_{S_{v_n}S_{w_n}S_{z_n}S_{u_n}}-\Phi_{uvwz},rm\right)\\ &\geq\left(\Upsilon_i\left(S_{v_n}-S_u,m\right)\right)^{k_1}*\left(\Upsilon_i\left(S_{w_n}-S_v,m\right)\right)^{k_2}*\left(\Upsilon_i\left(S_{z_n}-S_w,m\right)\right)^{k_3}*\left(\Upsilon_i\left(S_{u_n}-S_z,m\right)\right)^{k_4},\end{aligned}\tag{3.16}$$

$$\begin{aligned}&\Upsilon_i\left(S_v-S_{w_{n+1}},rm\right)=\Upsilon_i\left(\Phi_{S_{w_n}S_{z_n}S_{u_n}S_{v_n}}-\Phi_{vwzu},rm\right)\\ &\geq\left(\Upsilon_i\left(S_{w_n}-S_v,m\right)\right)^{k_1}*\left(\Upsilon_i\left(S_{z_n}-S_w,m\right)\right)^{k_2}*\left(\Upsilon_i\left(S_{u_n}-S_z,m\right)\right)^{k_3}*\left(\Upsilon_i\left(S_{v_n}-S_u,m\right)\right)^{k_4},\end{aligned}\tag{3.17}$$

$$\begin{aligned}&\Upsilon_i\left(S_w-S_{z_{n+1}},rm\right)=\Upsilon_i\left(\Phi_{S_{z_n}S_{u_n}S_{v_n}S_{w_n}}-\Phi_{wzuv},rm\right)\\ &\geq\left(\Upsilon_i\left(S_{z_n}-S_w,m\right)\right)^{k_1}*\left(\Upsilon_i\left(S_{u_n}-S_z,m\right)\right)^{k_2}*\left(\Upsilon_i\left(S_{v_n}-S_u,m\right)\right)^{k_3}*\left(\Upsilon_i\left(S_{w_n}-S_v,m\right)\right)^{k_4},\end{aligned}\tag{3.18}$$

$$\Upsilon_i\left(S_z - S_{u_{n+1}}, rm\right) = \Upsilon_i\left(\Phi_{S_{u_n}S_{v_n}S_{w_n}S_{z_n}} - \Phi_{zuvw}, rm\right)$$
$$\geq \left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_1} * \left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_4}. \tag{3.19}$$

Let

$$\Gamma_n(rm) = \Upsilon_i\left(S_{v_n} - S_u, rm\right) * \Upsilon_i\left(S_{w_n} - S_v, rm\right) * \Upsilon_i\left(S_{z_n} - S_w, rm\right) * \Upsilon_i\left(S_{u_n} - S_z, rm\right),$$

$\forall\ m > 0,\ n \geq 0$, and $\forall\ \Upsilon_i \in B$. From and (3.16)-(3.19), it is evident that

$$\Gamma_{n+1}(rm) = \Upsilon_i\left(S_{v_{n+1}} - S_u, m\right) * \Upsilon_i\left(S_{w_{n+1}} - S_v, m\right) * \Upsilon_i\left(S_{z_{n+1}} - S_w, m\right) * \Upsilon_i\left(S_{u_{n+1}} - S_z, m\right)$$
$$\geq \left(\left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_1} * \left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_4}\right)$$
$$* \left(\left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_1} * \left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_4}\right)$$
$$* \left(\left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_1} * \left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_4}\right)$$
$$* \left(\left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_1} * \left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_4}\right)$$

$$\Gamma_{n+1}(rm) = \left(\left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_1} * \left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_4} * \left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_2}\right)$$
$$* \left(\left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_1} * \left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_4} * \left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_3}\right)$$
$$* \left(\left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_1} * \left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_4}\right)$$
$$* \left(\left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_4} * \left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_1}\right).$$

It suggests that

$$\Gamma_{n+1}(rm) \geq \left(\left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_1}.\left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_4}.\left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_3}.\left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_2}\right)$$
$$* \left(\left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_2}.\left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_1}.\left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_4}.\left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_3}\right)$$
$$* \left(\left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_3}.\left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_2}.\left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_1}.\left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_4}\right)$$
$$* \left(\left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_4}.\left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_3}.\left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_2}.\left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_1}\right)$$
$$= \left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_1+k_2+k_3+k_4} * \left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_1+k_2+k_3+k_4} * \left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_1+k_2+k_3+k_4}$$
$$* \left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_1+k_2+k_3+k_4}$$
$$\geq \Upsilon_i\left(S_{v_n} - S_u, m\right) * \Upsilon_i\left(S_{w_n} - S_v, m\right) * \Upsilon_i\left(S_{z_n} - S_w, m\right) * \Upsilon_i\left(S_{u_n} - S_z, m\right) = \Gamma_n(m)$$

This suggests that $\Gamma_{n+1}(rm) \geq \Gamma_n(m)$, $\forall n \geq 0,\ m > 0$, and $\forall\ \Upsilon_i \in B$. Performing this procedure once again,

$$\Gamma_n(m) \geq \Gamma_{n-1}\left(\frac{m}{r}\right) \geq \Gamma_{n-2}\left(\frac{m}{r^2}\right) \geq \cdots \geq \Gamma_0\left(\frac{m}{r^n}\right), \tag{3.20}$$

$\forall n \geq 0,\ m > 0$, and $\forall\ \Upsilon_i \in B$. From (3.16)-(3.20), we get that

$$\Upsilon_i\left(S_u - S_{v_{n+1}}, rm\right) \geq \left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_1} * \left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_3}$$
$$* \left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_4} \geq \Gamma_n(m) \geq \Gamma_0\left(\frac{m}{r^n}\right); \tag{3.21}$$

$$\Upsilon_i\left(S_v - S_{w_{n+1}}, rm\right) \geq \left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_1} * \left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_3}$$
$$* \left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_4} \geq \Gamma_n(m) \geq \Gamma_0\left(\frac{m}{r^n}\right); \tag{3.22}$$

$$\Upsilon_i\left(S_w - S_{z_{n+1}}, rm\right) \geq \left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_1} * \left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_4} \geq \Gamma_n(m) \geq \Gamma_0\left(\frac{m}{r^n}\right); \tag{3.23}$$

$$\Upsilon_i\left(S_z - S_{u_{n+1}}, rm\right) \geq \left(\Upsilon_i\left(S_{u_n} - S_z, m\right)\right)^{k_1} * \left(\Upsilon_i\left(S_{v_n} - S_u, m\right)\right)^{k_2} * \left(\Upsilon_i\left(S_{w_n} - S_v, m\right)\right)^{k_3} * \left(\Upsilon_i\left(S_{z_n} - S_w, m\right)\right)^{k_4} \geq \Gamma_n(m) \geq \Gamma_0\left(\frac{m}{r^n}\right). \tag{3.24}$$

Hence,

$$\Upsilon_i\left(S_u - S_{v_{n+1}}, rm\right) \geq \Gamma_0\left(\frac{m}{r^n}\right),$$

$$\Upsilon_i\left(S_v - S_{w_{n+1}}, rm\right) \geq \Gamma_0\left(\frac{m}{r^n}\right),$$

$$\Upsilon_i\left(S_w - S_{z_{n+1}}, rm\right) \geq \Gamma_0\left(\frac{m}{r^n}\right),$$

$$\Upsilon_i\left(S_z - S_{u_{n+1}}, rm\right) \geq \Gamma_0\left(\frac{m}{r^n}\right).$$

$\forall n \geq 0$, $m > 0$, and $\forall\ \Upsilon_i \in B$. Considering the limit in (3.21)-(3.22) as $n \to \infty$ and since $\lim_{n\to\infty} \Gamma_0\left(\frac{m}{r^n}\right) = 1$, $\forall m > 0$, we have $\lim_{n\to\infty} S_{u_{n+1}} = S_z$, $\lim_{n\to\infty} S_{w_{n+1}} = S_v$, $\lim_{n\to\infty} S_{z_{n+1}} = S_w$, and $\lim_{n\to\infty} S_{v_{n+1}} = S_u$. Along with (3.15), this demonstrates that

$$\Phi_{zuvw} = S_z = \lim_{n\to\infty} S_{u_{n+1}} = u,$$

$$\Phi_{vwzu} = S_v = \lim_{n\to\infty} S_{w_{n+1}} = w,$$

$$\Phi_{wzuv} = S_w = \lim_{n\to\infty} S_{z_{n+1}} = z,$$

$$\Phi_{uvwz} = S_u = \lim_{n\to\infty} S_{v_{n+1}} = v.$$

Step 5. We will demonstrate that $u = v = w = z$. Let $\Lambda(m) = \Upsilon_i(u - v, m) * \Upsilon_i(v - w, m) * \Upsilon_i(w - z, m) * \Upsilon_i(z - u, m)$, $\forall m > 0$, and $\forall\ \Upsilon_i \in B$. In light of (3.1), we may then write

$$\Upsilon_i(u - v, rm) = \Upsilon_i(\Phi_{uvwz} - \Phi_{vwzu}, rm) \geq (\Upsilon_i(S_u - S_v, m))^{k_1} * (\Upsilon_i(S_v - S_w, m))^{k_2} * (\Upsilon_i(S_w - S_z, m))^{k_3} * (\Upsilon_i(S_z - S_u, m))^{k_4} = (\Upsilon_i(v - w, m))^{k_1} * (\Upsilon_i(w - z, m))^{k_2} * (\Upsilon_i(z - u, m))^{k_3} * (\Upsilon_i(u - v, m))^{k_4}; \tag{3.25}$$

$$\Upsilon_i(v - w, rm) = \Upsilon_i(\Phi_{vwzu} - \Phi_{wzuv}, rm) \geq (\Upsilon_i(S_v - S_w, m))^{k_1} * (\Upsilon_i(S_w - S_z, m))^{k_2} * (\Upsilon_i(S_z - S_u, m))^{k_3} * (\Upsilon_i(S_u - S_v, m))^{k_4} = (\Upsilon_i(w - z, m))^{k_1} * (\Upsilon_i(z - u, m))^{k_2} * (\Upsilon_i(u - v, m))^{k_3} * (\Upsilon_i(v - w, m))^{k_4}; \tag{3.26}$$

$$\Upsilon_i(w - z, rm) = \Upsilon_i(\Phi_{wzuv} - \Phi_{zuvw}, rm) \geq (\Upsilon_i(S_w - S_z, m))^{k_1} * (\Upsilon_i(S_z - S_u, m))^{k_2} * (\Upsilon_i(S_u - S_v, m))^{k_3} * (\Upsilon_i(S_v - S_w, m))^{k_4} = (\Upsilon_i(z - u, m))^{k_1} * (\Upsilon_i(u - v, m))^{k_2} * (\Upsilon_i(v - w, m))^{k_3} * (\Upsilon_i(w - v, m))^{k_4}; \tag{3.27}$$

$$\Upsilon_i(z - u, rm) = \Upsilon_i(\Phi_{zuvw} - \Phi_{uvwz}, rm) \geq (\Upsilon_i(S_z - S_u, m))^{k_1} * (\Upsilon_i(S_u - S_v, m))^{k_2} * (\Upsilon_i(S_v - S_w, m))^{k_3} * (\Upsilon_i(S_w - S_z, m))^{k_4} = (\Upsilon_i(u - v, m))^{k_1} * (\Upsilon_i(v - w, m))^{k_2} * (\Upsilon_i(w - z, m))^{k_3} * (\Upsilon_i(z - u, m))^{k_4}. \tag{3.28}$$

Combining the four inequalities mentioned before,$\forall \Upsilon_i \in B$, we get

$$\begin{aligned}
&\Lambda(rm) = \Upsilon_i(u-v,rm) * \Upsilon_i(v-w,rm) * \Upsilon_i(w-z,rm) * \Upsilon_i(z-u,rm)\\
&\geq \left((\Upsilon_i(v-w,m))^{k_1} * (\Upsilon_i(w-z,m))^{k_2} * (\Upsilon_i(z-u,m))^{k_3} * (\Upsilon_i(u-v,m))^{k_4}\right)\\
&* \left((\Upsilon_i(w-z,m))^{k_1} * (\Upsilon_i(z-u,m))^{k_2} * (\Upsilon_i(u-v,m))^{k_3} * (\Upsilon_i(v-w,m))^{k_4}\right)\\
&* \left((\Upsilon_i(z-u,m))^{k_1} * (\Upsilon_i(u-v,m))^{k_2} * (\Upsilon_i(v-w,m))^{k_3} * (\Upsilon_i(w-v,m))^{k_4}\right)\\
&* \left((\Upsilon_i(u-v,m))^{k_1} * (\Upsilon_i(v-w,m))^{k_2} * (\Upsilon_i(w-z,m))^{k_3} * (\Upsilon_i(z-u,m))^{k_4}\right)\\
&= \left((\Upsilon_i(u-v,m))^{k_4} * (\Upsilon_i(u-v,m))^{k_3} * (\Upsilon_i(u-v,m))^{k_2} * (\Upsilon_i(u-v,m))^{k_1}\right)\\
&* \left((\Upsilon_i(v-w,m))^{k_1} * (\Upsilon_i(v-w,m))^{k_4} * (\Upsilon_i(w-z,m))^{k_3} * (\Upsilon_i(v-w,m))^{k_2}\right)\\
&* \left((\Upsilon_i(w-z,m))^{k_2} * (\Upsilon_i(w-z,m))^{k_1} * (\Upsilon_i(w-z,m))^{k_4} * (\Upsilon_i(w-z,m))^{k_3}\right)\\
&* \left((\Upsilon_i(z-u,m))^{k_3} * (\Upsilon_i(z-u,m))^{k_2} * (\Upsilon_i(z-u,m))^{k_1} * (\Upsilon_i(z-u,m))^{k_4}\right)\\
&\geq \left((\Upsilon_i(u-v,m))^{k_4}.(\Upsilon_i(u-v,m))^{k_3}.(\Upsilon_i(u-v,m))^{k_2}.(\Upsilon_i(u-v,m))^{k_1}\right)\\
&* \left((\Upsilon_i(v-w,m))^{k_1}.(\Upsilon_i(v-w,m))^{k_4}.(\Upsilon_i(w-z,m))^{k_3}.(\Upsilon_i(v-w,m))^{k_2}\right)\\
&* \left((\Upsilon_i(w-z,m))^{k_2}.(\Upsilon_i(w-z,m))^{k_1}.(\Upsilon_i(w-z,m))^{k_4}.(\Upsilon_i(w-z,m))^{k_3}\right)\\
&* \left((\Upsilon_i(z-u,m))^{k_3}.(\Upsilon_i(z-u,m))^{k_2}.(\Upsilon_i(z-u,m))^{k_1}.(\Upsilon_i(z-u,m))^{k_4}\right)\\
&= (\Upsilon_i(u-v,m))^{k_1+k_2+k_3+k_4} * (\Upsilon_i(v-w,m))^{k_1+k_2+k_3+k_4} * (\Upsilon_i(w-z,m))^{k_1+k_2+k_3+k_4}\\
&* (\Upsilon_i(z-u,m))^{k_1+k_2+k_3+k_4}\\
&\geq \Upsilon_i(u-v,m) * \Upsilon_i(v-w,m) * \Upsilon_i(w-z,m) * \Upsilon_i(z-u,m) = \Lambda(m).
\end{aligned}$$

Thus, $\Lambda(rm) \geq \Lambda(m)$ and $\Lambda(m) \geq \Lambda\left(\frac{m}{r}\right) \geq \Lambda\left(\frac{m}{r^2}\right) \geq \cdots \geq \Lambda\left(\frac{m}{r^n}\right)$, $\forall m > 0$ and $n \geq 1$. Using (3.25)-(3.28), we have

$$\begin{aligned}
&\Upsilon_i(u-v,rm) \geq (\Upsilon_i(S_u-S_v,m))^{k_1} * (\Upsilon_i(S_v-S_w,m))^{k_2} * (\Upsilon_i(S_w-S_z,m))^{k_3} * (\Upsilon_i(S_z-S_u,m))^{k_4}\\
&= (\Upsilon_i(v-w,m))^{k_1} * (\Upsilon_i(w-z,m))^{k_2} * (\Upsilon_i(z-u,m))^{k_3} * (\Upsilon_i(u-v,m))^{k_4} \geq \Lambda(m)\\
&\geq \Lambda\left(\frac{m}{r^n}\right),
\end{aligned}$$

$$\begin{aligned}
&\Upsilon_i(v-w,rm) \geq (\Upsilon_i(S_v-S_w,m))^{k_1} * (\Upsilon_i(S_w-S_z,m))^{k_2} * (\Upsilon_i(S_z-S_u,m))^{k_3} * (\Upsilon_i(S_u-S_v,m))^{k_4}\\
&= (\Upsilon_i(w-z,m))^{k_1} * (\Upsilon_i(z-u,m))^{k_2} * (\Upsilon_i(u-v,m))^{k_3} * (\Upsilon_i(v-w,m))^{k_4}\\
&\geq \Lambda(m) \geq \Lambda\left(\frac{m}{r^n}\right),
\end{aligned}$$

$$\begin{aligned}
&\Upsilon_i(w-z,rm) =\geq (\Upsilon_i(S_w-S_z,m))^{k_1} * (\Upsilon_i(S_z-S_u,m))^{k_2} * (\Upsilon_i(S_u-S_v,m))^{k_3} * (\Upsilon_i(S_v-S_w,m))^{k_4}\\
&= (\Upsilon_i(z-u,m))^{k_1} * (\Upsilon_i(u-v,m))^{k_2} * (\Upsilon_i(v-w,m))^{k_3} * (\Upsilon_i(w-v,m))^{k_4}\\
&\geq \Lambda(m) \geq \Lambda\left(\frac{m}{r^n}\right),
\end{aligned}$$

$$\begin{aligned}
&\Upsilon_i(z-u,rm) = \Upsilon_i(\Phi_{zuvw} - \Phi_{uvwz}, rm) \geq (\Upsilon_i(S_z-S_u,m))^{k_1} * (\Upsilon_i(S_u-S_v,m))^{k_2} * (\Upsilon_i(S_v-S_w,m))^{k_3} *\\
&(\Upsilon_i(S_w-S_z,m))^{k_4} = (\Upsilon_i(u-v,m))^{k_1} * (\Upsilon_i(v-w,m))^{k_2} * (\Upsilon_i(w-z,m))^{k_3} * (\Upsilon_i(z-u,m))^{k_4}\\
&\geq \Lambda(m) \geq \Lambda\left(\frac{m}{r^n}\right).
\end{aligned}$$

As $n \to \infty$, we get $\lim_{n\to\infty} \Lambda\left(\frac{m}{r^n}\right) = 1,\ \forall n \geq 1$. Thus, it follows that $\Upsilon_i(u-v, rm) = \Upsilon_i(v-w, rm) = \Upsilon_i(w-z, rm) = \Upsilon_i(z-u, rm) = 1,\ \forall m > 0$, and $\forall \Upsilon_i \in B$, it is that, $u = v = w = z$. It follows from (3.1) that u is unique.■

Example(3.1). Let Q=$\mathbb{R}$ be a fuzzy Fréchet space with fuzzy semi-norm $\Upsilon : \mathbb{R} \times (0,\infty) \to [0,1]$ defined as follows:

$$\Upsilon(\mathrm{u}-\mathrm{v}, \mathrm{m}) = e^{-\frac{|u+v|}{m}},\ m > 0.$$

Consider $a, b, r \in \mathbb{R}$ and $a, b > 0,\ 0 < r < 1$ such that $8a \leq br$, that is, $\frac{a}{r} \leq \frac{b}{8}$. We define $\Phi : Q^4 \to Q$ and $S : Q \to Q$ by $\Phi_{uvwz} = \frac{a}{2}(u-v)$ and $S(u) = \frac{b}{2}u,\ \forall u,\ v,\ w,\ z \in Q$. It is obvious that Φ and S are commuting, S is fuzzy continuous, and $\Phi\left(Q^4\right) = \mathrm{Q} = \mathrm{S}(\mathrm{Q})$. Furthermore,$\Upsilon$ satisfies

$$\Upsilon\left(\Phi_{u_1v_1w_1z_1} - \Phi_{u_2v_2w_2z_2}, rm\right) = (e^{|(u_1-u_2)+(v_1-v_2)|})^{-\frac{a}{2rm}}$$

$$\geq \left(e^{-\frac{2\max\{|(u_1-u_2),(v_1-v_2)|\}}{2m}}\right)^{\frac{a}{r}} \geq \left(e^{-\frac{2\max\{|(u_1-u_2),(v_1-v_2)|\}}{2m}}\right)^{\frac{b}{8}}$$

$$\geq \left(e^{-\frac{b}{8m}}\right)^{\max\{|(u_1-u_2),(v_1-v_2)|\}} = \min\left\{e^{-\frac{|(u_1-u_2)|}{8m}},\ e^{-\frac{|(v_1-v_2)|}{8m}}\right\}$$

$$\geq \min\left\{e^{-\frac{|(u_1-u_2)|}{8m}},\ e^{-\frac{|(v_1-v_2)|}{8m}},\ e^{-\frac{|(w_1-w_2)|}{8m}},\ e^{-\frac{|(z_1-z_2)|}{8m}}\right\}$$

$$= \min\left\{e^{-\frac{|(u_1-u_2)|}{2(4m)}},\ e^{-\frac{|(v_1-v_2)|}{2(4m)}},\ e^{-\frac{|(w_1-w_2)|}{2(4m)}},\ e^{-\frac{|(z_1-z_2)|}{2(4m)}}\right\}$$

$$= \min\left\{(\Upsilon(S(u_1)-S(u_2), \mathrm{m}))^{\frac{1}{4}},\ (\Upsilon(S(v_1)-S(v_2), \mathrm{m}))^{\frac{1}{4}}, (\Upsilon(S(w_1)-S(w_2), \mathrm{m}))^{\frac{1}{4}}, (\Upsilon(S(z_1)-S(z_2), \mathrm{m}))^{\frac{1}{4}}\right\}$$

Therefore, we can infer that Φ and S have a quadruple coincidence point from theorem (3.1).

4. Application

In order to emphasize the significance of the theoretical findings and demonstrate how to apply them to determine whether the solution to a Lipschitzian quadruple system exists, this section was specially prepared.

Definition(4.1). Let$Q_1,\ Q_2$ are two fuzzy Fréchet space and $B_1 = \{\Upsilon_i\}_{i\in I}, B_2 = \{\rho_k\}_{k\in I}$ are the families of fuzzy semi-norms such that $B_1 = \{\Upsilon_i\}_{i\in I}, B_2 = \{\rho_k\}_{k\in I}$ generates $Q_1,\ Q_2$'s fuzzy topologies, respectively. The mapping $T : Q_1 \longrightarrow Q_2$ is said to be FF- Lipschitzian mapping if there exists a constant $r \geq 0$ such that for all $v, u \in Q_1$ and for all $m > 0$:

$$\rho_k(T(v) - T(u), m) \geq \Upsilon_i\left(v-u, \frac{m}{r}\right),\ \forall \Upsilon_i \in B_1, \forall \rho_k \in B_2.$$

Suppose that T_1, T_2, T_3, T_4:$\mathbb{R} \to \mathbb{R}$ are FF- Lipschitzian mappings and a_1, a_2, a_3, a_4 are real numbers. Let Θ:$\mathbb{R} \to \mathbb{R}$ be specified by $\Theta = \sum_{j=1}^{4} a_j T_j(u)$,

$\forall u \in \mathbb{R}$; Then, Θ is moreover an FF- Lipschitzian mapping and $r_\Theta \leq \sum_{j=1}^{4} |a_j| r_{T_j}$. Now, $\forall u, v, w, z \in \mathbb{R}$, define $\Phi : Q^4 \to Q$ as

$$\Phi_{uvwz} = a_1T_1(u) + a_2T_2(v) + a_3T_3(w) + a_4T_4(z)$$

Clearly, that $\forall u \in \mathbb{R}, \Phi_{uuuu} = \Theta_u$. In addition, $\forall m > 0$ and $\forall \Upsilon_i \in B$, we have $\Upsilon_i\left(\Phi_{u_1u_2u_3u_4} - \Phi_{v_1v_2v_3v_4},\ m\right) = \frac{m}{m+\sum_{j=1}^{4}|a_j||T_j(u_j)-T_j(v_j)|} \geq \frac{m}{m+\sum_{j=1}^{4}|a_j|\ .r_{T_j}.|u_j-v_j|} \geq \frac{m}{m+\alpha\max_{1\leq j\leq 4}|u_j-v_j|}$.

If $\alpha < 1$, then Φ satisfies inequality (3.1) with $S_u = u,\ \forall u \in \mathbb{R}$.

The following corollary can be stated in light of the data above.

Corollary (4.1). Suppose that T_1, T_2, T_3, T_4:$\mathbb{R} \to \mathbb{R}$ are FF-Lipschitzian mappings and a_1, a_2, a_3, a_4 are real numbers such that $\sum_{j=1}^{4} |a_j| r_{T_j} < 1$, then the system

$$\begin{cases} u = a_1T_1(u) + a_2T_2(v) + a_3T_3(w) + a_4T_4(z), \\ v = a_1T_1(v) + a_2T_2(w) + a_3T_3(z) + a_4T_4(u), \\ w = a_1T_1(w) + a_2T_2(z) + a_3T_3(u) + a_4T_4(v), \\ z = a_1T_1(z) + a_2T_2(u) + a_3T_3(v) + a_4T_4(w), \end{cases} \tag{4.1}$$

contains a single solution (u_0, u_0, u_0, u_0), where u_0 is the lone real solution of $u = \sum_{j=1}^{4} a_j T_j(u)$.
Example (4.1)**.** Consider the system

$$\begin{cases} 120u + (\frac{4}{1+w^2})^2 - 15\sin^{-1}z &= 24\cos u \ - \frac{18}{1+v^2} + 144, \\ 120v + (\frac{4}{1+z^2})^2 - 15\sin^{-1}u &= 24\cos v \ - \frac{18}{1+w^2} + 144, \\ 120w + (\frac{4}{1+u^2})^2 - 15\sin^{-1}v &= 24\cos w \ - \frac{18}{1+z^2} + 144, \\ 120z + (\frac{4}{1+v^2})^2 - 15\sin^{-1}w &= 24\cos z \ - \frac{18}{1+u^2} + 144. \end{cases} \tag{4.2}$$

If we put $T_1(u) = 6 + \cos u$, $T_2(u) = \frac{1}{1+u^2}$, $T_3(u) = (\frac{4}{1+u^2})^2$, and $T_4(u) = \sin^{-1}z$, thus T_1, T_2, T_3, and T_4 are FF- Lipschitzian mappings, and $r_{T_1} =$, $r_{T_2} = \frac{3\sqrt{3}}{8}$, $r_{T_3} = \frac{27}{64}$, and $r_{T_4} = 1$. Assume that $a_1 = \frac{1}{5}, a_2 = -\frac{3}{20}, a_3 = \frac{2}{15}$, and $a_4 = \frac{1}{8}$. Hence, $\sum_{j=1}^{4} |a_j| \, r_{T_j} = 0.479 < 1$ as system (4.2) is an instance of system (4.1) in particular. Thus, there is a unique solution (u_0, u_0, u_0, u_0) for system (4.2), where u_0 a unique solution of

$$120u + (\frac{4}{1+w^2})^2 - 15\sin^{-1}z \ = 24\cos u \ - \frac{18}{1+v^2} + 144.$$

The value can be approximated $u_0 = 1.26624$.

5. Conclusion

Recently, there has been significant interest in exploring fixed-point theorems in ordered fuzzy metric spaces under a contractively condition applicable to all related points in the partial order. This paper aims to present quadruple coincidence point results for commuting mappings without requiring a partially order set. To support our theoretical concepts, we discuss intriguing examples and applications for identifying a unique solution in the Lipschitzian quadruple system.

References

1. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations integrals, Fundamenta mathematicae, 3(1), 133-181,(1922).https://eudml.org/doc/213289.

2. L. A. Zadeh, Fuzzy sets, Information and control, 8(3), 338-353,(1965). https://doi.org/10.1016/S0019-9958(65)90241-X.

3. A. K. Katsaras, Fuzzy topological vector spaces II, Fuzzy sets and systems, 12(2), 143-154, (1984). https://www.sciencedirect.com/science/article/pii/0165011484900344.

4. 4. I. Sadeqi, and F. K. Solaty, Fuzzy Seminormed Linear Space, In First Joint Congress on Fuzzy and Intelligent Systems, Ferdowsi University of Mashhad, Iran : 29-31, (2007). https://en.civilica.com/l/3784/.

5. A. G. Jasim, and Z. D. Al-Nafie, Fuzzy Fréchet Manifold, In Journal of Physics: Conference Series (Vol. 1818, No. 1, p. 012064). IOP Publishing, (2021, March). https://iopscience.iop.org/article/10.1088/1742-6596/1818/1/012064.

6. R. P. Agarwal, M. A. El-Gebeily, and D. O'Regan, Generalized contractions in partially ordered metric spaces, Applicable Analysis, 87(1), 109-116, (2008). https://doi.org/10.1080/00036810701556151.

7. J. J. Nieto, and R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order, 22(3), 223-239, (2005). https://link.springer.com/article/10.1007/s11083-005-9018-5.

8. T. G. Bhaskar, and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear analysis: theory, methods , applications, 65(7), 1379-1393,(2006). https://doi.org/10.1016/j.na.2005.10.017.

9. V. Lakshmikantham, and L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis: Theory, Methods and Applications, 70(12), 4341-4349, (2009). https://doi.org/10.1016/j.na.2008.09.020.

10. M. E. Gordji, H.Baghani, and Y. J. Cho, Coupled fixed point theorems for contractions in intuitionistic fuzzy normed spaces, Mathematical and Computer Modelling, 54(9-10), 1897-1906, (2011). https://doi.org/10.1016/j.mcm.2011.04.014.

11. A. G., Jasim and Z. D. Al-Nafie, Some fixed point theorems in fuzzy Fréchet manifold, In AIP Conference Proceedings (Vol. 2398, No. 1). AIP Publishing, (2022, October). https://doi.org/10.1063/5.0095582.

12. A. G. Jasim, A. A. Sangoor, A. S. Mohammed, T. H. Dahess, and A. H. Kamil, Common fixed point theorem in fuzzy Fréchet space, Journal of Interdisciplinary Mathematics, 27(4), 843-847, (2024). https://doi.org/10.47974/JIM-1881.

13. E.A.Huzam, and et.al.Existence and Uniqueness of Solutions to Hemi Equilibrium Problem with Application.Journal of Advanced Research in Dynamical and Control Systems, 11(1), 1754-1758(2019).

14. J. H. Eidi, E. M. Hameed, and J. R. Kider, Fixed Point Theorems with its Applications in Fuzzy Complete Convex Fuzzy Metric Spaces, International Journal of Neutrosophic Science (IJNS), 25(3), (2025). https://doi.org/10.54216/IJNS.250306.

15. A. G. Jasim, I. Harbi and A. S. Mohammed, On coupled fixed point theorem in partially fuzzy Fréchet space, Journal of Interdisciplinary Mathematics, 28:3-A, 787-793, (2025). DOI: 10.47974/JIM-1974.

16. O. Hadzic, and E. Pap, Fixed point theory in probabilistic metric spaces, Springer Science and Business Media, Vol. 536, (2013). https://link.springer.com/book/10.1007/978-94-017-1560-7.

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