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An Analytical Innovation of Hilbert via δ-Approach Structure

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ABSTRACT: This paper presents novel results within the framework of the multicharacter approach to Hilbert algebras, focusing on a refined analysis of multipliers and their algebraic behavior. The investigation systematically explores the interplay between fixed sets and their structural roles in shaping the internal dynamics of Hilbert algebras. Emphasis is placed on establishing rigorous connections through formal theorems and illustrative examples that clarify the utility and implications of the multicharacter perspective. By extending classical insights, the study provides a broader and more nuanced understanding of how multipliers operate within and influence the overall architecture of Hilbert algebras.

Key Words: Approach space, Normed approach space, Hilbert algebra, Symmetric algebra.

Contents

1. Introduction

There exist numerous generalizations of the concept of amortizable spaces, as well as of metric spaces. Traditionally, the fundamental notions of topology, uniform structures, and metrics have been treated as distinct frameworks. In 1989, Lowen [21] investigated the concept of distance between sets and points within metric spaces, explored approach spaces, and examined the associated closure operator. "This theory addressed natural questions arising from the interaction between metric spaces and topological spaces by introducing a uniquely appropriate super category encompassing both MET and TOP found this relation by Lowen [22]. The inception of approach space, and more generally, the evolution of approach theory in its entirely stems from a simple yet profound observation: we can define a canonical metric for finite products of amortizable (topological or uniform) spaces, construct metrics for countable products, but no general metric exists for uncountable products. These seemingly straightforward facts underpin a significant body of mathematical development. The fundamental distinction between approach spaces and metric spaces lies in the fact that, in approach spaces, all point-to-set distances are specified, similarly to how point-to-point distances are defined in metric spaces. An approach space is called topological if generated by a topological space, it is labeled as topological if a metric space gives rise to it, it is called metric. Considering normed approach vector spaces, we define Banach approach structures and proceed to explore their properties within the framework of approach theory. "Kreeam and Hussein [19] introduced a new structure of random approach normed spaces via Banach spaces. Abed and Hussein [5] studied a new class of vectors using the s-proximity structure. Hussein and Washaych [8,11] established new results on Q-boundedness defined in the algebra of symmetric Δ-Banach spaces and constructed a state space of measurable functions in symmetric E-Banach algebras. Hussein and Abd [2] introduced normed approach spaces via the β-approach structure. In another work, Hussein [9] identified an equivalent locally martingale measure for the deflator process on ordered Banach algebras. Kadhim and Hussein [13] investigated the relationship between topology and normed spaces. Similarly, Jameson [12] studied the relation between topology and normed spaces. Wang [30] studied multipliers in case the Banach Algebra

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is commutative Fleming [7] introduced result of Hermition operator on set of all continuous operator on Banach algebra Kamowitz [15] introduced certain properties in algebra, While [17,16] investigated operator algebras by Kaplansky. Neamah and Hussein [28,27] introduced a new structure called a^*-normed approach space and proved that such a space is complete whenever it lacks an identity. Abbas and Hussein [1,3,10,4] explored the structure of Banach approach spaces. In the realm of approach spaces, Baekeland and Lowen [6] studied measures of separability and the Lindelöf property. Lowen [22] introduced approach spaces within the topological–uniform–metric triad. Li and Zhang [20] examined sober metric approach spaces. Other authors, in collaboration with Lowen such as Sion [23], Verbeeck [24,25], and Verwulgen [26] explored various properties of approach spaces. In our work, we introduced the foundational concept of an approach Hilbert space, and examined the relationship between Hilbert spaces and Hilbert algebras through the completion of approach Hilbert spaces. This allowed us to extend the normed space, imposing an additional condition on the norm structure namely, that the distance generated by the norm function is defined between subsets of the power set and points in the approach space. Furthermore, we introduced the notions of approach Cauchy sequences and approach convergent sequences, established their properties in the context of Hilbert spaces, and derived several significant results.

2. Approach Space

Definition 2.1 *(Linear Algebra) [14] Let $\mathcal{T}$ be linear space over the field $\mathcal{E} = \mathbb{C}$ or $\mathbb{R}$. $\mathcal{T}$ is called linear algebra, if satisfy the conditions:*

1. *$\mathcal{T}$ is non-empty set.*
2. *An operations of multiplicative define in $\mathcal{T}$ satisfies the conditions for all a, b and $c \in \mathcal{T}$ and $\alpha \in \mathcal{E}$.*
3. *$\alpha (ab) = (\alpha a) b = a(\alpha b)$.*
4. *$(ab) c = a(bc)$.*
5. *$(a + b) c = (ac + bc)$ and $a (b + c) = ab + ac$.*

Definition 2.2 *(Symmetric Algebra) [18] A set Z is called a Symmetric algebra if it is satisfy the following conditions*

1. *$\mathcal{Z}$ is algebra.*
2. *An operation is defined in $\mathcal{Z}$ which each element $a \in \mathcal{Z}$ the element $a^* \in \mathcal{Z}$ satisfies the following conditions :*
3. *$(\alpha a \ + \beta b)^* = \ \bar{\alpha} a^* + \bar{\beta} b^*$ for all $a, \ b \in \mathcal{Z}, \alpha, \ \in \ \mathcal{E}$.*
4. *$a^{**} = a$.*
5. *$(ab)^* = b^* a^*$.*

*$a \to a^*$ we shall name this operation involution on an algebra $\mathcal{Z}$ if satisfying **(i), (ii), (iii)** and the element a^* called the adjoin ofa.*

Definition 2.3 *(Normed Algebra) [29] Let $\mathcal{Z}$ is linear algebra, $\mathcal{Z}$ is called normed algebra if satisfy the conditions:*

1. *$\mathcal{Z}$ is a normed space.*
2. *$\mathcal{Z}$ is an algebra.*
3. *For each $a, b \in \mathcal{Z}$, $\|ab\| \leq \|a\| \|b\|$.*
4. *If $\mathcal{Z}$ algebra with identity, $\|e\| = 1$.*

The pair $(\mathcal{Z}, \|.\|)$ is called normed algebra and $\mathcal{Z}$ is called Banach algebra if it is a complete normed algebra.

Definition 2.4 *(Approach space) Let $\mathcal{Z}$ be a non-empty set. A function $\delta\colon \mathcal{Z} \times 2^{\mathcal{Z}} \to [-\infty,\ \infty]$ is called distance on $\mathcal{Z}$ if it satisfies the following properties:*

1. *For all $a \in \mathcal{Z}:\ \ \delta(a, a) = 0$.*
2. *For all $a \in \mathcal{Z}:\ \ \delta(a,\ \emptyset) = \infty$.*
3. *for all $a \in \mathcal{Z}:$ and $a \in \mathcal{T}$ $\delta(a,\ \mathcal{T})\ \geq 0$.*
4. *for all $a \in \mathcal{Z}:$ for each $\mathcal{T},\ L \in\ 2^{\mathcal{Z}}: \delta(a,\ \mathcal{T} \cap L) = \max(\delta(a,\ \mathcal{T}), \delta(a,\ L))$.*
5. *for all $a \in \mathcal{Z}:$ for each $\mathcal{T} \in 2^{Z}$, for each $\varepsilon \in [0,\ \infty]: \delta(a, \mathcal{T}) \leq \delta(a, \mathcal{T}^{\varepsilon}) + \varepsilon$. For any $\varepsilon \in [0, \infty]$.*

$\mathcal{T}^{\varepsilon} := \{a\ \in\ \mathcal{Z} |\ \delta(a,\ \mathcal{T})\ \leq\ \varepsilon\}$. A pair $(\mathcal{Z},\ \delta)$ where δ is a distance is called an approach space and denoted by Ap-spaces. Instead of (5) for all a belong $\mathcal{Z}$ and $\mathcal{T},\ L \in 2^{\mathcal{Z}}.(5')\delta(a, \mathcal{T}) \leq\ \delta(a, L) + \sup_u \in L\delta(u, \mathcal{T})(5')$ is equivalent to (5).

Definition 2.5 *(Normed approach space) Let $\mathcal{Z}$ be app-vector space. A triple $(\mathcal{Z}, \|.\|, \delta_{\|\cdot\|})$ said to be normed approach space if satisfy the following :*

1. *$\|a\| = 0$ if and only if $a = 0$.*
2. *$\|\alpha.a\| =\mid \alpha \mid .\|a\|$ for each $\alpha\ \in\ \mathcal{E}, a \in\ \mathcal{Z}$.*
3. *$\|a + b\| \leq \|a\| + \|b\|$ for each $a, b \in\ \mathcal{Z}$.*
4. *$\|a\| \geq 0$,for for each $a \in \mathcal{Z}$.*
5. *$\delta_{\|\cdot\|}(a,\ \mathcal{T}) = \sup_{a\in\mathcal{Z}} \inf_{\mathfrak{r}\in\mathcal{T}} \|a - \mathfrak{r}\|$.*

Definition 2.6 *(Contraction) Let $(\mathcal{Z}, \delta)$ and $(\mathcal{Z}', \delta')$ are app-spaces. A function $f : \mathcal{Z} \to \mathcal{Z}'$ is called contraction if for all $a \in \mathcal{Z}$, for all $\mathcal{T} \in\ 2^{\mathcal{Z}}, \delta'(f((a), f(\mathcal{T})) \leq \delta(a, \mathcal{T})$.*

3. Approach Hilbert Algebra

Definition 3.1 *(Approach Inner Product) Let $\mathcal{Z}$ be approach vector space approach inner product on Ap-vector space is a function $\langle\cdot,\cdot\rangle : \mathcal{Z} \times \mathcal{Z} \to \mathcal{E}$ satisfy the following. for each $a, b \in \mathcal{Z}$ and $\alpha \in \mathcal{E}$.*

1. *$\langle a, a\rangle_{\delta} \geq 0$.*
2. *$\langle a, a\rangle_{\delta} = 0$ if and only if $a = 0$.*
3. *$\overline{\langle a, b\rangle}_{\delta} = \langle b, a\rangle_{\delta}$.*
4. *$\langle \alpha a, b\rangle_{\delta} = \alpha \langle a, b\rangle_{\delta}$.*
5. *$\mid \langle a, \daleth(\mathcal{T})\rangle_{\delta} \mid= \delta(a, \mathcal{T})$. Where $\daleth : 2^{Z} \to \mathcal{Z}$ is choices function.*

the pair $(\mathcal{Z}, \langle\cdot,\cdot\rangle)$ is called Ap-inner product. Which denoted by ApIp

Definition 3.2 *(Aproach Hilbert algebra)*

1. *$\mathcal{Z}$ is symmetric approach a normed algebra.*
2. *$\mathcal{Z}$ a Hilbert space.*
3. *The norm in $\mathcal{Z}$ matches the norm in the Hilbert space.*
4. *$\langle ab, c\rangle_{\delta} = \langle b, a^{*}c\rangle_{\delta}$ for each a, b and c belong $\mathcal{Z}$.*

5. *$aa^* \neq 0$ for each $a \neq 0$.*

Definition 3.3 *(Multi-character) $\mathcal{T}$ be approach Hilbert algebra for all $a, b \in \mathcal{T}$ and $\alpha \in \mathcal{E}$ and φ be a function from $\mathcal{T}$ to $\mathcal{T}$ is called multicharacter if satisfy:*

1. $\varphi(\ \alpha a) = \ \alpha\varphi(a)$
2. $\varphi(a^*) = (\varphi(a))^*$.
3. $\varphi(a + b) = \varphi(a) + \varphi(b)$.
4. $\varphi(ab) = a\varphi(b)$.
5. $\varphi(e) = 1$.
6. *φ is contraction.*

Proposition 3.4 *Let be approach Hilbert algebra and $\text{ß}(H)$ be a set of all bounded operator on H. Then $\text{ß}(H)$ is approach Hilbert algebra.*

Proof: Let $\Finv, \wp$ and $\mathscr{K}$ belong to $\text{ß}(H)$ and β and α belong to $\mathcal{E}$ for all $\mathcal{T}$ and S belong to $2^{\text{ß}(H)}$ and for all $\mathfrak{r}$ and s belong to H such that $(\Finv + \wp)(\mathfrak{r}) = \Finv(\mathfrak{r}) + \wp(\mathfrak{r})$ for all $\mathfrak{r} \in \mathcal{T}$ and for all $\alpha \in \mathcal{E}$ then $(\alpha\Finv)(\mathfrak{r}) = \alpha(\Finv(\mathfrak{r}))$ and $(\Finv.\wp)(\mathfrak{r}) = \Finv(\mathfrak{r}).\wp(\mathfrak{r})$ to prove $(\text{ß}(H), \delta, *, \odot)$ is approach vector.

First

We must prove $\text{ß}(H)$ is approach space such that

I. $\delta(\Finv, \{\Finv\}) = \inf_{\Finv \in \text{ß}(H)} d(\Finv, \Finv) = 0$.

II. $\delta(\Finv, \mathcal{T}) = \inf_{\wp \in \mathcal{T}} d(\Finv, \wp) = |\Finv - \wp| \geq 0$.

III. If $\mathcal{T} = \emptyset$ then $\delta(\Finv, \mathcal{T}) = \delta(\Finv, \emptyset) = \infty$.

IV. $\delta(\Finv, \mathcal{T} \cap L) = \inf_{\wp \in \mathcal{T} \cap L} d(\Finv, \wp) = \max\{\inf_{\wp \in \mathcal{T}} d(\Finv, \wp), \inf_{\wp \in L} d(\Finv, \wp)\} = \max\{\delta(\Finv, \mathcal{T}), \delta(\Finv, L)\}$.

V. $\delta(\Finv, \mathcal{T}) \leq \delta(\Finv, \mathcal{T}) + \varepsilon = \delta(\Finv, \mathcal{T}^{\varepsilon}) + \varepsilon$.

Second

To prove $\text{ß}(H)$ is symmetric $\Finv : \text{ß}(H) \times \text{ß}(H) \to \mathcal{E}$ such that $\Finv^* = \overline{\Finv(\mathfrak{r})}$ and $\langle \Finv, \wp \rangle = \Finv(\mathfrak{r})\overline{\wp(\mathfrak{r})}$ for all $\Finv$ and $\wp$ belong to $\text{ß}(H)$

A. To prove $\text{ß}(H)$ is symmetric then

  I. $\text{ß}(H)$ is non-empty set.

  II. An operation of multiplicative define in $\text{ß}(H)$ satisfies the condition for all $\Finv, \wp$ and $\mathscr{K}$ belong to $\text{ß}(H)$ and α belong to $\mathcal{E}$

   i. $\alpha(\Finv.\wp)(\mathfrak{r}) = \alpha(\Finv.\wp)(\mathfrak{r}) = (\alpha\Finv(\mathfrak{r}))\wp(\mathfrak{r}) = (\Finv(\mathfrak{r})\alpha)\wp(\mathfrak{r}) = \Finv(\mathfrak{r})\alpha\wp(\mathfrak{r}) = \Finv(\mathfrak{r})(\alpha\wp(\mathfrak{r}))$.

   ii. $(\Finv.\wp)(\mathfrak{r}).\mathscr{K}(\mathfrak{r}) = \Finv(\mathfrak{r}).\wp(\mathfrak{r}).\mathscr{K}(\mathfrak{r}) = \Finv(\mathfrak{r}).(\wp.\mathscr{K})(\mathfrak{r})$.

   iii. $(\Finv + \wp)(\mathfrak{r}).\mathscr{K}(\mathfrak{r}) = (\Finv(\mathfrak{r}) + \wp(\mathfrak{r})).\mathscr{K}(\mathfrak{r}) = \Finv(\mathfrak{r}).\mathscr{K}(\mathfrak{r}) + \wp(\mathfrak{r}).\mathscr{K}(\mathfrak{r}) = \mathscr{K}(\mathfrak{r}).\Finv(\mathfrak{r}) + \mathscr{K}(\mathfrak{r}).\wp(\mathfrak{r}) = \mathscr{K}(\mathfrak{r}).(\Finv + \wp)(\mathfrak{r})$.

B. An operation is defined in $\text{ß}(H)$ which each element $\Finv$ belong to $\text{ß}(H)$ the element $\Finv^*$ belong to $\text{ß}(H)$ satisfies the following condition:

  I. $((\alpha\Finv + \beta\wp)(\mathfrak{r}))^* = \overline{(\alpha\Finv + \beta\wp)(\mathfrak{r})} = \overline{(\alpha\Finv(\mathfrak{r}) + \beta\wp(\mathfrak{r}))} = \overline{\alpha\Finv(\mathfrak{r})} + \overline{\beta\wp(\mathfrak{r})} = \overline{\alpha}\Finv^*(\mathfrak{r}) + \overline{\beta}\wp^*(\mathfrak{r})$.

  II. $\Finv^{**}(\mathfrak{r}) = \overline{\Finv}^*(\mathfrak{r}) = \overline{\overline{\Finv}}(\mathfrak{r}) = \Finv(\mathfrak{r})$.

  III. $(\Finv.\wp)^*(\mathfrak{r}) = \overline{(\Finv(\mathfrak{r}).\wp(\mathfrak{r}))} = \overline{\Finv(\mathfrak{r}).\wp(\mathfrak{r})} = \overline{\wp(\mathfrak{r}).\Finv(\mathfrak{r})} = \overline{\wp(\mathfrak{r})}.\overline{\Finv(\mathfrak{r})} = \wp^*(\mathfrak{r}).\Finv^*(\mathfrak{r})$.

Third

To prove $(\beta(H), \delta, *)$ is approach vector space

I. $(\beta(H), s)$ is an approach.

II. $(\beta(H), s)$ is a grope it is clear.

III. $* : \beta(H) \times \beta(H) \to \beta(H) : (\dashv, \wp) \to \dashv + \wp$

Forth

To prove $(\beta(H), \delta, *)$ is approach vector

I. $(\beta(H), \delta, *)$ is an approach.

II. $\beta\dashv(\imath)$ belong to $\beta(H)$.

III. $\beta(\dashv + \wp)(\imath) = \beta(\dashv(\imath) + \wp(\imath)) = \beta\dashv(\imath) + \beta\wp(\imath)$.

IV. $((\dashv + \wp)(\imath))\beta = (\dashv(\imath) + \wp(\imath))\beta = \dashv(\imath)\beta + \wp(\imath)\beta$.

V. $(\Theta\beta)\dashv(\imath) = \Theta\beta\dashv(\imath) = \Theta(\beta\dashv(\imath)) = \Theta(\beta\dashv)(\imath)$.

VI. $1.\dashv(\imath) = \dashv(\imath)$.

Fifth

To prove $(\beta(H), \delta, *)$ is normed approach, we define $\|\cdot\|$:$\beta(H) \to R$,by $\|\dashv\| = \sup_{a\in H}|\dashv(\imath)|$ then

I. $(\beta(H), \delta)$ is approach space.

II. $\|\dashv\| = 0 \to \|\dashv\| = \sup_{a\in H}|\dashv(a)| = 0 \to |\dashv(a)| = 0 \to \dashv(a) = 0$.

III. $\|\alpha\dashv\| = \sup_{a\in H}|\alpha\dashv(a)| = |\alpha|\sup_{a\in H}|\dashv(a)| = |\alpha|\dashv$ for all $\alpha \in \mathcal{E}$ and $\dashv \in \beta(H)$.

IV. $\|\dashv + \wp\| = \sup_{a\in H}|\dashv(a) + \wp(a)| \le \sup_{a\in H}|\dashv(a)| + \sup_{a\in H}|\wp(a)| = \|\dashv\| + \|\wp\|$.

V. $\delta_{\|\cdot|}(\dashv, \mathcal{T}) = \sup_{\dashv\in\beta(H)}\inf_{\wp\in\mathcal{T}}\|\dashv(\imath) - \wp(\imath)\|$.

1. $\delta_{\|\cdot\|}(\dashv, \mathcal{T}) = 0$ then $\sup_{\dashv\in\beta(H)}\inf_{\wp\in\mathcal{T}}\|\dashv(\imath) - \wp(\imath)\| = 0$ then $\sup_{\dashv\in\beta(H)}\inf_{\wp\in\mathcal{T}}\sup_{a\in H}|\dashv(a) - \wp(a)| = 0$ then $\dashv(a) - \wp(a) = 0$ then $\dashv(a) = \wp(a)$.
2. $\delta_{\|\cdot\|}(\dashv, \mathcal{T}) = \sup_{\dashv\in\beta(H)}\inf_{\wp\in\mathcal{T}}\|\dashv(a) - \wp(a)\| = \sup_{\dashv\in\beta(H)}\inf_{\wp\in\mathcal{T}}\sup_{a\in H}|\dashv(a) - \wp(a)| \ge 0$.
3. If $\mathcal{T} = \emptyset$ then $\delta_{\|\cdot\|}(\dashv, \mathcal{T}) = \infty$.
4. $\delta_{\|\cdot\|}(\dashv, \mathcal{T}\cap) = \sup_{\dashv\in\beta(H)}\inf_{\wp\in\mathcal{T}\cap L}\|\dashv(a) - \wp(a)\| = \sup_{\dashv\in\beta(H)}\inf_{\wp\in\mathcal{T}\cap L}\sup_{a\in H}|\dashv(a) - \wp(a)| = \max\left\{\sup_{\dashv\in\beta(H)}\inf_{\wp\in\mathcal{T}}\sup_{a\in H}|\dashv(a) - \wp(a)|, \sup_{\dashv\in\beta(H)}\inf_{\wp\in L}\sup_{\imath\in X}|\dashv(a) - \wp(a)|\right\}$ $= \max\left\{\delta_{\|\cdot\|}(\dashv, \mathcal{T}), \delta_{\|\cdot\|}(\dashv, S)\right\}$.
5. $\delta_{\|\cdot\|}(\dashv, \mathcal{T}) \le \delta_{\|\cdot\|}(\dashv, \mathcal{T}) + \varepsilon = \delta_{\|\cdot\|}(\dashv, \mathcal{T}^{\varepsilon}) + \varepsilon$.

Sixth

To prove $\beta(H)$ is approach inner product such that $\langle\cdot,\cdot\rangle : \beta(H) \times \beta(H) \to \beta(H)$ such that $\langle\dashv, \wp\rangle = \dashv(\imath)\overline{\wp(\imath)}$. Then $\dashv, \wp$ and $h \in \beta(H)$ and $\alpha, \beta \in Q$.

i. $\langle\dashv, \dashv\rangle = \dashv(\imath)\overline{\dashv(\imath)} \ge 0$.

ii. $\langle\dashv, \dashv\rangle = \dashv(\imath)\overline{\dashv(\imath)} = 0$ then $\dashv(a)\overline{\dashv(a)} = 0$ then either $(Rel(\dashv(a)))^2 + (Ima(\dashv(a)))^2 = 0$ then $(Rel(\dashv(a)))^2 = 0$ then $Rel(\dashv(a)) = 0$ and $(Ima(\dashv(a)))^2 = 0$ then $Ima(\dashv(a)) = 0$ then $\dashv(a) = 0$.

iii. $\overline{\langle\dashv, \wp\rangle} = \overline{\dashv(\imath)\overline{\wp(\imath)}} = \overline{\dashv(\imath)}\wp(\imath) = \wp(\imath)\overline{\dashv(\imath)} = \langle\wp, \dashv\rangle$.

iv. $\langle\alpha\dashv + \beta\wp, \mathcal{K}\rangle = (\alpha\dashv(\imath) + \beta\wp(\imath))\overline{\mathcal{K}(\imath)} = \alpha\dashv(\imath)\overline{\mathcal{K}(\imath)} + \beta\wp(\imath)\overline{\mathcal{K}(\imath)} = \alpha\dashv, \mathcal{K} + \beta\langle\wp, \mathcal{K}\rangle$.

Seventh

To prove $\left(\text{ß}(\text{Ħ}), \|\cdot\|, \delta_{\|\cdot\|}\right)$ is a normed approach space and $\|\cdot\| : \text{ß}(\text{Ħ}) \to R, \|\Finv\| = \sup_{\Finv\in \text{ß}(H)} |\Finv|$.

i. $\left(\text{ß}(\text{Ħ}), \delta_{\|\cdot\|}\right)$ is Ap-space.

ii. $\|\Finv\| = 0 \to \|\Finv\| = \sup_{\text{ꝛ}\in \mathcal{T}} |\Finv| = 0 \to |\Finv| = 0 \to \Finv = 0.$

iii. $\|\alpha\Finv\| = \sup_{\text{ꝛ}\in \mathcal{T}} |\alpha\Finv| = |\alpha| \sup_{\text{ꝛ}\in \mathcal{T}} |\Finv| = |\alpha| \|\Finv\|$ for each α belon to $\mathcal{E}$ and $\Finv \in \text{ß}(\text{Ħ})$.

iv. $\|\Finv + \wp\| = \sup_{\Finv\ ,\ \wp\in \text{ß}(H)} |\Finv + \wp| \le \sup_{\Finv\in \text{ß}(H)} |\Finv| + \sup_{\ \wp\in \text{ß}(H)} |\wp| = \|\Finv| + \|\wp|.$

v. $\delta_{\|\cdot\|}(\Finv, L) = \sup_{\Finv\in \text{ß}(H)} \inf_{\text{ꝛ}\in L} \|\Finv - \text{ꝛ}\|$ for all $L \in 2^{\text{ß}(H)}$ then

a. $If \delta_{\|\cdot\|}(\Finv, L) = 0$ then $\sup_{\Finv\in \text{ß}(H)} \inf_{\text{ꝛ}\in L} \|\Finv - \text{ꝛ}\| = 0$ then $\|\Finv - \text{ꝛ}\| = 0$ then $\Finv - \text{ꝛ} = 0$ then $\Finv = \text{ꝛ}$ but $\text{ꝛ} \in L$ then $\Finv \in L$.

b. $\delta_{\|\cdot\|}(\Finv, L) = \sup_{\Finv\in \text{ß}(H)} \inf_{\text{ꝛ}\in L} \|\Finv - \text{ꝛ}\| \ge 0.$

c. $\delta_{\|\cdot\|}(\Finv, \emptyset) = \infty.$

d. $\delta_{\|\cdot\|}(\Finv, \mathcal{T}\cap L) = \sup_{\Finv\in \text{ß}(H)} \inf_{\text{ꝛ}\in \mathcal{T}\cap L} W = \sup_{\Finv\in \text{ß}(H)} \inf_{\text{ꝛ}\in \mathcal{T}\cap L} \sup_{\Finv\in \text{ß}(H)} |\Finv - \text{ꝛ}| = \max\left\{ \sup_{\Finv\in \text{ß}(H)} \inf_{\text{ꝛ}\in L} \sup_{\Finv\in \text{ß}(H)} |\Finv - \text{ꝛ}|, \sup_{\Finv\in \text{ß}(H)} \inf_{\text{ꝛ}\in \mathcal{T}} \sup_{\Finv\in \text{ß}(H)} |\Finv - \text{ꝛ}| \right\}.$

e. $\delta_{\|\cdot\|}(\Finv, \mathcal{T}) = \sup_{\Finv\in \text{ß}(H)} \inf_{\text{ꝛ}\in \mathcal{T}} W \le \sup_{\Finv\in \text{ß}(H)} \inf_{\text{ꝛ}\in \mathcal{T}} \sup_{\Finv\in \text{ß}(H)} |\Finv - \text{ꝛ}| + \varepsilon = \sup_{\Finv\in \text{ß}(H)} \inf_{\text{ꝛ}\in \mathcal{T}^{\varepsilon}} \sup_{\Finv\in \text{ß}(H)} |\Finv - \text{ꝛ}| + \varepsilon = \delta_{\|\cdot\|}(\Finv, \mathcal{T}^{\varepsilon}).$

Eighth

To prove $\Finv\Finv^* \ne 0$ then $\Finv\Finv^* = \Finv\bar{\Finv} = (Rel(\Finv))^2 + (Img(\Finv))^2 \ne 0$ if $\Finv \ne 0$ then $Rel(\Finv) \ne 0$ *or* $Img(\Finv) \ne 0$ then $(Rel(\Finv))^2 + (Img(\Finv))^2 \ne 0$ then $\Finv\Finv^* \ne 0$.

Ninth

To prove $\langle Fg, \mathcal{K}\rangle = \langle \Finv, \wp^*\mathcal{K}\rangle = \langle \wp, KF^*\rangle$

$$\langle Fg, \mathcal{K}\rangle = \Finv(\text{ꝛ})\, \wp(\text{ꝛ})\, \overline{\mathcal{K}(\text{ꝛ})} \tag{3.1}$$

$$\langle Fg, \mathcal{K}\rangle = \Finv(\text{ꝛ})\, \overline{\overline{\wp(\text{ꝛ})}\mathcal{K}(\text{ꝛ})} = \Finv(\text{ꝛ})\, \wp(\text{ꝛ})\, \overline{\mathcal{K}(\text{ꝛ})} \tag{3.2}$$

$$\langle \wp, KF^*\rangle = \wp(\text{ꝛ})\, \overline{\mathcal{K}(\text{ꝛ})\, \overline{\Finv(\text{ꝛ})}} = \wp(\text{ꝛ})\, \overline{\mathcal{K}(\text{ꝛ})}\Finv(\text{ꝛ}) = \Finv(\text{ꝛ})\, \wp(\text{ꝛ})\, \overline{\mathcal{K}(\text{ꝛ})} \tag{3.3}$$

□

Proposition 3.5 *Let $\mathcal{T}$ be an approach Hilbert algebra. Then set of all multicharacter $MC(\mathcal{T})$ on $\mathcal{T}$ is approach Hilbert algebra.*

Proof:

First

Let $\Finv, \wp$ and $\mathcal{K}$ belong to $MC(\mathcal{T})$ and β and α belong to $\mathcal{E}$ for all ꝛ and y belong to $\mathcal{T}$ such that $(\Finv + \wp)(\text{ꝛ}) = \Finv(\text{ꝛ}) + \wp(\text{ꝛ})$ for all $\text{ꝛ} \in \mathcal{T}$ and for all $\alpha \in \mathcal{E}$ then $(\alpha\Finv)(\text{ꝛ}) = \alpha(\Finv(\text{ꝛ}))$ and $(\Finv.\wp)(\text{ꝛ}) = \Finv(\text{ꝛ}).\wp(\text{ꝛ})$ to prove $(MC(\mathcal{T}), \delta, *, \odot)$ is approach vector.

A. To prove $MC(\mathcal{T})$ is approach space $\delta\colon MC(\mathcal{T}) \times 2^{MC(\mathcal{T})} \to [-\infty, \infty]$ such that

$$\delta(\Finv, A) = \begin{cases} \inf_{\Finv\notin \mathcal{T}} |\Finv| & if\ \ \mathcal{T} \ne \emptyset. \\ \infty & if\ \ \mathcal{T} = \emptyset. \\ 0 & if\ \ A \ne \emptyset, \Finv \in \mathcal{T}. \end{cases}$$

1. We prove $\delta_e(a, \mathcal{T}) \ge 0$.

   I. If $\mathcal{T}$ is non-empty $\Finv \notin \mathcal{T}$ then $\delta_e(\Finv, \mathcal{T}) = 0$.

II. If $\mathcal{T}$ is empty then $\delta_e(\daleth, \mathcal{T}) = \infty > 0$.

III. If A is non-empty $\daleth \in A$ then $\delta_e(\daleth, \mathcal{T}) = \sup_{\daleth \in A} |\daleth| \geq 0$.

2. If $\delta_e(\daleth, \mathcal{T}) = 0$ then $\daleth \in \mathcal{T}$.
3. If $\mathcal{T} = \emptyset$ then $\daleth \notin \mathcal{T}$ and $\delta_e(\daleth, \mathcal{T}) = \infty > 0$.
4. For each $\mathcal{T} \cap b \in 2^{MC(\mathcal{T})}$ then $\delta(\daleth, \mathcal{T} \cap b)$.

I. If $\mathcal{T} \cap b \neq \emptyset, \delta(\daleth, \mathcal{T} \cap b) = \sup_{\daleth \notin \mathcal{T} \cap b} |\daleth| = \sup\{\sup_{\daleth \notin \mathcal{T}} |\daleth|, \sup_{\daleth \notin b} |\daleth|\} = \sup\{\delta(\daleth, \mathcal{T}), \delta(\daleth, b)\}$.

II. If $\mathcal{T} \cap b = \emptyset \delta(\daleth, \mathcal{T} \cap b) = \infty = \{\infty, \infty\} = \{\sup_{\daleth \notin \mathcal{T}} |\daleth|, \sup_{\daleth \notin b} |\daleth|\} = sup\{\sup_{\daleth \notin \mathcal{T}} |\daleth|, \sup_{\daleth \notin b} |\daleth|\}$ $= \sup\{\delta(\daleth, \mathcal{T}), \delta(\daleth, b)\}$.

5. For each $\daleth \in MC(\mathcal{T})$: *and foreach* $A \in 2^{MC(\mathcal{T})}$, $\delta(\daleth, \mathcal{T}) \leq \delta(\daleth, \mathcal{T}) + \varepsilon \leq \delta(\daleth, \mathcal{T}^{\varepsilon}) + \varepsilon$. Then $(MC(\mathcal{T}), \delta)$ is an approach.

B. $\Theta\daleth(\mathfrak{r})$ belong to $MC(\mathcal{T})$.

C. $\beta(\daleth + \wp)(\mathfrak{r}) = \beta(\daleth(\mathfrak{r}) + \wp(\mathfrak{r})) = \beta\daleth(\mathfrak{r}) + \beta\wp(\mathfrak{r})$.

D. $((\daleth + \wp)(\mathfrak{r}))\beta = (\daleth(\mathfrak{r}) + \wp(\mathfrak{r}))\beta = \daleth(\mathfrak{r})\beta + \wp(\mathfrak{r})\beta$.

E. $(\Theta\beta)\daleth(\mathfrak{r}) = \Theta(\beta\daleth(\mathfrak{r}))$.

F. $1.\daleth(\mathfrak{r}) = \daleth(\mathfrak{r})$.

Second

To prove $MC(\mathcal{T})$ is symmetric algebra

1. To prove $MC(\mathcal{T})$ algebra

i. $I : \mathcal{T} \to \mathcal{T}$ such that $I(\mathfrak{r}) = \mathfrak{r}$ for all $\mathfrak{r} \in \mathcal{T}$. To prove $I \in MC(\mathcal{T})$

a. $I(\alpha\mathfrak{r}) = (\alpha\mathfrak{r}) = (\alpha\mathfrak{r}) = \alpha I(\mathfrak{r})$.
b. $I(\mathfrak{r}^*) = \mathfrak{r}^* = (I)^* = (I(\mathfrak{r}))^*$.
c. $I(y + \mathfrak{r}) = (y + \mathfrak{r}) = y + \mathfrak{r} = I(y) + I(\mathfrak{r})$.
d. $I(yx) = yx = y(I) = yI(\mathfrak{r})$.
e. $I(e) = e = 1$.
f. *Let* $(MC(\mathcal{T}), \delta), (MC(\mathcal{T})', \delta')$ are approach spaces $I : MC(\mathcal{T}) \to MC(\mathcal{T})'$ such that $I(\daleth) = \daleth$ for each $\daleth \in MC(\mathcal{T})$

I. If $\mathcal{T} \neq \emptyset$ then $I(\mathcal{T}) \neq \emptyset$ then $\delta'(I(\daleth), I(\mathcal{T})) = \max|I(\daleth)| = \daleth \leq \max_{\daleth \notin \mathcal{T}} |\daleth| = \delta(\daleth, \mathcal{T})$.

II. If $\mathcal{T} = \emptyset$ then $I(\mathcal{T}) = \emptyset$ then $\delta'(I(\daleth), I(\mathcal{T})) = Max|I(\daleth)| = \infty \leq \infty = \max_{\daleth \notin \mathcal{T}} |\daleth| = \delta(\daleth, \mathcal{T})$.

III. if $\mathcal{T} \neq \emptyset, \daleth \in \mathcal{T}$ then $\delta'(I(\daleth), I(\mathcal{T})) = \daleth \leq \daleth = \delta(\daleth, \mathcal{T})$. Then $MC(\mathcal{T}) \neq \emptyset$.

IV. Any operation of multiplicative is define in $MC(\mathcal{T})$

Third

For each $\daleth$ and $\wp$ belong to $MC(\mathcal{T})$ to prove $\daleth.\wp$ belong to $MC(\mathcal{T})$

I. $(\daleth.\wp)(\beta\mathfrak{r}) = \daleth(\beta\mathfrak{r}).\wp(\beta\mathfrak{r}) = \beta\daleth(\mathfrak{r}).\beta\wp(\mathfrak{r}) = \beta(\daleth(\mathfrak{r}).\beta\wp(\mathfrak{r})) = \beta(\beta\daleth(\mathfrak{r}).\wp(\mathfrak{r})) = \beta(\daleth.\wp)(\beta\mathfrak{r})$.

II. $(\daleth.\wp)(\mathfrak{r}^*) = \daleth(\mathfrak{r}^*).\wp(\mathfrak{r}^*) = (\daleth(\mathfrak{r}))^*.(\wp(\mathfrak{r}))^* = ((\daleth.\wp)(\mathfrak{r}))^*$.

$(\daleth.\wp)(\mathfrak{r} + y) = \daleth(\mathfrak{r} + y).\wp(\mathfrak{r} + y) = \daleth(\mathfrak{r}) + \daleth(y) + \wp(\mathfrak{r}) + \wp(y) = \daleth(\mathfrak{r}) + \wp(\mathfrak{r}) + \daleth(y) + \wp(y) = (\daleth + \wp)(\mathfrak{r}) + (\daleth + \wp)(y)$.

III. $(\daleth.\wp)(yx) = \daleth(yx).\wp(yx) = y\daleth(\mathfrak{r}).yg(\mathfrak{r}) = y(\daleth(\mathfrak{r}).yg(\mathfrak{r})) = y(yF(\mathfrak{r}).\wp(\mathfrak{r})) = y(\daleth.\wp)(yx)$.

IV. $(\daleth.\wp)(e) = 1 = 1.1 = \daleth(e).\wp(e)$.

V. Let $(MC(\mathcal{T}), \delta), (MC(\mathcal{T})', \delta')$ are two approach spaces $P : MC(\mathcal{T}) \to MC(\mathcal{T})'$

1. If $\mathcal{T} \neq \emptyset$ then $P(\mathcal{T}) \neq \emptyset$ then $\delta'(P(\daleth.\wp), P(\mathcal{T})) = \sup|P(\daleth.\wp)| = 0 \leq \sup_{\daleth.\wp\notin\mathcal{T}}|\daleth.\wp| = \delta(\daleth.\wp, \mathcal{T})$.
2. If $\mathcal{T} = \emptyset$ then $P(f.\wp)(\mathcal{T}) = \emptyset$ then $\delta'(P(\daleth.\wp), P(\mathcal{T})) = \sup|P(\daleth.\wp)| = \infty \leq \infty = \sup_{\daleth.\wp\notin\mathcal{T}}|\daleth.\wp|$ $= \delta(\daleth.\wp, \mathcal{T})$.
3. if $\mathcal{T} \neq \emptyset, \daleth.\wp \in \mathcal{T}$ then $\delta'(P(\daleth.\wp), P(\mathcal{T})) = 0 \leq 0 = \delta((\daleth.\wp), \mathcal{T})$.

A. $\beta(\daleth.\wp)(\mathfrak{r}) = \beta(\daleth(\mathfrak{r}).\wp(\mathfrak{r})) = (\beta\daleth(\mathfrak{r}).\wp(\mathfrak{r})) = \daleth(\mathfrak{r}).\beta\wp(\mathfrak{r})$.

B. $((\daleth.\wp)(\mathfrak{r})).\mathcal{K}(\mathfrak{r}) = (\daleth(\mathfrak{r}).\wp(\mathfrak{r})).\mathcal{K}(\mathfrak{r}) = \daleth(\mathfrak{r}).\wp(\mathfrak{r}).\mathcal{K}(\mathfrak{r}) = \daleth(\mathfrak{r}).(\wp(\mathfrak{r}).\mathcal{K}(\mathfrak{r})) = \daleth(\mathfrak{r}).(\wp.\mathcal{K}(\mathfrak{r}))$.

C. $((\daleth + \wp)(\mathfrak{r})).\mathcal{K}(\mathfrak{r}) = (\daleth(\mathfrak{r}) + \wp(\mathfrak{r})).\mathcal{K}(\mathfrak{r}) = \daleth(\mathfrak{r}).\mathcal{K}(\mathfrak{r}) + \wp(\mathfrak{r}).\mathcal{K}(\mathfrak{r}) = \mathcal{K}(\mathfrak{r}).\daleth(\mathfrak{r}) + \mathcal{K}(\mathfrak{r}).\wp(\mathfrak{r}) = \mathcal{K}(\mathfrak{r})(\daleth(\mathfrak{r}) + \wp(\mathfrak{r}))$.

1. An operation is define in $MC(\mathcal{T})$ which for each $\daleth$ belong $MC(\mathcal{T})$ the element $\daleth^*$ belong to $MC(\mathcal{T})$
   a. $(\propto \daleth + \beta\wp)^*(\mathfrak{r}) = \left(\bar{\Theta}\daleth^* + \bar{\beta}\wp^*\right)(\mathfrak{r}) = \bar{\Theta}\daleth^*(\mathfrak{r}) + \bar{\beta}\wp^*(\mathfrak{r})$.
   b. $\daleth^{**} = \left(\bar{\daleth}\right)^* = \bar{\bar{\daleth}} = \daleth$.
   c. $(\daleth.\wp)^* = \bar{\daleth}.\bar{\wp} = \bar{\wp}.\bar{\daleth} = (\wp.\daleth)^*$.

Forth

To prove $\left(MC(\mathcal{T}), \|\cdot\|, \delta_{\|\cdot\|}\right)$ is a normed approach space and $\{MC(\mathcal{T}) : \daleth : \mathcal{T} \to R\ \daleth \text{ is bounded}\}.\|\cdot\| : MC(\mathcal{T}) \to R, \|\daleth\| = \sup_{\mathfrak{r}\in\mathcal{T}}|\daleth(\mathfrak{r})|$.

1. $\left(MC(\mathcal{T}), \delta_{\|\cdot\|}\right)$ is app-space.
2. $\|\daleth\| = 0 \to \|\daleth\| = \sup_{\mathfrak{r}\in\mathcal{T}}|\daleth(\mathfrak{r})| = 0 \to |\daleth(\mathfrak{r})| = 0 \to \daleth(\mathfrak{r}) = 0$.
3. $\|\alpha\daleth\| = \sup_{\mathfrak{r}\in\mathcal{T}}|\daleth(\mathfrak{r})| = |\alpha|\sup_{\mathfrak{r}\in\mathcal{T}}|\daleth(\mathfrak{r})| = |\alpha|\,\|\daleth\|$ for each α belong to $\mathcal{E}$ and $\daleth \in MC(\mathcal{T})$.
4. $\|\daleth + \wp| = \sup_{\mathfrak{r}\in\mathcal{T}}|\daleth(\mathfrak{r}) + \wp(\mathfrak{r})| \leq \sup_{\mathfrak{r}\in\mathcal{T}}|\daleth(\mathfrak{r})| + \sup_{\mathfrak{r}\in\mathcal{T}}|\wp(\mathfrak{r})| = \|\daleth\| + \|\wp\|$

Forth

To prove $MC(\mathcal{T})$ inner product We define $\langle\cdot,\cdot\rangle : MC(\mathcal{T}) \times MC(\mathcal{T}) \to \mathcal{E}$. For each $\daleth, \wp$ and $\mathcal{K} \in MC(\mathcal{T})$ and α and μ belong to Q Such that $\langle\daleth, \wp\rangle = \daleth\bar{\wp}$ then

a. $\langle\daleth, \daleth\rangle_\delta = \daleth\bar{\daleth} \geq 0$.

b. $\langle\daleth, \daleth\rangle_\delta = 0$ then $\daleth\bar{\daleth} = 0\,[Rel(\daleth)]^2 + [Img(\daleth)]^2 = 0$ then $[Rel(\daleth)]^2 = 0$ and $[Img(\daleth)]^2 = 0$ then $[Rel(\daleth)] = 0$ and $[Img(\daleth)] = 0$ then $\daleth = 0$.

c. $\overline{\langle\daleth, \wp\rangle}_\delta = \overline{\daleth\bar{\wp}} = \bar{\daleth}\wp = \wp\bar{\daleth} = \langle\wp, \daleth\rangle$

d. $\langle\alpha\daleth + \ \mu\wp, \mathcal{K}\rangle_\delta = (\alpha\daleth + \ \mu\wp)\bar{\mathcal{K}} = \alpha FK + \ \mu\wp\bar{\mathcal{K}} = \alpha\langle\daleth, \mathcal{K}\rangle_\delta + \mu\langle\wp, \mathcal{K}\rangle_\delta$

Sixth

The norm in the $MC(\mathcal{T})$ coincides with theorem in the Hilbert space.

Seventh

$\langle\daleth\ \wp, \mathcal{K}\rangle = \daleth\wp\bar{\mathcal{K}} = \ \wp\ \daleth\bar{\mathcal{K}} = \wp\bar{\mathcal{K}}\daleth = \wp\overline{\bar{\daleth}\mathcal{K}} = \langle\wp, \daleth^*\mathcal{K}\rangle$

Eighth

if $\daleth\daleth^* \neq 0$ then then $\daleth\bar{\daleth} \neq 0\,[Rel(\daleth)]^2 + [Img(\daleth)]^2 \neq 0$ then $[Rel(\daleth)]^2 \neq 0$ *and* $[Img(\daleth)]^2 \neq 0$ then $[Rel(\daleth)] \neq 0$ and $[Img(\daleth)] \neq 0$ then $\daleth \neq 0$.

Ninth

To prove a triple $(MC(\mathcal{T}), \delta, *)$ approach group

a. $(MC(\mathcal{T}), \delta)$ is an approach.

b. $(MC(\mathcal{T}), \delta)$ is a group for all $\daleth, \wp$ and $\mathcal{K}$ belong to $MC(\mathcal{T})$ and t belong to $\mathcal{T}$.

   i. $(\daleth + \wp)(\mathfrak{r}) = \daleth(\mathfrak{r}) + \wp(\mathfrak{r})\,belongtoMC(\mathcal{T})$.

ii. $(\daleth+\wp)(\imath)+\mathcal{K}(\imath)=\daleth(\imath)+\wp(\imath)+\mathcal{K}(\imath)=\daleth(\imath)+(\wp(\imath)+\mathcal{K}(\imath))=\daleth(\imath)+(\wp+\mathcal{K})(\imath)$.

iii. $\hat{0}$ belong to $MC(\mathcal{T})$ $\hat{0}+\daleth=\daleth+\hat{0}=\daleth$.

iv. For all $\daleth$ belong to $MC(\mathcal{T})$ there exist $-\daleth$ belong to $MC(\mathcal{T})$ then $\daleth+(-\daleth)=\hat{0}$

c. $*:\ MC(\mathcal{T})\times MC(\mathcal{T})\rightarrow MC(\mathcal{T})$ such that $(\daleth,\wp)\rightarrow\daleth+\wp$ is the $\delta-$ contraction.

i. If $\mathcal{T}$ is non empty $\daleth$ and $\wp\in\mathcal{T}$ then $\delta_e(\daleth,\mathcal{T})=\delta'((\daleth+\wp)(\imath),(\daleth+\wp)(\mathcal{T}))=\sup_{\daleth+\wp\notin\mathcal{T}}|\daleth+\wp|\leq\sup_{\daleth\notin\mathcal{T}}|\daleth|+\sup_{\daleth\notin\mathcal{T}}|\wp|=\delta(\daleth,\mathcal{T})+\delta(\wp,b)$.

ii. If $\mathcal{T}$ is non-empty $\daleth$ and $\wp\notin\mathcal{T}$ then $\daleth+\wp\notin\mathcal{T}$ and $\delta'((\daleth+\wp)(\imath),(\daleth+\wp)(\mathcal{T}))=0\leq\delta((\daleth,\wp)(\imath),(\daleth,\wp)(\mathcal{T}))$.

iii. If $\mathcal{T}$ is empty then $\delta'((\daleth+\wp)(\imath),(\daleth+\wp)(\mathcal{T}))=\infty\leq\infty=\delta((\daleth,\wp)(\imath),(\daleth,\wp)(\mathcal{T}))$.

□

Example 3.6 *Let M_2 consider the set formed by all 2×2 orthogonal matrices, and it denote by. M_2 can be seen as a subset of $\mathbb{R}^4$ with the Euclidean metric, then $(M_2,\ \delta(d),\odot)$ is approach group. In fact,*

$$d(\mathcal{T},L)=2\sqrt{1-\cos(\Theta-\beta)}.$$

Then $d(\mathcal{T},L)=d\left(\mathcal{T}^{-1},L^{-1}\right)$. Such that

$$\delta_e(a,\ \mathcal{T})=\begin{cases}\infty & if\ a\ \notin\mathcal{T}\ and\ \mathcal{T}\ is\ bounded.\\ 0 & if\ a\ \in\mathcal{T}\ and\ \mathcal{T}\ is\ bounded.\\ \inf_{\imath\in\mathcal{T}}d(a,\imath) & if\ \ \mathcal{T}\ is\ unbounded.\end{cases}$$

Solve:

1. To prove $\delta_e(\mathcal{T},\ L)=0$ if and only if $L\in\mathcal{T}$.

   I. If $\mathcal{T}$ is bounded then $L\in\mathcal{T}$.

   II. If $\mathcal{T}$ is unbounded then $\delta_e(\mathcal{T},\mathcal{T})=0$ then $\inf_{\imath\in\mathcal{T}}d(\mathcal{T},\imath)=0$ then

$$\inf_{\imath\in\ \mathcal{T}}d\left(\begin{bmatrix}\cos\sigma & \sin\sigma\\ \sin\sigma & -\cos\sigma\end{bmatrix},\begin{bmatrix}\cos\beta & \sin\beta\\ \sin\beta & -\cos\beta\end{bmatrix}\right)=0\ then\ \Theta=\beta.$$

2. To prove $\delta_e(a,\ \mathcal{T})\geq 0$.

   I. If $\mathcal{T}$ is bounded then $L\in\mathcal{T}$ then $\delta_e(a,\ \mathcal{T})=0$.

   II. If $\mathcal{T}$ is bounded then $L\in\mathcal{T}$ then $\delta_e(a,\ \mathcal{T})=0$.

   III. If $\mathcal{T}$ is bounded then $\mathcal{T}\notin\mathcal{T}$ then $\delta_e(\mathcal{T},L)=\infty>0$.

   IV. If $\mathcal{T}$ is unbounded then $\delta_e(\mathcal{T},L)=0$ then

$$\inf_{\imath\in\ \mathcal{T}}d(\mathcal{T},\imath)=\inf_{\imath\in\mathcal{T}}d\left(\begin{bmatrix}\cos\sigma & \sin\sigma\\ \sin\sigma & -\cos\sigma\end{bmatrix},\begin{bmatrix}\cos\beta & \sin\beta\\ \sin\beta & -\cos\beta\end{bmatrix}\right)\geq 0.$$

3. If $\mathcal{T}=\emptyset$ then $\mathcal{T}$ is bounded then

$$\delta_e(\mathcal{T},\ \emptyset)=\inf_{\imath\in\ \mathcal{T}}\ d\left(\begin{bmatrix}\cos\sigma & \sin\sigma\\ \sin\sigma & -\cos\sigma\end{bmatrix},\emptyset\right)=\infty>0$$

4. To prove $\delta_e(\mathcal{T},\ \mathcal{T}_2\cap S_2)=\max\{\delta_e(\mathcal{T},\ \mathcal{T}_2),\delta_e(\mathcal{T},\ S_2)\}$.

   I. If $\mathcal{T}_2\cap S_2=\emptyset$ then $\delta_e(\mathcal{T},\ \mathcal{T}_2\cap S_2)=\inf_{\emptyset\in\mathcal{T}_2\cap S_2}d(\mathcal{T},\emptyset)=\infty=\max\{\infty,\infty\}=\max\left\{\delta_e(\mathcal{T},\ \mathcal{T}_2),\ \delta_e(\mathcal{T},\ S_2)\right\}$.

II. If $\mathcal{T}_2 \cap S_2 = N_2 \neq \emptyset$ then

$$\begin{aligned}\delta_e(\mathcal{T}, \mathcal{T}_2 \cap S_2) &= \inf_{N_2 \in \mathcal{T}_2 \cap S_2} d(\mathcal{T}, N_2) = \left\{\inf_{N_2 \in \mathcal{T}_2} d(\mathcal{T}, N_2), \inf_{N_2 \in S_2} d(\mathcal{T}, N_2)\right\} \\ &= \max\left\{\inf_{N_2 \in \mathcal{T}_2}\left\|\begin{bmatrix}\cos\sigma & \sin\sigma \\ \sin\sigma & -\cos\sigma\end{bmatrix} - N_2\right\|, \inf_{N_2 \in S_2}\left\|\begin{bmatrix}\cos\sigma & \sin\sigma \\ \sin\sigma & -\cos\sigma\end{bmatrix} - N_2\right\|\right\} \\ &= \max\left\{\inf_{N_2 \in \mathcal{T}_2} d(\mathcal{T}, N_2), \inf_{N_2 \in S_2} d(\mathcal{T}, N_2)\right\} \\ &= \max\{\delta_e(\mathcal{T}, \mathcal{T}_2), \delta_e(\mathcal{T}, S_2)\}.\end{aligned}$$

5.

$$\begin{aligned}\delta_e(\mathcal{T}, \mathcal{T}_2) = \inf_{\mathcal{T} \in \mathcal{T}_2} d(\mathcal{T}, \mathcal{T}_2) &\leq \inf_{\mathcal{T} \in \mathcal{T}_2}\left(\begin{bmatrix}\cos\sigma & \sin\sigma \\ \sin\sigma & -\cos\sigma\end{bmatrix}, \mathcal{T}_2\right) \\ &+ \varepsilon \leq \inf_{\mathcal{T} \in \mathcal{T}_2}\left(\begin{bmatrix}\cos\sigma & \sin\sigma \\ \sin\sigma & -\cos\sigma\end{bmatrix}, \mathcal{T}_2\right) + \varepsilon \leq \inf_{\mathcal{T} \in \mathcal{T}_2^{\varepsilon}} d(\mathcal{T}, \mathcal{T}_2) \\ &+ \varepsilon = \delta_e(\mathcal{T}, \mathcal{T}_2^{\varepsilon}) + \varepsilon.\end{aligned}$$

Then (M_2, δ_e) is approach space.

4. Main Result

Proposition 4.1 *Let $MC(\mathcal{T})$ be a set of all multicharacter. Let $S(\mathcal{H})$ be a left ideal and $\check{R}(\mathcal{H})$ be right ideal of the $\beta(\mathcal{H})$. Then $\check{R}(\mathcal{H}) \cap S(\mathcal{H}) = MC(\mathcal{T})$.*

Proof: It is clear $\check{R}(\mathcal{H}) \cap S(\mathcal{H}) \subseteq MC(\mathcal{T})$. Let $\varphi \in MC(\mathcal{T})$ then $\varphi(ab) = a\varphi(b)$ then for all a, b and c belong to $\mathcal{H}$ then $\varphi(ab)\ c = ab\varphi(c) = a(b\varphi(c)) = a(b\varphi c) = a(\varphi b\ c)$ then φ belong to $\check{R}(\mathcal{H})$ and $\varphi(ab)\,c = ab\varphi(c) = a(b\varphi(c)) = a\varphi(b\ c) = \varphi a(bc)$ then φ belong to $S(\mathcal{H})$ then φ belong to $\check{R}(\mathcal{H}) \cap S(\mathcal{H})$ then $\check{R}(\mathcal{H}) \cap S(\mathcal{H}) = MC(\mathcal{T})$. □

Proposition 4.2 *Let $\mathcal{T}$ be approach Hilbert algebra. Then $MC(\mathcal{T})$ be a set of all multicharacter on $\mathcal{T}$ is a closed commutative approach Hilbert sub algebra.*

Proof: Let φ_n belong to $MC(\mathcal{T})$ and $\|\varphi_n - \varphi\| \to 0$ such that $n = 0, 1, 2, \ldots$ we not that for each a and b belong to $\mathcal{T}\|a(\varphi b) - (\varphi a)\,b\| = \|a(\varphi b) - \varphi_n(ab) + \varphi_n(ab) - (\varphi a)\,b\| \leq \|a(\varphi b) - \varphi_n(ab)\| + \|\varphi_n(ab) - (\varphi a)\,b\| = 2\|a\|\|b\|\|\varphi_n - \varphi\| = 0$ then $a(\varphi b) - (\varphi a)\,b = 0$ then $a(\varphi b) = (\varphi a)\,b$ then φ belong to $MC(\mathcal{T})$ and $MC(\mathcal{T})$ is closed. □

Proposition 4.3 *$MC(\mathcal{T})$ Is Maximal sub algebra of $\beta(\mathcal{T})$ if and only if $\mathcal{T}$ is commutative.*

Proof: *Let $\mathcal{T}$ is commutative then for all a belong to $\mathcal{T}$ then $[\mathcal{T}] = \{Fab : a \in \mathcal{T}\} = \{aFb : a \in \mathcal{T}\} \subseteq MC(\mathcal{T})$ suppose $\mathcal{T}$ is not maximal be some maximal commutative approach Hilbert sub algebra containing $\mathcal{T}$ we may pick $\mathcal{T}$ belong to $MC(\mathcal{T}), \varphi(a)\,b = \varphi(\varphi a)\,b = a(\varphi b)$ then $\varphi \in MC(\mathcal{T})$ on the other hand $\mathcal{T}$ belong to $MC(\mathcal{T})$ implies $\mathcal{T}$ is commutative with all element of $[\mathcal{T}]$. Conversely let $MC(\mathcal{T})$ be maximal commutative approach algebra thus $\mathcal{T}$ belong to $MC(\mathcal{T})$ and $ST = TS$ for all S belong to $MC(\mathcal{T})$ then $(\mathcal{T}_a S)b = a(Sb) = (aS)\,b = (Sa)\,b = (S\mathcal{T}_a)\,b$ for all $\mathcal{T}_a$ belong to $MC(\mathcal{T})$ then $(ab)\,c = \mathcal{T}_a(bc) = b(\mathcal{T}_a c) = bac$ for all a, b and c belong to $\mathcal{T}$ then for all $ab = ba$ then $\mathcal{T}$ is commutative.* □

Theorem 4.4 *Every multicharacter φ on commutative approach Hilbert algebra $\mathcal{T}$ linear operator with bounded norm.*

Proof: consider arbitrary elements $\daleth, \wp$ and $\mathcal{K}$ in $\mathcal{T}$ such that. Let α and μ arbitrary element of $\mathcal{E}$ then $\mathcal{K}\varphi\left(\alpha\daleth+\mu\wp\right)=\left(\alpha\daleth+\mu\wp\right)\mathcal{K}\varphi=\alpha FK\varphi+\mu gK\varphi=\alpha\daleth\varphi\mathcal{K}+\mu\wp\varphi\mathcal{K}=\alpha\varphi KF+\mu\varphi Kg=\left(\alpha\varphi\daleth+\mu\varphi\wp\right)\mathcal{K}$. Then we get $\varphi\left(\alpha\daleth+\mu\wp\right)=\alpha\varphi\daleth+\mu\varphi\wp$.

To prove φ is bounded let assume that $F_n\in\daleth$ for every $n_n\in N$ with properties that F_n converges to F and $\varphi.\daleth_n$ converges to g as n tends to infinity. Then

$$gK=\left(\lim_{n\to\infty}\varphi.\daleth_n\right)\mathcal{K}=\lim_{n\to\infty}\left(\varphi.\daleth_n K\right)=\lim_{n\to\infty}\varphi.\mathfrak{W}_n K=\wp\varphi\mathcal{K}.$$

Then $gK=\wp\varphi\mathcal{K}$ then φ is bounded. □

Lemma 4.5 *Let $\mathcal{T}$ be approach a Hilbert algebra expressed as a direct sum of two sided ideals that are closed in the associated topology $\{I_\Theta:\Theta\in z^+\}$ in $\mathcal{T}$. If φ belong to $MC(\mathcal{T})$, then φ maps for each I_Θ in to itself.*

Proof: $\mathcal{T}$ and S belong to $MC(\mathcal{T})$ and a belong to I_Θ for some $\Theta\in\mathcal{E}$ suppose that $(\varphi a)_\beta\neq 0$ projection of φa in to I_β for some $\beta\neq\Theta,\beta$ belong to $\mathcal{T}$. Let b belong to I_β and $b\neq 0$ such that

$$(\varphi a)b=(\varphi a)_\beta\, b=0.$$

If $(\varphi a)_\beta\, I_\beta=0$ then

$$(\varphi a)_\beta\,\varphi=\left((\varphi a)_\beta\right)\left(\oplus\sum_{\Theta\in\mathcal{E}}I_\Theta\right)=\left((\varphi a)_\beta\right)I_\beta=0$$

that is contraction. But on the other hand $\varphi(ab)=a\left(\varphi b\right)$ for all a and b belong to φ then $\varphi:I_\Theta\to I_\Theta$. □

Theorem 4.6 *Let $\mathcal{T}$ be an approach Hilbert algebra and $\{I_\Theta:\Theta\in\mathcal{E}\}$ consider the collection of all minimal closed two-sided ideals in $\mathcal{T}$ denote by M the topological space formed by this set ideal in $\mathcal{T}$ with the discrete topology. Then there exists a $*$-isomorphism which is at the same time on isometry of $MC(\mathcal{T})$ on to $C^{**}(\ M)$, the space of all bounded contentious complex function on M.*

Proof: Define $\emptyset:MC(\mathcal{T})\to C(\ M)$. the space the collection of all function taking values in C over M by $\emptyset(\mathcal{T})(\Theta)=t(\Theta)$ for each Θ belong to M. Then for all $\mathcal{T}$ and S belong to $MC(\mathcal{T})$then $\emptyset(\mathcal{T}+S)(\Theta)=(t+s)(\Theta)=t(\Theta)+s(\Theta)=\emptyset(\mathcal{T})(\Theta)+\emptyset(S)(\Theta)$ and $\emptyset(\ \alpha\mathcal{T})(\Theta)=\ \alpha t(\Theta)=\ \alpha\emptyset(\mathcal{T})(\Theta)$ Then $\emptyset$ is linear, and $\emptyset$ is multicharacter (that is $*$ operation for element in $C^{**}(\ M)$ and operator adjoint for element in $MC(\mathcal{T})$). To show that $\emptyset$ is isometric we observe

$$\|Ta\|^2=\left\|\mathcal{T}\left(\oplus\sum_{\Theta\in\mathcal{E}}a_\Theta\right)\right\|^2=\|\oplus\sum_{\Theta\in\mathcal{E}}\mathcal{T}_\Theta a_\Theta\|^2=\sum_{\Theta\in\mathcal{E}}\|\mathcal{T}_\Theta a_\Theta\|^2\leqq\|\emptyset(\mathcal{T})\|^2\|a\|^2,$$

and hence $\|\mathcal{T}\|\leqq\|\emptyset(\mathcal{T})\|$. conversely, we have for some non-empty $a_\Theta,|\emptyset(\mathcal{T})(\Theta)|=|t(\Theta)|=\frac{\|\mathcal{T}_\Theta a_\Theta\|}{\|a_\Theta\|}\leqq\|\mathcal{T}_\Theta\|\leqq\|\mathcal{T}\|$. Proving $\|\emptyset(\mathcal{T})\|\leqq\|\mathcal{T}\|$. Thus, $\emptyset$ is indeed an isometry, and being linear, it is one-to-one. On the other hand for each f belong to $C^{**}(M)\subseteqq C(M)$, Let $\mathcal{T}_\Theta=f(\Theta)P_\Theta$. It is readily seen that the mapping $\mathcal{T}$ determined by $\{\mathcal{T}_n\}$ belong to $MC(\mathcal{T})$ and satisfying $\emptyset(\mathcal{T})=f$. Thus, we conclude that $\emptyset$ is an isometric $*$isomorphism from $MC(\mathcal{T})$ onto $C^{**}(M)$. □

References

1. R. Abbas. *A New Structure of Banach Approach Space.* M.sc. thesis, University of Al-Qadisiyah, 2021.
2. R.K. Abbas and Boushra Y. Hussein. New results of normed approach space. *Iraqi Journal of Science*, 63(5):2103–2113, 2022.
3. R.K. Abbas and B.Y. Hussein. A new kind of topological vector space: Topological approach vector space. *AIP Conference Proceedings*, 2386(1):060008, 2022.

4. R.K. Abbas and B.Y. Hussein. New results of normed approach space. *Iraqi Journal of science*, 63(5):2103–2113, 2022.

5. S.S.A. Ali and B.Y. Hussein. New kind of vector space via ş-proximit structure. *E3S Web of Conferences*, 508:04012, 2024.

6. R. Baekeland and R. Lowen. Measures of lindelof and separability in approach spaces. *International Journal of Mathematics and Mathematical Sciences*, 17(3):597–606, 1994.

7. R.J. Fleming and J.E. Jamison. Hermitian operators on c (x, e) and the banach-stone theorem. *Mathematische Zeitschrift*, 170:77–84, 1980.

8. Boushra Y. Hussein and Hadeer A. Wshayeh. New results of q-bounded functional in symmetric δ-banach algebra. *AIP Conference Proceedings*, 2845(1):050040, 2023.

9. B.Y. Hussein. Equivalent locally martingale measure for the deflator process on ordered banach algebra. *Journal of Mathematics*, 2020(1):5785098, 2020.

10. B.Y Hussein and R.K. Abbas. New results of completion normed approach space. *AIP Conference Proceedings*, 2845(1):050036, 2023.

11. B.Y. Hussein and Huda A.A. Wshayeh. On state space of measurable function in symmetric δ-banach algebra with new results. *AIP Conference Proceedings*, 2398(1):060064, 2022.

12. G. J. O. Jameson. *Topology and Normed Spaces*. Chapman and Hall, London, 1974.

13. D.A. Kadhim and B.Y. Hussein. New kind of topological vector space via proximity structure. *Journal of Interdisciplinary Mathematics*, 26(6):1065–1075, 2023.

14. R.V. Kadison and John R. Ringrose. *Fundamentals of the Theory of Operator Algebras: Elementary Theory*. Academic Press, Inc., London, 1983.

15. H. Kamowitz and S. Scheinberg. Some properties of endomorphisms of lipschitz algebras. *Studia Mathematica*, 96:255–261, 1990.

16. I. Kaplansky. *Algebraic and Analytic Aspects of Operator Algebras*. American Mathematical Society, Providence, RI, USA, 1970.

17. I. Kaplansky. *Fields and Rings*. The University of Chicago Press, Chicago, IL, USA, 2nd edition, 1972.

18. C. Kellogg. Centralizers and h^*-algebras. *Pacific Journal of Mathematics*, 17(1):121–129, 1966.

19. A.A. Kream and B.Y. Hussein. A new structure of random approach normed space via banach space. *Iraqi Journal of Science*, 65(10):5617–5628, 2024.

20. W. Li and D. Zhang. Sober metric approach spaces. *Topology and its Applications*, 233:67–88, 2018.

21. R. Lowen. Approach spaces a common supercategory of top and met. *Mathematische Nachrichten*, 141(1):183–226, 1989.

22. R. Lowen. *Approach Spaces: The Missing Link in the Topology—Uniformity—Metric Triad*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 1997.

23. R. Lowen, D. Vaughan, and M. Sioen. Completing quasi metric spaces: An alternative approach. *Houston Journal of Mathematics*, 29(1):113–136, 2003.

24. R. Lowen and C. Verbeeck. Local compactness in approach spaces i. *International Journal of Mathematics and Mathematical Sciences*, 21(3):429–438, 1998.

25. R. Lowen and C. Verbeeck. Local compactness in approach spaces ii. *International Journal of Mathematics and Mathematical Sciences*, 2003(2):109–117, 2003.

26. R. Lowen and S. Verwulgen. Approach vector spaces. *Houston Journal of Mathematics*, 30(4):1127–1142, 2004.

27. M.A. Neamah and B.Y. Hussein. New result of t^w normed approach space. *Journal of Interdisciplinary Mathematics*, 26(6):1109–1211, 2023.

28. Maysoon A Neamah and Boushra Y Hussein. Some new results of completion t^ω-normed approach space. *Periodicals of Engineering and Natural Sciences (PEN)*, 10(5):82–89, 2022.

29. W. Rudin. *Functional Analysis*. McGraw-Hill, New York, 1991.

30. J.K. Wang. Multipliers of commutative banach algebra. *Pacific Journal of Mathematics*, 11:1131–1149, 1961.

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