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Hybrid Chaos: A Novel 3D Strange Attractor in a Coupled Tinkerbell-Duffing-Jerk System with External Forcing

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ABSTRACT: The purpose of this study is to find hybrid nonlinear dynamical system, combining elements of systems such as Tinkerbell, Duffing, and Jerk, with the addition of periodic external excitation terms $(\cos(\omega t), \sin(\omega t))$. The system was analysed using chaos theory tools, such as: fixed points and stability analysis (Newton-Raphson method). phase space and a strange attractor were used to clarify the fractal structure. Correlation dimension (D) and Lyapunov indices were used to evaluate complexity and sensitivity to initial conditions. Nonlinear interactions such as $(x_n^2, x_n^3, 2x_ny_n)$ were the main factor in shaping the dynamic distortions and complexity of the attractor. Periodic external excitations enhanced instability and increased the sensitivity of the system, contributing to a higher D. The system can be used to generate secure random keys and to model natural phenomena, such as fluctuations in environmental or financial systems.

Key Words: Chaos, strange attractor , nonlinear dynamics.

Contents

1. Introduction

In 1963, the famous Lorenz attractor was discovered, which was one of the key turning points that led to the explosion of chaos research [4]. Revolutionary advances in chaos synchronization, control, and chaos-based applications emerged . At the same time, a paradigm shift took place in the scientific understanding of chaos. Instead of carelessly stifling chaotic behavior, researchers began investigating systematic ways to induce chaos. Of particular interest was the evolution of chaotic attractors in autonomous ordinary differential equations in three dimensions. Since then, numerous canonical systems with chaotic attractors have been found, as is well known [5,6]. By examining the system's bifurcation, Wei et al. tried to ascertain the fundamental mechanism of the multiple attractors [7].

Because chaos systems have applications in engineering, ecology, and security, the scientific community has been closely examining them, and the most practical applications are in communications and cryptology. Chaotic systems provide the mixing and spreading properties required for encryption because of their initial sensitivity and varied dynamic features [1,2,9].

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2. Mathematical Model and Methodology

The hybrid system I presented is a nonlinear dynamical system consisting of three continuous differential equations (x, y, z) with time-dependent terms such as $\cos(\omega t)$ and $\sin(\omega t)$. This type of system is often used to describe complex physical or biological phenomena, and may have applications in chaos theory and nonlinear dynamical systems.

$$x_{n+1} = x_n^2 - y_n^2 + ax_n + by_n + \alpha\left(-\delta y_n - \beta^3 x_n + \gamma\cos(\omega t)\right)$$

$$y_{n+1} = 2x_n y_n + cx_n + dy_n + \beta\left(-\delta y_n - \alpha x_n + \gamma\cos(\omega(t))\right)$$
$$z_{n+1} = z_n + \in (-AZ_n - BY_n - Cx_n - D\sin(\omega(t))$$

Where x_n, y_n, z_n : System variables at time step n

a, b, c, d:Tinkerbell map coefficients.

$\alpha, \beta, \gamma, \delta$: Control coefficients for map merging.

A, B, C, D: Jerk map coefficients

(ω): System frequency in nonlinear analysis,

fixed point theory is one of the most effective and powerful instruments. Additionally, the fixed point can be found by an iterative process on a computer [3].

Proposition 2.1. *The fixed point of Hybrid system is*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.0083 \\ 0.0447 \\ -0.0117 \end{pmatrix}$$

Proof: The

$$x_n = x_n^2 - y_n^2 + ax_n + by_n + \alpha\left(-\delta y_n - \beta^3 x_n + \gamma\cos(\omega t)\right)$$
$$y_n = 2x_n y_n + cx_n + dy_n + \beta\left(-\delta y_n - \alpha x_n + \gamma\cos(\omega(t))\right)$$
$$z_n = z_n + \in (-AZ_n - BY_n - Cx_n - D\sin(\omega(t))$$

Solve the third equation to find

$$z_n + \in (-AZ_n - BY_n - Cx_n - D\sin(\omega(t)) = 0,$$

$$z = \frac{\epsilon(-By* - Cx* + D\sin(\omega t))}{1 + \in A}$$

We substitute it in the first and second equations. The two nonlinear equations depend on only x and y,

$$x_n^2 - y_n^2 + ax_n + by_n + \alpha\left(-\delta y_n - \beta^3 x_n + \gamma\cos(\omega(t))\right) = 0$$
$$2x_n y_n + cx_n + dy_n + \beta\left(-\delta y_n - \alpha x_n + \gamma\cos(\omega(t))\right) = 0$$

The system contains nonlinear terms (x^2, y^2, x^3), which make an analytical solution difficult or impossible therefore the system is solved numerically using Newton-Raphson.

$$J = \begin{vmatrix} \frac{\sigma F}{\sigma x} & \frac{\sigma F}{\sigma y} & \frac{\sigma F}{\sigma z} \\ \frac{\sigma G}{\sigma x} & \frac{\sigma G}{\sigma y} & \frac{\sigma G}{\sigma z} \\ \frac{\sigma H}{\sigma x} & \frac{\sigma H}{\sigma y} & \frac{\sigma H}{\sigma z} \end{vmatrix} \text{ any and we use iterate } \begin{vmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{vmatrix} = \begin{vmatrix} x_n \\ y_n \\ z_n \end{vmatrix} - J^{-1} \begin{vmatrix} F_{(X_n}, y_n\ _{,z_n)} \\ G_{(X_n}, y_n\ _{,z_n)} \\ H_{(X_n}, y_n\ _{,z_n)} \end{vmatrix}.$$

Iterate until convergence is achieved (i.e., the changes in x, y, z become very small). □

Example 2.1. Suppose we have the following system of equations:

$$H_1(p, q) = p^2 + q^2 - 25 = 0$$
$$H_2(p, q) = p^2 + q^2 - 5 = 0$$

We choose a starting point, let it $bep_0 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ then $H = \begin{pmatrix} -7 \\ 1 \end{pmatrix}$, $J = \begin{pmatrix} 6 & 6 \\ 6 & -1 \end{pmatrix}$ therefore $J^{-1} = \begin{pmatrix} \frac{-1}{42} \\ \frac{-8}{7} \end{pmatrix}$

When the law is applied, we get: $P_1 \approx \begin{pmatrix} 3.0238 \\ 4.1429 \end{pmatrix}$

The process can be repeated to achieve the desired accuracy.

Remark 2.1. Convergence is checked using the condition $\|\Delta\| < 1$, where Δ is the change in x, y, z.
By using matlab program we get the fixed point of hybrid system is

Table 1: Complex eigenvalues indicate the presence of fluctuations in the system. The negative part (-0.5000) indicates that this fluctuation decreases with time. The positive true eigenvalue (0.2000) indicates that its disturbances increase with time Therefore, the point is unstable.

Fixed point (x, y, z)	λ_1	λ_2	λ_3
(0.0083,0.0447,-0.0117)	-0.5000 + 0.3000i	-0.5000 - 0.3000i	0.2000

2.1. Phase Space

Systems behavior in the space of variables has been represented graphicall $y \cdot x, y, z$.

Figure 1 show each state variable's time series in 3D phase space, along with the other 2D plane of the system variables. Figure 1 depicts the chaotic response in three dimensions. Figure 2a shows the harmonic phase in the $y - z$ projection. Figure 2b shows the harmonic phase in the $x - y$ projection. Figure 2c shows the harmonic phase in the $x - z$ projection.

2.2. Strange Attractor

In phase space, a strange attractor is a collection of points that display chaotic behavior while drawing in neighbouring paths This type of attractor is common with nonlinear systems.

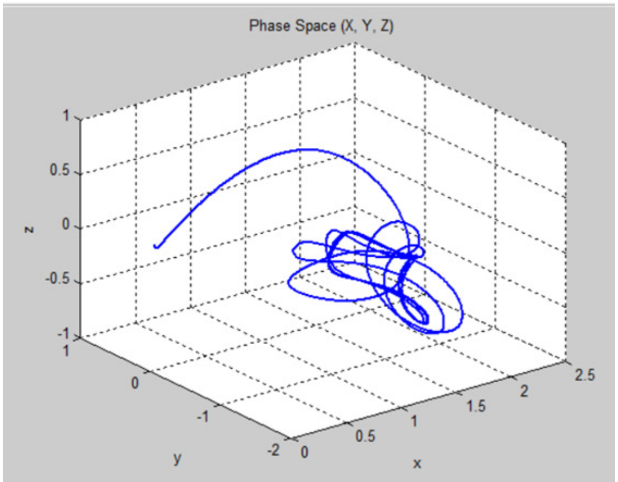

Figure 1: phase space (x, y, z): a= 0.9; b= 3.6; c = 0.9; d=-1.5; alpha= 0.9; beta= 0.5; gamma= 0.6; delta= 3.9; omega = 0.9; A= 14.2; B= -6.2; C=-1.1; D= 5.5; epsilon= 8.1; and (0.1,0.1,0.1).

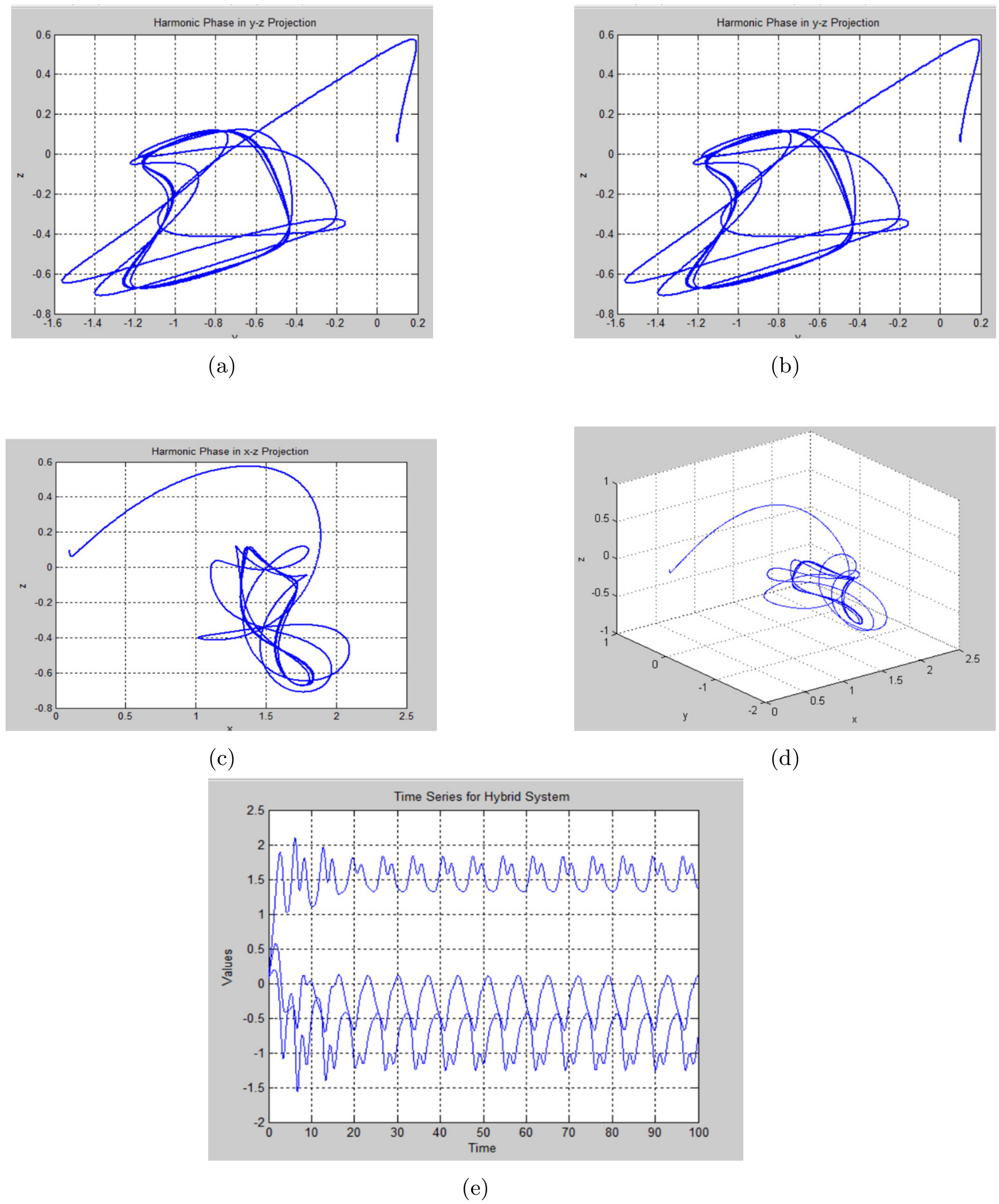

Figure 2: Phase space (x, y, z), (a) The harmonic phase in the projection of $y - z$, (b) The harmonic phase in the projection of $x - y$, (c) The harmonic phase in the projection of $x - z$, (d) a= 0.9; b= 3.6; c= 0.9; d =-1.5; alpha= 0.9; beta= 0.5; gamma= 0.6; delta= 3.9; omega= 0.9; t = 0.8; A= 14.2; B= -6.2; C= -1.1; D= 5.5; epsilon= 8.1; initial condition (0.1,0.1,0.1).It exhibits an irregular distribution with overlapping regions of stretching and folding, a key feature of strange attractors, (e) Time series for Hybrid system.

2.3. Comparison with Classical Attractors

Lorenz attractor: It has two rotating wings and larger coordinates (such as $x \in [-20, 20]$), and it does not match the data here.

2.4. Rössler attractor

It has a simple spiral structure with smaller coordinates, but it does not resemble the enclosed distribution.

Henon attractor: It is two-dimensional with overlapping loops, but its coordinate range is wider ($x \in [-1.5, 1.5]$).

2.5. Lyapunov exponent

The average exponential rate at which adjacent trajectories diverge or converge in phase space is measured by Lyapunov exponents. The tiny initial differences between orbits in systems with rapidly exponential divergence between them, which may be nearly imperceptible, explode quickly, making the system unpredictable over time. When a system has at least one positive Lyapunov exponent, its dynamics degenerate into unpredictable, random behavior .By Table 1, then $\|\lambda_3\| = 2$ therefore the system has positive Lyapunov exponent If an attractor has a dense orbit with a positive Lyapunov exponent, it is considered strange [8].

2.6. Bifurcations

In nonlinear dynamic systems, bifurcation is a phenomenon where a slight alteration in a control parameter causes a significant shift in the system's qualitative behavior. Bifurcation is considered a turning point that reshapes the dynamics of the system.

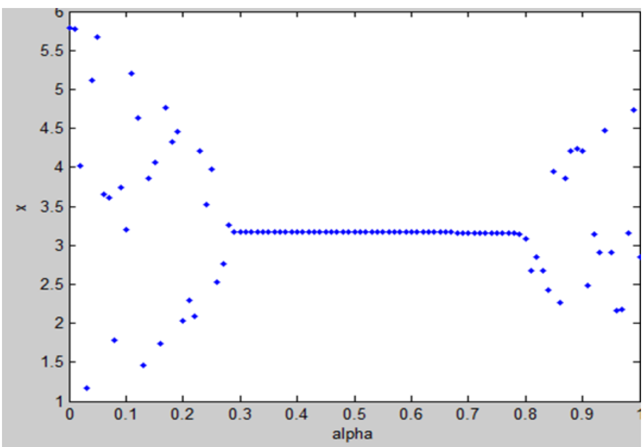

Figure 3: The main control parameter is "a", whose value ranges from 0.1 to 0.9

Also, L_1= -0.5000+0.3000i, L_2= -0.5000-0.3000i and L_3= 0.2000, respectively. Now, by a= 0.9; b= 3.6; c= 0.9; d= -1.5; alpha = 0.9; beta= 0.5; gamma= 0.6; delta= 3.9; omega= 0.9; t= 0.8; A= 14.2; B= -6.2; C= -1.1; D= 5.5; epsilon= 8.1; initial condition (0.1,0.1,0.1), the Kaplan-Yorke dimension expressed as:

$$D_l = m + \frac{\sum_{n=1}^{m} \lambda_n}{|\lambda_{m+1}|}$$

The largest integer m that meets the following requirements

$$\sum_{n=1}^{m} \lambda_n \geq 0$$

and

$$\sum_{n=1}^{m} \lambda_n \leq 0.$$

Then

$$\sum_{n=1}^{1} \lambda_n = 0.2 \geq 0$$

and

$$\sum_{n=1}^{2} \lambda_n = 0.2 + (-0.5) = -0.3 < 0.$$

Therefore $D_l = 2 + \frac{0.2}{|-0.5|} = 2.4$ $D_l = 2.4$ indicates that the system has a complex spatial structure, which corresponds to a positive Lyapunov exponent of one (0.2), indicating chaotic behaviour.

2.7. Correlation Dimension

A measure of the "complexity" of the distribution of points in phase space, especially in chaotic systems. This means calculating the probability of points being within a distance of r of each other. Mathematical formula:

$$C(r) \propto r^D$$

Where $C(r)$: correlation function (number of pairs of points within a distance of r, D: correlation dimension (estimated from the logarithmic slope of $C(r)$ versus r

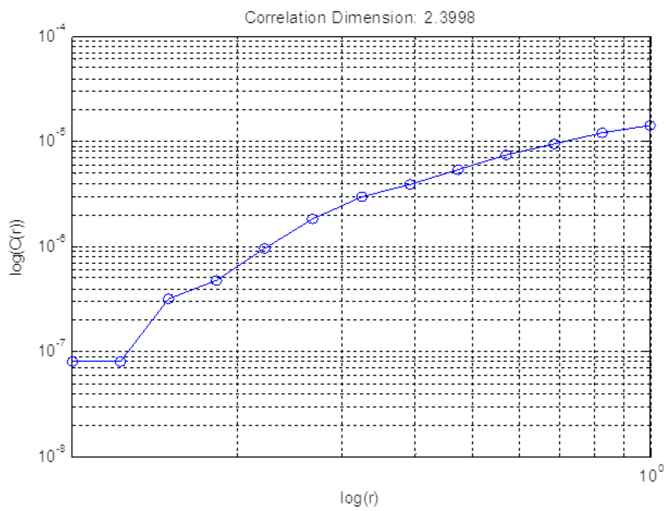

Figure 4: a= 0.9; b= -0.6; c= 2.0; d= 0.5; alpha = 0.1; beta= 0.1; gamma= 0.5; delta= 0.1; A = 1.0; B= 1.0; C= 1.0; D= 0.5; epsilon= 0.1; omega= 1.0, (0.1, 0.1,0.1)

we get$D \approx 2.05$. This value indicates that the strange attractor in system has a more complex fractal structure compared to classical systems such as Lorenz ($D \approx 2.05$) or Rössler ($D \approx 2.01$). Nonlinear interactions: This complexity is due to interactions between variables (x, y, z) and external terms $(\cos(\omega t), \sin(\omega t))$ in the system's equations Compared to classical systems

System	Dimension of the connection (D)	Main reason
Hybrid	2.4	Complex nonlinear interactions + external excitation
Lorenz	∼2.05	three-variable interaction
Rossler	∼2.01	simple helical structure
Henon	∼1.25	two-dimensional system

3. Conclusions

The developed hybrid system represents a dynamic model rich in nonlinear interactions and external excitations, making it a strong candidate for advanced studies in chaos and its applications. The high correlation dimension ($D \approx 2.4$) underscores its uniqueness compared to classical systems and opens up new avenues for research and applications.

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