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Exact Solution of Certain Fractional Fredholm and Volterra Singular Integral Equations

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ABSTRACT: In this article, the author exposed some capabilities of integral transforms, as well as special functions, to readers interested in this topic through various examples. It is shown the applications of integral transforms in evaluating series and integrals involving the Bessel functions, as well as solving fractional differential equations, the partial fractional differential equation and the fractional singular integral equation, where the fractional derivatives are in the Caputo-Fabrizio sense. It should be emphasized that none of the issues raised in this work are found in previous references.

Key Words: Laplace transform, Fourier transform, Stieltjes transform, Bessel function, Newmann function, fractional derivative of Caputo-Fabrizio.

1. Introduction And Notations

One of the most useful and applied topics in mathematics, which is very extensive and useful and has diverse applications in applied mathematics, mathematical physics, and engineering mathematics, is integral transforms and special functions. This work provides a concise exposition of the basic ideas of the theory of integral transforms and special functions and its applications to fractional calculus. Methods in which techniques are used in applications are illustrated, and many examples are included. For the sake of clarity in the presentation, we recall some definitions and results related to operational calculus that may be found in [1-4, 7, 8].

Definition 1.1. Let us assume that the function $\phi(t)$ is of exponential order, then the Laplace transform of $\phi(t)$ is as follows[4,8,9]

$$\mathcal{L}\{\phi(t); t \to s\} = \Phi(s) = \int_0^{+\infty} e^{-st}\phi(t)dt,$$

provided that the above integral is convergent.

Lemma 1.1. Let us assume that $\mathcal{L}\{\phi(t); s\} = \Phi(s)$ then we have

$$1. \qquad \mathcal{L}\{\phi\{\frac{1}{t}); s\} = \frac{1}{\sqrt{s}}\int_0^{+\infty} \sqrt{\xi} J_1(2\sqrt{s\xi})\Phi(\xi)d\xi.$$

$$2. \qquad \mathcal{L}\{\phi(t^2); s\} = \frac{1}{2\sqrt{\pi}}\int_0^{+\infty} e^{-\frac{s^2}{4\xi}}\xi^{-\frac{1}{2}}\Phi(\xi)d\xi.$$

Proof. See[4] In the sequel, we give some new integrals by means of the above Lemma 1.1.

Lemma 1.2. The following integral identity holds

$$\int_0^{+\infty} \frac{(\gamma + \ln \xi)J_1(2\sqrt{s\xi})}{\sqrt{\xi}}d\xi = -\frac{(\gamma + \ln s)}{\sqrt{s}}.$$

Proof. Let us take $\phi(t) = \ln t$ then we have $\ln t = -\ln(\frac{1}{t})$ and $\mathcal{L}\{\ln t; s\} = -\mathcal{L}\{\ln(\frac{1}{t})\} = -\frac{\gamma+\ln s}{s}$, on the other hand, by using first part of the Lemma 1.1. we have

$$\mathcal{L}\{\ln(\frac{1}{t}); \} = -\frac{1}{\sqrt{s}}\int_0^{+\infty} \sqrt{\xi} J_1(2\sqrt{s\xi})[\frac{\gamma + \ln \xi}{\xi}]d\xi = -\mathcal{L}\{\ln t; s\} = \frac{\gamma + \ln s}{s},$$

finally

$$\int_0^{+\infty} J_1(2\sqrt{s\xi})[\frac{\gamma + \ln \xi}{\sqrt{\xi}}]d\xi = -\frac{\gamma + \ln s}{\sqrt{s}},$$

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where $\gamma = -\Gamma'(1)$ is Euler constant. In the last integral let us set $s = 1$ we get

$$\int_0^{+\infty} J_1(2\sqrt{\xi})[\frac{\gamma + \ln \xi}{\sqrt{\xi}}]d\xi = -\gamma.$$

Lemma 1.3. The following integral identity holds

$$\int_0^{+\infty} \frac{\gamma + \ln \xi}{\xi\sqrt{\xi}} e^{-\frac{s^2}{4\xi}} d\xi = \frac{2(\gamma + \ln s)}{s}.$$

Proof. Let us take $\phi(t) = \ln t$ then we have
$\ln t^2 = 2 \ln t$ and $\mathcal{L}\{\ln t; s\} = \frac{1}{2}\mathcal{L}\{\ln(t^2)\} = -\frac{\gamma+\ln s}{s}$, on the other hand, by using second part of the Lemma 1.1. we have

$$\mathcal{L}\{\ln(t^2)\} = \frac{-1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-\frac{s^2}{4\xi}} \xi^{-\frac{1}{2}} [\frac{\gamma + \ln \xi}{\xi}]d\xi = -2[\frac{\gamma + \ln s}{s}],$$

finally

$$\int_0^{+\infty} e^{-\frac{s^2}{4\xi}} [\frac{\gamma + \ln \xi}{\xi\sqrt{\xi}}]d\xi = 2[\frac{\gamma + \ln s}{s}],$$

In the last integral let us set $s = 1$ we have

$$\int_0^{+\infty} e^{-\frac{1}{4\xi}} [\frac{\gamma + \ln \xi}{\xi\sqrt{\xi}}]d\xi = 2\gamma.$$

Definition 1.3. The inverse Laplace transform of the function $\Phi(s)$ is as follows

$$\mathcal{L}^{-1}\{\Phi(s); t\} = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} e^{ts}\Phi(s)ds,$$

the above complex integration along vertical line $x = c$ in the complex plane, is known as Bromwich integral [4,7].

Lemma 1.3. We have the following identity

$$S = \sum_{n=0}^{+\infty} \frac{e(-1)^n Ln(at)}{n!} = I_0(2\sqrt{at}),$$

where $Ln(.)$ and $I_0(.)$ stand for the Laguerre polynomial and the modified Bessel function of the first kind of order zero respectively.

Proof. By taking the Laplace transform of the left hand side and using the fact that $\mathcal{L}\{Ln(at); s\} = \frac{1}{s}(1 - \frac{a}{s})^n$, we get

$$\mathcal{L}\{S; t \to s\} = \sum_{n=0}^{+\infty} \frac{e(-1)^n}{n!} \mathcal{L}\{Ln(at)\} = \sum_{n=0}^{+\infty} \frac{e(-1)^n}{n!} [\frac{1}{s}(1 - \frac{a}{s})^n] = ..$$

$$.. = \frac{e}{s} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} (1 - \frac{a}{s})^n = \frac{e}{s} e^{-(1-\frac{a}{s})} = \frac{e^{\frac{a}{s}}}{s},$$

taking the inverse Laplace transform leads to

$$S = \mathcal{L}^{-1}\{\frac{e^{\frac{a}{s}}}{s}; t\} = I_0(2\sqrt{at}).$$

Lemma 1.4. We have the following identity

$$S = \sum_{k=0}^{+\infty} (-1)^k (\frac{t}{\lambda})^{\nu k} J_{2\nu k}(2\sqrt{at}) = J_0(2\sqrt{at}) - \int_0^t J_0(2\sqrt{a(t-\eta)})\eta^{2\nu-1} E_{2\nu,2\nu}(-\eta^{2\nu})d\eta,$$

where $J_\nu(.)$ stands for the Bessel function of the first kind of order ν and $E_{\alpha,\beta}(.)$ is Mittag-Leffler functin in two parameters.
Proof. By taking the Laplace transform of the left hand side of the above sum and using the fact that

$$\mathcal{L}\{(\frac{t}{a})^\nu J_{2\nu}(2\sqrt{at}); s\} = \frac{e^{-\frac{a}{s}}}{s^{2\nu+1}},$$

we have

$$\mathcal{L}\{S; t\to s\} = \sum_0^{+\infty}(-1)^k\frac{e^{-\frac{a}{s}}}{s^{2\nu k+1}} = e^{-\frac{a}{s}}[\sum_{k=0}^{+\infty}\frac{(-1)^k}{s^{2\nu k+1}}] = ..$$

$$.. = e^{\frac{a}{s}}.\frac{\frac{1}{s}}{1+\frac{1}{s^{2\nu}}} = e^{-\frac{a}{s}}.\frac{s^{2\nu-1}}{s^{2\nu}+1} = \frac{e^{-\frac{a}{s}}}{s}.\frac{s^{2\nu}}{s^{2\nu}+1} = ..$$

$$.. = \frac{e^{-\frac{a}{s}}}{s} - \frac{1}{s^{2\nu}+1}.\frac{e^{-\frac{a}{s}}}{s},$$

at this point, taking the inverse Laplace transform leads to

$$S = J_0(2\sqrt{at}) - \int_0^t J_0(2\sqrt{a(t-\eta)})F(\eta)d\eta,$$

where $F(t) = \mathcal{L}^{-1}\{\frac{1}{s^{2\nu}+1}\} = t^{2\nu-1}E_{2\nu,2\nu}(-t^{2\nu})$, in fact we have

$$\mathcal{L}^{-1}\{\frac{s^{\alpha-\beta}}{s^\alpha+\lambda}; t\} = t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha).$$

Example 1.1. Let us show that

$$\mathcal{L}\{\phi(t); t\to s\} = \mathcal{L}\{\int_0^{+\infty} ber(2\sqrt{at\cosh(\xi)})d\xi; t\to s\} = -\frac{\pi}{2s}Y_0(\frac{a}{s}),$$

where $ber(.)$ and $Y_0(.)$ stand for the Kelvin function and Newmann function of order zero respectively.
Solution. Let us make a change of variable $a\cosh(\xi) = u$, after simplifying we get

$$\phi(t) = \int_a^{+\infty}\frac{ber(2\sqrt{tu})}{\sqrt{u^2-a^2}}du,$$

taking the Laplace transform of $\phi(t)$ and using the fact that
$\mathcal{L}\{ber(2\sqrt{at}); t\to s\} = \frac{1}{s}\cos(\frac{a}{s})$, we obtain

$$\mathcal{L}\{\phi(t); s\} = \Phi(s) = \frac{1}{s}\int_a^{+\infty}\frac{\cos(\frac{u}{s})}{\sqrt{u^2-a^2}}du = -\frac{\pi}{2s}[-\frac{2}{\pi}\int_a^{+\infty}\frac{\cos(\frac{u}{s})}{\sqrt{u^2-a^2}}du] = -\frac{\pi}{2s}Y_0(\frac{a}{s}).$$

In the last step we used an integral representation of the Newmann function $Y_0(.)$.
Lemma 1.5. The following integral identity involving the Bessel functions holds

$$\int_0^1(\frac{\xi}{a})^\nu J_{2\nu}(2\sqrt{a\xi})(\frac{1-\xi}{a})^\tau I_{2\tau}(2\sqrt{a(1-\xi)})d\xi = \frac{1}{\Gamma(2(\nu+\tau+1))}.$$

Proof. Let us take $f(t) = (\frac{t}{a})^\nu J_{2\nu}(2\sqrt{at})$ and $g(t) = [(\frac{t}{a})^\tau I_{2\tau}(2\sqrt{at})$ then we have

$$F(s) = \mathcal{L}\{(\frac{t}{a})^\nu J_{2\nu}(2\sqrt{at}); s\} = \frac{e^{-\frac{a}{s}}}{s^{2\nu+1}},$$

and

$$G(s) = \mathcal{L}\{(\frac{t}{a})^\tau I_{2\tau}(2\sqrt{at}); s\} = \frac{e^{\frac{a}{s}}}{s^{2\tau+1}},$$

from which we deduce that

$$F(s)G(s) = \frac{1}{s^{2(\nu+\tau+1)}},$$

at this stage, taking the inverse Laplace transform of the above relation we have

$$\mathcal{L}^{-1}\{F(s)G(s); s \to t\} = \int_0^t (\frac{\xi}{a})^\nu J_{2\nu}(2\sqrt{a\xi})(\frac{t-\xi}{a})^\tau I_{2\tau}(2\sqrt{a(t-\xi)})d\xi = ..$$

$$.. = \mathcal{L}^{-1}\{\frac{1}{s^{2(\nu+\tau+1)}}\} = \frac{t^{2(\nu+\tau)+1}}{\Gamma(2(\nu+\tau+1))},$$

or

$$\int_0^t (\frac{\xi}{a})^\nu J_{2\nu}(2\sqrt{a\xi})(\frac{t-\xi}{a})^\tau I_{2\tau}(2\sqrt{a(t-\xi)})d\xi = \frac{t^{2(\nu+\tau)+1}}{\Gamma(2(\nu+\tau+1))},$$

by taking $t = 1$ after simplifying we have

$$\int_0^1 (\frac{\xi}{a})^\nu J_{2\nu}(2\sqrt{a\xi})(\frac{1-\xi}{a})^\tau I_{2\tau}(2\sqrt{a(1-\xi)})d\xi = \frac{1}{\Gamma(2(\nu+\tau+1))}.$$

Eample 1.2. Let us show that the following relation holds

$$\mathcal{L}^{-1}\{\Phi(s); t\} = \phi(t) = \frac{t}{\sqrt{t^2 - a^2}},$$

where

$$\Phi(s) = \int_0^{+\infty} e^{-s\sqrt{\xi^2+a^2}} d\xi.$$

Solution. By using bromwich integral we get

$$\phi(t) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} e^{ts}[\int_0^{+\infty} e^{-s\sqrt{\xi^2+a^2}} d\xi]ds,$$

changing the order of integration, we have

$$\phi(t) = \int_0^{+\infty} [\frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} e^{(t-\sqrt{\xi^2+a^2})s} ds]d\xi,$$

the value of the inner integral is $\delta(t - \sqrt{\xi^2 + a^2})$, therefore we obtain

$$\phi(t) = \int_0^{+\infty} \delta(t - \sqrt{\xi^2 + a^2})d\xi,$$

at this point, making a change of variable $u = t - \sqrt{\xi^2 + a^2}$ in the above integral and after simplifying and in view of the elementary property of the Dirac-delta function, we get finally

$$\phi(t) = \int_{-\infty}^{t-a} \frac{(t-u)\delta(u)}{\sqrt{(t-u)^2 - a^2}} du = \frac{t}{\sqrt{t^2 - a^2}}, \quad t > a.$$

Definition 1.4. Fractional integral of the function $\phi(t)$ of order α is defined as follows

$$\mathcal{J}^\alpha\{\phi(t)\} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1}\phi(\xi)d\xi, \quad 0 \le \alpha \le 1.$$

Definition 1.5. The Laplace transform of fractional integral of order $\alpha, 0 < \alpha < 1$ of the given function $\phi(t)$ is defined as follows

$$\mathcal{L}\{\mathcal{J}^\alpha[\phi(t)]; s\} = \frac{\Phi(s)}{s^\alpha}.$$

Definition 1.6. Fractional derivative of the function$\psi(t)$ belong to $\mathcal{C}^1$ of order α with $0<\alpha<1$ in the sense of Caputo-Fabrizio is defined as follows

$$D_{0,t}^{C.F,\alpha}\psi(t)=\frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha(t-\xi)}{1-\alpha}}\psi'(\xi)d\xi=\frac{e^{-\frac{\alpha t}{1-\alpha}}}{1-\alpha}\int_0^t e^{\frac{\alpha\xi}{1-\alpha}}\psi'(\xi)d\xi, \qquad D_{0,t}^{C.F,\alpha}\psi(0)=0.$$

In special case $\alpha=0.5$ we have

$$\frac{1}{2}D_{0,t}^{C.F,\frac{1}{2}}\psi(t)=\int_0^t e^{-(t-\xi)}\psi'(\xi)d\xi=e^{-t}\int_0^t e^{\xi}\psi'(\xi)d\xi,$$

for example

$$\frac{1}{2}D_{0,t}^{C.F,\frac{1}{2}}[e^{-t}]=-\int_0^t e^{-(t-\xi)}e^{-\xi}d\xi=-te^{t}.$$

and

$$\frac{1}{2}D_{0,t}^{C.F,\frac{1}{2}}[\mathcal{H}(t-\lambda)]=-\int_0^t e^{-(t-\xi)}e^{-\xi}\delta(\xi-\lambda)d\xi=-e^{-t},$$

where $\mathcal{H}(.)$ and $\delta(.)$ stand for the Heaviside unit step function and Dirac-delta function respectively. We have also

$$\lim_{\alpha\to 1}D_{0,t}^{C.F,\alpha}\psi(t)=\psi'(t).$$

Let us assume that $\mathcal{L}\{\psi(t);t\to s\}=\Psi(s)$, then the Laplace transform of the factional operator $D_{0,t}^{C.F,\alpha}$ is given as below

$$\mathcal{L}\{D_{0,t}^{C.F,\alpha}\psi(t);t\to s\}=\frac{s\Psi(s)-\psi(0)}{s(1-\alpha)+\alpha}, \quad 0<\alpha<1,$$

and in special case $\alpha=\frac{1}{2}$ we have

$$\mathcal{L}\{\frac{1}{2}D_{0,t}^{C.F,\frac{1}{2}}\psi(t);t\to s\}=\frac{s\Psi(s)-\psi(0)}{s+1}.$$

Problem 1.1. Let us solve the following fractional differential equation, where fractional derivative is in the Caputo-Fabrizio sense

$$\frac{1}{2}D_{0,t}^{C.F,\frac{1}{2}}\psi(t)+\lambda\psi(t)=\sinh(t), \quad \psi(0)=u_0=0.$$

Solution. By taking the Laplace transform of the fractional differential equation and using boundary condition, we get

$$[\frac{s}{s+1}+\lambda]\Psi(s)=\frac{1}{s^2-1},$$

from which we deduce that

$$\Psi(s)=\frac{1}{(\lambda+1)(s-1)(s+\frac{\lambda}{\lambda+1})},$$

at this stage taking the inverse Laplace transform and in view of the convolution theorem for the Laplace transform we arrive at

$$\psi(t)=\frac{1}{\lambda+1}\int_0^t e^{t-\eta}e^{-\frac{\lambda\eta}{\lambda+1}}d\eta=\frac{e^t}{\lambda+1}\int_0^t e^{-(\frac{2\lambda+1}{\lambda+1})\eta}d\eta,$$

after simplifying we obtain

$$\psi(t)=\frac{e^t}{2\lambda+1}[1-e^{-(\frac{2\lambda+1}{\lambda+1})t}], \qquad \psi(0)=0.$$

In special case $\lambda=0$ we get $\psi(t)=e^t-1$ from which we deduce that

$$\frac{1}{2}D_{0,t}^{C.F,\frac{1}{2}}[e^t-1]=\sinh(t).$$

2. Solution To Certain Integral Equations With Trigonometric And Bessel Function As Kernel

Problem 2.1. Let us solve the following fractional singular integral equation, where the fractional derivative is in the Caputo-Fabrizio sense

$$\frac{1}{2}D_{0,t}^{C.F,\frac{1}{2}}\phi(t)-\int_t^{+\infty}\phi(\xi)d\xi=1-e^{-t},\quad \phi(0)=u_0=1,\quad \int_0^{+\infty}\phi(t)dt=\Phi(0)=0.$$

Solution. By taking the Laplace transform of the fractional differential equation and using boundary condition we get

$$\frac{s\Phi(s)-\phi(0)}{s+1}+\frac{\Phi(s)}{s}=\frac{1}{s}-\frac{1}{s+1},$$

after simplifying we get

$$\Phi(s)=\frac{s+1}{s^2+s+1}=\frac{s}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2)}+\frac{1}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2)},$$

taking the inverse Laplace transform of the above relation leads to

$$\phi(t)=e^{-\frac{t}{2}}[\cos(\frac{\sqrt{3}t}{2})+\sin(\frac{\sqrt{3}t}{2})],\quad \phi(0)=1.$$

Problem 2.2. Let us solve the following singular integral equation with trigonometric kernel

$$\frac{1}{\sqrt{\pi t}}\int_0^{+\infty}\cos(2\sqrt{t\xi})\psi(\xi)d\xi=Erfc(\frac{a}{2\sqrt{t}}).$$

Solution. By taking the Laplace transform of the above singulae integral equation, we arrive at

$$\mathcal{L}\{\frac{1}{\sqrt{\pi t}}\int_0^{+\infty}\cos(2\sqrt{t\xi})\psi(\xi)d\xi\}=\frac{1}{\sqrt{s}}\Psi(\frac{1}{s})=\mathcal{L}\{Erfc(\frac{a}{2\sqrt{t}})\}=\frac{e^{-a\sqrt{s}}}{s},$$

or

$$\frac{1}{\sqrt{s}}\Psi(\frac{1}{s})=\frac{e^{-a\sqrt{s}}}{s},$$

in the above equation let us make a substitution $s\to\frac{1}{s}$, we get

$$\Psi(s)=\sqrt{s}e^{-\frac{a}{\sqrt{s}}},$$

the transformed function $\Psi(s)$ has $s=0$ as branch point, therefore by using Gros-Levi inversion formula we have

$$\psi(t)=\frac{1}{\pi}\int_0^{+\infty}e^{-tr}\mathcal{I}m\{\lim_{\beta\to-\pi}\sqrt{re^{i\beta}}e^{-\frac{a}{\sqrt{re^{i\beta}}}}\}dr,$$

after simplifying we have

$$\psi(t)=\frac{1}{\pi}\int_0^{+\infty}e^{-tr}\sqrt{r}\cos(\frac{a}{\sqrt{r}})dr,$$

finally a change of variable $\frac{1}{\sqrt{r}}=\xi$ leads to

$$\psi(t)=\frac{2}{\pi}\int_0^{+\infty}e^{-\frac{t}{\xi^2}}\cos(a\xi)\frac{d\xi}{\xi^4}.$$

Problem 2.3. Let us solve the following fractional singular integral equation with Bessel function as a kernel

$$\int_0^t(\frac{\xi}{t-\xi})^{\frac{1}{4}}J_{-\frac{1}{2}}(2\sqrt{\xi(t-\xi)})\phi(\xi)d\xi=J_0(t).$$

Solution. Let us recall a useful Laplace transform identity as follows

$$\mathcal{L}\{\int_0^t (\frac{t-\xi}{\xi})^{\nu} J_{2\nu}(2\sqrt{\xi(t-\xi)})\phi(\xi)d\xi\} = \frac{1}{s^{2\nu+1}}\Psi(s+\frac{1}{s}).$$

Now, by taking the Laplace transform of the integral equation and using the above identity with $\nu = -\frac{1}{4}$ we get

$$\frac{1}{\sqrt{s}}\Psi(s+\frac{1}{s}) = \frac{1}{\sqrt{s^2+1}} = \frac{1}{\sqrt{s}\sqrt{s+\frac{1}{s}}},$$

after simplifying we get finally

$$\Psi(s+\frac{1}{s}) = \frac{1}{\sqrt{s+\frac{1}{s}}} \to \Psi(s) = \frac{1}{\sqrt{s}} \to \psi(t) = \frac{1}{\sqrt{\pi t}},$$

substitution of the obtained solution in the integral equation leads to

$$\int_0^t (\frac{\xi}{t-\xi})^{\frac{1}{4}} J_{-\frac{1}{2}}(2\sqrt{\xi(t-\xi)})\frac{d\xi}{\sqrt{\pi\xi}} = J_0(t),$$

at this stage let us take $t = 1$ after simplifying we get

$$\int_0^t (\frac{1}{\xi(1-\xi)})^{\frac{1}{4}} J_{-\frac{1}{2}}(2\sqrt{\xi(1-\xi)})d\xi = \sqrt{\pi}J_0(1),$$

3. Stieltjes And Fourier Transforms With Applications

Definition 3.1. Let us assume that the function $\phi(t)$ is in $\mathcal{S}(R)$ Schwartz space (space of the rapidly decresing functions, i.e $e^{-x^2}, e^{-|x|}, Ai(x)$), then the Stieltjes transform of $\phi(t)$ is as below [4,7]

$$\mathcal{L}\{\mathcal{L}\{\phi(t);u\};s\} = S\{\phi(t);s\} = \Phi(s) = \int_0^{+\infty} \frac{\phi(t)}{t+s}dt,$$

provided that the above integral is convergent. The second itrate of the Laplace transform, is the Stieltjes transform.

Definition 3.2. The inverse Stieltjes transform of the function $\Phi(s)$ is defined as below

$$S^{-1}\{\Phi(s);t\} = \frac{1}{\pi}\mathcal{I}_m\{\lim_{\beta\to-\pi} \Phi(te^{i\beta})\}.$$

Example 3.1. Let us show that

$$\int_0^{+\infty} \frac{\cos(2\sqrt{at})J_0(2\sqrt{bt})}{t+s}dt = 2\cosh(2\sqrt{as})K_0(2\sqrt{bs}).$$

Solution. We need to show that

$$S^{-1}\{2\cosh(2\sqrt{as})K_0(2\sqrt{bs});t\} = \cos(2\sqrt{at})J_0(2\sqrt{bt}),$$

by means of the inversion formula for the Stieltjes transform we have

$$S^{-1}\{2\cosh(2\sqrt{as})K_0(2\sqrt{bs});t\} = \frac{1}{\pi}\mathcal{I}m\{2\cosh(2\sqrt{ate^{-i\pi}})K_0(2\sqrt{bte^{-i\pi}})\} = ..$$

$$. = \frac{1}{\pi}\mathcal{I}m\{2\cosh(2i\sqrt{at}K_0(2i\sqrt{bt}\},$$

using the fact that $K_0(z) = \frac{i\pi}{2}[J_0(ze^{\frac{i\pi}{2}}) + iY_0(ze^{\frac{i\pi}{2}})$ we conclude that

$$S^{-1}[2\cosh(2\sqrt{as})K_0(2\sqrt{bs});t] = \frac{2\cos(2\sqrt{at})}{\pi}\mathcal{I}m\{\frac{i\pi}{2}[J_0(-2\sqrt{bt}) + iY_0(-2\sqrt{bt})\} = \cos(2\sqrt{at})J_0(2\sqrt{bt}).$$

Lemma 3.1. Let us consider the following singular integral equation of Stieltjes type

$$\int_0^{+\infty} \frac{\phi(t)}{t+s} dt = \sqrt[n]{s} K_\nu(\tau s),$$

then the above singular integral equation has the following formal solution

$$\phi(t) = \frac{\sqrt[n]{t}}{\pi} \sin(\nu\pi + \frac{\pi}{n}) K_\nu(\tau t).$$

Proof. In view of the Stieltjes transform and its inversion formula, the solution of the above singular equation can be obtained as follows

$$\phi(t) = S^{-1}\{ \sqrt[n]{s} K_\nu(\tau s)\} = \frac{1}{\pi} \mathcal{I}_m\{ \lim_{\beta \to -\pi} \sqrt[n]{t e^{i\beta}} K_\nu(\tau t e^{i\beta})\},$$

or

$$\phi(t) = \frac{1}{\pi} \mathcal{I}_m\{ \sqrt[n]{-t} K_\nu(-\tau t)\} = \frac{1}{\pi} \mathcal{I}_m\{(-1)^{\frac{1}{n}} \sqrt[n]{t} (-1)^\nu K_\nu(\tau t)\} = ..$$

$$. = \frac{\sqrt[n]{t}}{\pi} \mathcal{I}_m\{ e^{i(\nu + \frac{1}{n})\pi} K_\nu(\tau t)\},$$

finally

$$\phi(t) = \frac{\sqrt[n]{t}}{\pi} \sin(\nu\pi + \frac{\pi}{n}) K_\nu(\tau t).$$

Definition 3.3. Let us assume that the function $\psi(x)$ is absolutely integrable, then the Fourier transform of $\psi(x)$ is defined as follows

$$\mathcal{F}\{\psi(t); w\} = \Psi(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iwx} \psi(x) dx,$$

then, the transformed function $\Psi(w)$ is bounded.
Example 3.2.

$$\mathcal{F}\{\frac{e^{-ax}\mathcal{H}(x)}{\sqrt{\pi x}}; w\} = \frac{1}{\sqrt{a - iw}}, \quad a > 0.$$

where $\mathcal{H}(x)$stands for the Heaviside unit step function.
Solution. We have

$$\mathcal{F}\{\frac{e^{-ax}\mathcal{H}(x)}{\sqrt{\pi x}}; w\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iwx} [\frac{e^{-ax}\mathcal{H}(x)}{\sqrt{\pi x}}] dx = ..$$

$$.. = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{iwx} \frac{e^{-ax}}{\sqrt{\pi x}} dx,$$

or

$$\mathcal{F}\{\frac{e^{-ax}\mathcal{H}(x)}{\sqrt{\pi x}}; w\} = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-(a-iw)x} \frac{1}{\sqrt{\pi x}} dx = \frac{1}{\sqrt{a - iw}}.$$

Definition 3.4. The inverse Fourier transform of the function $\Psi(w)$ is defined as follows

$$\mathcal{F}^{-1}\{\Psi(w); x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixw} \Psi(w) dw.$$

Example 3.3. Let us show that

$$\mathcal{F}^{-1}\{\frac{e^{-|y|\sqrt{w^2+a^2}}}{\sqrt{w^2+a^2}}; w \to x\} = \phi(x) = \sqrt{\frac{2}{\pi}} K_0(a\sqrt{x^2+y^2}).$$

Solution. The left hand side can be written as follows

$$\mathcal{F}^{-1}\{\frac{e^{-|y|\sqrt{w^2+a^2}}}{\sqrt{w^2+a^2}}; w \to x\} = \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixw} \frac{e^{-|y|\sqrt{w^2+a^2}}}{\sqrt{w^2+a^2}} dw,$$

or

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{ixw-|y|\sqrt{w^2+a^2}}}{\sqrt{w^2+a^2}} dw,$$

at this stage, we introduce a change of variable $w = a\sinh(\tau)$, we arrive at

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{iax\sinh(\tau)-|y|\sqrt{a^2\sinh^2(\tau)+a^2}}}{\sqrt{a^2\sinh^2\tau + a^2}} a\cosh(\tau) d\tau = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iax\sinh(\tau)-a|y|\cosh(\tau)} d\tau.$$

after simplifying we get

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{iax}{2}(e^{\tau}-e^{-\tau})-\frac{a|y|}{2}(e^{\tau}+e^{-\tau})} d\tau = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-(\frac{a|y|-iax}{2}e^{\tau}+\frac{a|y|+iax}{2}e^{-\tau})} d\tau,$$

at this point making a change of variable $e^{\tau} = \xi$ leads to

$$\phi(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} e^{-(\frac{a|y|-iax}{2})\xi+(\frac{a|y|+iax}{2})\xi^{-1}} \frac{d\xi}{2\xi} = \sqrt{\frac{2}{\pi}} K_0(a\sqrt{x^2+y^2}),$$

in the last step we used an integral representation for the modified Bessel function $K_0(.)$ as follows

$$K_0(2\sqrt{\alpha\beta}) = \int_{0}^{+\infty} e^{-(\alpha\xi+\beta\xi^{-1})} \frac{d\xi}{2\xi}.$$

Lemma 3.2. If $\int_{-\infty}^{+\infty} \psi(x)dx = \Psi(0) \neq 0$ then we have

$$\mathcal{F}\{\int_{-\infty}^{x} \psi(\xi)d\xi; x \to w\} = \frac{i}{w}\Psi(w) + \pi\Psi(0)\delta(w).$$

Proof. See [3, 5, 7].

Corollary 3.1. We have the following relation

$$\mathcal{F}\{\int_{-\infty}^{x} Ai(\xi)d\xi; x \to w\} = \frac{i}{w} e^{\frac{iw^3}{3}} + \pi\delta(w).$$

In the above relation $Ai(.)$ stands for the Airy function [3 - 6, 9].

4. Main Results: Solution to Fractional Homogeneous Fredholm and Volterra Singular Integral Equations with Fractional Derivative in the Caputo-Fabrizio sense

Problem 4.1. Let us solve the following fractional singular integral

$$\frac{1}{2} D_{0,t}^{C.F,\frac{1}{2}} \psi(t) = \frac{1}{\sqrt{\pi t}} \int_{0}^{+\infty} \cos(2\sqrt{t\xi})\psi(\xi)d\xi, \quad \psi(0) = 1.$$

Solution. Taking the Laplace transform of the above integral equation leads to

$$\frac{s\Psi(s) - 1}{s+1} = \frac{1}{\sqrt{s}} \Psi(\frac{1}{s}),$$

in the above relation we change s to $\frac{1}{s}$ to get

$$\frac{\Psi(\frac{1}{s}) - s}{s+1} = \sqrt{s}\Psi(s),$$

from which we get

$$\Psi(\frac{1}{s}) = s + (s+1)\sqrt{s}\Psi(s).$$

by setting this value in the first relation, we obtain

$$\frac{s\Psi(s)-1}{s+1} = \frac{1}{\sqrt{s}}[s + (s+1)\sqrt{s}\Psi(s)] = \sqrt{s} + (s+1)\Psi(s),$$

finally we have

$$\Psi(s) = \frac{1+\sqrt{s}(s+1)}{s-(s+1)^2} = -\frac{s\sqrt{s}+\sqrt{s}+1}{s^2+s+1} = \frac{1}{\sqrt{s}(s^2+s+1)} - \frac{1}{\sqrt{s}} - \frac{1}{s^2+s+1},$$

at this point, taking the inverse Laplace transform, yields

$$\psi(t) = \int_0^t \frac{e^{\frac{\xi}{2}}\sin(\frac{\sqrt{3}\xi}{2})}{\sqrt{\pi(t-\xi)}}d\xi - \frac{1}{\sqrt{\pi t}} - e^{\frac{t}{2}}\sin(\frac{\sqrt{3}t}{2}).$$

Problem 4.2. Let us solve the following convolution type fractional Volterra singular integral equation

$$\int_0^t \frac{1}{2}D_{0,t}^{C.F,\frac{1}{2}}\psi(\xi).D_{0,t}^{C,\beta}\psi(t-\xi)d\xi = \cosh(t), \quad \psi(0)=0.$$

Solution. Taking the Laplace transform of the above integral equation leads to

$$\frac{s\Psi(s)}{s+1}.s^{\beta}\Psi(s) = \frac{s}{s^2-1},$$

after simplifying we get

$$\Psi(s) = \frac{1}{s^{\frac{\beta}{2}}\sqrt{s-1}},$$

finally, by taking the inverse Laplace transform of the above equation, we obtain

$$\psi(t) = \frac{1}{\Gamma(\frac{\beta}{2})}\int_0^t (t-\xi)^{\frac{\beta}{2}-1}\frac{e^{\xi}}{\sqrt{\pi\xi}}d\xi.$$

5. Conclusion

In this study, It is shown the applications of integral transforms in evaluating series and integrals, as well as solving fractional differential equations, the partial fractional differential equation and the fractional singular integral equation, where the fractional derivatives are in the Caputo-Fabrizio sense. Some new integral relations involving the Bessel functions are also provided.

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