



Generalized S -function and Its Application to Statistical Distribution

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ABSTRACT: In this paper, we present $p + q + m$ and $p + q + 2m$ parametric S -functions, exploring the relationships between these two types of functions. We derive their fundamental properties, including generating functions, recurrence relations, differential formulas and integral representations. From these results, we establish several notable corollaries. Furthermore, we present a distribution function involving the S -function and compute the Laplace transform of its density function. Additionally, we explore the connections between the S -function and other significant transcendental functions.

Key Words: Generalized S -function, generalized Hypergeometric function, Mittag-Leffler function, wright function, generating functions, recurrence relations, differentiation formulas, integral representations.

Contents

1 Introduction and Preliminaries	1
2 Generating Functions	3
3 Recurrence Relations	5
4 Differentiation Formulas	9
5 Integral Representations	10
6 Relations between $p + q + m$ and $p + q + 2m$ Parametric S-Functions	12
7 Application of S-Function to Statistical Distribution	13
8 Relations with other Transcendental Functions	15
9 Future Scope	18
10 Conclusion	18

1. Introduction and Preliminaries

Special functions, characterized by their unique properties and versatile applications, play pivotal roles in a variety of fields including mathematical physics, signal processing, number theory, quantum mechanics, and beyond. These functions, each with unique features designed for their specific purposes, have been the subject of comprehensive investigation and analysis. Their extensive study has unveiled a diverse array of behaviours and functionalities, allowing researchers to utilize their capabilities for a wide range of applications. Depending on the intricacies of the context and the particular field of study, these special functions are often designated by specific symbols or nomenclature, serving as indispensable tools for modeling, analysis, and problem-solving in a wide spectrum of scientific disciplines, (see for example, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]).

In this paper, we introduce the generalized S -function in connection with special functions, characterized

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by $p + q + 2m$ parameters, as a power series with p parameters in the numerator and $q + 2m$ parameters in the denominator.

$$\begin{aligned}
{}_pS_{q,(d_j),(\alpha_\tau,\beta_\tau)}^{(c_i),m}(z) &= {}_pS_q^m \left[\begin{matrix} (c_p) \\ (d_q) \end{matrix} \middle| (\alpha_\tau, \beta_\tau); z \right] \\
&= {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] \\
&= \sum_{k=0}^{\infty} \frac{(c_1)_k, \dots, (c_p)_k}{(d_1)_k, \dots, (d_q)_k} \frac{z^k}{\prod_{\tau=1}^m \Gamma(\alpha_\tau k + \beta_\tau) k!},
\end{aligned} \tag{1.1}$$

where $\alpha_\tau, \beta_\tau, z \in \mathbb{C}(\tau = \overline{1, m}; m \in \mathbb{N})$, $c_i \in \mathbb{C}(i = \overline{1, p}; p \in \mathbb{N})$, $d_j \in \mathbb{C}(j = \overline{1, q}; q \in \mathbb{N})$, $\Re(\alpha_\tau) > 0$, $\Re(\beta_\tau) > 0$.

Here $(\zeta)_k$ is well known Pochhammer symbol defined by

$$\begin{aligned}
(\zeta)_k &= \begin{cases} 1 & k = 0, \zeta \neq 0 \\ \zeta(\zeta + 1) \cdots (\zeta + k - 1) & k \in \mathbb{N}, \zeta \in \mathbb{C} \end{cases} \\
&= \frac{\Gamma(\zeta + k)}{\Gamma(\zeta)}, \quad k \in \mathbb{N} \cup \{0\}.
\end{aligned} \tag{1.2}$$

wherein familiar Gamma function, denoted as $\Gamma(z)$, finds its definition as

$$\Gamma(z) = \int_0^\infty \xi^{z-1} e^{-\xi} d\xi \quad (\Re(z) > 0). \tag{1.3}$$

In series (1.1), none of the denominator parameter d_j should be zero or negative integer. Additionally, if any numerator parameter c_i is zero or negative integer, the series terminates as a polynomial in z . For cases other than these,

- (i) If $\Re\left(\sum_{\tau=1}^m \alpha_\tau\right) \geq p - q$, the series (1.1) converges absolutely for every finite value of z .
- (ii) If $\Re\left(\sum_{\tau=1}^m \alpha_\tau\right) < p - q - 1$, the series diverges.
- (iii) If $\Re\left(\sum_{\tau=1}^m \alpha_\tau\right) = p - q - 1$, the series is absolutely convergent when $|z| < \prod_{\tau=1}^m \{\Re(\alpha_\tau)\}^{\Re(\alpha_\tau)}$ and divergent when $|z| > \prod_{\tau=1}^m \{\Re(\alpha_\tau)\}^{\Re(\alpha_\tau)}$.
- (iv) If $\alpha_\tau \in \mathbb{N}(\tau = \overline{1, m})$, $\sum_{\tau=1}^m \alpha_\tau = p - q - 1$ and $|z| = \prod_{\tau=1}^m (\alpha_\tau)^{\alpha_\tau}$, it converges absolutely when $\Re\left(\sum_{i=1}^{\alpha_1} \left(\frac{\beta_1 + i_1 - 1}{\alpha_1}\right) + \cdots + \sum_{i_m=1}^{\alpha_m} \left(\frac{\beta_m + i_m - 1}{\alpha_m}\right) + \sum_{j=1}^q d_j - \sum_{i=1}^p c_i\right) > 0$.

On setting $\beta_\tau = 1(\tau = \overline{1, m}; m \in \mathbb{N})$, $p + q + 2m$ parametric S -function reduces to $p + q + m$ parametric S -function, with p parameters in the numerator and $q + m$ in the denominator, as shown below:

$$\begin{aligned}
{}_pS_{q,(d_j),(\alpha_\tau)}^{(c_i),m}(z) &= {}_pS_q^m \left[\begin{matrix} (c_p) \\ (d_q) \end{matrix} \middle| \alpha_\tau; z \right] \\
&= {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| \alpha_1, \dots, \alpha_m; z \right] \\
&= \sum_{k=0}^{\infty} \frac{(c_1)_k, \dots, (c_p)_k}{(d_1)_k, \dots, (d_q)_k} \frac{z^k}{\prod_{\tau=1}^m \Gamma(\alpha_\tau k + 1) k!},
\end{aligned} \tag{1.4}$$

where $\alpha_\tau, z \in \mathbb{C}(\tau = \overline{1, m}; m \in \mathbb{N}), c_i \in \mathbb{C}(i = \overline{1, p}; p \in \mathbb{N}), d_j \in \mathbb{C}(j = \overline{1, q}; q \in \mathbb{N}), \Re(\alpha_\tau) > 0$. If $\alpha_\tau \in \mathbb{N}(\tau = \overline{1, m}), \sum_{\tau=1}^m \alpha_\tau = p - q - 1$ and $|z| = \prod_{\tau=1}^m (\alpha_\tau)^{\alpha_\tau}$, it converges absolutely when

$$\Re \left(\sum_{i=1}^{\alpha_1} \left(\frac{i_1}{\alpha_1} \right) + \cdots + \sum_{i_m=1}^{\alpha_m} \left(\frac{i_m}{\alpha_m} \right) + \sum_{j=1}^q d_j - \sum_{i=1}^p c_i \right) > 0.$$

rest of the convergence conditions remain same as stated with (1.1).

2. Generating Functions

In many fields of applied mathematics and mathematical physics, generating functions are crucial for studying important properties of the sequences they generate. These functions are employed to derive specific properties and formulas for numbers and polynomials, playing a significant role in diverse research areas, including modern combinatorics (see, for example, [40], [41], [42]). In this section, we introduce several generating functions related to S -functions.

First, we revisit the definition of the generalized binomial coefficient, denoted as $\binom{\theta}{\mu}$, which can be defined for real or complex parameters θ and μ as follows:

$$\binom{\theta}{\mu} = \frac{\Gamma(\theta + 1)}{\Gamma(\mu + 1)\Gamma(\theta - \mu + 1)} = \binom{\theta}{\theta - \mu} \quad (\theta, \mu \in \mathbb{C}). \quad (2.1)$$

In the particular instance when $\mu = n$ ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$), we get

$$\binom{\theta}{n} = \frac{\theta(\theta - 1) \cdots (\theta - n + 1)}{n!} = \frac{(-1)^n (-\theta)_n}{n!} \quad (n \in \mathbb{N}_0). \quad (2.2)$$

Here, $(\theta)_n$ ($\theta \in \mathbb{C}$) represents the Pochhammer symbol, as defined in equation (1.2).

Theorem 2.1 *If the convergence conditions of (1.1) are satisfied and $\lambda \in \mathbb{C}$, then the following generating function is valid:*

$$\begin{aligned} & \sum_{n=0}^{\infty} \binom{\lambda + n - 1}{n} {}_pS_q^m \left[\begin{matrix} \lambda + n, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] t^n \\ &= (1 - t)^{-\lambda} {}_pS_q^m \left[\begin{matrix} \lambda, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); \frac{z}{1 - t} \right] \quad (|t| < 1). \end{aligned} \quad (2.3)$$

Proof: Let $\mathcal{L}_1$ represent the left-hand side of equation (2.3). Utilizing (1.1), we derive:

$$\mathcal{L}_1 = \sum_{n=0}^{\infty} \binom{\lambda + n - 1}{n} \left(\sum_{k=0}^{\infty} \frac{(\lambda + n)_k (c_2)_k, \dots, (c_p)_k}{(d_1)_k, (d_2)_k, \dots, (d_q)_k} \frac{z^k}{\prod_{\tau=1}^m \Gamma(\alpha_\tau k + \beta_\tau) k!} \right) t^n,$$

changing the order of summations, using formula (2.1) and making little simplifications, we obtain

$$\mathcal{L}_1 = \sum_{k=0}^{\infty} \frac{(\lambda)_k (c_2)_k, \dots, (c_p)_k}{(d_1)_k, (d_2)_k, \dots, (d_q)_k \prod_{\tau=1}^m \Gamma(\alpha_\tau k + \beta_\tau)} \left[\sum_{n=0}^{\infty} \binom{\lambda + n + k - 1}{n} t^n \right] \frac{z^k}{k!}.$$

Now applying following generalized binomial expansion formula

$$\sum_{n=0}^{\infty} \binom{\lambda + n - 1}{n} t^n = (1 - t)^{-\lambda} \quad (|t| < 1, \lambda \in \mathbb{C}), \quad (2.4)$$

This leads to the desired conclusion (2.3) of Theorem 2.1. $\square$

On setting $\beta_\tau = 1$ ($\tau = \overline{1, m}; m \in \mathbb{N}$), we obtain following corollary in terms of $p + q + m$ parametric S -function,

Corollary 2.1 *If the convergence conditions of (1.4) are satisfied and $\lambda \in \mathbb{C}$, then the following generating function is valid:*

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{\lambda + n - 1}{n} {}_pS_q^m \left[\begin{matrix} \lambda + n, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| \alpha_1, \dots, \alpha_m; z \right] t^n \\ = (1-t)^{-\lambda} {}_pS_q^m \left[\begin{matrix} \lambda, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| \alpha_1, \dots, \alpha_m; \frac{z}{1-t} \right] \quad (|t| < 1). \end{aligned} \quad (2.5)$$

Definition 2.1 We introduce sequence $\left\{ {}_p\mathscr{W}_{q,n,(d_j),(\alpha_\tau,\beta_\tau)}^{(\lambda,\rho),(c_i),m}(z) \right\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} {}_p\mathscr{W}_{q,n,(d_j),(\alpha_\tau,\beta_\tau)}^{(\lambda,\rho),(c_i),m}(z) &= {}_p\mathscr{W}_{q,n}^{(\lambda,\rho),m} \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] \\ &= {}_{\varrho+p}S_q^m \left[\begin{matrix} \Delta(\varrho, \lambda + n), c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right], \\ &(\lambda \in \mathbb{C}, \varrho \in \mathbb{N}) \end{aligned} \quad (2.6)$$

for convenience, we use $\Delta(\varrho; \lambda)$ to denote the array of ϱ parameters.

$$\frac{\lambda}{\varrho}, \frac{\lambda+1}{\varrho}, \dots, \frac{\lambda+\varrho-1}{\varrho}; \quad (\varrho \in \mathbb{N}).$$

Definition 2.2 If we set $\beta_\tau = 1$ ($\tau = \overline{1, m}; m \in \mathbb{N}$) in (2.6), we obtain sequence $\left\{ {}_p\mathscr{W}_{q,n,(d_j),(\alpha_\tau)}^{(\lambda,\rho),(c_i),m}(z) \right\}_{n=0}^{\infty}$ in terms of $p + q + m$ parametric S -function, defined as

$$\begin{aligned} {}_p\mathscr{W}_{q,n,(d_j),(\alpha_\tau)}^{(\lambda,\rho),(c_i),m}(z) &= {}_p\mathscr{W}_{q,n}^{(\lambda,\rho),m} \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| \alpha_1, \dots, \alpha_m; z \right] \\ &= {}_{\varrho+p}S_q^m \left[\begin{matrix} \Delta(\varrho, \lambda + n), c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| \alpha_1, \dots, \alpha_m; z \right], \\ &(\lambda \in \mathbb{C}, \varrho \in \mathbb{N}) \end{aligned} \quad (2.7)$$

where $\Delta(\varrho; \lambda)$ is same as given with (2.6).

We thus obtain the following generating functions.

Theorem 2.2 *For $\lambda \in \mathbb{C}$, $\varrho \in \mathbb{N}$, then the following generating function is valid:*

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{\lambda + \nu + n - 1}{n} {}_p\mathscr{W}_{q,\nu+n,(d_j),(\alpha_\tau,\beta_\tau)}^{(\lambda,\rho),(c_i),m}(z) t^n \\ = (1-t)^{-(\lambda+\nu)} {}_p\mathscr{W}_{q,\nu,(d_j),(\alpha_\tau,\beta_\tau)}^{(\lambda,\rho),(c_i),m} \left(\frac{z}{(1-t)^\varrho} \right) \quad (\nu \in \mathbb{N}_0, |t| < 1). \end{aligned} \quad (2.8)$$

Proof: Let $\mathcal{L}_2$ represent the left-hand side of (2.8). By applying the definitions in (2.6) and (1.1), and rearranging the order of summation, we arrive at:

$$\begin{aligned} \mathcal{L}_2 &= \sum_{k=0}^{\infty} \left(\frac{\lambda + \nu}{\varrho} \right)_k \left(\frac{\lambda + \nu + 1}{\varrho} \right)_k \dots \left(\frac{\lambda + \nu + \varrho - 1}{\varrho} \right)_k \\ &\quad \times \frac{(c_1)_k, \dots, (c_p)_k}{(d_1)_k, \dots, (d_q)_k \prod_{\tau=1}^m \Gamma(\alpha_{\tau} k + \beta_{\tau})} \\ &\quad \times \left[\sum_{n=0}^{\infty} \left(\frac{\lambda + \nu + \varrho k + n - 1}{n} \right) t^n \right] \frac{z^k}{k!}. \end{aligned} \quad (2.9)$$

Now, by applying the generalized binomial expansion formula (2.4), we readily obtain the desired result (2.8) of Theorem 2.2. $\square$

On setting $\beta_{\tau} = 1$ ($\tau = \overline{1, m}; m \in \mathbb{N}$) in (2.8), we obtain following corollary involving the sequence as defined in (2.7).

Corollary 2.2 *For $\lambda \in \mathbb{C}$, $\varrho \in \mathbb{N}$, then the following generating function is valid:*

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\frac{\lambda + \nu + n - 1}{n} \right) {}_p\mathcal{W}_{q, \nu+n, (d_j), (\alpha_{\tau})}^{(\lambda, \rho), (c_i), m}(z) t^n \\ &= (1-t)^{-(\lambda+\nu)} {}_p\mathcal{W}_{q, \nu, (d_j), (\alpha_{\tau})}^{(\lambda, \rho), (c_i), m} \left(\frac{z}{(1-t)^{\varrho}} \right) \quad (\nu \in \mathbb{N}_0, |t| < 1). \end{aligned} \quad (2.10)$$

3. Recurrence Relations

Recurrence relations of special functions play important role in various scientific and engineering fields. For instance, Özbey et al. [43] explored recurrence relations of the multiplicative Bessel function and utilized them to analyze its orthogonality and norm. Similarly, Sachan et al. [5] derived novel recurrence relations for the generalized Mittag-Leffler function and, by applying Riemann-Liouville fractional calculus operators, established new connections with Fox-Wright functions. These studies underscore the fundamental significance of recurrence relations of special functions in diverse scientific and engineering applications. In this sections, we derive several noteworthy recurrence relations for the function (1.1) with respect to its parameters, as outlined below:

Theorem 3.1 *Let $\alpha_{\tau}, \beta_{\tau}, c_i, d_j, z \in \mathbb{C}$ ($\tau = \overline{1, m}, i = \overline{1, p}; j = \overline{1, q}$), with $\Re(\alpha_{\tau}) > 0, \Re(\beta_{\tau}) > 0$, ($m, n \in \mathbb{N}$) and all d_j strictly positive, then*

$$\begin{aligned} &{}_p S_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] = \frac{z^n}{n!} \left(\frac{\prod_{i=1}^p (c_i)_n}{\prod_{j=1}^q (d_j)_n} \right) \\ &\quad \times {}_{p+1} S_{q+1}^m \left[\begin{matrix} 1, c_1 + n, \dots, c_p + n \\ n + 1, d_1 + n, \dots, d_q + n \end{matrix} \middle| (\alpha_1, n\alpha_1 + \beta_1), \dots, (\alpha_m, n\alpha_m + \beta_m); z \right] \\ &\quad + \sum_{k=0}^{n-1} \left(\frac{\prod_{i=1}^p (c_i)_k}{\prod_{j=1}^q (d_j)_k} \right) \frac{z^k}{\prod_{\tau=1}^m \Gamma(\alpha_{\tau} k + \beta_{\tau}) k!}. \end{aligned} \quad (3.1)$$

Proof: To prove (3.1), we start from,

$$\begin{aligned}
& {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] \\
&= \sum_{k=0}^{n-1} \left(\frac{\prod_{i=1}^p (c_i)_k}{\prod_{j=1}^q (d_j)_k} \right) \frac{z^k}{\prod_{\tau=1}^m \Gamma(\alpha_\tau k + \beta_\tau) k!} \\
&= \sum_{k=n}^{\infty} \left(\frac{\prod_{i=1}^p (c_i)_k}{\prod_{j=1}^q (d_j)_k} \right) \frac{z^k}{\prod_{\tau=1}^m \Gamma(\alpha_\tau k + \beta_\tau) k!},
\end{aligned}$$

replacing k by $k + n$, then

$$\begin{aligned}
& \sum_{k=n}^{\infty} \left(\frac{\prod_{i=1}^p (c_i)_k}{\prod_{j=1}^q (d_j)_k} \right) \frac{z^k}{\prod_{\tau=1}^m \Gamma(\alpha_\tau k + \beta_\tau) k!} \\
&= \sum_{k=0}^{\infty} \left(\frac{\prod_{i=1}^p (c_i)_{k+n}}{\prod_{j=1}^q (d_j)_{k+n}} \right) \frac{z^{k+n}}{\prod_{\tau=1}^m \Gamma(\alpha_\tau (k+n) + \beta_\tau) (k+n)!},
\end{aligned}$$

using formula $(c)_{k+n} = (c)_n (c+n)_k$ and doing simple manipulations, we obtain desired result (3.1). $\square$

On setting $n = 1, 2, 3 \dots$ in (3.1), we obtain following corollaries:

Corollary 3.1 Let $\alpha_\tau, \beta_\tau, c_i, d_j, z \in \mathbb{C}$ ($\tau = \overline{1, m}; i = \overline{1, p}; j = \overline{1, q}$), with $\Re(\alpha_\tau) > 0, \Re(\beta_\tau) > 0, m \in \mathbb{N}$ and all d_j strictly positive, then

$$\begin{aligned}
& {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] = z \left(\frac{\prod_{i=1}^p (c_i)}{\prod_{j=1}^q (d_j)} \right) \\
& \times {}_{p+1}S_{q+1}^m \left[\begin{matrix} 1, c_1 + 1, \dots, c_p + 1 \\ 2, d_1 + 1, \dots, d_q + 1 \end{matrix} \middle| (\alpha_1, \alpha_1 + \beta_1), \dots, (\alpha_m, \alpha_m + \beta_m); z \right] + \frac{1}{\prod_{\tau=1}^m \Gamma(\beta_\tau)}. \quad (3.2)
\end{aligned}$$

Corollary 3.2 Let $\alpha_\tau, \beta_\tau, c_i, d_j, z \in \mathbb{C}$ ($\tau = \overline{1, m}; i = \overline{1, p}; j = \overline{1, q}$), with $\Re(\alpha_\tau) > 0, \Re(\beta_\tau) > 0, m \in \mathbb{N}$ and all d_j strictly positive, then

$$\begin{aligned}
{}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] &= \frac{z^2}{2!} \left(\frac{\prod_{i=1}^p (c_i)_2}{\prod_{j=1}^q (d_j)_2} \right) \\
&\times {}_{p+1}S_{q+1}^m \left[\begin{matrix} 1, c_1 + 2, \dots, c_p + 2 \\ 3, d_1 + 2, \dots, d_q + 2 \end{matrix} \middle| (\alpha_1, 2\alpha_1 + \beta_1), \dots, (\alpha_m, 2\alpha_m + \beta_m); z \right] \\
&+ \left(\frac{\prod_{i=1}^p (c_i)}{\prod_{j=1}^q (d_j)} \right) \frac{z}{\prod_{\tau=1}^m \Gamma(\alpha_\tau + \beta_\tau)} + \frac{1}{\prod_{\tau=1}^m \Gamma(\beta_\tau)}.
\end{aligned} \tag{3.3}$$

Corollary 3.3 Let $\alpha_\tau, \beta_\tau, c_i, d_j, z \in \mathbb{C}$ ($\tau = \overline{1, m}, i = \overline{1, p}; j = \overline{1, q}$), with $\Re(\alpha_\tau) > 0, \Re(\beta_\tau) > 0, m \in \mathbb{N}$ and all d_j strictly positive, then

$$\begin{aligned}
{}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] &= \frac{z^3}{3!} \left(\frac{\prod_{i=1}^p (c_i)_3}{\prod_{j=1}^q (d_j)_3} \right) \\
&\times {}_{p+1}S_{q+1}^m \left[\begin{matrix} 1, c_1 + 3, \dots, c_p + 3 \\ 4, d_1 + 3, \dots, d_q + 3 \end{matrix} \middle| (\alpha_1, 3\alpha_1 + \beta_1), \dots, (\alpha_m, 3\alpha_m + \beta_m); z \right] \\
&+ \left(\frac{\prod_{i=1}^p (c_i)_2}{\prod_{j=1}^q (d_j)_2} \right) \frac{z^2}{2 \prod_{\tau=1}^m \Gamma(2\alpha_\tau + \beta_\tau)} + \left(\frac{\prod_{i=1}^p (c_i)}{\prod_{j=1}^q (d_j)} \right) \frac{z}{\prod_{\tau=1}^m \Gamma(\alpha_\tau + \beta_\tau)} + \frac{1}{\prod_{\tau=1}^m \Gamma(\beta_\tau)}.
\end{aligned} \tag{3.4}$$

and so on.

Theorem 3.2 Let $\alpha_\tau, \beta_\tau, c_i, d_j, z \in \mathbb{C}$ ($\tau = \overline{1, m}, i = \overline{1, p}; j = \overline{1, q}$), with $\Re(\alpha_\tau) > 0, \Re(\beta_\tau) > 0, m \in \mathbb{N}$ and all d_j strictly positive, then

$$\begin{aligned}
&{}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1 - 1), (\alpha_2, \beta_2), \dots, (\alpha_m, \beta_m); z \right] \\
&= \alpha_1 c_{1p} {}_pS_q^m \left[\begin{matrix} c_1 + 1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] \\
&- (1 + \alpha_1 c_1 - \beta_1) {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] \quad \Re(\beta_1) > 1.
\end{aligned} \tag{3.5}$$

Proof: Let $\mathcal{R}_1$ represent the right-hand side of (3.5). Using the definition of S -function (1.1), we obtain

$$\begin{aligned}\mathcal{R}_1 &= \sum_{k=0}^{\infty} \frac{(c_2)_k, \dots, (c_p)_k}{(d_1)_k, \dots, (d_q)_k} \frac{[\alpha_1 c_1 (c_1 + 1)_k - (1 + \alpha_1 c_1 - \beta_1)(c_1)_k] z^k}{\prod_{\tau=1}^m \Gamma(\alpha_{\tau} k + \beta_{\tau}) k!} \\ &= \sum_{k=0}^{\infty} \frac{(c_1)_k, \dots, (c_p)_k}{(d_1)_k, \dots, (d_q)_k} \frac{z^k}{\Gamma(\alpha_1 k + \beta_1 - 1) \prod_{\tau=2}^m \Gamma(\alpha_{\tau} k + \beta_{\tau}) k!} \\ &= {}_p S_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1 - 1), (\alpha_2, \beta_2), \dots, (\alpha_m, \beta_m); z \right].\end{aligned}$$

This leads the completion of Theorem 3.2. $\square$

Theorem 3.3 Let $\alpha_{\tau}, \beta_{\tau}, c_i, d_j, z \in \mathbb{C}$ ($\tau = \overline{1, m}, i = \overline{1, p}; j = \overline{1, q}$), with $\Re(\alpha_{\tau}) > 0, \Re(\beta_{\tau}) > 0, \Re(\beta_1 - \alpha_1) > 0, m \in \mathbb{N}$ and all d_j strictly positive, then

$$\begin{aligned}z \left(\frac{\prod_{i=2}^p (c_i)}{\prod_{j=1}^q (d_j)} \right) {}_p S_q^m \left[\begin{matrix} c_1, c_2 + 1, \dots, c_p + 1 \\ d_1 + 1, d_2 + 1, \dots, d_q + 1 \end{matrix} \middle| (\alpha_1, \beta_1), (\alpha_2, \beta_2 + \alpha_2), \dots, (\alpha_m, \beta_m + \alpha_m); z \right] \\ = {}_p S_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1 - \alpha_1), (\alpha_2, \beta_2), \dots, (\alpha_m, \beta_m); z \right] \\ - {}_p S_q^m \left[\begin{matrix} c_1 - 1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1 - \alpha_1), (\alpha_2, \beta_2), \dots, (\alpha_m, \beta_m); z \right].\end{aligned}\tag{3.6}$$

Proof: Let $\mathcal{R}_2$ represent the right-hand side of (3.6). Using the definition of S -function (1.1), we obtain

$$\mathcal{R}_2 = \sum_{k=0}^{\infty} \frac{(c_2)_k, \dots, (c_p)_k}{(d_1)_k, \dots, (d_q)_k} \frac{[(c_1)_k - (c_1 - 1)_k] z^k}{\Gamma(\alpha_1 k + \beta_1 - \alpha_1) \prod_{\tau=2}^m \Gamma(\alpha_{\tau} k + \beta_{\tau}) k!},$$

writing $(c_1)_k - (c_1 - 1)_k = \frac{k(c_1 - 1)_k}{c_1 - 1}$, we obtain

$$\mathcal{R}_2 = \sum_{k=1}^{\infty} \frac{k(c_1 - 1)_k (c_2)_k, \dots, (c_p)_k}{(c_1 - 1) (d_1)_k, \dots, (d_q)_k} \frac{z^k}{\Gamma(\alpha_1 k + \beta_1 - \alpha_1) \prod_{\tau=2}^m \Gamma(\alpha_{\tau} k + \beta_{\tau}) k!},$$

replacing k by $k + 1$ and using formula $(a)_{m+n} = (a)_n (a + n)_m$, we have

$$\mathcal{R}_2 = z \left(\frac{\prod_{i=2}^p (c_i)}{\prod_{j=1}^q (d_j)} \right) {}_p S_q^m \left[\begin{matrix} c_1, c_2 + 1, \dots, c_p + 1 \\ d_1 + 1, d_2 + 1, \dots, d_q + 1 \end{matrix} \middle| (\alpha_1, \beta_1), (\alpha_2, \beta_2 + \alpha_2), \dots, (\alpha_m, \beta_m + \alpha_m); z \right].$$

This leads the completion of Theorem 3.3. $\square$

Theorem 3.4 Let $\alpha_\tau, \beta_\tau, c_i, d_j, z \in \mathbb{C}$ ($\tau = \overline{1, m}, i = \overline{1, p}; j = \overline{1, q}$), with $\Re(\alpha_\tau) > 0, \Re(\beta_\tau) > 0, m \in \mathbb{N}$ and all d_j strictly positive, then

$$\begin{aligned} & {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] \\ &= \beta_1 {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1 + 1), (\alpha_2, \beta_2), \dots, (\alpha_m, \beta_m); z \right] \\ &+ \alpha_1 z \frac{d}{dz} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1 + 1), (\alpha_2, \beta_2), \dots, (\alpha_m, \beta_m); z \right]. \end{aligned} \quad (3.7)$$

Proof: Let $\mathcal{R}_3$ represent the right-hand side of (3.7). Using the definition of S -function (1.1), we obtain

$$\begin{aligned} \mathcal{R}_3 &= \beta_1 {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1 + 1), (\alpha_2, \beta_2), \dots, (\alpha_m, \beta_m); z \right] \\ &+ \alpha_1 z \frac{d}{dz} \left[\sum_{k=0}^{\infty} \frac{(c_1)_k, \dots, (c_p)_k}{(d_1)_k, \dots, (d_q)_k} \frac{z^k}{\Gamma(\alpha_1 k + \beta_1 + 1) \prod_{\tau=2}^m \Gamma(\alpha_\tau k + \beta_\tau) k!} \right], \\ &= \beta_1 {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1 + 1), (\alpha_2, \beta_2), \dots, (\alpha_m, \beta_m); z \right] \\ &+ \sum_{k=0}^{\infty} \frac{(c_1)_k, \dots, (c_p)_k}{(d_1)_k, \dots, (d_q)_k} \frac{(\alpha_1 k + \beta_1 - \beta_1) z^k}{\Gamma(\alpha_1 k + \beta_1 + 1) \prod_{\tau=2}^m \Gamma(\alpha_\tau k + \beta_\tau) k!}, \\ &= {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right]. \end{aligned}$$

This leads the completion of Theorem 3.4. $\square$

4. Differentiation Formulas

In this section, we introduce formulas for higher-order derivatives of the function (1.1), presented within the scope of the following theorems.

Theorem 4.1 Let $\alpha_\tau, \beta_\tau, c_i, d_j, z \in \mathbb{C}$ ($\tau = \overline{1, m}, i = \overline{1, p}; j = \overline{1, q}$), with $\Re(\alpha_\tau) > 0, \Re(\beta_\tau) > 0, m, n \in \mathbb{N}$ and all d_j strictly positive, then following differential formula holds true:

$$\begin{aligned} & \left(\frac{d}{dz} \right)^n {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] \\ &= \left(\frac{\prod_{i=1}^p (c_i)_n}{\prod_{j=1}^q (d_j)_n} \right) {}_pS_q^m \left[\begin{matrix} c_1 + n, c_2 + n, \dots, c_p + n \\ d_1 + n, d_2 + n, \dots, d_q + n \end{matrix} \middle| (\alpha_1, n\alpha_1 + \beta_1), \dots, (\alpha_m, n\alpha_m + \beta_m); z \right]. \end{aligned} \quad (4.1)$$

Proof: Expressing S -function in its series form and differentiating term by term n times in succession, we have

$$\begin{aligned} & \left(\frac{d}{dz} \right)^n {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] \\ &= \sum_{k=n}^{\infty} \frac{(c_1)_k, \dots, (c_p)_k}{(d_1)_k, \dots, (d_q)_k} \frac{z^{k-n}}{(k-n)! \prod_{\tau=1}^m \Gamma(\alpha_\tau k + \beta_\tau)}, \end{aligned}$$

setting $k = n + r$ and using formula $(a)_{n+r} = (a)_n(a+n)_r$, we can easily obtain desired result. This leads the completion of Theorem 4.1. $\square$

Theorem 4.2 Let $\alpha_\tau, \beta_\tau, c_i, d_j, z \in \mathbb{C}$ ($\tau = \overline{1, m}, i = \overline{1, p}; j = \overline{1, q}$), with $\Re(\alpha_\tau) > 0, \Re(\beta_\tau) > 0, m, n \in \mathbb{N}$ and all d_j strictly positive, then following differential formula holds true:

$$\begin{aligned} & \left(\frac{d}{dz} \right)^n \left\{ z^{c_1+n-1} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] \right\} \\ &= (c_1)_n z^{c_1-1} {}_pS_q^m \left[\begin{matrix} c_1 + n, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right]. \end{aligned} \quad (4.2)$$

Proof: Expressing S -function in its series form and differentiating term by term n times in succession, we have

$$\begin{aligned} & \left(\frac{d}{dz} \right)^n \left\{ z^{c_1+n-1} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] \right\} \\ &= \sum_{k=0}^{\infty} \frac{(c_1)_k, (k+c_1)_n, (c_2)_n, \dots, (c_p)_k}{(d_1)_k, \dots, (d_q)_k} \frac{z^{k+c_1-1}}{\prod_{\tau=1}^m \Gamma(\alpha_\tau k + \beta_\tau) k!}, \end{aligned}$$

using formula $(a)_{n+r} = (a)_n(a+n)_r$, we can easily obtain desired result. This leads the completion of Theorem 4.2. $\square$

Theorem 4.3 Let $\alpha_\tau, \beta_\tau, c_i, d_j, w, z \in \mathbb{C}$ ($\tau = \overline{1, m}, i = \overline{1, p}; j = \overline{1, q}$), with $\Re(\alpha_\tau) > 0, \Re(\beta_\tau) > 0, \Re(\beta_1 - n) > 0, m, n \in \mathbb{N}$ and all d_j strictly positive, then following differential formula holds true:

$$\begin{aligned} & \left(\frac{d}{dz} \right)^n \left\{ z^{\beta_1-1} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); wz^{\alpha_1} \right] \right\} \\ &= z^{\beta_1-n-1} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1 - n), (\alpha_2, \beta_2), \dots, (\alpha_m, \beta_m); wz^{\alpha_1} \right]. \end{aligned} \quad (4.3)$$

Proof: Expressing S -function in its series form and differentiating term by term n times in succession, we have

$$\begin{aligned} & \left(\frac{d}{dz} \right)^n \left\{ z^{\beta_1-1} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); wz^{\alpha_1} \right] \right\} \\ &= z^{\beta_1-n-1} \sum_{k=0}^{\infty} \frac{(c_1)_k, (c_2)_n, \dots, (c_p)_k}{(d_1)_k, \dots, (d_q)_k} \frac{(wz^{\alpha_1})^k}{\Gamma(\alpha_1 k + \beta_1 - n) \prod_{\tau=2}^m \Gamma(\alpha_\tau k + \beta_\tau) k!}, \\ &= z^{\beta_1-n-1} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1 - n), (\alpha_2, \beta_2), \dots, (\alpha_m, \beta_m); wz^{\alpha_1} \right]. \end{aligned}$$

This completes the proof of Theorem 4.3. $\square$

5. Integral Representations

In this section, we present the derivation of several important integrals for the function (1.1), focusing on its parameters and the variable z , as detailed below:

Theorem 5.1 Let $\alpha_\tau, \beta_\tau, c_i, d_j, z \in \mathbb{C}$ ($\tau = \overline{1, m}, i = \overline{1, p}; j = \overline{1, q}$), with $\Re(\alpha_\tau) > 0, \Re(\beta_\tau) > 0, \Re(c_1) > 0, m \in \mathbb{N}$ and all d_j strictly positive, then following integral formula holds true:

$$\begin{aligned} & \frac{1}{\Gamma(c_1)} \int_0^\infty e^{-t} t^{c_1-1} {}_{p-1}S_q^m \left[\begin{matrix} -, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); zt \right] dt \\ &= {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right]. \end{aligned} \quad (5.1)$$

Proof: Let $\mathcal{L}_3$ denote the left-hand side of (5.1). By applying the definition of the S -function in (1.1) and interchanging the order of integration and summation, we obtain:

$$\mathcal{L}_3 = \frac{1}{\Gamma(c_1)} \sum_{k=0}^{\infty} \frac{(c_2)_k, \dots, (c_p)_k}{(d_1)_k, \dots, (d_q)_k} \frac{z^k}{\prod_{\tau=1}^m \Gamma(\alpha_{\tau}k + \beta_{\tau})k!} \int_0^{\infty} e^{-t} t^{c_1+k-1} dt,$$

using Gamma function formula $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$, $\Re(z) > 0$, we obtain desired result. This leads the completion of Theorem 5.1. $\square$

Theorem 5.2 Let $\alpha_{\tau}, \beta_{\tau}, c_i, d_j, z \in \mathbb{C} (\tau = \overline{1, m}, i = \overline{1, p}; j = \overline{1, q})$, with $\Re(\alpha_{\tau}) > 0, \Re(\beta_{\tau}) > 0, m \in \mathbb{N}$ and all d_j strictly positive, then following integral formula holds true:

$$\begin{aligned} & \frac{1}{\Gamma(\alpha_1)} \int_0^z t^{\beta_1-1} (z-t)^{\alpha_1-1} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); \lambda t^{\alpha_1} \right] dt \\ &= z^{\beta_1+\alpha_1-1} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1 + \alpha_1), (\alpha_2, \beta_2), \dots, (\alpha_m, \beta_m); \lambda z^{\alpha_1} \right]. \end{aligned} \quad (5.2)$$

Proof: Let $\mathcal{L}_4$ represent the left-hand side of (5.2). Expressing the S -function into its series form using (1.1) and integrating term by term, we arrive at the following expression:

$$\mathcal{L}_4 = \frac{1}{\Gamma(\alpha_1)} \sum_{k=0}^{\infty} \frac{(c_1)_k, \dots, (c_p)_k}{(d_1)_k, \dots, (d_q)_k} \frac{\lambda^k}{\prod_{\tau=1}^m \Gamma(\alpha_{\tau}k + \beta_{\tau})k!} \int_0^z t^{\alpha_1 k + \beta_1 - 1} (z-t)^{\alpha_1 - 1} dt,$$

substituting $t = z\nu$ and using Beta function formula $\int_0^1 z^{m-1} (1-z)^{n-1} dz = B(m, n)$, $\Re(m) > 0, \Re(n) > 0$ we obtain desired result. This leads the completion of Theorem 5.2. $\square$

Theorem 5.3 Let $\alpha_{\tau}, \beta_{\tau}, c_i, d_j, z \in \mathbb{C} (\tau = \overline{1, m}, i = \overline{1, p}; j = \overline{1, q})$, with $\Re(\alpha_{\tau}) > 0, \Re(\beta_{\tau}) > 0, m \in \mathbb{N}$ and all d_j strictly positive, then following integral formula holds true:

$$\begin{aligned} & {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] = \frac{1}{B(c_1, d_1 - c_1)} \\ & \times \int_0^1 t^{c_1-1} (1-t)^{d_1-c_1-1} {}_{p-1}S_{q-1}^m \left[\begin{matrix} -, c_2, \dots, c_p \\ -, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); zt \right] dt. \end{aligned} \quad (5.3)$$

Proof: It is not difficult to verify that

$$\frac{(c_1)_k}{(d_1)_k} = \frac{1}{B(c_1, d_1 - c_1)} B(c_1 + k, d_1 - c_1), \quad (5.4)$$

using Beta function formula $B(m, n) = \int_0^1 z^{m-1} (1-z)^{n-1} dz$, $\Re(m) > 0, \Re(n) > 0$, equation (5.4) can be written as

$$\frac{(c_1)_k}{(d_1)_k} = \frac{1}{B(c_1, d_1 - c_1)} \int_0^1 t^{c_1+k-1} (1-t)^{d_1-c_1-1} dt. \quad (5.5)$$

Now, by the use of (5.5), we have

$$\begin{aligned} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] &= \frac{1}{B(c_1, d_1 - c_1)} \\ &\times \sum_{k=0}^{\infty} \frac{(c_2)_k, \dots, (c_p)_k}{(d_2)_k, \dots, (d_q)_k} \frac{z^k}{\prod_{\tau=1}^m \Gamma(\alpha_{\tau}k + \beta_{\tau})k!} \int_0^1 t^{c_1+k-1} (1-t)^{d_1-c_1-1} dt, \end{aligned}$$

rearranging the order of summation and integration, we arrive at the desired result. This leads the completion of Theorem 5.3. $\square$

6. Relations between $p + q + m$ and $p + q + 2m$ Parametric S -Functions

If we set $\beta_{\tau} = 1, (\tau = \overline{1, m})$ in (3.1), (3.7), (4.1) and (5.2), we obtain following relations between $p + q + m$ and $p + q + 2m$ parametric S -functions.

Corollary 6.1 *Let $\alpha_{\tau}, c_i, d_j, z \in \mathbb{C} (i = \overline{1, p}; j = \overline{1, q})$, with $\Re(\alpha_{\tau}) > 0$, $(m, n \in \mathbb{N})$ and all d_j strictly positive, then*

$$\begin{aligned} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| \alpha_1, \alpha_2, \dots, \alpha_m; z \right] &= \frac{z^n}{n!} \left(\frac{\prod_{i=1}^p (c_i)_n}{\prod_{j=1}^q (d_j)_n} \right) \\ &\times {}_{p+1}S_{q+1}^m \left[\begin{matrix} 1, c_1 + n, \dots, c_p + n \\ n + 1, d_1 + n, \dots, d_q + n \end{matrix} \middle| (\alpha_1, n\alpha_1 + 1), \dots, (\alpha_m, n\alpha_m + 1); z \right] \\ &+ \sum_{k=0}^{n-1} \left(\frac{\prod_{i=1}^p (c_i)_k}{\prod_{j=1}^q (d_j)_k} \right) \frac{z^k}{\prod_{\tau=1}^m \Gamma(\alpha_{\tau}k + 1)k!}. \end{aligned} \quad (6.1)$$

Corollary 6.2 *Let $\alpha_{\tau}, c_i, d_j, z \in \mathbb{C} (i = \overline{1, p}; j = \overline{1, q})$, with $\Re(\alpha_{\tau}) > 0$, $m \in \mathbb{N}$ and all d_j strictly positive, then*

$$\begin{aligned} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| \alpha_1, \alpha_2, \dots, \alpha_m; z \right] \\ &= {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, 2), (\alpha_2, 1), \dots, (\alpha_m, 1); z \right] \\ &+ \alpha_1 z \frac{d}{dz} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, 2), (\alpha_2, 1), \dots, (\alpha_m, 1); z \right]. \end{aligned} \quad (6.2)$$

Corollary 6.3 *Let $\alpha_{\tau}, c_i, d_j, z \in \mathbb{C} (i = \overline{1, p}; j = \overline{1, q})$, with $\Re(\alpha_{\tau}) > 0$, $m, n \in \mathbb{N}$ and all d_j strictly positive, then following differential formula holds true:*

$$\begin{aligned} &\left(\frac{d}{dz} \right)^n {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| \alpha_1, \alpha_2, \dots, \alpha_m; z \right] \\ &= \left(\frac{\prod_{i=1}^p (c_i)_n}{\prod_{j=1}^q (d_j)_n} \right) {}_pS_q^m \left[\begin{matrix} c_1 + n, c_2 + n, \dots, c_p + n \\ d_1 + n, d_2 + n, \dots, d_q + n \end{matrix} \middle| (\alpha_1, n\alpha_1 + 1), \dots, (\alpha_m, n\alpha_m + 1); z \right]. \end{aligned} \quad (6.3)$$

Corollary 6.4 Let $\alpha_\tau, c_i, d_j, z \in \mathbb{C}$ ($i = \overline{1, p}; j = \overline{1, q}$), with $\Re(\alpha_\tau) > 0$, $m \in \mathbb{N}$ and all d_j strictly positive, then following integral formula holds true:

$$\begin{aligned} & \frac{1}{\Gamma(\alpha_1)} \int_0^z (z-t)^{\alpha_1-1} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| \alpha_1, \alpha_2, \dots, \alpha_m; \lambda t^{\alpha_1} \right] dt \\ &= z^{\alpha_1} {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha_1, \alpha_1 + 1), (\alpha_2, 1), \dots, (\alpha_m, 1); \lambda z^{\alpha_1} \right]. \end{aligned} \quad (6.4)$$

7. Application of S -Function to Statistical Distribution

Several authors have explored distribution functions (for definition, refer to [44]) using Mittag-Leffler (ML) type functions, as seen in [45, 46, 47, 48, 49]. In this section, we present a distribution function involving S -function and additionally derive the Laplace transform (LT) of the density function.

S -Function Statistical Distribution: The distribution function associated with the S -function ($p + q + m$ parametric), given in (1.4), is defined as follows:

$$\begin{aligned} F_y(y) &= 1 - {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| \alpha_1, \dots, \alpha_m; -y^{\alpha_1} \right], \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{y^{\alpha_1 k}}{\prod_{\tau=1}^m \Gamma(\alpha_\tau k + 1) k!}, \quad 0 < \alpha_1 \leq 1, y > 0 \end{aligned} \quad (7.1)$$

and $F_y(y) = 0$ for $y \leq 0$.

Theorem 7.1 Let $y, \alpha_\tau \in \mathbb{R}^+$ with $0 < \alpha_1 \leq 1$, $y > 0$ and $\Re(c_i) > 0$ ($i = \overline{1, p}$), $\Re(d_j) > 0$ ($j = \overline{1, q}$), $m \in \mathbb{N}$. If

$$F_y(y) = 1 - {}_pS_q^m \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| \alpha_1, \dots, \alpha_m; -y^{\alpha_1} \right]$$

then the density function $f(y)$ of distribution function $F_y(y)$ is given by

$$\begin{aligned} f(y) &= y^{\alpha_1-1} \left(\frac{\prod_{i=1}^p (c_i)}{\prod_{j=1}^q (d_j)} \right) \\ &\times {}_{p+1}S_{q+1}^m \left[\begin{matrix} c_1 + 1, \dots, c_p + 1, 1 \\ d_1 + 1, \dots, d_q + 1, 2 \end{matrix} \middle| (\alpha_1, \alpha_1), (\alpha_2, \alpha_2 + 1), \dots, (\alpha_m, \alpha_m + 1); -y^{\alpha_1} \right]. \end{aligned} \quad (7.2)$$

Proof: By taking the derivative of equation (7.1) with respect to y , we obtain the expression for the density function $f(y)$ as follows:

$$\begin{aligned} f(y) &= \frac{d}{dy} F_y(y) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{\alpha_1 k y^{\alpha_1 k - 1}}{\prod_{\tau=1}^m \Gamma(\alpha_\tau k + 1) k!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{y^{\alpha_1 k - 1}}{\Gamma(\alpha_1 k) \prod_{\tau=2}^m \Gamma(\alpha_\tau k + 1) k!}, \end{aligned}$$

replacing k by $k + 1$ and using formula $(a)_{k+1} = a(a+1)_k$, we easily obtain

$$f(y) = y^{\alpha_1-1} \left(\frac{\prod_{i=1}^p (c_i)}{\prod_{j=1}^q (d_j)} \right) \times {}_{p+1}S_{q+1}^m \left[\begin{matrix} c_1 + 1, \dots, c_p + 1, 1 \\ d_1 + 1, \dots, d_q + 1, 2 \end{matrix} \middle| (\alpha_1, \alpha_1), (\alpha_2, \alpha_2 + 1) \dots, (\alpha_m, \alpha_m + 1); -y^{\alpha_1} \right].$$

where ${}_pS_q^m \left[\begin{matrix} (c_p) \\ (d_q) \end{matrix} \middle| (\alpha_\tau, \beta_\tau); z \right]$ is generalized S -function ($p+q+2m$ parametric), defined by (1.1). Thus, the proof of Theorem 7.1 is concluded. $\square$

Theorem 7.2 Let $\alpha_\tau \in \mathbb{R}^+$ with $0 < \alpha_1 \leq 1$ and $\Re(c_i) > 0$ ($i = \overline{1, p}$), $\Re(d_j) > 0$ ($j = \overline{1, q}$), $m \in \mathbb{N}$. Then the LT of the density function (7.2) is given by

$$\mathcal{L}\{f(x)\} = 1 - {}_pS_q^{m-1} \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| -\alpha_2, \dots, \alpha_m; -t^{-\alpha_1} \right]. \quad (7.3)$$

Proof: To prove (7.3), we initiate the proof by considering LT formula.

$$\mathcal{L}\{f(x)\} = f(t) = \int_0^\infty e^{-tx} f(x) dx, \quad (7.4)$$

using (7.2) in (7.4), we have

$$\begin{aligned} \mathcal{L}\{f(x)\} &= \int_0^\infty e^{-tx} x^{\alpha_1-1} \left(\frac{\prod_{i=1}^p (c_i)}{\prod_{j=1}^q (d_j)} \right) \\ &\quad \times {}_{p+1}S_{q+1}^m \left[\begin{matrix} c_1 + 1, \dots, c_p + 1, 1 \\ d_1 + 1, \dots, d_q + 1, 2 \end{matrix} \middle| (\alpha_1, \alpha_1), (\alpha_2, \alpha_2 + 1) \dots, (\alpha_m, \alpha_m + 1); -x^{\alpha_1} \right] dx, \end{aligned}$$

by rewriting the S -function in its series form and exchanging the order of integration and summation, we obtain the following expression.

$$\begin{aligned} \mathcal{L}\{f(x)\} &= \left(\frac{\prod_{i=1}^p (c_i)}{\prod_{j=1}^q (d_j)} \right) \sum_{k=0}^\infty \frac{(-1)^k (c_1 + 1)_k \cdots (c_p + 1)_k (1)_k}{(d_1 + 1)_k \cdots (d_q + 1)_k (2)_k \Gamma(\alpha_1 k + \alpha_1) \prod_{\tau=2}^m \Gamma(\alpha_\tau k + \alpha_\tau + 1) k!} \\ &\quad \times \int_0^\infty e^{-tx} x^{\alpha_1 k + \alpha_1 - 1} dx, \end{aligned}$$

setting $tx = s$, we have

$$\begin{aligned} \mathcal{L}\{f(x)\} &= \left(\frac{\prod_{i=1}^p (c_i)}{\prod_{j=1}^q (d_j)} \right) \sum_{k=0}^\infty \frac{(-1)^k (c_1 + 1)_k \cdots (c_p + 1)_k (1)_k t^{-\alpha_1 k - \alpha_1}}{(d_1 + 1)_k \cdots (d_q + 1)_k (2)_k \Gamma(\alpha_1 k + \alpha_1) \prod_{\tau=2}^m \Gamma(\alpha_\tau k + \alpha_\tau + 1) k!} \\ &\quad \times \int_0^\infty e^{-s} s^{\alpha_1 k + \alpha_1 - 1} ds, \end{aligned}$$

using (1.3) and replacing k by $k - 1$, we obtain

$$\begin{aligned}\mathcal{L}\{f(x)\} &= - \sum_{k=1}^{\infty} \frac{(c_1)_k \cdots (c_p)_k (-t^{-\alpha_1})^k}{(d_1)_k \cdots (d_q)_k \prod_{\tau=2}^m \Gamma(\alpha_{\tau}k + 1) k!} \\ &= 1 - {}_pS_q^{m-1} \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| -t^{-\alpha_1} \right]\end{aligned}$$

where ${}_pS_q^m \left[\begin{matrix} (c_p) \\ (d_q) \end{matrix} \middle| \alpha_{\tau}; z \right]$ is $p + q + m$ parametric S -function, defined by (1.4). With that, the proof of Theorem 7.2 is concluded. $\square$

8. Relations with other Transcendental Functions

A wide range of basic and special functions can be obtained from the generalized S -function ${}_pS_{q,(d_j),(\alpha_{\tau},\beta_{\tau})}^{(c_i),m}(z)$. In this section, we aim to explore connections with several well-known higher transcendental functions.

- (i) If we set $m = 1, c_p = 1, \alpha_1 = \beta_1 = 1$, then S -function (1.1) reduces to a particular case of generalized hypergeometric function.

$$\begin{aligned}{}_pS_q^1 \left[\begin{matrix} c_1, c_2, \dots, c_{p-1}, 1 \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (1, 1); z \right] &= \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_{p-1})_k}{(d_1)_k \cdots (d_q)_k} \frac{z^k}{k!} \\ &= {}_{p-1}F_q \left[\begin{matrix} c_1, c_2, \dots, c_{p-1} \\ d_1, d_2, \dots, d_q \end{matrix} ; z \right]\end{aligned}\tag{8.1}$$

- (ii) If we set $m = 1, p = 3, q = 1, \alpha_1 = \beta_1 = 1$, then S -function (1.1) reduces to Gauss hypergeometric function.

$$\begin{aligned}{}_3S_1^1 \left[\begin{matrix} a, b, 1 \\ c \end{matrix} \middle| (1, 1); z \right] &= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \\ &= {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; z \right]\end{aligned}\tag{8.2}$$

Likewise, we can easily obtain Kummer's function (or confluent hypergeometric function).

- (iii) If we set $p = m + 1, q = m$ and $d_1 = d_2 \dots = d_m = 1$ in S -function (1.1), we obtain $3m$ -parametric ML function due to J. Paneva-Kanovska, see [13], [14].

$$\begin{aligned}{}_{m+1}S_m^m \left[\begin{matrix} c_1, c_2, \dots, c_m, 1 \\ 1, 1, \dots, 1 \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] &= \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_m)_k}{\prod_{\tau=1}^m \Gamma(\alpha_{\tau}k + \beta_{\tau})} \frac{z^k}{(k!)^m} \\ &= E_{(\alpha_{\tau}),(\beta_{\tau})}^{(c_{\tau}),m}(z)\end{aligned}\tag{8.3}$$

- (iv) If we set $p = 1, q = 0, c_1 = 1$ in (1.1), we obtain ML function due Al-Bassam and Luchko , see [15].

$$\begin{aligned}{}_1S_0^m \left[\begin{matrix} 1 \\ - \end{matrix} \middle| (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z \right] &= \sum_{k=0}^{\infty} \frac{z^k}{\prod_{\tau=1}^m \Gamma(\alpha_{\tau}k + \beta_{\tau})} \\ &= E_{(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)}(z)\end{aligned}\tag{8.4}$$

- (v) If we set $p = 1, q = 0, c_1 = 1, \alpha_\tau = \frac{1}{\rho_\tau}, (\rho_\tau > 0)$ in (1.1), then we obtain multiindex (m-tuple, multiple) ML function due to Kiryakova, see [16].

$${}_1S_0^m \left[\begin{matrix} 1 \\ - \end{matrix} \middle| \left(\frac{1}{\rho_1}, \beta_1 \right), \dots, \left(\frac{1}{\rho_m}, \beta_m \right); z \right] = \sum_{k=0}^{\infty} \frac{z^k}{\prod_{\tau=1}^m \Gamma \left(\frac{k}{\rho_\tau} + \beta_\tau \right)} \quad (8.5)$$

$$= E_{(1/\rho_\tau), (\beta_\tau)}(z)$$

- (vi) If we set $m = 1$ then S -function (1.1) reduces to R -function due to Desai and Shukla, see [17].

$${}_pS_q^1 \left[\begin{matrix} c_1, c_2, \dots, c_p \\ d_1, d_2, \dots, d_q \end{matrix} \middle| (\alpha, \beta); z \right] = \sum_{k=0}^{\infty} \frac{(c_1)_k, \dots, (c_p)_k z^k}{(d_1)_k, \dots, (d_q)_k \Gamma(\alpha k + \beta) k!} \quad (8.6)$$

$$= {}_pR_q(\alpha, \beta; z).$$

- (vii) If we set $p = 1, q = 0$ and $m = 2$ then S -function (1.1) reduces to five parameter function ML introduced by Özarslan and Fernandez, see [18].

$${}_1S_0^2 \left[\begin{matrix} c \\ - \end{matrix} \middle| (\alpha_1, \beta_1), (\alpha_2, \beta_2); z \right] = \sum_{k=0}^{\infty} \frac{(c)_k z^k}{\Gamma(\alpha_1 k + \beta_1) \Gamma(\alpha_2 k + \beta_2) k!} \quad (8.7)$$

$$= E_{\alpha_1, \alpha_2, \beta_1, \beta_2}^c(z).$$

- (viii) If we set $p = 0, q = 0, m = 1, \alpha_1 = 1, \beta_1 = 1 + v$ and z is replaced by $-\frac{z^2}{4}$ then S -function (1.1) reduces to Bessel function of first kind, see [19], [20].

$${}_0S_0^1 \left[\begin{matrix} - \\ - \end{matrix} \middle| (1, 1 + v); -\frac{z^2}{4} \right] = \left(\frac{z}{2} \right)^{-v} J_v(z) \quad (8.8)$$

where

$$J_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2} \right)^{v+2k}}{\Gamma(k + v + 1) k!}$$

is the Bessel function of first kind.

- (ix) If we set $p = 0, q = 0, m = 1, \alpha_1 = \alpha, \beta_1 = 1 + v$ and z is replaced by $-z$ in (1.1), then we have

$${}_0S_0^1 \left[\begin{matrix} - \\ - \end{matrix} \middle| (\alpha, v + 1); -z \right] = J_v^\alpha(z) \quad (8.9)$$

where

$$J_v^\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{\Gamma(\alpha k + v + 1) k!}$$

is the Bessel-Maitland function, see [21].

- (x) If we set $p = 1, q = 0, m = 2, \alpha_1 = 1, \alpha_2 = \alpha, \beta_1 = r + 1, \beta_2 = r + v + 1$ and z is replaced by $-\frac{z^2}{4}$ in (1.1), then we have

$${}_1S_0^2 \left[\begin{matrix} 1 \\ - \end{matrix} \middle| (1, r + 1), (\alpha, r + v + 1); -\frac{z^2}{4} \right] = \left(\frac{z}{2} \right)^{-v-2r} J_{v,r}^\alpha(z) \quad (8.10)$$

where

$$J_{v,r}^\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2} \right)^{v+2r+2k}}{\Gamma(k + r + 1) \Gamma(\alpha k + r + v + 1)}$$

is the generalized Bessel-Maitland function, see [21], [26].

- (xi) If we set $p = 1, q = 0, m = 1, c_1 = r, \alpha_1 = v, \beta_1 = rv - \xi$ and z is replaced by $w(z - a)^v$ in (1.1), then we have

$${}_1S_0^1 \left[\begin{matrix} r \\ - \end{matrix} \middle| (v, rv - \xi); w(z - a)^v \right] = (z - a)^{-rv + \xi + 1} G_{v, \xi, r}(w, a, z) \quad (8.11)$$

where

$$G_{v, \xi, r}(w, a, z) = \sum_{k=0}^{\infty} \frac{(r)_k w^k (z - a)^{kv + rv - \xi - 1}}{\Gamma(kv + rv - \xi) k!}$$

is the Lorenzo-Hartley function, see [27], [28].

- (xii) If we set $p = 1, q = 0, m = 2, \alpha_1 = \alpha_2 = c_1 = 1, \beta_1 = \frac{3}{2}, \beta_2 = v + \frac{3}{2}$ and z is replaced by $-\frac{z^2}{4}$ in (1.1), then we have

$${}_1S_0^2 \left[\begin{matrix} 1 \\ - \end{matrix} \middle| \left(1, \frac{3}{2}\right), \left(1, v + \frac{3}{2}\right); -\frac{z^2}{4} \right] = \left(\frac{z}{2}\right)^{-v-1} H_v(z) \quad (8.12)$$

where

$$H_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+v+1}}{\Gamma(k + \frac{3}{2}) \Gamma(k + v + \frac{3}{2})}$$

is Struve function, see [19], [29].

- (xiii) If we set $p = 1, q = 0, m = 2, \alpha_1 = \alpha_2 = c_1 = 1, \beta_1 = \frac{\xi+r}{2} + \frac{3}{2}, \beta_2 = \frac{\xi-r}{2} + \frac{3}{2}$ and z is replaced by $-\frac{z^2}{4}$ in (1.1), then we have

$${}_1S_0^2 \left[\begin{matrix} 1 \\ - \end{matrix} \middle| \left(1, \frac{\xi+r}{2} + \frac{3}{2}\right), \left(1, \frac{\xi-r}{2} + \frac{3}{2}\right); -\frac{z^2}{4} \right] = \frac{4z^{-\xi-1}}{\Gamma\left(\frac{1+\xi+r}{2}\right) \Gamma\left(\frac{1+\xi-r}{2}\right)} S_{\xi, r}(z) \quad (8.13)$$

where

$$S_{\xi, r}(z) = \frac{z^{\xi+1} \Gamma\left(\frac{1+\xi+r}{2}\right) \Gamma\left(\frac{1+\xi-r}{2}\right)}{4} \sum_{k=0}^{\infty} \frac{\left(-\frac{z^2}{4}\right)^k}{\Gamma\left(\frac{\xi+r}{2} + k + \frac{3}{2}\right) \Gamma\left(\frac{\xi-r}{2} + k + \frac{3}{2}\right)}$$

is Lommel function, see [29].

- (xiv) If we set $p = 1, q = 0, m = 2, c_1 = 1$ in (1.1), then we have

$${}_1S_0^2 \left[\begin{matrix} 1 \\ - \end{matrix} \middle| (\alpha_1, \beta_1), (\alpha_2, \beta_2); z \right] = W_{(\alpha_1, \beta_1), (\alpha_2, \beta_2)}(z) \quad (8.14)$$

where

$$W_{(\alpha_1, \beta_1), (\alpha_2, \beta_2)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k + \beta_1) \Gamma(\alpha_2 k + \beta_2)}$$

is the four parametric Wright function, see [30].

- (xv) If we set $p = 0, q = 0, m = 1$ in (1.1), then we have

$${}_0S_0^1 \left[\begin{matrix} - \\ - \end{matrix} \middle| (\alpha, \beta); z \right] = W_{(\alpha, \beta)}(z) \quad (8.15)$$

where

$$W_{(\alpha, \beta)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta) k!}$$

is the Wright function, see [30].

- (xvi) If we set $p = q = 1, m = 1$ then generalized S -function in (1.1) reduces to generalized ML function in four parameters due to Salim, see [31].

$${}_2S_1^1 \left[\begin{matrix} \gamma, 1 \\ \eta \end{matrix} \middle| (\alpha, \beta); z \right] = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{(\eta)_k \Gamma(\alpha k + \beta)} = E_{\alpha, \beta}^{\gamma, \eta}(z). \quad (8.16)$$

- (xvii) If we set $p = 1, q = 0, m = 1$ in (1.1), then we obtain generalized ML function due to Prabhakar, see [32], [5].

$${}_1S_0^1 \left[\begin{matrix} \gamma \\ - \end{matrix} \middle| (\alpha, \beta); z \right] = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta) k!} = E_{\alpha, \beta}^{\gamma}(z). \quad (8.17)$$

- (xviii) If we set $p = 1, q = 0, m = 1$ in (1.1), then we obtain Wiman function, see [5], [33], [34], [35].

$${}_1S_0^1 \left[\begin{matrix} 1 \\ - \end{matrix} \middle| (\alpha, \beta); z \right] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = E_{\alpha, \beta}(z). \quad (8.18)$$

- (xix) If we set $p = 1, q = 0, m = 1$ $\alpha_1 = \alpha$ and $\beta_1 = 1$ in (1.1), then we obtain classical ML function, see [5], [36], [37], [38], [39].

$${}_1S_0^1 \left[\begin{matrix} 1 \\ - \end{matrix} \middle| (\alpha, 1); z \right] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} = E_{\alpha}(z). \quad (8.19)$$

Remark 8.1 The Coulomb function, Cunningham functions, exponential integral, Hermite polynomials, Laguerre polynomials, parabolic cylinder functions, Poisson-Charlier functions, Meixner polynomials, Meixner-Pollaczek polynomials, Toronto functions, Mainardi function, Legendre functions, elliptic modular functions, incomplete gamma functions, incomplete beta functions, Jacobi polynomials, Chebyshev polynomials, Gegenbauer polynomials, as well as the Miller-Ross and Robotnov functions, all represent specific instances of the generalized S -function.

9. Future Scope

Fractional calculus has been widely studied, including its basic theory, numerical methods, and applications in physics and engineering (see, for example, [5], [7], [9], [22], [23], [24], [25]). The S -function is a generalization of the Mittag-Leffler function, which plays a key role in fractional calculus. Due to this connection, the S -function can also be used to solve problems in fractional calculus.

10. Conclusion

This paper introduces $p + q + m$ and $p + q + 2m$ parametric S -functions, providing a comprehensive exploration of their interrelations and fundamental properties. Through the derivation of generating functions, recurrence relations, differential formulas, and integral representations, we establish a solid mathematical foundation for these functions. The resulting corollaries further highlight the significance of these findings. Additionally, the incorporation of a distribution function involving the S -function and the computation of Laplace transform of density function extend the utility of these functions to broader applications. Finally, the demonstrated connections between S -functions and other transcendental functions underscore their relevance and potential for future research in mathematical and applied contexts.

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