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Fuzzy Soft Graph Structure with Application

Umair Amin and Aliya Fahmi*

ABSTRACT: Graph structures provide a powerful framework for addressing combinatorial problems in diverse areas of artificial intelligence and computer systems. Extending this idea, the fuzzy graph structure has been widely studied. In this work, we introduce the concept of a *fuzzy soft graph structure* ($\mathcal{FSGS}$) as a generalization of fuzzy soft graphs. We define and investigate several notions associated with $\mathcal{FSGS}$, including strong $\mathcal{FSGS}$, complete $\mathcal{FSGS}$, uniform vertex $\mathcal{FSGS}$, uniform edge $\mathcal{FSGS}$, the complement of $\mathcal{FSGS}$, the μ-complement of $\mathcal{FSGS}$, regular $\mathcal{FSGS}$, and totally regular $\mathcal{FSGS}$. Fundamental properties of these concepts are established, along with illustrative examples. Furthermore, we explore an application of $\mathcal{FSGS}$ in decision-making, specifically the selection of an optimal courier service. An algorithm is proposed to outline the general steps of this application. The study aims to contribute to enhancing efficiency and productivity in routine decision-making processes.

Key Words: Fuzzy Soft Graph Structure ($\mathcal{FSGS}$), Strong $\mathcal{FSGS}$, Complete $\mathcal{FSGS}$, Uniform Vertex $\mathcal{FSGS}$, Uniform Edge $\mathcal{FSGS}$, Complement of $\mathcal{FSGS}$, μ-Complement of $\mathcal{FSGS}$, Regular $\mathcal{FSGS}$, Totally Regular $\mathcal{FSGS}$.

Contents

1. Introduction

The journey of graph theory started with the publication of Swiss mathematician Leonhard Euler, who solved the well-known "Königsberg bridge problem," giving birth to graph theory. Euler is laterally known as the "Father of Graph Theory." Although graph theory is an old subject, there is an extremely large number of growing applications in several fields such as genetics, traffic planning, linguistics, architecture, operational research, and many more, which keep this subject young. In current days, it is used in computer designs, social networks, and communication systems [1]. Graph theory is a very interesting subject and useful for modeling a broad range of problems. There are a significant number of mathematicians, engineers, computer scientists, chemists, geologists, and many more who show interest in this subject and play a huge role in various varieties of graph theory [2].

Before the middle of the 20th century, probability theory was the only theory to deal with all kinds of uncertainties based on the logic of Aristotle [3], which is YES or NO logic. Many human inventions such as computers, vehicles, switches, and more depend on this logic. In 1965, a new type of set known as a fuzzy set was introduced in the seminal paper by Lotfi A. Zadeh, which led to the development of

* Corresponding author.
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fuzzy logic [4]. Zadeh changed the binary logic of YES or NO to a degree of membership ranging from 0 to 1 for each element in a fuzzy set. Operations like union, intersection, and complements were later introduced in fuzzy logic.

Fuzzy graph theory was initially introduced by Kauffman in 1973 [5], but significant progress was made by Rosenfeld in his research article [6], where various connectivity parameters and structural ideas were discussed. Concepts from graph theory such as paths, cycles, trees, and connectedness were also explored. Subsequent developments in fuzzy graph theory have been built upon Rosenfeld's earlier work. The applications of graph theory can be seen in pattern classification, database theory, social sciences, decision analysis, group structure, and various other areas [7,8,9].

Another mathematical tool dealing with uncertainties and vagueness is the concept of soft set theory, initiated in 1999 by D. Molodtsov [10]. Soft sets have many potential applications in various directions such as game theory, operational research, Riemann integration, person integration, probability theory, and measurement theory [11]. Later on, many worldwide researchers showed their attention to soft sets. The idea of fuzzy soft sets, which is a combination of fuzzy set and soft set, was developed by Maji, Roy, and Biswas in 2001 [12,13]. In 2015, Mohinta and Samanta [14] examined the ideas of union and intersection of fuzzy soft graphs, with properties related to finite union and intersection. Additionally, Akram et al. [15,16,17,18] presented various notions, concepts, and discussed the applications of fuzzy soft graphs in networking. Al-Masarwah and Abu Qamar [19] worked on new ideas of fuzzy soft graphs, including the complement of fuzzy soft graphs and μ-complement of fuzzy soft graphs, along with their properties. They also elaborated on results related to strong fuzzy soft graphs, complete fuzzy soft graphs, and isolated fuzzy soft graphs.

The concept of graph structures was introduced by Sampathkumar [20]. Graph structures are useful in the study of structures like graphs, signed graphs, edge-like graphs, semi-graphs, and edge-labeled graphs. They also have advantages in the study of different domains of computer science and computational intelligence. Dinesh [21] introduced the concept of fuzzy graph structures and some related notions. Fuzzy graph structures have characteristics that deal with uncertainty and impreciseness in many real-life applications, making them more valuable than graph structures. Ramakrishnan and Dinesh [22,23,24] focused on generalized fuzzy graph structures. In 2015, Akram et al. [25,26] introduced various concepts of m-polar fuzzy graph structures. Akram and Sitara [27] introduced certain concepts of fuzzy graph structures and further discussed the degree and total degree of these fuzzy graph structures. They also discussed an application of fuzzy graph structures and developed an algorithm to outline the general procedure of the application.

1.1. Motivation

Akram and Sitara [27] introduced certain concepts of fuzzy graph structures and further discussed the degree and total degree of these fuzzy graph structures. They also discussed an application of fuzzy graph structures and developed an algorithm to outline the general procedure of the application. Massa'deh et al. [28] developed the ideas of bipolar fuzzy imda i-cycles and ϕ-complement of bipolar fuzzy soft graph structure and examined ϕ-complement, strong self-complementary and self-complementary bipolar fuzzy graph structures. Jiang et al. [29] examined the important product operations of cubic fuzzy graph structure and presented the application of cubic fuzzy graph structure in the diagnosis of brain lesions. Kosari et al. [30] introduced the ideas of vertex regularity in cubic fuzzy graph structures and studied some of its characteristics. Further, in the last of the research work, they present the application of cubic fuzzy graph structure. Rao et al. [31] developed the maximal product in cubic fuzzy graph structures and finally discussed the application of cubic fuzzy graph structure in project management.

In this paper, we introduce fuzzy soft graph structures and certain notions related to them, as well as their properties. We then develop an application of fuzzy soft graph structures in decision-making problems, such as identifying the best courier service. Additionally, we introduce an algorithm to outline the general steps of our application.

2. Preliminaries

First, we introduce some key definitions that will be useful throughout this research work.

Definition 2.1 *A graph $\dot{G} = (W, E)$ is a pair of two sets W and E such that:*

- *W represents a set, and its elements are called* vertices/nodes/points*,*
- *E represents a relation on W, and its elements are called* edges/arcs/paths.

The sets W and E are called the vertex set *and the* edge set *of $\dot{G}$, respectively.*

Definition 2.2 *A* fuzzy set *F on a set W is represented by its* membership function

$$\ddot{\mu}_F : W \to [0, 1],$$

where, for each $w \in W$, $\ddot{\mu}_F(w)$ denotes the degree of membership of w in F. A fuzzy relation *is a fuzzy subset of $W \times W$.*

Definition 2.3 *For a non-empty set W, let*

$$\ddot{\sigma} : W \to [0, 1] \quad \textit{and} \quad \ddot{\mu} : W \times W \to [0, 1]$$

be a pair of mappings. For all $w_1, w_2 \in W$, if

$$\ddot{\mu}(w_1, w_2) \leq \ddot{\sigma}(w_1) \wedge \ddot{\sigma}(w_2),$$

then $G = (W, \ddot{\sigma}, \ddot{\mu})$ is called a fuzzy graph (FG)*. Here, $\ddot{\sigma}$ and $\ddot{\mu}$ are referred to as the* fuzzy vertex *and* fuzzy edge*, respectively.*

Definition 2.4 *Let $G = (W, \ddot{\sigma}, \ddot{\mu})$ be a fuzzy graph. The* order *and* size *of G are defined as*

$$\ddot{O}(G) = \sum_{w \in W} \ddot{\sigma}(w), \qquad \ddot{S}(G) = \sum_{(w_1, w_2) \in W \times W} \ddot{\mu}(w_1, w_2).$$

Definition 2.5 *Let $G = (W, \ddot{\sigma}, \ddot{\mu})$ be a $\mathcal{FG}$. Then G is a* complete $\mathcal{FG}$ *if for all $w_1, w_2 \in W$,*

$$\ddot{\mu}(w_1, w_2) = \ddot{\sigma}(w_1) \wedge \ddot{\sigma}(w_2),$$

and is a strong $\mathcal{FG}$ *if for all $(w_1, w_2) \in W \times W$,*

$$\ddot{\mu}(w_1, w_2) = \ddot{\sigma}(w_1) \wedge \ddot{\sigma}(w_2).$$

Definition 2.6 *Let $G = (W, \ddot{\sigma}, \ddot{\mu})$ be a $\mathcal{FG}$ and let*

$$\ddot{\sigma}' = \{\ddot{\sigma}(w) > 0 \ : \ w \in W\}.$$

Then G is a uniform vertex $\mathcal{FG}$ *if*

$$\ddot{\sigma}(w) = h \quad \forall w \in \ddot{\sigma}',$$

where h is some positive real number such that $0 < h \leq 1$.

Definition 2.7 *Let $G = (W, \ddot{\sigma}, \ddot{\mu})$ be a $\mathcal{FG}$ and let*

$$\ddot{\mu}' = \{\ddot{\mu}(w_1, w_2) > 0 \ : \ (w_1, w_2) \in W \times W\}.$$

Then G is a uniform edge $\mathcal{FG}$ *if*

$$\ddot{\mu}(w_1, w_2) = h \quad \forall (w_1, w_2) \in \ddot{\mu}',$$

where h is some positive real number such that $0 < h \leq 1$.

Definition 2.8 *Let $G = (W, \ddot{\sigma}, \ddot{\mu})$ be a $\mathcal{FG}$. Then the* complement *of G is defined by*

$$\overline{G} = (W, \overline{\ddot{\sigma}}, \overline{\ddot{\mu}})$$

such that

$$\overline{\ddot{\sigma}(w)} = \ddot{\sigma}(w), \quad \overline{\ddot{\mu}(w_1, w_2)} = \ddot{\sigma}(w_1) \wedge \ddot{\sigma}(w_2) - \ddot{\mu}(w_1, w_2),$$

for all $w_1, w_2 \in W$.

*The $\ddot{\mu}$-*complement *of G is defined by*

$$\overline{G} = (W, \overline{\ddot{\sigma}}, (\ddot{\mu})^{|\mu|})$$

such that

$$\overline{\ddot{\mu}(w_1, w_2)} = \ddot{\sigma}(w_1) \wedge \ddot{\sigma}(w_2) - \ddot{\mu}(w_1, w_2), \quad \forall (w_1, w_2) \in W \times W,$$

and

$$\overline{\ddot{\mu}(w_1, w_2)} = 0, \quad \forall (w_1, w_2) \notin W \times W.$$

Definition 2.9 *Let $G = (W, \ddot{\sigma}, \ddot{\mu})$ be a $\mathcal{FG}$. Then the* degree *of a vertex w in G is defined as*

$$\deg_G(w) = \deg(w) = \sum_{(w_1, w_2) \in W \times W} \ddot{\mu}(w_1, w_2),$$

and the total degree *of w in G is defined as*

$$t \deg_G(w) = t \deg(w) = \sum_{(w_1, w_2) \in W \times W} \ddot{\mu}(w_1, w_2) + \ddot{\sigma}(w_1).$$

Definition 2.10 *Let $G = (W, \ddot{\sigma}, \ddot{\mu})$ be a $\mathcal{FG}$. Then G is called an h_1-*regular *$\mathcal{FG}$ if for all $w \in W$,*

$$\deg_G(w) = h_1,$$

i.e., each vertex of G has the same degree h_1.

*Similarly, G is called an h_2-*totally regular *$\mathcal{FG}$ if for all $w \in W$,*

$$t \deg_G(w) = h_2,$$

i.e., each vertex has the same total degree h_2, where $h_1, h_2 \in (0, 1]$ are positive real numbers.

Definition 2.11 *Consider a universal set W and $\ddot{P}$ as a set of parameters. Let I^W represent the collection of all fuzzy subsets of W and let Q be a subset of $\ddot{P}$. Then a pair (H, Q) is said to be a* soft set *over W, where*

$$H : Q \to I^W.$$

Definition 2.12 *A* fuzzy soft graph (FSG) *$G = (\dot{G}, \ddot{\sigma}, \ddot{\mu}, \ddot{P})$ is a 4-tuple such that:*

1. *$\dot{G} = (W, E)$ is a simple graph,*
2. *$\ddot{P}$ is a non-empty set of parameters,*
3. *$(\ddot{\sigma}, \ddot{P})$ is a fuzzy soft set over W where*

$$\ddot{\sigma} : \ddot{P} \to E(W)$$

such that for $\rho \in \ddot{P}$, $\ddot{\sigma}(\rho) = \ddot{\sigma}_\rho$ (say), with

$$\ddot{\sigma}_\rho : W \to [0, 1],$$

and for $w_i \in W$, $(\ddot{\sigma}_\rho, \ddot{P})$ is a fuzzy soft vertex.

4. $(\ddot{\mu}, \ddot{P})$ *is a fuzzy soft set over* E *such that for* $\rho \in \ddot{P}$, $\ddot{\mu}(\rho) = \ddot{\mu}_\rho$ *(say), with*

$$\ddot{\mu}_\rho : E \to [0,1],$$

and for $(w_i, w_j) \in E$, *we have* $(w_i, w_j) \mapsto \ddot{\mu}_\rho(w_i, w_j)$, *where* $(\ddot{\mu}_\rho, \ddot{P})$ *is a* fuzzy soft edge.

Then a graph

$$G((\ddot{P}, \ddot{\sigma}_\rho), (\ddot{P}, \ddot{\mu}_\rho))$$

is an $\mathcal{FSG}$ *if and only if*

$$\ddot{\mu}_\rho(w_1, w_2) \leq \ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2), \quad \forall \rho \in \ddot{P},\ w_1, w_2 \in W.$$

We denote this as

$$G_{(\ddot{P},W)} = ((\ddot{P}, \ddot{\sigma}_\rho), (\ddot{P}, \ddot{\mu}_\rho)).$$

Example 2.1 *Let*

$$G_{(\ddot{P},W)} = \big((\ddot{P}, \ddot{\sigma}_\rho), (\ddot{P}, \ddot{\mu}_\rho)\big)$$

be an $\mathcal{FSG}$ *over a simple graph* $\dot{G} = (W, E)$, *where*

$$W = \{w_1, w_2, w_3\}, \quad E = \{(w_1, w_2), (w_2, w_3), (w_3, w_1)\}.$$

For a set of parameters

$$\ddot{P} = \{\rho_1, \rho_2, \rho_3\},$$

we have $(\ddot{\sigma}_\rho, \ddot{P})$ *as a fuzzy soft set over* W *and* $(\ddot{\mu}_\rho, \ddot{P})$ *as a fuzzy soft set over* E, *as given in Table 1 and Figure 1.*

Table 1: Fuzzy soft vertex and edge sets

$\ddot{\sigma}_\rho$	w_1	w_2	w_3
ρ_1	0.2	0.5	0.4
ρ_2	0.6	0.3	0.4
ρ_3	0.5	0.7	0.4

$\ddot{\mu}_\rho$	(w_1, w_2)	(w_2, w_3)	(w_3, w_1)
ρ_1	0.2	0.0	0.2
ρ_2	0.2	0.1	0.3
ρ_3	0.4	0.3	0.4

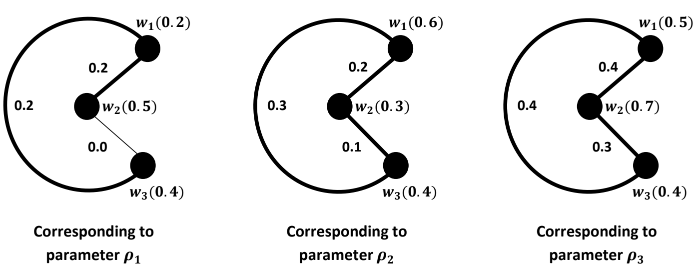

Figure 1: $\mathcal{FSG}$ $G_{(\ddot{P},W)} = \big((\ddot{P}, \ddot{\sigma}_\rho), (\ddot{P}, \ddot{\mu}_\rho)\big)$

Definition 2.13 *The order of a fuzzy soft graph (FSG) $G_{(\ddot{P},W)}$ is defined as:*

$$O(G_{(\ddot{P},W)}) = \sum_{\rho\in\ddot{P}} \sum_{w\in W} \ddot{\sigma}_\rho(w)$$

Definition 2.14 *The size of a fuzzy soft graph (FSG) $G_{(\ddot{P},W)}$ is defined as:*

$$S(G_{(\ddot{P},W)}) = \sum_{\rho\in\ddot{P}} \sum_{(w_1,w_2)\in W\times W} \ddot{\mu}_\rho(w_1,w_2)$$

Example 2.2 *The order and size of the FSG $G_{(\ddot{P},W)}$ given in Table 1 and Figure 1 are:*

$$\begin{aligned} O(G_{(\ddot{P},W)}) &= \ddot{\sigma}_\rho(w_1) + \ddot{\sigma}_\rho(w_2) + \ddot{\sigma}_\rho(w_3) \\ &= (0.2+0.6+0.5) + (0.5+0.3+0.7) + (0.4+0.4+0.4) \\ &= 4.0 \end{aligned}$$

$$\begin{aligned} S(G_{(\ddot{P},W)}) &= \ddot{\mu}_\rho(w_1,w_2) + \ddot{\mu}_\rho(w_2,w_3) + \ddot{\mu}_\rho(w_3,w_1) \\ &= (0.2+0.2+0.4) + (0.0+0.1+0.3) + (0.2+0.3+0.4) \\ &= 2.1 \end{aligned}$$

Definition 2.15 *For a fuzzy soft graph (FSG) $G_{(\ddot{P},W)} = ((\ddot{P},\ddot{\sigma}_\rho),(\ddot{P},\ddot{\mu}_\rho))$ over a simple graph $\ddot{G} = (W,E)$, $G_{(\ddot{P},W)}$ is called a* strong FSG *if $\ddot{G}$ is strong; i.e.,*

$$\ddot{\mu}_\rho(w_1,w_2) = \ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2) \quad \textit{for all } \rho\in\ddot{P} \textit{ and } w_1,w_2\in W.$$

Example 2.3 *Let the set of parameters be $\ddot{P} = \{\rho_1,\rho_2\}$, and consider a fuzzy soft graph $G_{(\ddot{P},W)} = ((\ddot{P},\ddot{\sigma}_\rho),(\ddot{P},\ddot{\mu}_\rho))$ over $\ddot{G} = (W,E)$ where*

$$W = \{w_1,w_2,w_3,w_4,w_5\}, \quad E = \{(w_1,w_2),(w_2,w_3),(w_3,w_4),(w_4,w_5),(w_5,w_1)\}.$$

Then, a strong fuzzy soft graph is given as in Table 2 and Figure 2.

Table 2: Fuzzy soft vertex and edge set

σ_ρ	w_1	w_2	w_3	w_4	w_5
ρ_1	0.2	0.3	0.4	0.2	0.1
ρ_2	0.4	0.1	0.4	0.3	0.2

$\ddot{\mu}_\rho$	(w_1,w_2)	(w_2,w_3)	(w_3,w_4)	(w_4,w_5)	(w_5,w_1)
ρ_1	0.2	0.3	0.2	0.1	0.1
ρ_2	0.1	0.1	0.3	0.2	0.2

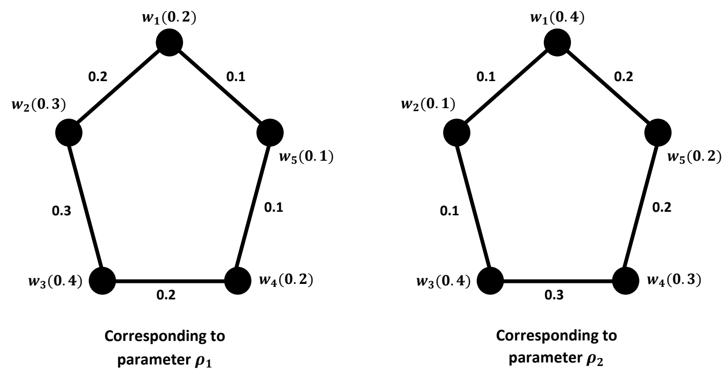

Figure 2: Strong $\mathcal{FSG}$ $G_{(\ddot{P},W)}$

Definition 2.16 *For a fuzzy soft graph (FSG) $G_{(\ddot{P},W)} = ((\ddot{P},\ddot{\sigma}_\rho),(\ddot{P},\ddot{\mu}_\rho))$ over a simple graph $\ddot{G} = (W,E)$, $G_{(\ddot{P},W)}$ is called a* complete FSG *if $\ddot{G}$ is complete; i.e.,*

$$\ddot{\mu}_\rho(w_1,w_2) = \ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2) \quad \text{for all } \rho \in \ddot{P} \text{ and } (w_1,w_2) \in W \times W.$$

Example 2.4 *Let the set of parameters be $\ddot{P} = \{\rho_1,\rho_2\}$, and consider a fuzzy soft graph $G_{(\ddot{P},W)} = ((\ddot{P},\ddot{\sigma}_\rho),(\ddot{P},\ddot{\mu}_\rho))$ over $\ddot{G} = (W,E)$ where*

$$\begin{aligned} W &= \{w_1,w_2,w_3,w_4,w_5\}, \\ E &= \{(w_1,w_2),(w_1,w_3),(w_1,w_4),(w_1,w_5), \\ &\quad (w_2,w_3),(w_2,w_4),(w_2,w_5), \\ &\quad (w_3,w_4),(w_3,w_5),(w_4,w_5)\}. \end{aligned}$$

Then, a complete fuzzy soft graph is given as in Table 3 and Figure 3.

Table 3: Fuzzy soft vertex and edge set

σ_ρ	w_1	w_2	w_3	w_4	w_5
ρ_1	0.2	0.3	0.4	0.2	0.1
ρ_2	0.4	0.1	0.4	0.3	0.2

$\ddot{\mu}_\rho$	(w_1,w_2)	(w_1,w_3)	(w_1,w_4)	(w_1,w_5)	(w_2,w_3)	(w_2,w_4)	(w_2,w_5)	(w_3,w_4)	(w_3,w_5)	(w_4,w_5)
ρ_1	0.2	0.2	0.2	0.1	0.3	0.1	0.1	0.1	0.1	0.1
ρ_2	0.1	0.4	0.3	0.2	0.2	0.1	0.1	0.3	0.2	0.2

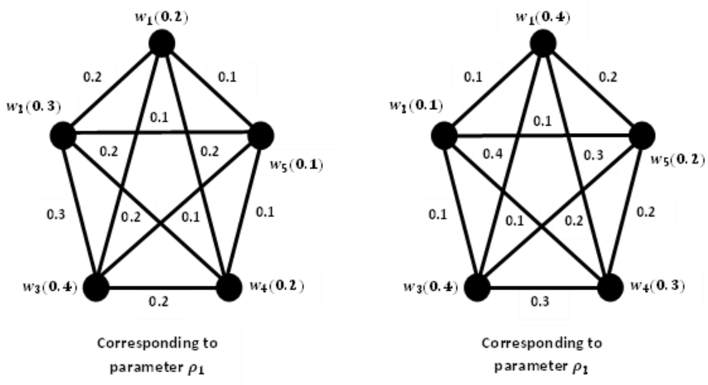

Figure 3: Complete $\mathcal{FSG}$ $G_{(\ddot{P},W)}$

Definition 2.17 *A fuzzy soft graph (FSG) $G_{(\ddot{P},W)} = ((\ddot{P}, \ddot{\sigma}_\rho), (\ddot{P}, \ddot{\mu}_\rho))$ is called a* uniform vertex FSG *if, for all $\rho \in \ddot{P}$, the graph G is a uniform vertex fuzzy graph.*

Example 2.5 *Let the set of parameters be $\ddot{P} = \{\rho_1, \rho_2, \rho_3\}$, and consider a fuzzy soft graph $G_{(\ddot{P},W)} = ((\ddot{P}, \ddot{\sigma}_\rho), (\ddot{P}, \ddot{\mu}_\rho))$ over $\ddot{G} = (W, E)$ where*

$$W = \{w_1, w_2, w_3, w_4\}, \quad E = \{(w_1, w_2), (w_2, w_3), (w_2, w_4), (w_3, w_4), (w_4, w_1)\}.$$

Then, a uniform vertex fuzzy soft graph is given as in Table 4 and Figure 4.

Table 4: Fuzzy soft vertex and edge set

$\ddot{\sigma}_\rho$	w_1	w_2	w_3	w_4
ρ_1	0.4	0.4	0.4	0.4
ρ_2	0.6	0.6	0.6	0.6
ρ_3	0.7	0.7	0.7	0.7

$\ddot{\mu}_\rho$	(w_1, w_2)	(w_2, w_3)	(w_2, w_4)	(w_3, w_4)	(w_4, w_1)
ρ_1	0.2	0.3	0.4	0.4	0.3
ρ_2	0.3	0.5	0.4	0.5	0.6
ρ_3	0.7	0.6	0.3	0.2	0.4

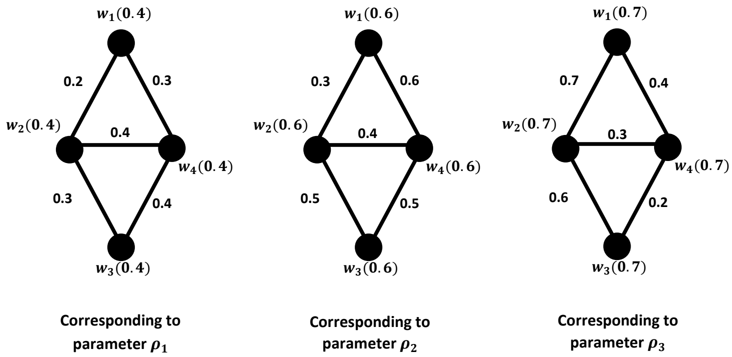

Figure 4: Uniform vertex of $\mathcal{FSG}$ $G_{(\ddot{P},W)}$

Definition 2.18 *A fuzzy soft graph (FSG)* $G_{(\ddot{P},W)} = ((\ddot{P}, \ddot{\sigma}_\rho), (\ddot{P}, \ddot{\mu}_\rho))$ *is called a* uniform edge FSG *if, for all* $\rho \in \ddot{P}$*, the graph* G *is a uniform edge fuzzy graph.*

Example 2.6 *Let the set of parameters be* $\ddot{P} = \{\rho_1, \rho_2, \rho_3\}$*, and consider a fuzzy soft graph* $G_{(\ddot{P},W)} = ((\ddot{P}, \ddot{\sigma}_\rho), (\ddot{P}, \ddot{\mu}_\rho))$ *over* $\ddot{G} = (W, E)$ *where*

$$W = \{w_1, w_2, w_3, w_4\}, \quad E = \{(w_1, w_2), (w_2, w_3), (w_3, w_4), (w_4, w_1)\}.$$

Then, a uniform edge fuzzy soft graph is given as in Table 5 and Figure 5.

Table 5: Fuzzy soft vertex and edge set

$\ddot{\sigma}_\rho$	w_1	w_2	w_3	w_4
ρ_1	0.4	0.4	0.5	0.3
ρ_2	0.5	0.4	0.4	0.6
ρ_3	0.2	0.3	0.7	0.4

$\ddot{\mu}_\rho$	(w_1, w_2)	(w_2, w_3)	(w_2, w_4)	(w_3, w_4)	(w_4, w_1)
ρ_1	0.3	0.3	0.3	0.3	0.3
ρ_2	0.4	0.4	0.4	0.4	0.4
ρ_3	0.2	0.2	0.2	0.2	0.2

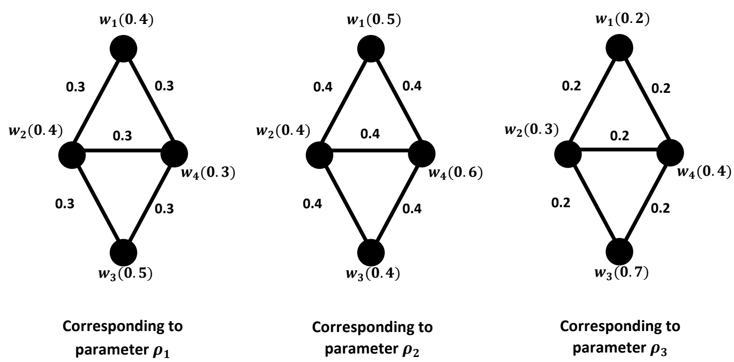

Figure 5: Uniform of edge $\mathcal{FSG}$ $G_{(\ddot{P},W)}$

Definition 2.19 *Let* $G_{(\ddot{P},W)} = ((\ddot{P}, \ddot{\sigma}_\rho), (\ddot{P}, \ddot{\mu}_\rho))$ *be a fuzzy soft graph (FSG). Then the* complement *of* $G_{(\ddot{P},W)}$ *is defined as*

$$\overline{G_{(\ddot{P},W)}} = ((\ddot{P}, \ddot{\sigma}_\rho), (\ddot{P}, \overline{\ddot{\mu}}_\rho))$$

where

$$\overline{\ddot{\mu}}_\rho(w_1, w_2) = \ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2) - \ddot{\mu}_\rho(w_1, w_2), \quad \forall w_1, w_2 \in W,\ \rho \in \ddot{P}.$$

Example 2.7 *Let* $G_{(\ddot{P},W)} = ((\ddot{P}, \ddot{\sigma}_\rho), (\ddot{P}, \ddot{\mu}_\rho))$ *be a fuzzy soft graph as shown in Figure 2.1. Then its complement,* $\overline{G_{(\ddot{P},W)}}$*, is given in Table 6 and shown in Figure 6.*

Table 6: Fuzzy soft vertex and edge set

$\ddot{\sigma}_\rho$	w_1	w_2	w_3
ρ_1	0.2	0.5	0.4
ρ_2	0.6	0.3	0.4
ρ_3	0.5	0.7	0.4

$\overline{\ddot{\mu}}_\rho$	(w_1, w_2)	(w_2, w_3)	(w_3, w_1)
ρ_1	0.0	0.4	0.0
ρ_2	0.1	0.3	0.1
ρ_3	0.3	0.1	0.0

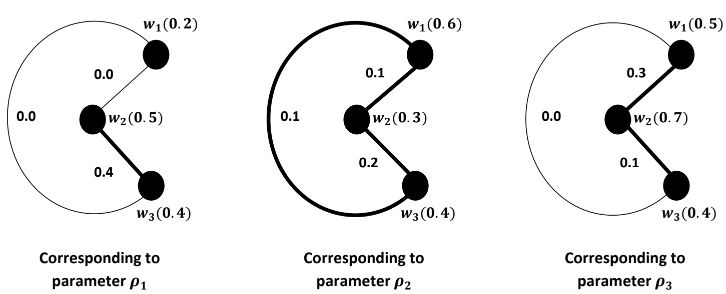

Figure 6: Complement of $\mathcal{FSG}$ $G_{(\ddot{P},W)}$

Definition 2.20 *Let $G_{(\ddot{P},W)} = ((\ddot{P},\ddot{\sigma}_\rho),(\ddot{P},\ddot{\mu}_\rho))$ be a fuzzy soft graph (FSG). Then the μ-*complement *of $G_{(\ddot{P},W)}$ is defined as*

$$G^{\mu}_{(\ddot{P},W)} = ((\ddot{P},\ddot{\sigma}_\rho),(\ddot{P},\ddot{\mu}_\rho)^{\mu})$$

where, for all $(w_1,w_2) \in W \times W$ and $\rho \in \ddot{P}$,

$$\ddot{\mu}^{\mu}_{\rho}(w_1,w_2) = \begin{cases} \ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2) - \ddot{\mu}_\rho(w_1,w_2), & \text{if } \ddot{\mu}_\rho(w_1,w_2) > 0, \\ 0, & \text{if } \ddot{\mu}_\rho(w_1,w_2) = 0. \end{cases}$$

Example 2.8 *Let $G_{(\ddot{P},W)} = ((\ddot{P},\ddot{\sigma}_\rho),(\ddot{P},\ddot{\mu}_\rho))$ be a fuzzy soft graph as shown in Figure 2.1. Then its μ-complement, $G^{\mu}_{(\ddot{P},W)}$, is given in Table 7 and shown in Figure 7.*

Table 7: Fuzzy soft vertex and edge set

$\ddot{\sigma}_\rho$	w_1	w_2	w_3
ρ_1	0.2	0.5	0.4
ρ_2	0.6	0.3	0.4
ρ_3	0.5	0.7	0.4

$\ddot{\mu}^{\mu}_{\rho}$	(w_1,w_2)	(w_2,w_3)	(w_3,w_1)
ρ_1	0.0	0.0	0.2
ρ_2	0.1	0.3	0.1
ρ_3	0.3	0.1	0.0

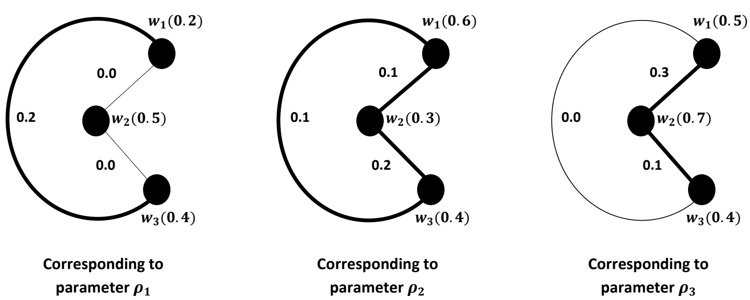

Figure 7: $\ddot{\mu}$ -complement of $\mathcal{FSG}\, G_{(\ddot{P},W)}$

Definition 2.21 *A fuzzy soft graph (FSG) $G_{(\ddot{P},W)} = ((\ddot{P},\ddot{\sigma}_\rho),(\ddot{P},\ddot{\mu}_\rho))$ is called a* regular FSG *if, for all $\rho \in \ddot{P}$, the graph $G_{(\ddot{P},W)}$ is a regular fuzzy graph (i.e., each vertex has the same degree in each parameter of the FSG).*

*If the degree of $G_{(\ddot{P},W)}$ is h_1 for all $\rho \in \ddot{P}$, then it is called an h_1-*regular FSG*, where h_1 is a positive real number such that $h_1 \in (0,1]$.*

Definition 2.22 *A fuzzy soft graph (FSG) $G_{(\ddot{P},W)} = ((\ddot{P},\ddot{\sigma}_\rho),(\ddot{P},\ddot{\mu}_\rho))$ is called a* totally regular FSG *if, for all $\rho \in \ddot{P}$, the graph $G_{(\ddot{P},W)}$ is a totally regular fuzzy graph (i.e., each vertex has the same total degree in each parameter of the FSG).*

*If the total degree of $G_{(\ddot{P},W)}$ is h_2 for all $\rho \in \ddot{P}$, then it is called an h_2-*totally regular FSG*, where h_2 is a positive real number such that $h_2 \in (0,1]$.*

Example 2.9 *Let the set of parameters be* $\ddot{P} = \{\rho_1, \rho_2\}$*, and consider a fuzzy soft graph* $G_{(\ddot{P},W)} = ((\ddot{P}, \ddot{\sigma}_\rho), (\ddot{P}, \ddot{\mu}_\rho))$ *over* $\ddot{G} = (W, E)$ *where*

$$W = \{w_1, w_2, w_3, w_4, w_5\}, \quad E = \{(w_1, w_2), (w_2, w_3), (w_3, w_4), (w_4, w_5), (w_5, w_1), (w_1, w_3), (w_1, w_4)\}.$$

The FSG is given in Table 8 and Figure 8.

- According to parameter ρ_1*, the degree and total degree of* $G_{(\ddot{P},W)}$ *are* 0.6 *and* 1.0*, respectively. - According to parameter* ρ_2*, the degree and total degree of* $G_{(\ddot{P},W)}$ *are* 0.8 *and* 1.3*, respectively.*

Table 8: Fuzzy soft vertex and edge set

$\ddot{\sigma}_\rho$	w_1	w_2	w_3	w_4	w_5
ρ_1	0.4	0.4	0.4	0.4	0.4
ρ_2	0.5	0.5	0.5	0.5	0.5

$\ddot{\mu}_\rho$	(w_1, w_2)	(w_2, w_3)	(w_3, w_4)	(w_4, w_5)	(w_5, w_1)	(w_1, w_3)	(w_1, w_4)
ρ_1	0.2	0.4	0.1	0.4	0.2	0.1	0.1
ρ_2	0.3	0.5	0.2	0.5	0.3	0.1	0.1

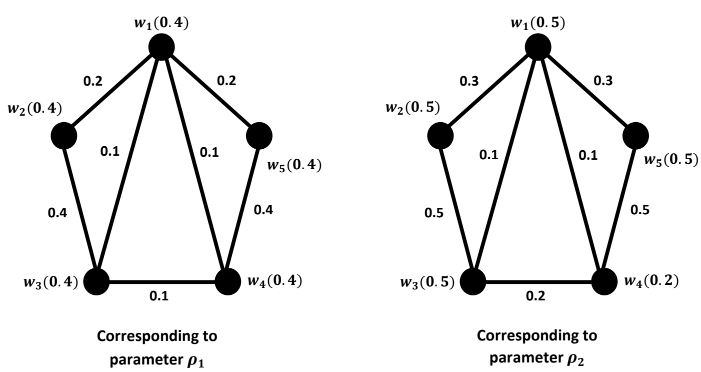

Figure 8: Regular and Totally Regular $\mathcal{FSG}$ $G_{(\ddot{P},W)}$

Definition 2.23 *A* graph structure *($\mathcal{GS}$) is defined as*

$$G = (W, R_i), \quad 1 \le i \le n,$$

where each R_i *is a relation that is mutually disjoint with the others, symmetric, and irreflexive. Each edge in the graph is labeled according to its corresponding* R_i*.*

Example 2.10 *Consider a graph structure*

$$G = (W, R_1, R_2, R_3),$$

where

$$W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}\},$$

and the relations are defined as:

$$R_1 = \{(w_1, w_2), (w_1, w_5), (w_1, w_7), (w_1, w_9)\},$$
$$R_2 = \{(w_2, w_3), (w_4, w_5), (w_6, w_7), (w_9, w_{10})\},$$
$$R_3 = \{(w_3, w_4), (w_3, w_8), (w_3, w_{10}), (w_4, w_6), (w_4, w_8)\}.$$

Clearly, R_1, R_2, and R_3 are mutually disjoint, symmetric, and irreflexive relations. Thus, $G = (W, R_1, R_2, R_3)$ is a graph structure, as shown in Figure 9.

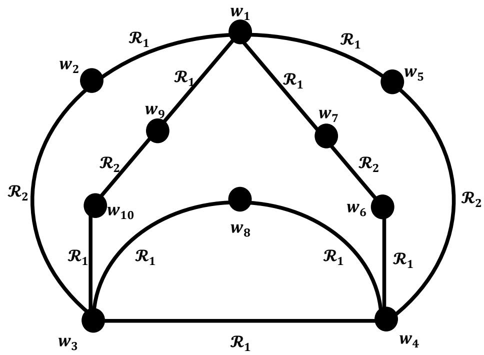

Figure 9: Graph Structure ($\mathcal{GS}$)

Definition 2.24 *Consider fuzzy vertex and edge sets $\ddot{\sigma}$ and $\ddot{\mu}_i$ defined on W and on each R_i, respectively. If*

$$0 \leq \ddot{\mu}_i(w_i, w_j) \leq \ddot{\sigma}(w_i) \wedge \ddot{\sigma}(w_j), \quad \forall w_i, w_j \in W, \quad i = 1, 2, \ldots, n,$$

then

$$G = (\ddot{\sigma}, \ddot{\mu}_i)$$

is called a fuzzy graph structure *($\mathcal{FGS}$) with underlying crisp graph structure*

$$G = (W, R_i).$$

Example 2.11 *Consider the graph structure*

$$G = (W, R_1, R_2, R_3)$$

shown in Figure 9. Let $\ddot{\sigma}$ be a fuzzy vertex set on W as given in Table 9, and let $\ddot{\mu}_1$, $\ddot{\mu}_2$, $\ddot{\mu}_3$ be fuzzy edge sets on R_1, R_2, R_3, respectively, as given in Table 10.

Then

$$G = (\ddot{\sigma}, \ddot{\mu}_1, \ddot{\mu}_2, \ddot{\mu}_3)$$

is a fuzzy graph structure ($\mathcal{FGS}$), as shown in Figure 10.

Table 9: Fuzzy vertex set $\ddot{\sigma}$ on W

Elements	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}
$\ddot{\sigma}(w_i)$	0.4	0.3	0.5	0.4	0.3	0.6	0.4	0.4	0.5	0.6

Table 10: Fuzzy edge sets $\ddot{\mu}_1$, $\ddot{\mu}_2$, and $\ddot{\mu}_3$ on relations R_1, R_2, and R_3

Fuzzy edge set $\ddot{\mu}_1$ on R_1		Fuzzy edge set $\ddot{\mu}_2$ on R_2		Fuzzy edge set $\ddot{\mu}_3$ on R_3	
Elements	$\ddot{\mu}_1(w_i, w_j)$	Elements	$\ddot{\mu}_2(w_i, w_j)$	Elements	$\ddot{\mu}_3(w_i, w_j)$
(w_1, w_2)	0.2	(w_2, w_3)	0.3	(w_3, w_{10})	0.4
(w_1, w_9)	0.4	(w_9, w_{10})	0.4	(w_3, w_8)	0.2
(w_1, w_7)	0.3	(w_6, w_7)	0.3	(w_3, w_4)	0.3
(w_1, w_5)	0.3	(w_4, w_5)	0.3	(w_4, w_6)	0.3
				(w_4, w_8)	0.2

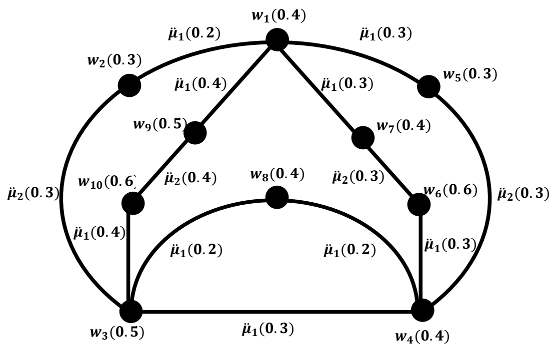

Figure 10: Fuzzy Graph Structure ($\mathcal{FGS}$)

3. Fuzzy soft graph structure

Definition 3.1 *A fuzzy soft graph structure ($\mathcal{FSGS}$) is a 4-tuple*

$$G_{(\ddot{P},W)}(S) = (\ddot{G}, \ddot{\sigma}, \ddot{\mu}, \ddot{P}),$$

such that:

1. *$\ddot{G} = (W, R_i)$ is a generalized soft graph ($\mathcal{GS}$),*
2. *$\ddot{P}$ is a non-empty set of parameters,*
3. *$(\ddot{\sigma}, \ddot{P})$ is a fuzzy soft set over W, where for each $\rho \in \ddot{P}$,*

$$\ddot{\sigma}(\rho) = \ddot{\sigma}_\rho \quad \text{with} \quad \ddot{\sigma}_\rho : W \to [0, 1],$$

   and for every $w_i \in W$, the pair $\left((\ddot{\sigma}_\rho, \ddot{P})\right)_s$ defines a fuzzy soft vertex set.

4. *$(\ddot{\mu}, \ddot{P})$ is a fuzzy soft set over R_i, where each relation R_i $(1 \leq i \leq n)$ is mutually disjoint, symmetric, and irreflexive. The pair $\left((\ddot{\mu}_\rho, \ddot{P})\right)_s$ defines a fuzzy soft edge set.*

Then, for all $\rho \in \ddot{P}$, $w_1, w_2 \in W$, and each relation R_i $(1 \leq i \leq n)$, the graph

$$G_{(\ddot{P},W)}(S) = \left\{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\right\}$$

over a generalized soft graph $\ddot{G} = (W, R_i)$ is called a fuzzy soft graph structure ($\mathcal{FSGS}$) if and only if

$$\ddot{\mu}_\rho(w_1, w_2) \leq \ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2).$$

Example 3.1 *Consider a fuzzy soft graph structure* ($\mathcal{FSGS}$)

$$G_{(\ddot{P},W)}(S) = \left\{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\right\}$$

over a generalized soft graph $\ddot{G} = (W, R_1, R_2)$*, where*

$$W = \{w_1, w_2, w_3, w_4\},$$

and relations are defined as

$$R_1 = \{(w_1, w_2), (w_2, w_3)\}, \quad R_2 = \{(w_3, w_4), (w_4, w_1)\}.$$

Let $\ddot{P} = \{\rho_1, \rho_2\}$ *be a set of parameters. Then* $(\ddot{\sigma}_\rho, \ddot{P})_s$ *is a fuzzy soft vertex set over* W*, and* $(\ddot{\mu}_\rho, \ddot{P})_s$ *is a fuzzy soft edge set over* R_i*.*

The corresponding ($\mathcal{FSGS}$) *is given in Table 11 and illustrated in Figure 11. Hence, G is a fuzzy soft graph structure* ($\mathcal{FSGS}$)*.*

Table 11: Fuzzy soft vertex set $\ddot{\sigma}$ over W and fuzzy soft edge sets $\ddot{\mu}_1, \ddot{\mu}_2$ over relations R_1, R_2

$\ddot{\sigma}$	w_1	w_2	w_3	w_4
ρ_1	0.5	0.4	0.3	0.2
ρ_2	0.4	0.7	0.6	0.5

$\ddot{\mu}_1$	(w_1, w_2)	(w_2, w_3)
ρ_1	0.4	0.2
ρ_2	0.3	0.3

$\ddot{\mu}_2$	(w_3, w_4)	(w_4, w_1)
ρ_1	0.1	0.1
ρ_2	0.5	0.0

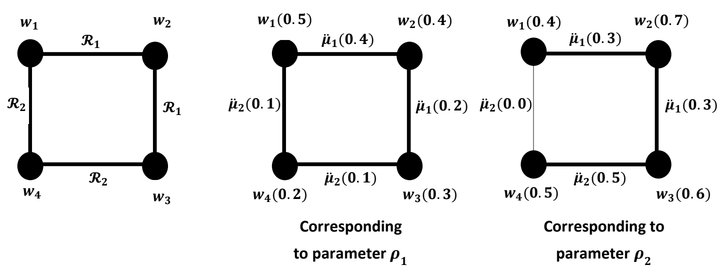

Figure 11: Fuzzy Soft Graph Structure ($\mathcal{FSGS}$)

Definition 3.2 *The order of a fuzzy soft graph structure* ($\mathcal{FSGS}$)

$$G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$$

over a graph structure

$$\mathcal{G}\ddot{S} = (W, R_i),$$

where each relation R_i *for* $i = 1, 2, \ldots, n$ *is mutually disjoint, symmetric, and irreflexive, is defined as*

$$O\ddot{r}(G_{(\ddot{P},W)}(S)) = \sum_{\rho_i \in \ddot{P}} \left(\sum_{w_i \in W} \ddot{\sigma}_{\rho_i}(w_i) \right), \quad \forall \rho \in \ddot{P}.$$

Example 3.2 *The order of the $\mathcal{FSGS}$, as illustrated in Table 11 and Figure 11, is computed as:*

$$\begin{aligned} O\ddot{r}(G_{(\ddot{P},W)}(S)) &= \ddot{\sigma}_\rho(w_1) + \ddot{\sigma}_\rho(w_2) + \ddot{\sigma}_\rho(w_3) \\ &= (0.5 + 0.2 + 0.3 + 0.1) + (0.4 + 0.7 + 0.6 + 0.5) \\ &= 3.3. \end{aligned}$$

Definition 3.3 *The size of a fuzzy soft graph structure ($\mathcal{FSGS}$)*

$$G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$$

over a graph structure

$$\mathcal{G}\ddot{S} = (W, R_i),$$

where each relation R_i for $i = 1, 2, \ldots, n$ is mutually disjoint, symmetric, and irreflexive, is defined as

$$S\ddot{z}(G_{(\ddot{P},W)}(S)) = \sum_{\rho_i \in \ddot{P}} \left(\sum_{(w_i,w_j) \in R_i} \ddot{\mu}_{\rho_i}(w_i, w_j) \right), \quad \forall \rho \in \ddot{P}.$$

Example 3.3 *The size of the $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$ given in Table 11 and Figure 11 is computed as:*

$$\begin{aligned} S\ddot{z}(G(S)) &= \ddot{\mu}_\rho(w_1, w_2) + \ddot{\mu}_\rho(w_2, w_3) + \ddot{\mu}_\rho(w_3, w_1) \\ &= (0.3 + 0.2 + 0.2 + 0.1) + (0.4 + 0.6 + 0.5 + 0.4) \\ &= 2.7. \end{aligned}$$

Definition 3.4 *A fuzzy soft graph structure ($\mathcal{FSGS}$)*

$$G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$$

over $\mathcal{G}\ddot{S} = (W, R_i)$, where each R_i for $i = 1, 2, \ldots, n$ is mutually disjoint, symmetric, and irreflexive, is said to be a strong FSGS if for all $\rho_i \in \ddot{P}$ we have

$$\ddot{\mu}_\rho(w_i, w_j) = \ddot{\sigma}_\rho(w_i) \wedge \ddot{\sigma}_\rho(w_j), \quad \forall w_i, w_j \in W.$$

Example 3.4 *Consider the $\mathcal{FSGS}$*

$$G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$$

over $\ddot{G} = (W, R_1, R_2, R_3)$, where

$$W = \{w_1, w_2, w_3, w_4, w_5, w_6\},$$

$$R_1 = \{(w_1, w_2), (w_2, w_3)\}, \quad R_2 = \{(w_3, w_4), (w_4, w_5)\}, \quad R_3 = \{(w_1, w_2), (w_2, w_3)\}.$$

Let $\ddot{P} = \{\rho_1, \rho_2\}$ be a set of parameters, where $(\ddot{\sigma}_\rho, \ddot{P})_s$ is a fuzzy soft vertex set over W and $(\ddot{\mu}_\rho, \ddot{P})_s$ is a fuzzy soft edge set over R_i. Then the $\mathcal{FSGS}$ is given in Table 12 and Table 13, as shown in Figure 12. Hence, G is a strong $\mathcal{FSGS}$.

Table 12: Fuzzy set $\ddot{\sigma}$ on vertex set W

$\ddot{\sigma}_\rho$	w_1	w_2	w_3	w_4	w_5	w_6
ρ_1	0.5	0.2	0.3	0.1	0.4	0.3
ρ_2	0.4	0.7	0.6	0.5	0.1	0.2

Table 13: Fuzzy sets $\ddot{\mu}_1, \ddot{\mu}_2, \ddot{\mu}_3$ on relations R_1, R_2, R_3

$\ddot{\mu}_1$ on $R_1 = \{(w_1, w_2), (w_2, w_3)\}$

	(w_1, w_2)	(w_2, w_3)
ρ_1	0.2	0.2
ρ_2	0.4	0.6

$\ddot{\mu}_2$ on $R_2 = \{(w_3, w_4), (w_4, w_5)\}$

	(w_3, w_4)	(w_4, w_5)
ρ_1	0.1	0.1
ρ_2	0.5	0.1

$\ddot{\mu}_3$ on $R_3 = \{(w_5, w_6), (w_6, w_1)\}$

	(w_5, w_6)	(w_6, w_1)
ρ_1	0.3	0.3
ρ_2	0.1	0.2

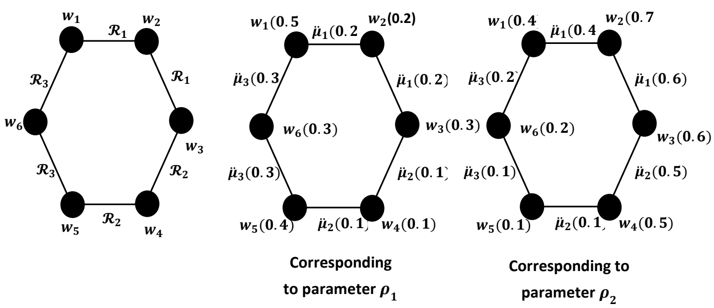

Figure 12: Strong Fuzzy Soft Graph Structure ($\mathcal{FSGS}$)

Definition 3.5 *A fuzzy soft graph structure (FSGS)*

$$G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$$

over

$$\mathcal{G}\ddot{S} = (W, R_i),$$

where each R_i for $i = 1, 2, \ldots, n$ is mutually disjoint, symmetric, and irreflexive, is said to be a complete FSGS if for all $\rho_i \in \ddot{P}$, we have

$$\ddot{\mu}_\rho(w_i, w_j) = \ddot{\sigma}_\rho(w_i) \wedge \ddot{\sigma}_\rho(w_j), \quad \forall (w_i, w_j) \in R_i.$$

Example 3.5 *Consider the $\mathcal{FSGS}$*

$$G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$$

over $\ddot{G} = (W, R_1, R_2, R_3)$, where

$$W = \{w_1, w_2, w_3, w_4, w_5, w_6\},$$

$$R_1 = \{(w_1, w_2), (w_1, w_3), (w_1, w_4), (w_1, w_5), (w_1, w_6)\},$$

$$R_2 = \{(w_2, w_3), (w_2, w_4), (w_2, w_5), (w_2, w_6), (w_3, w_4)\},$$

$$R_3 = \{(w_3, w_5), (w_3, w_6), (w_4, w_5), (w_4, w_6), (w_5, w_6)\}.$$

Let $\ddot{P} = \{\rho_1, \rho_2\}$ *be a set of parameters, where* $(\ddot{\sigma}_\rho, \ddot{P})_s$ *is a fuzzy soft vertex set over* W *and* $(\ddot{\mu}_\rho, \ddot{P})_s$ *is a fuzzy soft edge set over* R_i.

Then the $\mathcal{FSGS}$ *is given in Table 14 and Table 15, as shown in Figure 13. Hence, G is a complete* $\mathcal{FSGS}$.

Table 14: Fuzzy set $\ddot{\sigma}$ on vertex set W

$\ddot{\sigma}_\rho$	w_1	w_2	w_3	w_4	w_5	w_6
ρ_1	0.1	0.4	0.2	0.3	0.2	0.3
ρ_2	0.5	0.1	0.7	0.6	0.5	0.2

Table 15: Fuzzy sets $\ddot{\mu}_1, \ddot{\mu}_2, \ddot{\mu}_3$ on relations R_1, R_2, R_3

$\ddot{\mu}_1$ on $R_1 = \{(w_1, w_2), (w_1, w_3), (w_1, w_4), (w_1, w_5), (w_1, w_6)\}$

	(w_1, w_2)	(w_1, w_3)	(w_1, w_4)	(w_1, w_5)	(w_1, w_6)
ρ_1	0.1	0.1	0.1	0.1	0.1
ρ_2	0.1	0.2	0.5	0.5	0.2

$\ddot{\mu}_2$ on $R_2 = \{(w_2, w_3), (w_2, w_4), (w_2, w_5), (w_2, w_6), (w_3, w_4)\}$

	(w_2, w_3)	(w_2, w_4)	(w_2, w_5)	(w_2, w_6)	(w_3, w_4)
ρ_1	0.2	0.3	0.2	0.3	0.2
ρ_2	0.1	0.1	0.1	0.1	0.3

$\ddot{\mu}_3$ on $R_3 = \{(w_3, w_5), (w_3, w_6), (w_4, w_5), (w_4, w_6), (w_5, w_6)\}$

	(w_3, w_5)	(w_3, w_6)	(w_4, w_5)	(w_4, w_6)	(w_5, w_6)
ρ_1	0.2	0.2	0.2	0.3	0.2
ρ_2	0.5	0.2	0.2	0.2	0.2

Definition 3.6 *A* $\mathcal{FSGS}$

$$G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$$

over a $\mathcal{GS}$ $\ddot{G} = (W, R_i)$ *such that each* R_i *for* $i = 1, 2, \ldots, n$ *are mutually disjoint, symmetric, and irreflexive relations, is called a* ***uniform vertex*** $\mathcal{FSGS}$ *if* $G_{(\ddot{P},W)}$ *is a uniform vertex* $\mathcal{FSG}$ *for all* $\rho \in \ddot{P}$.

Example 3.6 *Consider a* $\mathcal{FSGS}$

$$G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$$

over $\ddot{G} = (W, R_1, R_2)$, *where*

$$W = \{w_1, w_2, w_3, w_4, w_5, w_6\},$$

$$R_1 = \{(w_1, w_2), (w_1, w_6), (w_2, w_6), (w_2, w_3)\},$$

$$R_2 = \{(w_3, w_4), (w_4, w_5), (w_5, w_6)\}.$$

Let $\ddot{P} = \{\rho_1, \rho_2\}$ *be a set of parameters. Here,* $(\ddot{\sigma}_\rho, \ddot{P})_s$ *is the fuzzy soft vertex set over* W*, and* $(\ddot{\mu}_\rho, \ddot{P})_s$ *is the fuzzy soft edge set over* R_i.

Then, the $\mathcal{FSGS}$ *is given in Table 16 and Table 17 as shown in Figure 14. Hence, G is a uniform vertex* $\mathcal{FSGS}$.

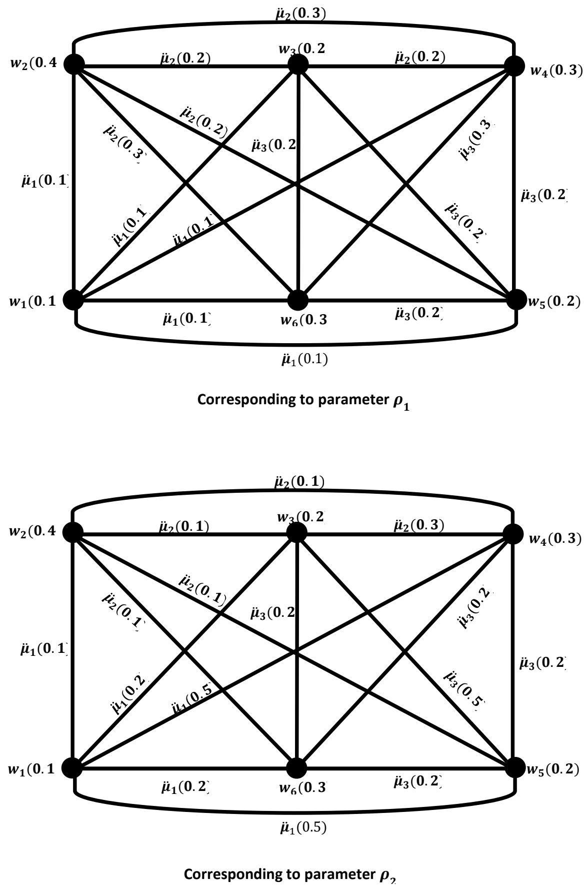

Figure 13: Complete Fuzzy Soft Graph Structure ($\mathcal{FSGS}$)

Table 16: Fuzzy set $\ddot{\sigma}$ on vertex set W

$\ddot{\sigma}_\rho$	w_1	w_2	w_3	w_4	w_5	w_6
ρ_1	0.3	0.3	0.3	0.3	0.3	0.3
ρ_2	0.4	0.4	0.4	0.4	0.4	0.4

Table 17: Fuzzy sets $\ddot{\mu}_1, \ddot{\mu}_2$ on relations R_1, R_2

$\ddot{\mu}_1$	(w_1, w_2)	(w_1, w_6)	(w_2, w_6)	(w_2, w_3)
ρ_1	0.3	0.2	0.2	0.1
ρ_2	0.2	0.1	0.3	0.4

$\ddot{\mu}_2$	(w_3, w_4)	(w_4, w_5)	(w_5, w_6)
ρ_1	0.2	0.1	0.1
ρ_2	0.2	0.3	0.2

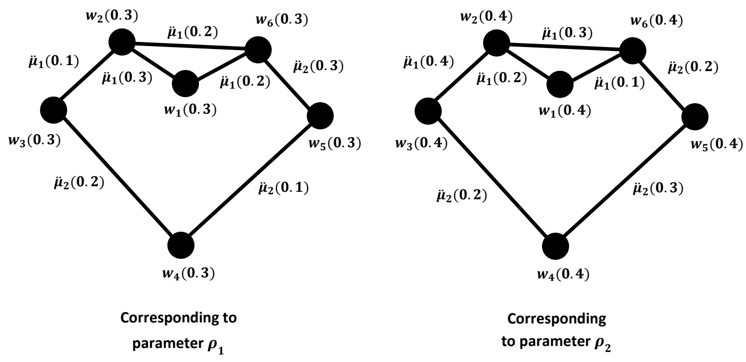

Figure 14: Uniform Vertex Fuzzy Soft Graph Structure ($\mathcal{FSGS}$)

Definition 3.7 *A $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$ over a $\mathcal{GS}$ $\ddot{G} = (W, R_i)$ such that each R_i for $i = 1, 2, \ldots, n$ are mutually disjoint, symmetric, and irreflexive relations, is called a* ***uniform edge*** *$\mathcal{FSGS}$ if $\mathcal{FSG}$ $G_{(\ddot{P},W)}$ is a uniform edge $\mathcal{FSG}$ for all $\rho \in \ddot{P}$.*

Example 3.7 *Consider a $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$ over $\ddot{G} = (W, R_1, R_2)$ with*

$$W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$$

and relations

$$R_1 = \{(w_1, w_2), (w_1, w_6), (w_2, w_6), (w_2, w_3)\},$$

$$R_2 = \{(w_3, w_4), (w_4, w_5), (w_5, w_6)\}.$$

Let $\ddot{P} = \{\rho_1, \rho_2\}$ be a set of parameters. Here, $(\ddot{\sigma}_\rho, \ddot{P})_s$ is the fuzzy soft vertex set over W and $(\ddot{\mu}_\rho, \ddot{P})_s$ is the fuzzy soft edge set over R_i.

*Then, the $\mathcal{FSGS}$ is given in **Table 18 and Table 19** as shown in **Figure 15**. Hence, G is a* ***uniform edge $\mathcal{FSGS}$****.*

Table 18: Fuzzy set $\ddot{\sigma}$ on vertex set W

$\ddot{\sigma}_\rho$	w_1	w_2	w_3	w_4	w_5	w_6
ρ_1	0.3	0.4	0.5	0.2	0.5	0.2
ρ_2	0.4	0.2	0.3	0.1	0.4	0.3

Table 19: Fuzzy sets $\ddot{\mu}_1, \ddot{\mu}_2$ on relations R_1, R_2

$\ddot{\mu}_1$	(w_1, w_2)	(w_1, w_6)	(w_2, w_6)	(w_2, w_3)
ρ_1	0.2	0.2	0.2	0.2
ρ_2	0.1	0.1	0.1	0.1

$\ddot{\mu}_2$	(w_3, w_4)	(w_4, w_5)	(w_5, w_6)
ρ_1	0.2	0.2	0.2
ρ_2	0.1	0.1	0.1

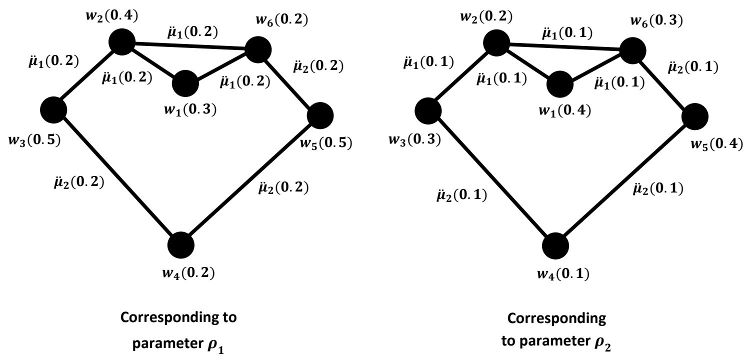

Figure 15: Uniform Edge Fuzzy Soft Graph Structure ($\mathcal{FSGS}$)

Definition 3.8 *Let*

$$G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$$

be a $\mathcal{FSGS}$ *over a* $\mathcal{GS}$ $\ddot{G} = (W, R_i)$ *such that each* R_i *for* $i = 1, 2, \ldots, n$ *are mutually disjoint, symmetric, and irreflexive relations.*

Then the ***complement*** *of* $G_{(\ddot{P},W)}(S)$ *is defined as*

$$\overline{G_{(\ddot{P},W)}(S)} = \{(\ddot{\sigma}_\rho, \ddot{P})_s, \overline{(\ddot{\mu}_\rho, \ddot{P})_s}\}$$

such that

$$\overline{\ddot{\mu}_\rho(w_1, w_2)} = \ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2) - \ddot{\mu}_\rho(w_1, w_2),$$

for all $w_1, w_2 \in W$ *and* $\rho \in \ddot{P}$.

Example 3.8 *Consider a* $\mathcal{FSGS}$

$$G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$$

given in Table 11 and Figure 11.

The complement of $\mathcal{FSGS}$ *is*

$$\overline{G_{(\ddot{P},W)}(S)}$$

*which is presented in **Table 20** and illustrated in **Figure 16**.*

Table 20: Fuzzy set $\ddot{\sigma}$ on vertex set W and $\ddot{\mu}_1, \ddot{\mu}_2$ on relations R_1, R_2

$\ddot{\sigma}$	w_1	w_2	w_3	w_4
ρ_1	0.5	0.4	0.3	0.2
ρ_2	0.4	0.7	0.6	0.5

$\ddot{\mu}_1$	(w_1, w_2)	(w_2, w_3)
ρ_1	0.0	0.1
ρ_2	0.1	0.3
$\ddot{\mu}_2$	(w_3, w_4)	(w_4, w_1)
ρ_1	0.1	0.1
ρ_2	0.0	0.4

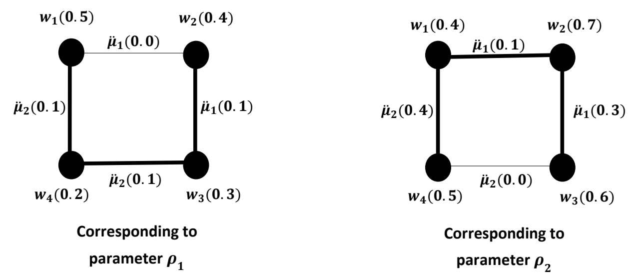

Figure 16: Compliment of Fuzzy Soft Graph Structure ($\mathcal{FSGS}$)

Definition 3.9 *Let*

$$G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$$

be a $\mathcal{FSGS}$ over a $\mathcal{GS}$ $\ddot{G} = (W, R_i)$ such that each R_i for $i = 1, 2, \ldots, n$ are mutually disjoint, symmetric, and irreflexive relations.

*Then the μ-**complement** of $G_{(\ddot{P},W)}(S)$ is defined as*

$$G^{\mu}_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, ((\ddot{\mu}_\rho, \ddot{P})_s)^{\mu}\},$$

such that for all $(w_1, w_2) \in W \times W$ and $\rho \in \ddot{P}$:

$$\ddot{\mu}^{\mu}_{\rho}(w_1, w_2) = \begin{cases} \ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2) - \ddot{\mu}_\rho(w_1, w_2), & \text{if } \ddot{\mu}_\rho(w_1, w_2) > 0, \\ 0, & \text{if } \ddot{\mu}_\rho(w_1, w_2) = 0. \end{cases}$$

Example 3.9 *Consider a $\mathcal{FSGS}$*

$$G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$$

given in Table 11 and Figure 11.

The μ-complement of $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$ is

$$G^{\mu}_{(\ddot{P},W)}(S),$$

as given in Table 21 and shown in Figure 17.

Table 21: Fuzzy set $\ddot{\sigma}$ on vertex set W and $\ddot{\mu}_1, \ddot{\mu}_2$ on relations R_1, R_2 (μ-complement)

$\ddot{\sigma}$	w_1	w_2	w_3	w_4
ρ_1	0.5	0.4	0.3	0.2
ρ_2	0.4	0.7	0.6	0.5

$\ddot{\mu}_1$	(w_1, w_2)	(w_2, w_3)
ρ_1	0.0	0.1
ρ_2	0.1	0.3

$\ddot{\mu}_2$	(w_3, w_4)	(w_4, w_1)
ρ_1	0.1	0.1
ρ_2	0.0	0.0

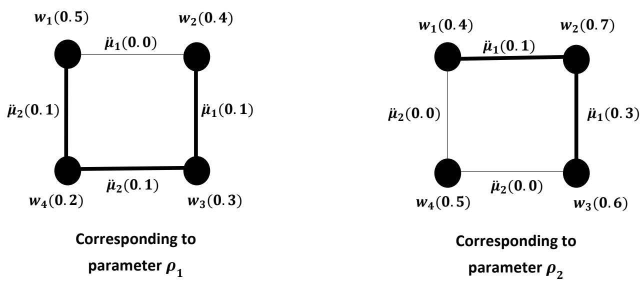

Figure 17: μ-complement of Fuzzy Soft Graph Structure ($\mathcal{FSGS}$)

Definition 3.10 *Let $G_{\ddot{P},W}(S) = \{\langle\ddot{\sigma}_\rho, \ddot{P}\rangle_s, \langle\ddot{\mu}_\rho, \ddot{P}\rangle_s\}$ be $\mathcal{FSGS}$ over a $\mathcal{GS}$ $\ddot{G} = (W, R_i)$ such that each R_i for $i = 1, 2, \ldots, n$ are mutually disjoint, symmetric and irreflexive relations. Then for all $\rho \in \ddot{P}$, $\mathcal{FSGS}$ $G_{\ddot{P},W}(S)$ is said to be a regular $\mathcal{FSGS}$ if the degree of each vertex of $\mathcal{FSG}$ $G_{\ddot{P},W}$ is h_1, then it is called a h_1-regular $\mathcal{FSGS}$*

Example 3.10 *Consider the fuzzy soft graph structure*

$$G_{(\ddot{P},W)}(S) = \Big\{(\ddot{\sigma}_\rho, \ddot{P})_s,\ (\ddot{\mu}_\rho, \ddot{P})_s\Big\}$$

over

$$\ddot{G} = (W, R_1, R_2, R_3),$$

where

$$W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\},$$

and the relations are given as

$$R_1 = \{(w_1, w_2), (w_3, w_4), (w_5, w_6), (w_7, w_8)\},$$

$$R_2 = \{(w_1, w_7), (w_3, w_5), (w_4, w_6), (w_2, w_8)\},$$

$$R_3 = \{(w_1, w_3), (w_2, w_4), (w_6, w_8), (w_5, w_7)\}.$$

Let

$$\ddot{P} = \{\rho_1, \rho_2\}$$

be a set of parameters, where $(\ddot{\sigma}_\rho, \ddot{P})_s$ is a fuzzy soft vertex set over W, and $(\ddot{\mu}_\rho, \ddot{P})_s$ is a fuzzy soft edge set over R_i.

Then, the fuzzy soft graph structure $\mathcal{FSGS}$ is represented in Table 22 and Table 23, and illustrated in Figure 18.

Hence, G is a ***regular fuzzy soft graph structure****.*

Table 22: Fuzzy set $\ddot{\sigma}$ on vertex set W

$\ddot{\sigma}$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
ρ_1	0.5	0.3	0.4	0.2	0.4	0.7	0.4	0.4
ρ_2	0.4	0.5	0.4	0.2	0.4	0.7	0.3	0.2

Table 23: Fuzzy sets $\ddot{\mu}_1, \ddot{\mu}_2, \ddot{\mu}_3$ on relations R_1, R_2, R_3

$\ddot{\mu}_1$	(w_1, w_2)	(w_3, w_4)	(w_5, w_6)	(w_7, w_8)
ρ_1	0.3	0.2	0.2	0.3
ρ_2	0.4	0.3	0.3	0.4
$\ddot{\mu}_2$	(w_1, w_7)	(w_3, w_5)	(w_4, w_6)	(w_2, w_8)
ρ_1	0.2	0.3	0.3	0.2
ρ_2	0.3	0.4	0.4	0.3
$\ddot{\mu}_3$	(w_1, w_3)	(w_2, w_4)	(w_6, w_8)	(w_5, w_7)
ρ_1	0.1	0.1	0.1	0.1
ρ_2	0.2	0.2	0.2	0.2

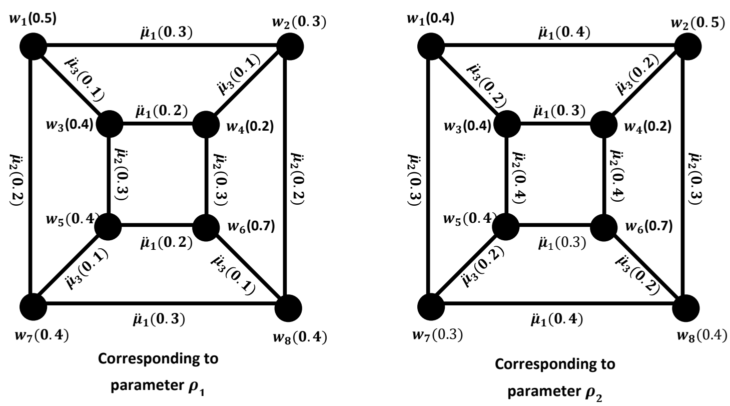

Figure 18: Fuzzy Soft Graph Structure ($\mathcal{FSGS}$)

Definition 3.11 *Let*

$$G_{(\ddot{P},W)}(S) = \left\{(\ddot{\sigma}_\rho, \ddot{P})_s,\ (\ddot{\mu}_\rho, \ddot{P})_s\right\}$$

be a fuzzy soft graph structure ($\mathcal{FSGS}$) over a generalized structure

$$\ddot{G} = (W, R_i),$$

such that each relation R_i for $i = 1, 2, \ldots, n$ is mutually disjoint, symmetric and irreflexive.

Then, for all $\rho \in \ddot{P}$, the fuzzy soft graph structure $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$ is said to be a ***totally regular fuzzy soft graph structure*** *if the* total degree *of each vertex of the fuzzy soft graph $\mathcal{FSG}$ $G_{(\ddot{P},W)}$ is h_2.*

*In this case, it is called an h_2-**totally regular fuzzy soft graph structure**.*

Example 3.11 *Consider the fuzzy soft graph structure*

$$G_{(\ddot{P},W)}(S) = \left\{(\ddot{\sigma}_\rho, \ddot{P})_s,\ (\ddot{\mu}_\rho, \ddot{P})_s\right\}$$

over

$$\ddot{G} = (W, R_1, R_2, R_3),$$

where

$$W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\},$$

and the relations are given as

$$R_1 = \{(w_1, w_2), (w_1, w_3), (w_1, w_6), (w_1, w_7)\},$$

$$R_2 = \{(w_2, w_3), (w_2, w_4), (w_2, w_7), (w_3, w_4), (w_3, w_5)\},$$

$$R_3 = \{(w_4, w_5), (w_4, w_6), (w_5, w_6), (w_5, w_7), (w_6, w_7)\}.$$

Let

$$\ddot{P} = \{\rho_1, \rho_2\}$$

be a set of parameters, where $(\ddot{\sigma}_\rho, \ddot{P})_s$ *is a fuzzy soft vertex set over* W*, and* $(\ddot{\mu}_\rho, \ddot{P})_s$ *is a fuzzy soft edge set over* R_i*.*

Then, the fuzzy soft graph structure $\mathcal{FSGS}$ *is represented in Table 24 and Table 25, and illustrated in Figure 19.*

Hence, G is a ***totally regular fuzzy soft graph structure****.*

Table 24: Fuzzy set $\ddot{\sigma}$ on vertex set W

$\ddot{\sigma}$	w_1	w_2	w_3	w_4	w_5	w_6	w_7
ρ_1	0.4	0.3	0.1	0.3	0.5	0.4	0.2
ρ_2	0.3	0.2	0.4	0.1	0.3	0.1	0.1

Table 25: Fuzzy sets $\ddot{\mu}_1, \ddot{\mu}_2, \ddot{\mu}_3$ on relations R_1, R_2, R_3

$\ddot{\mu}_1$	(w_1, w_2)	(w_1, w_3)	(w_1, w_6)	(w_1, w_7)
ρ_1	0.1	0.4	0.2	0.4
ρ_2	0.3	0.1	0.3	0.1

$\ddot{\mu}_2$	(w_2, w_3)	(w_2, w_4)	(w_2, w_7)	(w_3, w_4)	(w_3, w_5)
ρ_1	0.5	0.3	0.3	0.3	0.1
ρ_2	0.4	0.4	0.3	0.4	0.2

$\ddot{\mu}_3$	(w_4, w_5)	(w_4, w_6)	(w_5, w_6)	(w_5, w_7)	(w_6, w_7)
ρ_1	0.3	0.4	0.3	0.2	0.2
ρ_2	0.3	0.2	0.1	0.3	0.4

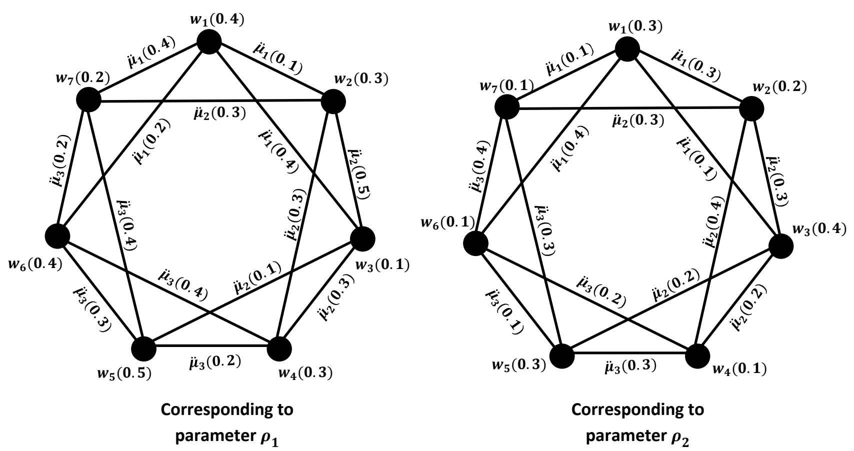

Figure 19: Fuzzy Soft Graph Structure ($\mathcal{FSGS}$)

Note. From Example 3.18, we observe that the graph is a regular fuzzy soft graph structure ($\mathcal{FSGS}$) but not a totally regular fuzzy soft graph structure. On the other hand, in Example 3.20, the graph is a totally regular fuzzy soft graph structure but not a regular fuzzy soft graph structure.

Hence, these examples justify and motivate the forthcoming theorems.

4. Main Result

Theorem 4.1 *Let*

$$G_{(\ddot{P},W)}(S) = \left\{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\right\}$$

be a fuzzy soft graph structure ($\mathcal{FSGS}$) over a generalized structure

$$\ddot{G} = (W, R_i),$$

such that each R_i for $i = 1, 2, \ldots, n$ is mutually disjoint, symmetric, and irreflexive.

Then, for all $\rho \in \ddot{P}$, if $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$ is regular and $\ddot{H}$ is a constant function in $\ddot{G}$, then $G_{(\ddot{P},W)}(S)$ is a totally regular fuzzy soft graph structure.

Proof Suppose $G_{(\ddot{P},W)}(S)$ is a regular fuzzy soft graph structure ($\mathcal{FSGS}$) and $\ddot{H}$ is a constant function. Then, for all $\rho \in \ddot{P}$, $w \in W$, and $i = 1, 2, 3, \ldots, n$, we have

$$\ddot{H}_{\rho_i}(w_i) = b_i,$$

for some positive real number b_i, and

$$\deg_{G_{(\ddot{P},W)}(S)}(w) = k_i$$

in the generalized structure $\ddot{G}$, for some positive real number k_i.

Now, the total degree of w in $G_{(\ddot{P},W)}(S)$ is

$$t\deg_{G_{(\ddot{P},W)}(S)}(w) = \deg_{G_{(\ddot{P},W)}(S)}(w) + \ddot{H}_{\rho_i}(w) = k_i + b_i,$$

which is constant for all $w \in W$ and $\rho \in \ddot{P}$.

Therefore, $G_{(\ddot{P},W)}(S)$ is a totally regular fuzzy soft graph structure.

Theorem 4.2 *Let*

$$G_{(\ddot{P},W)}(S) = \left\{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\right\}$$

be a fuzzy soft graph structure ($\mathcal{FSGS}$) over a generalized structure

$$\ddot{G} = (W, R_i),$$

such that each R_i for $i = 1, 2, \ldots, n$ is mutually disjoint, symmetric, and irreflexive.

Then, for all $\rho \in \ddot{P}$, if $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$ is totally regular and $\ddot{H}$ is a constant function in $\ddot{G}$, then $G_{(\ddot{P},W)}(S)$ is a regular fuzzy soft graph structure.

Proof Suppose $G_{(\ddot{P},W)}(S)$ is a totally regular fuzzy soft graph structure ($\mathcal{FSGS}$) and $\ddot{H}$ is a constant function. Then, for all $\rho \in \ddot{P}$, $w \in W$, and $i = 1, 2, 3, \ldots, n$, we have

$$\ddot{H}_{\rho_i}(w_i) = b_i,$$

for some positive real number b_i, and

$$t\deg_{G_{(\ddot{P},W)}(S)}(w) = k_i$$

in the generalized structure $\ddot{G}$, for some positive real number k_i.

Since

$$t\deg_{G_{(\ddot{P},W)}(S)}(w) = \deg_{G_{(\ddot{P},W)}(S)}(w) + \ddot{H}_{\rho_i}(w),$$

for all $\rho \in \ddot{P}$, $i = 1, 2, 3, \ldots, n$, and $w \in W$, we obtain

$$\deg_{G_{(\ddot{P},W)}(S)}(w) = t\deg_{G_{(\ddot{P},W)}(S)}(w) - \ddot{H}_{\rho_i}(w).$$

Thus,

$$\deg_{G_{(\ddot{P},W)}(S)}(w) = k_i - b_i,$$

which is constant for all $\rho \in \ddot{P}$ and $w \in W$.

Hence, $G_{(\ddot{P},W)}(S)$ is a regular fuzzy soft graph structure.

Theorem 4.3 *Let*

$$G_{(\ddot{P},W)}(S) = \left\{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\right\}$$

be a fuzzy soft graph structure ($\mathcal{FSGS}$) over a generalized structure

$$\ddot{G} = (W, R_i),$$

such that each R_i for $i = 1, 2, \ldots, n$ is mutually disjoint, symmetric, and irreflexive, and suppose it is both a totally regular $\mathcal{FSGS}$ and a regular $\mathcal{FSGS}$. Then, for all $\rho \in \ddot{P}$, $i = 1, 2, 3, \ldots, n$, and $w \in W$, $\ddot{H}$ is a constant function in $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$.

Proof Suppose $G_{(\ddot{P},W)}(S)$ is both totally regular and regular. Then, for all $\rho \in \ddot{P}$, $i = 1, 2, 3, \ldots, n$, and $w \in W$, we have

$$\deg_{G_{(\ddot{P},W)}(S)}(w) = u_i, \quad t\deg_{G_{(\ddot{P},W)}(S)}(w) = v_i,$$

for some positive real numbers u_i and v_i in $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$.

Since

$$\deg_{G_{(\ddot{P},W)}(S)}(w) + \ddot{H}_{\rho_i}(w) = t\deg_{G_{(\ddot{P},W)}(S)}(w) = v_i,$$

for all $i = 1, 2, 3, \ldots, n$ and $w \in W$, we obtain

$$u_i + \ddot{H}_{\rho_i}(w) = v_i,$$

which gives

$$\ddot{H}_{\rho_i}(w) = v_i - u_i,$$

a constant in $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$.

Thus, $\ddot{H}$ is a constant function in $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$.

Remark. The converse of Theorem 4.3 is not true in general. If $\ddot{H}$ is a constant function, then $G_{(\ddot{P},W)}(S)$ may or may not be both regular and totally regular.

Theorem 4.4 *Let*

$$G_{(\ddot{P},W)}(S) = \left\{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\right\}$$

be a fuzzy soft graph structure ($\mathcal{FSGS}$) over a generalized structure

$$\ddot{G} = (W, R_i),$$

such that each R_i for $i = 1, 2, \ldots, n$ is mutually disjoint, symmetric, and irreflexive, and suppose it is both a regular $\mathcal{FSGS}$ and a uniform vertex $\mathcal{FSGS}$. Then, the complement of $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$ is also a regular $\mathcal{FSGS}$.

Proof Suppose $G_{(\ddot{P},W)}(S)$ is both regular and a uniform vertex $\mathcal{FSGS}$. Then, for all $\rho \in \ddot{P}$, $i = 1,2,3,\dots,n$, and $w \in W$, we have

$$\deg_{G_{(\ddot{P},W)}(S)}(w) = u_i, \quad \ddot{\sigma}_\rho(w_i) = v_i,$$

for some positive real numbers u_i and v_i.

By the definition of the complement of $\mathcal{FSGS}$,

$$\deg_{\overline{G_{(\ddot{P},W)}(S)}}(w_1) = \sum_{w_1 \neq w_2} \left[\ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2) - \ddot{\mu}_\rho(w_1,w_2)\right].$$

This can be simplified as

$$\deg_{\overline{G_{(\ddot{P},W)}(S)}}(w_1) = \sum_{w_1 \neq w_2} \ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2) - \sum_{w_1 \neq w_2} \ddot{\mu}_\rho(w_1,w_2).$$

Since $\ddot{\sigma}_\rho(w_i) = v_i$ and $\deg_{G_{(\ddot{P},W)}(S)}(w_1) = u_i$, we have

$$\deg_{\overline{G_{(\ddot{P},W)}(S)}}(w_1) = (n-1)v_i - u_i.$$

Thus, the complement of $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$ is also a regular fuzzy soft graph structure.

Theorem 4.5 *Let*

$$G_{(\ddot{P},W)}(S) = \left\{(\ddot{\sigma}_\rho, \ddot{P})_s,\ (\ddot{\mu}_\rho, \ddot{P})_s\right\}$$

be an $\mathcal{FSGS}$ over a $\mathcal{GS}$ $\ddot{G} = (W, R_i)$ such that each R_i for $i = 1,2,\dots,n$ are mutually disjoint, symmetric, and irreflexive relations. If $G_{(\ddot{P},W)}(S)$ is both a totally regular $\mathcal{FSGS}$ and a uniform vertex $\mathcal{FSGS}$, then the complement of $G_{(\ddot{P},W)}(S)$ is also a totally regular $\mathcal{FSGS}$.

Proof Suppose $G_{(\ddot{P},W)}(S)$ is a totally regular $\mathcal{FSGS}$ and a uniform vertex $\mathcal{FSGS}$. Then, for all $\rho \in \ddot{P}$, $i = 1,2,3,\dots,n$, and $w \in W$, we have

$$t\deg_{G_{(\ddot{P},W)}(S)}(w) = u_i, \qquad \ddot{\sigma}_\rho(w) = v_i,$$

for some positive real numbers u_i and v_i in the $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$.

By the definition of the complement of an $\mathcal{FSGS}$, we obtain

$$\deg_{\overline{G_{(\ddot{P},W)}(S)}}(w_1) = \sum_{w_1 \neq w_2} \Big[\ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2) - \ddot{\mu}_\rho(w_1,w_2)\Big] + \ddot{\sigma}_\rho(w_1).$$

This expands as

$$\begin{aligned}\deg_{\overline{G_{(\ddot{P},W)}(S)}}(w_1) &= \sum_{w_1 \neq w_2} \Big(\ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2)\Big) - \sum_{w_1 \neq w_2} \ddot{\mu}_\rho(w_1,w_2) - \ddot{\sigma}_\rho(w_1) + \ddot{\sigma}_\rho(w_1) + \ddot{\sigma}_\rho(w_1) \\ &= \sum_{w_1 \neq w_2} \Big(\ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2)\Big) - \left(\sum_{w_1 \neq w_2} \ddot{\mu}_\rho(w_1,w_2) + \ddot{\sigma}_\rho(w_1)\right) + 2\ddot{\sigma}_\rho(w_1).\end{aligned}$$

Since $\ddot{\sigma}_\rho(w_i) = v_i$ and $t\deg_{G_{(\ddot{P},W)}(S)}(w_1) = u_i$, we get

$$\deg_{\overline{G_{(\ddot{P},W)}(S)}}(w_1) = \sum_{w_1 \neq w_2} \Big(v_i - u_i\Big) + 2v_i = (n+1)v_i - u_i.$$

Therefore, for all $\rho \in \ddot{P}$, $i = 1,2,\dots,n$, and $w \in W$,

$$\deg_{\overline{G_{(\ddot{P},W)}(S)}}(w_1) = (n+1)v_i - u_i.$$

Thus, the complement of $G_{(\ddot{P},W)}(S)$ is also a totally regular $\mathcal{FSGS}$.

Theorem 4.6 *Let*

$$G_{(\ddot{P},W)}(S) = \Big\{(\ddot{\sigma}_\rho, \ddot{P})_s,\ (\ddot{\mu}_\rho, \ddot{P})_s\Big\}$$

be an $\mathcal{FSGS}$ over a $\mathcal{GS}$ $\ddot{G} = (W, R_i)$ such that each R_i for $i = 1, 2, \dots, n$ are mutually disjoint, symmetric, and irreflexive relations, and suppose it is a uniform edge $\mathcal{FSGS}$. Then $G_{(\ddot{P},W)}(S)$ is a regular $\mathcal{FSGS}$.

Proof Suppose $G_{(\ddot{P},W)}(S)$ is a uniform edge $\mathcal{FSGS}$. Then, for all $\rho \in \ddot{P}$, $i = 1, 2, 3, \dots, n$ and $(w_1, w_2) \in W \times W$,

$$\ddot{\mu}_\rho(w_1, w_2) = u_i,$$

for some positive real number u_i. Then

$$\deg_{G_{(\ddot{P},W)}(S)}(w_1) = \sum_{w_1 \neq w_2} \ddot{\mu}_\rho(w_1, w_2) = \sum_{w_1 \neq w_2} u_i = (n-1)u_i,$$

for all $\rho \in \ddot{P}$, $i = 1, 2, 3, \dots, n$ and $w \in W$. Hence $G_{(\ddot{P},W)}(S)$ is a regular $\mathcal{FSGS}$.

Theorem 4.7 *Let*

$$G_{(\ddot{P},W)}(S) = \Big\{(\ddot{\sigma}_\rho, \ddot{P})_s,\ (\ddot{\mu}_\rho, \ddot{P})_s\Big\}$$

be an $\mathcal{FSGS}$ over a $\mathcal{GS}$ $\ddot{G} = (W, R_i)$ such that each R_i for $i = 1, 2, \dots, n$ are mutually disjoint, symmetric, and irreflexive relations, and suppose it is both a uniform vertex $\mathcal{FSGS}$ and a uniform edge $\mathcal{FSGS}$. Then $G_{(\ddot{P},W)}(S)$ is a totally regular $\mathcal{FSGS}$.

Proof Suppose $G_{(\ddot{P},W)}(S)$ is both a uniform vertex $\mathcal{FSGS}$ and a uniform edge $\mathcal{FSGS}$. Then, for all $\rho \in \ddot{P}$, $i = 1, 2, 3, \dots, n$, $w \in W$, and $(w_1, w_2) \in W \times W$, we have

$$\ddot{\sigma}_\rho(w) = u_i, \qquad \ddot{\mu}_\rho(w_1, w_2) = v_i,$$

for some positive real numbers u_i and v_i. Then

$$t\deg_{G_{(\ddot{P},W)}(S)}(w_1) = \sum_{w_1 \neq w_2} \ddot{\mu}_\rho(w_1, w_2) + \ddot{\sigma}_\rho(w_1) = \sum_{w_1 \neq w_2} v_i + u_i = (n-1)v_i + u_i,$$

for all $\rho \in \ddot{P}$, $i = 1, 2, 3, \dots, n$, and $w \in W$. Thus $G_{(\ddot{P},W)}(S)$ is a totally regular $\mathcal{FSGS}$.

Theorem 4.8 *Let*

$$G_{(\ddot{P},W)}(S) = \Big\{(\ddot{\sigma}_\rho, \ddot{P})_s,\ (\ddot{\mu}_\rho, \ddot{P})_s\Big\}$$

be a complete $\mathcal{FSGS}$ over a $\mathcal{GS}$ $\ddot{G} = (W, R_i)$ such that each R_i for $i = 1, 2, \dots, n$ is mutually disjoint, symmetric, and irreflexive, and suppose it is a uniform vertex $\mathcal{FSGS}$. Then, for all $\rho \in \ddot{P}$, $i = 1, 2, 3, \dots, n$, and $w \in W$, $G_{(\ddot{P},W)}(S)$ is a regular $\mathcal{FSGS}$.

Proof Suppose $G_{(\ddot{P},W)}(S)$ is a complete $\mathcal{FSGS}$ and a uniform vertex $\mathcal{FSGS}$. Then, for all $\rho \in \ddot{P}$, $i = 1, 2, 3, \dots, n$, $w \in W$ and $(w_1, w_2) \in W \times W$, we have

$$\ddot{\mu}_\rho(w_1, w_2) = \ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2), \qquad \ddot{\sigma}_\rho(w) = u_i,$$

for some positive real number u_i. Then

$$\deg_{G_{(\ddot{P},W)}(S)}(w_1) = \sum_{w_1 \neq w_2} \ddot{\mu}_\rho(w_1, w_2) = \sum_{w_1 \neq w_2} \big(\ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2)\big) = (n-1)u_i.$$

Thus $\deg_{G_{(\ddot{P},W)}(S)}(w_1) = (n-1)u_i$ for all $\rho \in \ddot{P}$, $i = 1, 2, \dots, n$, and $w \in W$. Hence $G_{(\ddot{P},W)}(S)$ is a regular $\mathcal{FSGS}$.

Theorem 4.9 *Let $G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$ be a complete $\mathcal{FSGS}$ over a $\mathcal{GS}$ $\ddot{G} = (W, R_i)$ such that each R_i for $i = 1, 2, \ldots, n$ are mutually disjoint, symmetric and irreflexive relations and be a uniform vertex $\mathcal{FSGS}$. Then, for all $\rho \in \ddot{P}$, $i = 1, 2, \ldots, n$ and $w \in W$, $G_{(\ddot{P},W)}(S)$ is totally regular $\mathcal{FSGS}$.*

Proof Suppose $G_{(\ddot{P},W)}(S)$ is a complete $\mathcal{FSGS}$ and a uniform vertex $\mathcal{FSGS}$. Then, for all $\rho \in \ddot{P}$, $i = 1, 2, \ldots, n$, $w \in W$ and $(w_1 \times w_2) \in W \times W$, we have $\ddot{\mu}_\rho(w_1, w_2) = \ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2)$ and $\ddot{\sigma}_\rho(w_i) = u_i$ for some positive real numbers u_i in a $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$. Then, by definition,

$$t\deg_{G_{(\ddot{P},W)}(S)}(w_1) = \sum_{w_1 \neq w_2} \big(\ddot{\mu}_\rho(w_1, w_2) + \ddot{\sigma}_\rho(w_1)\big) = \sum_{w_1 \neq w_2} \big(\ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2)\big) + u_i = (n-1)u_i + u_i = nu_i.$$

Thus, for all $\rho \in \ddot{P}$, $i = 1, 2, \ldots, n$, and $w \in W$, $\deg_{G_{(\ddot{P},W)}(S)}(w_1) = nu_i$. Hence, $G_{(\ddot{P},W)}(S)$ is a totally regular $\mathcal{FSGS}$.

Theorem 4.10 *Let $G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$ be a regular $\mathcal{FSGS}$ over a $\mathcal{GS}$ $\ddot{G} = (W, R_i)$ such that each R_i for $i = 1, 2, \ldots, n$ are mutually disjoint, symmetric, and irreflexive relations and be a uniform vertex $\mathcal{FSGS}$. Then, for all $\rho \in \ddot{P}$, $i = 1, 2, \ldots, n$ and $w \in W$, $G_{(\ddot{P},W)}(S)$ is a totally regular $\mathcal{FSGS}$.*

Proof Suppose $G_{(\ddot{P},W)}(S)$ is a regular $\mathcal{FSGS}$ and a uniform vertex $\mathcal{FSGS}$. Then, for all $\rho \in \ddot{P}$, $i = 1, 2, \ldots, n$, $w \in W$ and $(w_1 \times w_2) \in W \times W$, we have $\deg_{G_{(\ddot{P},W)}(S)}(w) = u_i$ and $\ddot{\sigma}_\rho(w_i) = v_i$ for some positive real numbers u_i, v_i in a $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$. Then, by definition,

$$t\deg_{G_{(\ddot{P},W)}(S)}(w_1) = \sum_{w_1 \neq w_2} \big(\ddot{\mu}_\rho(w_1, w_2) + \ddot{\sigma}_\rho(w_1)\big) = \sum_{w_1 \neq w_2} u_i + v_i = (n-1)u_i + v_i.$$

Thus, $\deg_{G_{(\ddot{P},W)}(S)}(w_1) = (n-1)u_i + v_i$. Hence, $G_{(\ddot{P},W)}(S)$ is a totally regular $\mathcal{FSGS}$.

Theorem 4.11 *Let $G_{(\ddot{P},W)}(S) = \{(\ddot{\sigma}_\rho, \ddot{P})_s, (\ddot{\mu}_\rho, \ddot{P})_s\}$ be a $\mathcal{FSGS}$ over a $\mathcal{GS}$ $\ddot{G} = (W, R_i)$ such that each R_i for $i = 1, 2, \ldots, n$ are mutually disjoint, symmetric, and irreflexive relations. Then, for all $\rho \in \ddot{P}$, if $\mathcal{FSGS}$ $G_{(\ddot{P},W)}(S)$ is a regular $\mathcal{FSGS}$ and $\ddot{H}$ is a constant function on $\ddot{G}$, then $G_{(\ddot{P},W)}(S)$ is a totally regular $\mathcal{FSGS}$.*

Proof Suppose $G_{(\ddot{P},W)}(S)$ is a regular $\mathcal{FSGS}$ and $\ddot{H}$ is a constant function. Then, for all $\rho \in \ddot{P}$, $w \in W$, $i = 1, 2, \ldots, n$, we have $\ddot{H}_{\rho_i}(w_i) = b_i$ for some positive real number b_i and $\deg_{G_{(\ddot{P},W)}(S)}(w) = k_i$ in $\mathcal{GS}$ $\ddot{G}$, for some positive real number k_i, for all $\rho \in \ddot{P}$, $i = 1, 2, \ldots, n$ and $w \in W$. Since

$$t\deg_{G_{(\ddot{P},W)}(S)}(w) = \deg_{G_{(\ddot{P},W)}(S)}(w) + \ddot{H}_{\rho_i}(w) = k_i + b_i$$

in $\mathcal{GS}$ $\ddot{G}$, for all $\rho \in \ddot{P}$, $i = 1, 2, \ldots, n$ and $w \in W$. Thus, $G_{(\ddot{P},W)}(S)$ is a totally regular $\mathcal{FSGS}$.

5. Application

Identification of the Best Courier Service

Transportation has been an essential part of human life for thousands of years, occurring simultaneously with economic improvements. The invention of the wheel made transportation easier. People started making carts with the help of wheels, which saved time and energy. With the passage of time, engines were invented and vehicles were created using wheels and engines. Initially, coal engines were invented, followed by engines that worked with petrol and gas. Trains were invented, which significantly reduced travel time. Later, airplanes were invented, which became a much better and more convenient mode of transportation. Airplanes made international transportation much more convenient.

A company, organization, or person that conveys a message, package, or letter from one place or person to another is called a *courier*. The word courier comes from the Latin word "currere," which means "to run." In ancient times, when the courier service started, the means of delivery was initially through running. A person would deliver the message and have to run to reach their destination. Eventually, as time went on, this service evolved to include the use of horses, camels, stagecoaches, pigeons, dogs, rabbits, trains, bicycles, and automobiles.

Nowadays, the courier industry is flush with options for the purpose of delivering products. We can deliver any product these days, such as medicine, electronics, documents, time-sensitive materials, mail, and many more. This industry can deliver packages within one day and within the same day if sent out by a definite time and the destination is within a definite distance. This industry works within a wide range, from certain regions or territories to other regions of the country and other countries of the world. Also, the world of business works on large scales and they want to send their products in a short time, so they use courier services. Today, this industry provides unique services by using courier software that gives electronic proof and electronic tracking information of packages or parcels, making it much easier to deliver packages.

Shopping through the internet is trending all around the world. Nowadays, people want to buy products from their homes. They want everything delivered to their doorsteps. They feel frustrated when they have to buy offline. Some customers see it as a waste of time, so they prefer online shopping. You just need a phone to approach your favorite stores. Now, such people send their packages to their friends or relatives and get their products through courier services. In short, couriers are the safest, easiest, and quickest method of delivering any type of packages.

There are many courier industries that facilitate people in every country. When a customer wants to send a package from one city to another, they have to consider many factors in choosing a courier service such as service charges, time slot, courier facilities, etc. Depending on these factors, a customer will select the best courier service for them.

A *Fuzzy Soft Graph Structure* (FSGS) of courier services can be utilized to find which courier service is the best for delivering their packages in various parameters such as time, service charges, and facilities. This FSGS of courier services will show which courier industry is more effective. By keeping this in view, one can choose their reliable courier industry. It will also represent the best courier industry. By keeping this in view, other industries can also find their own standard and identify localities where they need to improve their service in order to become popular.

Consider an FSGS

$$G_{(\ddot{P},W)}(S) = \left\{(\ddot{\sigma}_\rho, \ddot{P})_s,\ (\ddot{\mu}_\rho, \ddot{P})_s\right\}$$

over the graph structure

$$G' = (W, R_1, R_2, R_3, R_4, R_5, R_6),$$

where

$$\begin{aligned} W = \{&\text{Faisalabad (FSD), Lahore (LHR), Toba Tek Singh (TTS), Dera Ghazi Khan (DGK),} \\ &\text{Islamabad (ISB), Sargodha (SGD), Sheikhupura (SKP), Jhang (JHG)}\}, \end{aligned}$$

and relations

$$\begin{aligned} R_1 &= \text{Courier service provided by TCS,} \\ R_2 &= \text{Courier service provided by LEOPARDS,} \\ R_3 &= \text{Courier service provided by DHL,} \\ R_4 &= \text{Courier service provided by PO,} \\ R_5 &= \text{Courier service provided by SPEEDEX,} \\ R_6 &= \text{Courier service provided by OCS.} \end{aligned}$$

Thus, it will be understood that an element belongs to that particular relation for which its membership value is higher than that of other relations. Let

$$\ddot{P} = \{\rho_1, \rho_2\}$$

be a set of parameters such that ρ_1 can be regarded as the evaluation of a six-month period from 1st January to 30th June, and ρ_2 can be regarded as the evaluation of a six-month period from 1st July to 31st December. Then,

$$(\ddot{\sigma}_\rho, \ddot{P})_s$$

is a fuzzy soft vertex set over W, as given in Table 26.

Further,

$$(\ddot{\mu}_\rho, \ddot{P})_s$$

is a fuzzy soft edge set over R_i, representing the membership values of various courier industries between each pair of cities, as given in Tables 27, 28, 29, 30, 31, 32, 33, 34, and 35. Many relations may be defined on W. Now, let us define the relations on the set of cities W.

Table 26: Fuzzy soft set $\ddot{\sigma}$ on vertex set W

Corresponding to parameter ρ_1		Corresponding to parameter ρ_2	
City	**Membership Value**	**City**	**Membership Value**
Faisalabad (FSD)	0.7	Faisalabad (FSD)	0.7
Lahore (LHR)	0.6	Lahore (LHR)	0.8
Toba Tek Singh (TTS)	0.8	Toba Tek Singh (TTS)	0.6
Dera Ghazi Khan (DGK)	0.8	Dera Ghazi Khan (DGK)	0.7
Islamabad (ISB)	0.6	Islamabad (ISB)	0.6
Sargodha (SGD)	0.7	Sargodha (SGD)	0.8
Sheikhupura (SKP)	0.8	Sheikhupura (SKP)	0.7
Jhang (JHG)	0.6	Jhang (JHG)	0.4

Table 27: Fuzzy soft set on courier industries giving service from Sheikhupura (SKP) to other cities

Courier Agency	**Corresponding to parameter ρ_1**				**Corresponding to parameter ρ_2**			
	SKP–FSD	SKP–LHR	SKP–TTS	SKP–SGD	SKP–FSD	SKP–LHR	SKP–TTS	SKP–SGD
TCS	0.2	0.2	0.7	0.4	0.7	0.1	0.4	0.6
LEOPARDS	0.1	0.1	0.7	0.1	0.5	0.2	0.3	0.5
DHL	0.4	0.4	0.5	0.2	0.1	0.2	0.3	0.4
PO	0.7	0.1	0.1	0.2	0.6	0.7	0.5	0.5
SPEEDEX	0.5	0.2	0.4	0.3	0.2	0.4	0.3	0.5
OCS	0.1	0.2	0.2	0.4	0.3	0.3	0.4	0.4

Table 28: Fuzzy soft set on courier industries giving service from Dera Ghazi Khan (DGK) to other cities

Courier Agency	**Corresponding to parameter ρ_1**			**Corresponding to parameter ρ_2**		
	DGK–ISB	DGK–SGD	DGK–SKP	DGK–ISB	DGK–SGD	DGK–SKP
TCS	0.4	0.4	0.7	0.2	0.7	0.4
LEOPARDS	0.5	0.3	0.6	0.1	0.5	0.3
DHL	0.4	0.4	0.3	0.4	0.4	0.7
PO	0.3	0.5	0.5	0.5	0.3	0.6
SPEEDEX	0.6	0.4	0.3	0.6	0.4	0.5
OCS	0.2	0.2	0.4	0.2	0.2	0.5

Table 29: Fuzzy soft set on courier industries giving service from Lahore (LHR) to other cities

Courier Agency	**Corresponding to parameter ρ_1**			**Corresponding to parameter ρ_2**			
	LHR–ISB	LHR–TTS	LHR–DGK	LHR–ISB	LHR–JHG	LHR–TTS	LHR–DGK
TCS	0.2	0.5	0.2	0.4	0.6	0.4	0.6
LEOPARDS	0.4	0.3	0.1	0.3	0.4	0.3	0.4
DHL	0.5	0.5	0.4	0.3	0.5	0.5	0.5
PO	0.2	0.6	0.5	0.5	0.5	0.2	0.4
SPEEDEX	0.1	0.4	0.6	0.3	0.5	0.3	0.5
OCS	0.4	0.5	0.2	0.4	0.4	0.5	0.5

Table 30: Fuzzy soft set on courier industries giving service from Toba Tek Singh (TTS) to other cities

Courier Agency	**Corresponding to parameter ρ_1**				**Corresponding to parameter ρ_2**			
	TTS–ISB	TTS–SGD	TTS–FSD	TTS–JHG	TTS–ISB	TTS–SGD	TTS–FSD	TTS–JHG
TCS	0.4	0.3	0.7	0.5	0.1	0.2	0.5	0.3
LEOPARDS	0.5	0.5	0.5	0.3	0.0	0.3	0.3	0.2
DHL	0.1	0.2	0.2	0.5	0.0	0.2	0.5	0.4
PO	0.2	0.2	0.5	0.6	0.1	0.2	0.4	0.3
SPEEDEX	0.4	0.1	0.4	0.4	0.5	0.1	0.5	0.5
OCS	0.4	0.4	0.5	0.5	0.3	0.4	0.5	0.1

Table 31: Fuzzy soft set on courier industries giving service from Sargodha (SGD) to other cities

Courier Agency	**Corresponding to parameter ρ_1**				**Corresponding to parameter ρ_2**			
	SGD–FSD	SGD–SHK	SGD–JHG	SGD–LHR	SGD–FSD	SGD–SHK	SGD–JHG	SGD–LHR
TCS	0.4	0.1	0.5	0.2	0.7	0.5	0.2	0.3
LEOPARDS	0.1	0.1	0.4	0.3	0.5	0.5	0.2	0.3
DHL	0.5	0.2	0.5	0.5	0.2	0.4	0.3	0.4
PO	0.6	0.1	0.4	0.2	0.5	0.5	0.5	0.8
SPEEDEX	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3
OCS	0.5	0.0	0.4	0.4	0.5	0.4	0.4	0.4

Table 32: Fuzzy soft set on courier industries giving service from Islamabad (ISB) to other cities

Courier Agency	**Corresponding to parameter ρ_1**				**Corresponding to parameter ρ_2**			
	ISB–SGD	ISB–SKP	ISB–JHG	ISB–FSD	ISB–SGD	ISB–SKP	ISB–JHG	ISB–FSD
TCS	0.3	0.6	0.4	0.1	0.6	0.5	0.3	0.1
LEOPARDS	0.0	0.2	0.4	0.3	0.2	0.1	0.0	0.1
DHL	0.2	0.1	0.5	0.5	0.1	0.3	0.2	0.0
PO	0.0	0.1	0.6	0.1	0.1	0.4	0.3	0.0
SPEEDEX	0.3	0.2	0.4	0.4	0.2	0.5	0.5	0.4
OCS	0.0	0.0	0.5	0.0	0.0	0.2	0.0	0.0

Table 33: Fuzzy soft set on courier industries giving service from Faisalabad (FSD) to other cities

Courier Agency	**Corresponding to parameter ρ_1**			**Corresponding to parameter ρ_2**		
	FSD–LHR	FSD–TTS	FSD–DGK	FSD–LHR	FSD–TTS	FSD–DGK
TCS	0.3	0.6	0.5	0.7	0.4	0.7
LEOPARDS	0.0	0.2	0.1	0.5	0.1	0.4
DHL	0.4	0.7	0.3	0.1	0.5	0.3
PO	0.1	0.4	0.1	0.4	0.3	0.3
SPEEDEX	0.5	0.3	0.6	0.5	0.5	0.5
OCS	0.0	0.0	0.2	0.2	0.5	0.3

Table 34: Fuzzy soft set on courier industries giving service from Jhang (JHG) to other cities

Courier Agency	**Corresponding to parameter ρ_1**			**Corresponding to parameter ρ_2**		
	JHG–TTS	JHG–DGK	JHG–LHR	JHG–TTS	JHG–DGK	JHG–LHR
TCS	0.4	0.7	0.3	0.1	0.1	0.1
LEOPARDS	0.1	0.4	0.0	0.1	0.4	0.7
DHL	0.5	0.3	0.7	0.4	0.1	0.4
PO	0.4	0.1	0.4	0.1	0.5	0.3
SPEEDEX	0.4	0.5	0.3	0.5	0.3	0.3
OCS	0.3	0.0	0.0	0.0	0.1	0.4

Table 35: Fuzzy soft set on courier industries giving service between several cities

Courier Agency	**Corresponding to parameter ρ_1**			**Corresponding to parameter ρ_2**		
	FSD–ISB	JHG–SKP	TTS–DGK	FSD–ISB	JHG–SKP	TTS–DGK
TCS	0.1	0.6	0.5	0.1	0.4	0.7
LEOPARDS	0.4	0.7	0.1	0.0	0.1	0.4
DHL	0.1	0.4	0.3	0.0	0.5	0.3
PO	0.5	0.4	0.7	0.4	0.7	0.3
SPEEDEX	0.1	0.1	0.4	0.1	0.4	0.5
OCS	0.1	0.5	0.3	0.5	0.3	0.2

From the above information, members in each relation are paired with their membership values. These sets are fuzzy sets on R_1, R_2, R_3, R_4, R_5 and R_6 respectively, named as $\ddot{\mu}^1_\rho, \ddot{\mu}^2_\rho, \ddot{\mu}^3_\rho, \ddot{\mu}^4_\rho, \ddot{\mu}^5_\rho$ and $\ddot{\mu}^6_\rho$.

According to parameter ρ_1, let

$$\begin{aligned}
R_1 &= \{(\text{DGK TO SKP}), (\text{TTS TO FSD}), (\text{SGD TO JHG}), (\text{ISB TO SKP})\},\\
R_2 &= \{(\text{SKP TO TTS}), (\text{TTS TO ISB}), (\text{JHG TO SKP})\},\\
R_3 &= \{(\text{LHR TO ISB}), (\text{ISB TO FSD}), (\text{JHG TO LHR})\},\\
R_4 &= \{(\text{SKP TO FSD}), (\text{DGK TO SGD}),\\
&\quad (\text{TTS TO JHG}), (\text{SGD TO FSD}), (\text{ISB TO JHG}), (\text{TTS TO DGK})\},\\
R_5 &= \{(\text{DGK TO ISB}), (\text{LHR TO DGK}), (\text{FSD TO DGK})\},\\
R_6 &= \{(\text{LHR TO TTS}), (\text{JHG TO SKP})\}.
\end{aligned}$$

The related fuzzy sets are given as:

$$\begin{aligned}
\ddot{\mu}^1_{(\rho_1)} &= \{(\text{DGK TO SKP}, 0.7), (\text{TTS TO FSD}, 0.7), (\text{SGD TO JHG}, 0.5), (\text{ISB TO SKP}, 0.6)\},\\
\ddot{\mu}^2_{(\rho_1)} &= \{(\text{SKP TO TTS}, 0.7), (\text{TTS TO ISB}, 0.5), (\text{JHG TO SKP}, 0.7)\},\\
\ddot{\mu}^3_{(\rho_1)} &= \{(\text{LHR TO ISB}, 0.5), (\text{ISB TO FSD}, 0.5), (\text{JHG TO LHR}, 0.7)\},\\
\ddot{\mu}^4_{(\rho_1)} &= \{(\text{SKP TO FSD}, 0.7), (\text{DGK TO SGD}, 0.5), (\text{TTS TO JHG}, 0.6), (\text{SGD TO FSD}, 0.6),\\
&\quad (\text{ISB TO JHG}, 0.6), (\text{TTS TO DGK}, 0.7)\},\\
\ddot{\mu}^5_{(\rho_1)} &= \{(\text{DGK TO ISB}, 0.6), (\text{LHR TO DGK}, 0.6), (\text{FSD TO DGK}, 0.6)\},\\
\ddot{\mu}^6_{(\rho_1)} &= \{(\text{LHR TO TTS}, 0.5)\}.
\end{aligned}$$

According to parameter ρ_2, let

$R_1 = \{(\text{SKP TO FSD}), (\text{DGK TO SGD}), (\text{LHR TO ISB}), (\text{FSD TO LHR}), (\text{TTS TO DGK})\}$,
$R_2 = \{(\text{SKP TO SGD})\}$,
$R_3 = \{(\text{DGK TO SKP}), (\text{LHR TO JHG}), (\text{FSD TO TTS})\}$,
$R_4 = \{(\text{SKP TO LHR}), (\text{SGD TO LHR}), (\text{JHG TO DGK}), (\text{JHG TO SKP})\}$,
$R_5 = \{(\text{DGK TO ISB}), (\text{TTS TO ISB}), (\text{ISB TO SKP}), (\text{FSD TO DGK}), (\text{JHG TO TTS})\}$.

$$R_6 = \{(\text{LHR TO DGK}), (\text{TTS TO FSD}), (\text{FSD TO ISB})\}.$$

The related fuzzy sets are given as:

$\ddot{\mu}^1_{(\rho_2)} = \{(\text{SKP TO FSD}, 0.7), (\text{DGK TO SGD}, 0.7), (\text{LHR TO ISB}, 0.8), (\text{FSD TO LHR}, 0.7), (\text{TTS TO DGK}, 0.5)\}$,
$\ddot{\mu}^2_{(\rho_2)} = \{(\text{SKP TO SGD}, 0.5)\}$,
$\ddot{\mu}^3_{(\rho_2)} = \{(\text{DGK TO SKP}, 0.7), (\text{LHR TO JHG}, 0.5), (\text{FSD TO TTS}, 0.5)\}$,
$\ddot{\mu}^4_{(\rho_2)} = \{(\text{SKP TO LHR}, 0.7), (\text{SGD TO LHR}, 0.8), (\text{JHG TO DGK}, 0.5), (\text{JHG TO SKP}, 0.7)\}$,
$\ddot{\mu}^5_{(\rho_2)} = \{(\text{DGK TO ISB}, 0.6), (\text{TTS TO ISB}, 0.5), (\text{ISB TO SKP}, 0.5), (\text{FSD TO DGK}, 0.5), (\text{JHG TO TTS}, 0.5)\}$,
$\ddot{\mu}^6_{(\rho_2)} = \{(\text{LHR TO DGK}, 0.5), (\text{TTS TO FSD}, 0.5), (\text{FSD TO ISB}, 0.5)\}$.

It is clear that

$$\{\ddot{\sigma}_{(\rho_2)}, \ddot{\mu}^1_{(\rho_2)}, \ddot{\mu}^2_{(\rho_2)}, \ddot{\mu}^3_{(\rho_2)}, \ddot{\mu}^4_{(\rho_2)}, \ddot{\mu}^5_{(\rho_2)}, \ddot{\mu}^6_{(\rho_2)}\} \quad \text{and} \quad \{\ddot{\sigma}_{(\rho_1)}, \ddot{\mu}^1_{(\rho_1)}, \ddot{\mu}^2_{(\rho_1)}, \ddot{\mu}^3_{(\rho_1)}, \ddot{\mu}^4_{(\rho_1)}, \ddot{\mu}^5_{(\rho_1)}, \ddot{\mu}^6_{(\rho_1)}\}$$

are $\mathcal{FSGS}$ according to parameters ρ_1 and ρ_2, as shown in Figure 20 and Figure 21.

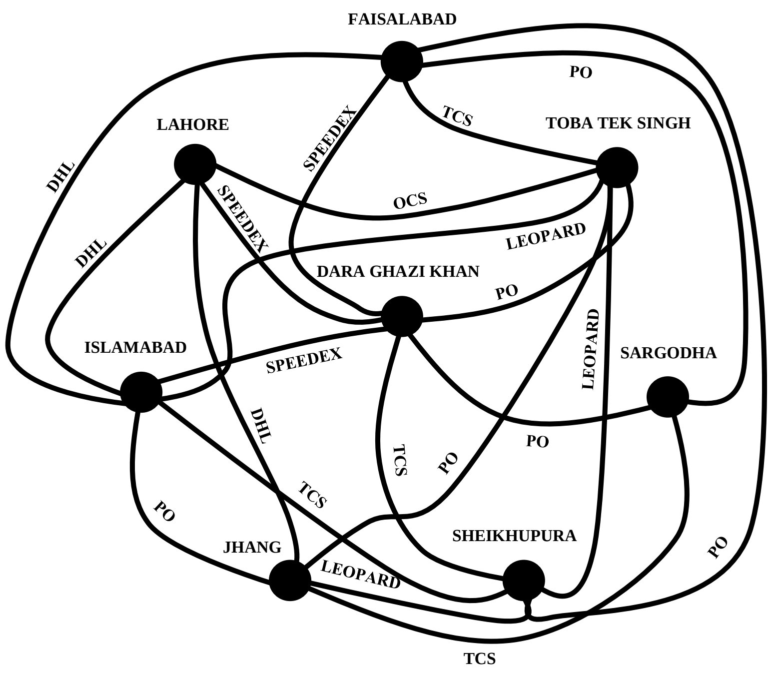

Figure 20: $\mathcal{FSGS}$ presenting best courier service according to parameter ρ_1

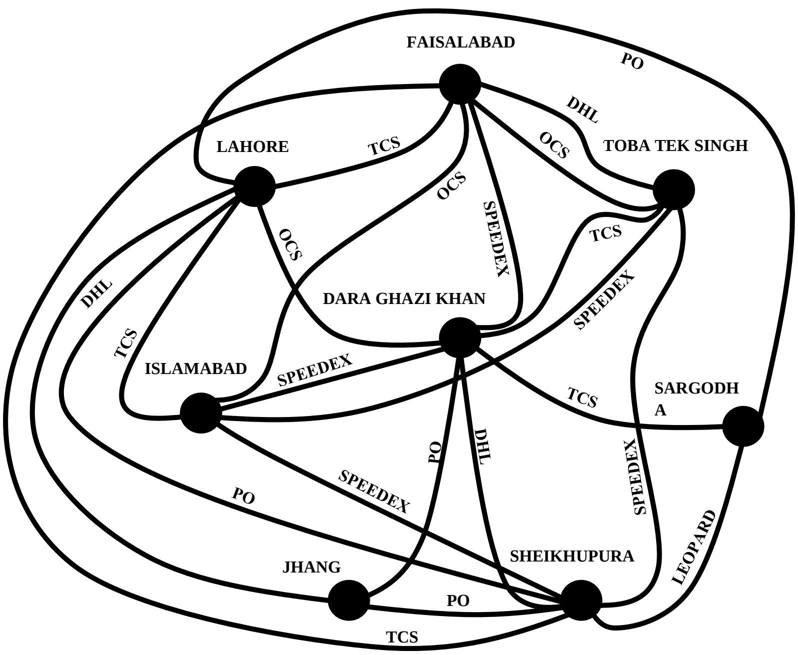

Figure 21: $\mathcal{FSGS}$ presenting best courier service according to parameter ρ_2

In the $\mathcal{FSGS}$ shown in Figure 20, each edge represents the best courier service between the linked cities. For example, the best courier service from Dera Ghazi Khan to Sheikhupura, Toba Tek Singh to Faisalabad, Sargodha to Jhang, and Islamabad to Sheikhupura is **TCS** according to parameter ρ_1. Similarly, the best courier service from Sheikhupura to Faisalabad, Dera Ghazi Khan to Sargodha, Lahore to Islamabad, Faisalabad to Lahore, and Toba Tek Singh to Dera Ghazi Khan is also **TCS** according to parameter ρ_2.

It is keenly observed that the vertices Dera Ghazi Khan, Sheikhupura, and Toba Tek Singh have the highest membership values of relation **TCS**, indicating that Dera Ghazi Khan, Sheikhupura, and Toba Tek Singh have TCS as the best courier service according to parameter ρ_1. Similarly, the vertices Lahore and Sargodha have the highest membership value of relation **TCS**, indicating that Lahore and Sargodha have TCS as the best courier service according to parameter ρ_2.

In the same way, we can find the best courier services in other cities. We can easily deduce that **TCS** is the best courier service at this time in this territory of eight cities.

Algorithm: Best Courier Service Selection Using $\mathcal{FSGS}$

1. Input the set of vertices (cities) such that

$$W = \{w_1, w_2, \ldots, w_n\}$$

and the fuzzy soft set $\ddot{\sigma}_\rho$, according to each parameter $\rho_i \in \ddot{P}$ defined on set W.

2. Input the fuzzy soft set $\ddot{\mu}_\rho$ according to each parameter $\rho_i \in \ddot{P}$ of courier agencies that provide services from one city to other cities and compute its membership values by applying

$$\ddot{\mu}_\rho(w_1, w_2) \leq \ddot{\sigma}_\rho(w_1) \wedge \ddot{\sigma}_\rho(w_2), \quad \forall\, \rho_i \in \ddot{P}.$$

3. Repeat Step 2 for the set of vertices W.

4. Introduce $\{R_i : 1 \leq i \leq n\}$ where each R_i for the set of vertices W are mutually disjoint, irreflexive, and symmetric according to each parameter $\rho_i \in \ddot{P}$, and nominate the relations as courier agencies.

5. Elect a courier agency as the **best courier agency** for delivering packages from one city to another, having membership values higher than those of all other courier agencies.

6. Build a fuzzy soft graph structure on the set of vertices W with relations $\{R_i : 1 \leq i \leq n\}$ according to each parameter $\rho_i \in \ddot{P}$, selecting those pairs having the same best courier agency as elements of the same relation.

7. Write the membership values of the relations $\{R_i : 1 \leq i \leq n\}$ with their membership values. Then
$$\{\ddot{\mu}_i : 1 \leq i \leq n\}$$
are fuzzy soft sets on the relations $\{R_i : 1 \leq i \leq n\}$, according to each parameter $\rho_i \in \ddot{P}$, respectively. Hence
$$\{\ddot{\sigma}_\rho, \ddot{\mu}^1_\rho, \ddot{\mu}^2_\rho, \ddot{\mu}^3_\rho, \ldots, \ddot{\mu}^n_\rho\}$$
is a fuzzy soft graph structure.

8. Draw the fuzzy soft graph structure according to each parameter $\rho_i \in \ddot{P}$ where every edge identifies the best courier agency for delivering packages from one city to another.

6. Comparison of Fuzzy Soft Graph and Fuzzy Soft Graph Structure

In mathematical modeling, fuzzy graphs, which are a combination of fuzzy sets and graphs, have significant value in representing relationships between various topics of interest. As science has evolved and encountered complex problems with multiple relationships between variables, the need for fuzzy graphs with multiple relationships has emerged.

Fuzzy soft sets are a combination of two soft computing models: fuzzy sets and soft sets. The introduction of a fuzzy soft graph structure provides better flexibility than traditional graphs in solving complex problems. By combining a fuzzy graph structure with fuzzy soft sets, a fuzzy soft graph structure was introduced, enhancing decision-making capabilities for complicated problems involving vagueness and uncertainties.

Moreover, we will elaborate on these two concepts: fuzzy soft graphs and fuzzy soft graph structures with an example. A fuzzy soft graph is given in Table 1 and presented in Figure 1, as well as a fuzzy soft graph structure given in Table 11 and presented in Figure 11.

7. Conclusion

The concepts of graph theory are useful for study and have many applications in various areas. However, some concepts of graph theory may be uncertain. To handle such uncertainties, the methods of fuzzy sets or fuzzy soft graph structures are more convenient than fuzzy sets. Therefore, we apply the idea of fuzzy soft graphs to graph structures. We have investigated some properties of fuzzy soft graph structures.

7.1. Future Work

We are extending our work to the following directions: bipolar fuzzy soft graph structures, intuitionistic fuzzy soft graph structures, rough fuzzy soft graph structures, and roughness in fuzzy soft graph structures.

Credit Authorship Contribution Statement

Aliya Fahmi: Software, Project administration, Data curation, Conceptualization.
Umair Amin: Project administration, Data curation, Conceptualization.

Declaration of Competing Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Compliance with Ethical Standards

This work has been carried out in compliance with ethical research standards.

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Umair Amin,
Department of Mathematics,
The University of Faisalabad,
Faisalabad, Pakistan.
E-mail address: umairmath90@gmail.com

and

Aliya Fahmi,
Department of Mathematics,
The University of Faisalabad,
Faisalabad, Pakistan.
E-mail address: fahmi.aliyai@yahoo.com