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A Numerical Simulation of Fractional Order Influenza Disease Model

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ABSTRACT: In this work, our purpose is to present the fractional model of the transmission dynamics of influenza virus in the aspect of drug resistance. Here, we construct a numerical algorithm based on the homotopy analysis transformation method to achieve a fractional form solution of the influenza virus transmission dynamics model. The fixed point theory is employed to investigate whether a solution exists, and the uniqueness of the solution of the influenza virus model is also analyzed. The numerical simulation of an influenza disease model is performed to observe the effect of different treatment parameters on the progression of the disease. The findings for the fractional influenza model indicate that the suggested method is quite precise and efficient.

Key Words: Influenza, homotopy analysis, Laplace transform, convergence, uniqueness.

Contents

1. Introduction

Influenza is a contagious respiratory disease that can cause illness in individuals of all ages, results in repeated infections throughout life and is responsible for annual global epidemics of varying severity. Influenza infection spreads throughout the world as a seasonal epidemic every year. The influenza virus infects the nose, throat, and lungs and it can cause mild to severe illness and sometimes lead to hospitalization and death. Getting a flu vaccine every year is the best way to prevent from influenza. Seasonal influenza is characterized by the sudden onset of fever, cough, headache, muscle and joint pain, feeling unwell, sore throat and a runny nose. The cough can be severe and can last two or more weeks. Seasonal influenza is an acute respiratory infection caused by influenza viruses that spread in all parts of the world. The best way to reduce the chance of becoming infected from influenza virus is vaccination.

The World Health Organization [1] estimates that influenza affects 3 to 5 million individuals each year, causing 250,000 to 0.5 million deaths . There are 4 types of influenza viruses : type A, B, C and D. Influenza A and B virus can cause seasonal epidemics of disease. Influenza A virus is further classified into subtypes according to the combinations of the Hemagglutinin (HA) and the Neuraminidase (NA), the proteins on the surface of the virus. Currently sub-type H1N1 and H3N2 infecting people. The exchange of gene segments between influenza A virus occurs when virus can-infect the same cell, due to the segmented nature of the virus genome. Influenza B virus can be broken down into lineages. Influenza C virus is detected less frequently and usually causes mild infections. Influenza D virus primarily affect cattle and are not known to infect or cause illness in people [2,25].

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This disease can be avoided by receiving a vaccination annually. People at greater risk of severe disease or complications when infected are: pregnant women, children under 59 months, individuals with chronic medical conditions (such as chronic cardiac, pulmonary, renal, metabolic, liver or hematologic diseases) and individuals with immunosuppressive conditions (such as HIV, receiving chemotherapy or steroids, or malignancy). For diagnosis of influenza virus infection several tests are available that have been performed in the clinical laboratories [6,28].

In terms of transmission, seasonal influenza spreads easily, with rapid transmission in crowded areas including schools and nursing homes. The typical ways that influenza viruses spread among individuals include the release of large droplets and the coughs of an infected person [7].

Recently, numerous researchers have proposed mathematical models to describe the transmission of influenza. Hussain et al. [7] analyzed the transmission dynamics of swine influenza model with deterministic approach. Islam et al. [8] suggested a SEIR mathematical model for the dynamics of influenza A virus transmission related to data of Bangladesh. Srivastava et al. [3,5,13,14,15,16,24] suggested a fractional form of diabetes model with its resulting complications. Khondaker [12] determine a mathematical model of influenza with preventive measures and optimum control analysis which reveals that the interventions reduce the number of exposed and infected individuals.

Alazman et al. [22] analyzed the infection and diffusion coefficient in an SIR Model by Using Generalized Fractional Derivative. Kumar et al. [11] examined a fractional SIRS-SI model for malaria transmission and evaluated how the order of the Caputo-Fabrizio fractional derivative impacts various segments of the human population and mosquito population. Dadhich et al. [18] analyzed the fractional form of ordinary differential equations for diabetes mellitus using Fractional Homotopy Analysis Transform Method. Althubiti et al. [20] suggested a SIAQR model of fractional form with time dependent infection rate. Sharma et al. [21] described the fractional model for omicron variant of corona virus using fractional derivatives.

Alazman et al. [29] analyzed a fractional model of rabies and applied the Laplace transform approach to the Atangana–Baleanu fractional operator iteratively. Areshi et al. [30] investigated the disparities in the Modified Bergman's glucose–insulin model between Yang–Abdel–Cattani derivative and Caputo–Fabrizio derivative.

Mishra et al. [17,19,23,26,31] examined the general fractional derivative of the fractional order in the context of an incomplete treatment of tuberculosis (TB). Alqahtani et al. [32] suggested a mathematical model for Streptococcus Suis infection and studied the given mathematical model of the illness using the Riemann Liouville's fractional derivative and Laplace transformation. Mishra et al. [34] suggested a Mathematical model of growth of tumor cells with chemotherapeutic cells.They established mathematical and graphical results using Yang–Abdel–Cattani fractional derivative operator.

Agarwal et al. [36] analyzes groundwater flow in finite fractured constrained flow using the dual-porosity model. The movement of water within geological formations known as aquifers is described by the Caputo Fabrizio fractional derivative with non-singular kernel because of their memory effect. Kumar et al. [37] fractionalize the Boussinesq equation in time variable, which describes the groundwater flow problem, using fractional derivatives in the sense of Caputo and Caputo–Fabrizio. Agarwal et al. [35] Used the Caputo-Fabrizio fractional derivative, the longitudinal dispersion phenomena in the flow of two miscible fluids across porous media was examined. The fixed point theorem was used to demonstrate the solution's existence and uniqueness.

2. Mathematical Preliminaries

Caputo's definition of fractional derivative of a function $f : [a, b] \to \mathbb{R}$ of order α and $t > a$ is given by

$$ {}_a^C D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha+1-n}}, & (n-1) < \alpha < n, \\ \frac{d^n f(t)}{dt^n}, & \alpha = n. \end{cases} $$

here $\Gamma(.)$ is the Gamma function. This operator was introduced by Caputo [33, Eq. 5, p. 530].
Let $f \in H^{'}(a, b),\ b > a$ then, definition of Caputo fractional derivative, given by Caputo and Fabrizio [9]

as

$$ {}_{0}^{CF}D_t^{\alpha}\psi(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t \psi^{'}(\tau)\exp\left(-\eta\frac{t-\tau}{1-\alpha}\right)d\tau, \tag{2.1}$$

where $M(\alpha)$ satisfies the normalized condition such that $M(0) = M(1) = 1$.
${}_{0}^{CF}D_t^{\alpha}$ is Caputo-Fabrizio fractional operator of order $\alpha \in (0, 1]$. According to the definition Caputo derivative have singularity at $t = \tau$ but Caputo-Fabrizio fractional derivative have not.
The Caputo-Fabrizio fractional derivative's integral operator of fractional order is defined as [10]

$$ {}_{0}^{CF}I_t^{\eta}\psi(t) = \frac{2(1-\eta)}{(2-\eta)M(\eta)}\psi(t) + \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t \psi(\zeta)d\zeta,\ t \geq 0 \tag{2.2}$$

Laplace transform of the Caputo Fabrizio fractional derivative is given by Caputo and Fabrizio [9] as

$$ L\left[{}_{a}^{CF}D_t^{(\alpha+n)}f(t)\right] = \frac{p^{n+1}\bar{f}(p) - p^n f(0) - p^{n-1}f^{'}(0)\ldots - f^{(n)}(0)}{p+\alpha(1-p)} \tag{2.3}$$

3. Fractional Model For Influenza Virus

In this manuscript, we fractionalize the mathematical model of the dynamics of influenza virus transmission in the aspect of drug resistance, which was previously given by Kanyiri et al. [4]. The mathematical model of influenza virus is given below :

$$\begin{aligned}
\frac{dS}{dt} =& \Lambda + \Theta R - (\gamma + \mu + \mu_1 + \mu_2)S(t),\\
\frac{dV}{dt} =& \gamma S(t) - ((1-\beta)\mu_1 + (1-\beta)\mu_2 + \mu)V(t),\\
\frac{dI_w}{dt} =& \mu_1 S(t) + (1-\beta)\mu_1 V(t) - (b + \mu + a_w + \delta_w)I_w(t),\\
\frac{dI_R}{dt} =& \mu_2 S(t) + (1-\beta)\mu_2 V(t) + bI_w(t) - (\mu + \delta_r + a_r)I_R(t),\\
\frac{dR}{dt} =& \delta_w I_w(t) + \delta_r I_R(t) - (\Theta + \mu)R(t).
\end{aligned} \tag{3.1}$$

Where $\mu_1 = \sigma_w I_w$ and $\mu_2 = \sigma_r(1+b^2)I_R$. The model subdivides the total population into five compartments: Susceptible (S), Vaccinated (V), Infected with wild-type strain (I_w), Infected with Resistant strain (I_R), and Recovered (R) population. The explanation of the different model parameters is given as follows

σ_w	Transmission rate of wild-type strain
σ_r	Transmission rate of resistant strain
β	Vaccine efficacy
γ	Vaccination rate
b	Rate of developing drug resistance
δ_w	Recovery rate for individuals in I_w class
δ_r	Recovery rate for individuals in I_R class
Θ	Rate of losing immunity
a_w	Death rate due to infection with wild-type strain
$\frac{1}{\mu}$	Average human lifespan
Λ	Recruitment rate
a_r	Death rate due to infection with resistant strain

In this section, we represent the fractional form of mathematical model of influenza virus (3.1) which is given as

$$\begin{aligned} {}_0^{CF}D_t^{\eta}S(t) &= \Lambda + \Theta R - (\gamma + \mu + \mu_1 + \mu_2)S(t), \\ {}_0^{CF}D_t^{\eta}V(t) &= \gamma S(t) - ((1-\beta)\mu_1 + (1-\beta)\mu_2 + \mu)V(t), \\ {}_0^{CF}D_t^{\eta}I_w(t) &= \mu_1 S(t) + (1-\beta)\mu_1 V(t) - (b + \mu + a_w + \delta_w)I_w(t), \\ {}_0^{CF}D_t^{\eta}I_R(t) &= \mu_2 S(t) + (1-\beta)\mu_2 V(t) + bI_w(t) - (\mu + \delta_r + a_r)I_R(t), \\ {}_0^{CF}D_t^{\eta}R(t) &= \delta_w I_w(t) + \delta_r I_R(t) - (\Theta + \mu)R(t), \end{aligned} \tag{3.2}$$

with the initial conditions $S(0) > 0,\ V(0) \geq 0,\ I_w(0) \geq 0,\ I_R(0) \geq 0,\ R(0) \geq 0$. Let B be a Banach space of continuous real functions defined on the interval I with corresponding supremum norm.

$$\|S,\ V,\ I_w,\ I_R,\ R\| = \|S\| + \|V\| + \|I_w\| + \|I_R\| + \|R\|. \tag{3.3}$$

In equation (3.3), we have

$$\begin{aligned} \|S\| &= \sup(|S(t)| : t\epsilon I), \\ \|V\| &= \sup(|V(t)| : t\epsilon I), \\ \|I_w\| &= \sup(|I_w(t)| : t\epsilon I), \\ \|I_R\| &= \sup(|I_R(t)| : t\epsilon I), \\ \|R\| &= \sup(|R(t)| : t\epsilon I), \end{aligned} \tag{3.4}$$

specifically $B = E(I) * E(I) * E(I) * E(I) * E(I)$, where $E(I)$ denotes the Banach space of continuous real valued functions on I and the associated supremum norm.

4. Existence and Uniqueness

In this section, we will check existence and uniqueness of the given model (3.2) by applying the fractional integral operator (2.2) and obtain

$$\begin{aligned} S(t) - S(0) &= {}_0^{CF}I_t^{\eta}[\Lambda + \Theta R - (\gamma + \mu + \mu_1 + \mu_2)S(t)], \\ V(t) - V(0) &= {}_0^{CF}I_t^{\eta}[\gamma S(t) - ((1-\beta)\mu_1 + (1-\beta)\mu_2 + \mu)V(t)], \\ I_w(t) - I_w(0) &= {}_0^{CF}I_t^{\eta}[\mu_1 S(t) + (1-\beta)\mu_1 V(t) - (b + \mu + a_w + \delta_w)I_w(t)], \\ I_R(t) - I_R(0) &= {}_0^{CF}I_t^{\eta}[\mu_2 S(t) + (1-\beta)\mu_2 V(t) + bI_w(t) - (\mu + \delta_r + a_r)I_R(t)], \\ R(t) - R(0) &= {}_0^{CF}I_t^{\eta}[\delta_w I_w(t) + \delta_r I_R(t) - (\Theta + \mu)R(t)]. \end{aligned} \tag{4.1}$$

Further, using the definition (2.2), we obtain

$$\begin{aligned} S(t) - S(0) = &\frac{2(1-\eta)}{(2-\eta)M(\eta)}[\Lambda + \Theta R - (\gamma + \mu + \mu_1 + \mu_2)S(t)] \\ &+ \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t [\Lambda + \Theta R(\zeta) - (\gamma + \mu + \mu_1 + \mu_2)S(\zeta)]d\zeta, \end{aligned}$$

$$\begin{aligned} V(t) - V(0) = &\frac{2(1-\eta)}{(2-\eta)M(\eta)}[\gamma S(t) - ((1-\beta)\mu_1 + (1-\beta)\mu_2 + \mu)V(t)] \\ &+ \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t [\gamma S(\zeta) - ((1-\beta)\mu_1 + (1-\beta)\mu_2 + \mu)V(\zeta)]d\zeta, \end{aligned}$$

$$\begin{aligned} I_w(t) - I_w(0) = &\frac{2(1-\eta)}{(2-\eta)M(\eta)}[\mu_1 S(t) + (1-\beta)\mu_1 V(t) - (b + \mu + a_w + \delta_w)I_w(t)] \\ &+ \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t [\mu_1 S(\zeta) + (1-\beta)\mu_1 V(\zeta) - (b + \mu + a_w + \delta_w)I_w(\zeta)]d\zeta, \end{aligned} \tag{4.2}$$

$$I_R(t)-I_R(0)=\frac{2(1-\eta)}{(2-\eta)M(\eta)}[\mu_2 S(t)+(1-\beta)\mu_2 V(t)+bI_w(t)-(\mu+\delta_r+a_r)I_R(t)]$$
$$+\frac{2\eta}{(2-\eta)M(\eta)}\int_0^t[\mu_2 S(\zeta)+(1-\beta)\mu_2 V(\zeta)+bI_w(\zeta)-(\mu+\delta_r+a_r)I_R(\zeta)]d\zeta$$

$$R(t)-R(0)=\frac{2(1-\eta)}{(2-\eta)M(\eta)}[\delta_w I_w(t)+\delta_r I_R(t)-(\Theta+\mu)R(t)]$$
$$+\frac{2\eta}{(2-\eta)M(\eta)}\int_0^t[\delta_w I_w(\zeta)+\delta_r I_R(\zeta)-(\Theta+\mu)R(\zeta)]d\zeta.$$

and the kernels are defined as

$$\begin{aligned}
\omega_1(t,S)=&\Lambda+\Theta R-(\gamma+\mu+\mu_1+\mu_2)S(t),\\
\omega_2(t,V)=&\gamma S(t)-((1-\beta)\mu_1+(1-\beta)\mu_2+\mu)V(t),\\
\omega_3(t,I_w)=&\mu_1 S(t)+(1-\beta)\mu_1 V(t)-(b+\mu+a_w+\delta_w)I_w(t),\\
\omega_4(t,I_R)=&\mu_2 S(t)+(1-\beta)\mu_2 V(t)+bI_w(t)-(\mu+\delta_r+a_r)I_R(t),\\
\omega_5(t,R)=&\delta_w I_w(t)+\delta_r I_R(t)-(\Theta+\mu)R(t),
\end{aligned}\tag{4.3}$$

Theorem 4.1 *The kernels* $\omega_1(t,S),\ \omega_2(t,V),\ \omega_3(t,I_w),\ \omega_4(t,I_R),\ \omega_5(t,R)$ *fulfill the Lipschitz condition and contraction if* $0\le\gamma+\mu+\mu_1+\mu_2<1$ *or* $0\le\gamma+\mu+\sigma_w k_3+\sigma_r(1+b^2)k_4<1$.

Proof: We start with $\omega_1(t,S)$. For two functions S and S_1

$$\|\omega_1(t,S)-\omega_1(t,S_1)\|=\|(\gamma+\mu+\mu_1+\mu_2)(S(t)-S_1(t))\|\tag{4.4}$$

Further, applying properties of norms to equation (4.4), we obtain

$$\begin{aligned}
\|\omega_1(t,S)-\omega_1(t,S_1)\|\le&\|(\gamma+\mu+\mu_1+\mu_2)(S(t)-S_1(t))\|\\
\le&\|(\gamma+\mu)(S(t)-S_1(t))\|+\|(\sigma_w I_w+\sigma_r(1+b^2)I_R)(S(t)-S_1(t))\|\\
\le&(\gamma+\mu)\|(S(t)-S_1(t))\|+(\sigma_w k_3+\sigma_r(1+b^2)k_4)\|(S(t)-S_1(t))\|\\
\le&(\gamma+\mu+\sigma_w k_3+\sigma_r(1+b^2)k_4)\|(S(t)-S_1(t))\|\\
\le&m_1\|(S(t)-S_1(t))\|
\end{aligned}\tag{4.5}$$

Similarly, $\omega_2(t,V),\ \omega_3(t,I_w),\ \omega_4(t,I_R),\ \omega_5(t,R)$ satisfy the Lipschitz conditions,

$$\begin{aligned}
\|\omega_2(t,V)-\omega_2(t,V_1)\|\le&m_2\|(V(t)-V_1(t))\|,\\
\|\omega_3(t,I_w)-\omega_3(t,I_{w_1})\|\le&m_3\|(I_w(t)-I_{w_1}(t)\|,\\
\|\omega_4(t,I_R)-\omega_4(t,I_{R_1})\|\le&m_4\|(I_R(t)-I_{R_1}(t)\|,\\
\|\omega_5(t,R)-\omega_5(t,R_1)\|\le&m_5\|(R(t)-R_1(t)\|.
\end{aligned}\tag{4.6}$$

Using the kernel notation above (4.3), equation (4.2) reduces to the system

$$\begin{aligned}
S(t)=&S(0)+\frac{2(1-\eta)}{(2-\eta)M(\eta)}\omega_1(t,S)+\frac{2\eta}{(2-\eta)M(\eta)}\int_0^t\omega_1(\zeta,S)d\zeta,\\
V(t)=&V(0)+\frac{2(1-\eta)}{(2-\eta)M(\eta)}\omega_2(t,V)+\frac{2\eta}{(2-\eta)M(\eta)}\int_0^t\omega_2(\zeta,V)d\zeta,\\
I_w(t)=&I_w(0)+\frac{2(1-\eta)}{(2-\eta)M(\eta)}\omega_3(t,I_w)+\frac{2\eta}{(2-\eta)M(\eta)}\int_0^t\omega_3(\zeta,I_w)d\zeta,\\
I_R(t)=&I_R(0)+\frac{2(1-\eta)}{(2-\eta)M(\eta)}\omega_4(t,I_R)+\frac{2\eta}{(2-\eta)M(\eta)}\int_0^t\omega_4(\zeta,I_R)d\zeta,\\
R(t)=&R(0)+\frac{2(1-\eta)}{(2-\eta)M(\eta)}\omega_5(t,R)+\frac{2\eta}{(2-\eta)M(\eta)}\int_0^t\omega_5(\zeta,R)d\zeta.
\end{aligned}\tag{4.7}$$

Further, we construct following recursive formulas

$$\begin{aligned} S_n(t) =& \frac{2(1-\eta)}{(2-\eta)M(\eta)}\omega_1(t, S_{n-1}) + \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t \omega_1(\zeta, S_{n-1})d\zeta, \\ V_n(t) =& \frac{2(1-\eta)}{(2-\eta)M(\eta)}\omega_2(t, V_{n-1}) + \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t \omega_2(\zeta, V_{n-1})d\zeta, \\ I_{w_n}(t) =& \frac{2(1-\eta)}{(2-\eta)M(\eta)}\omega_3(t, I_{w_{n-1}}) + \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t \omega_3(\zeta, I_{w_{n-1}})d\zeta, \\ I_{R_n}(t) =& \frac{2(1-\eta)}{(2-\eta)M(\eta)}\omega_4(t, I_{R_{n-1}}) + \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t \omega_4(\zeta, I_{R_{n-1}})d\zeta, \\ R_n(t) =& \frac{2(1-\eta)}{(2-\eta)M(\eta)}\omega_5(t, R_{n-1}) + \frac{2\eta}{(2-\eta)M(\eta)}\int_0^t \omega_5(\zeta, R_{n-1})d\zeta, \end{aligned} \tag{4.8}$$

along with the initial conditions, $S(0) > 0,\ V(0) \geq 0,\ I_w(0) \geq 0,\ I_R(0) \geq 0,\ R(0) \geq 0.$ We express the difference between the succession terms as

$$\begin{aligned} \varpi_{1n}(t) &= S_n(t) - S_{n-1}(t), \\ &= \frac{2(1-\eta)}{(2-\eta)M(\eta)}[\omega_1(t, S_{n-1}) - \omega_1(t, S_{n-2})] \\ &\quad + \frac{2\eta}{(2-\eta)M(\eta)}\textstyle\int_0^t[\omega_1(\zeta, S_{n-1}) - \omega_1(\zeta, S_{n-2})]d\zeta, \\ \varpi_{2n}(t) &= V_n(t) - V_{n-1}(t), \\ &= \frac{2(1-\eta)}{(2-\eta)M(\eta)}[\omega_2(t, V_{n-1}) - \omega_2(t, V_{n-2})] \\ &\quad + \frac{2\eta}{(2-\eta)M(\eta)}\textstyle\int_0^t[\omega_2(\zeta, V_{n-1}) - \omega_2(\zeta, V_{n-2})]d\zeta, \\ \varpi_{3n}(t) &= I_{w_n}(t) - I_{w_{n-1}}(t), \\ &= \frac{2(1-\eta)}{(2-\eta)M(\eta)}[\omega_3(t, I_{w_{n-1}}) - \omega_3(t, I_{w_{n-2}})] \\ &\quad + \frac{2\eta}{(2-\eta)M(\eta)}\textstyle\int_0^t[\omega_3(\zeta, I_{w_{n-1}}) - \omega_3(\zeta, I_{w_{n-2}})]d\zeta, \\ \varpi_{4n}(t) &= I_{R_n}(t) - I_{R_{n-1}}(t), \\ &= \frac{2(1-\eta)}{(2-\eta)M(\eta)}[\omega_4(t, I_{R_{n-1}}) - \omega_4(t, I_{R_{n-2}})] \\ &\quad + \frac{2\eta}{(2-\eta)M(\eta)}\textstyle\int_0^t[\omega_4(\zeta, I_{R_{n-1}}) - \omega_4(\zeta, I_{R_{n-2}})]d\zeta, \\ \varpi_{5n}(t) &= R_n(t) - R_{n-1}(t) \\ &= \frac{2(1-\eta)}{(2-\eta)M(\eta)}[\omega_5(t, R_{n-1}) - \omega_5(t, R_{n-2})] \\ &\quad + \frac{2\eta}{(2-\eta)M(\eta)}\textstyle\int_0^t[\omega_5(\zeta, R_{n-1}) - \omega_5(\zeta, R_{n-2})]d\zeta, \end{aligned} \tag{4.9}$$

it is worth observing that

$$\begin{aligned} S_n(t) =& \Sigma_{i=0}^n \varpi_{1i}(t), \\ V_n(t) =& \Sigma_{i=0}^n \varpi_{2i}(t), \\ I_{w_n}(t) =& \Sigma_{i=0}^n \varpi_{3i}(t), \\ I_{R_n}(t) =& \Sigma_{i=0}^n \varpi_{4i}(t), \\ R_n(t) =& \Sigma_{i=0}^n \varpi_{5i}(t). \end{aligned} \tag{4.10}$$

Further, we have

$$\begin{aligned}\|\varpi_{1n}(t)\| =&\|S_n(t)-S_{n-1}(t)\|,\\ =&\|\frac{2(1-\eta)}{(2-\eta)M(\eta)}[\omega_1(t,S_{n-1})-\omega_1(t,S_{n-2})]\\ &+\frac{2\eta}{(2-\eta)M(\eta)}\int_0^t[\omega_1(\zeta,S_{n-1})-\omega_1(\zeta,S_{n-2})]d\zeta\|.\end{aligned} \tag{4.11}$$

Applying triangular inequality to equation (4.11), we get

$$\begin{aligned}\|S_n(t)-S_{n-1}(t)\| \leq&\frac{2(1-\eta)}{(2-\eta)M(\eta)}\|[\omega_1(t,S_{n-1})-\omega_1(t,S_{n-2})]\|\\ &+\frac{2\eta}{(2-\eta)M(\eta)}\|\int_0^t[\omega_1(\zeta,S_{n-1})-\omega_1(\zeta,S_{n-2})]d\zeta\|.\end{aligned} \tag{4.12}$$

Since the kernels satisfy the Lipschitz condition given in equation (4.5,4.6), hence equation (4.12) gives

$$\begin{aligned}\|S_n(t)-S_{n-1}(t)\| \leqslant& \frac{2(1-\eta)}{(2-\eta)M(\eta)}m_1\|S_{n-1}-S_{n-2}\|\\ &+\frac{2\eta}{(2-\eta)M(\eta)}m_1\int_0^t\|S_{n-1}-S_{n-2}\|d\zeta.\end{aligned} \tag{4.13}$$

Consequently, we arrive at the subsequent results

$$\begin{aligned}\|\varpi_{1n}(t)\| \leq&\frac{2(1-\eta)}{(2-\eta)M(\eta)}m_1\|\varpi_{1n-1}(t)\|+\frac{2\eta}{(2-\eta)M(\eta)}m_1\int_0^t\|\varpi_{1n-1}(t)\|d\zeta,\\ \|\varpi_{2n}(t)\| \leq&\frac{2(1-\eta)}{(2-\eta)M(\eta)}m_2\|\varpi_{2n-1}(t)\|+\frac{2\eta}{(2-\eta)M(\eta)}m_2\int_0^t\|\varpi_{2n-1}(t)\|d\zeta,\\ \|\varpi_{3n}(t)\| \leq&\frac{2(1-\eta)}{(2-\eta)M(\eta)}m_3\|\varpi_{3n-1}(t)\|+\frac{2\eta}{(2-\eta)M(\eta)}m_3\int_0^t\|\varpi_{3n-1}(t)\|d\zeta,\\ \|\varpi_{4n}(t)\| \leq&\frac{2(1-\eta)}{(2-\eta)M(\eta)}m_4\|\varpi_{4n-1}(t)\|+\frac{2\eta}{(2-\eta)M(\eta)}m_4\int_0^t\|\varpi_{4n-1}(t)\|d\zeta,\\ \|\varpi_{5n}(t)\| \leq&\frac{2(1-\eta)}{(2-\eta)M(\eta)}m_5\|\varpi_{5n-1}(t)\|+\frac{2\eta}{(2-\eta)M(\eta)}m_5\int_0^t\|\varpi_{5n-1}(t)\|d\zeta.\end{aligned} \tag{4.14}$$

Taking equation (4.14) into account, for the model under consideration, we derive the existence of the solution.. □

Theorem 4.2 *The SVI_wI_RR model involving the Caputo-Fabrizio fractional operator (3.2) has a solution if there exists t_0 such that*

$$\frac{2(1-\eta)}{(2-\eta)M(\eta)}m_1+\frac{2\eta}{(2-\eta)M(\eta)}m_1t_0<1. \tag{4.15}$$

Proof: As we know the functions $S(t),\ V(t),\ I_w(t),\ I_R(t),\ R(t)$ are bounded. Using the results presented in (4.14) and using the recursive algorithm, we get

$$\begin{aligned}
\|\varpi_{1n}(t)\| \leq& \|S_n(0)\| \left[\frac{2(1-\eta)}{(2-\eta)M(\eta)}m_1 + \frac{2\eta}{(2-\eta)M(\eta)}m_1 t\right]^n,\\
\|\varpi_{2n}(t)\| \leq& \|V_n(0)\| \left[\frac{2(1-\eta)}{(2-\eta)M(\eta)}m_2 + \frac{2\eta}{(2-\eta)M(\eta)}m_2 t\right]^n,\\
\|\varpi_{3n}(t)\| \leq& \|I_{w_n}(0)\| \left[\frac{2(1-\eta)}{(2-\eta)M(\eta)}m_3 + \frac{2\eta}{(2-\eta)M(\eta)}m_3 t\right]^n,\\
\|\varpi_{4n}(t)\| \leq& \|I_{R_n}(0)\| \left[\frac{2(1-\eta)}{(2-\eta)M(\eta)}m_4 + \frac{2\eta}{(2-\eta)M(\eta)}m_4 t\right]^n,\\
\|\varpi_{5n}(t)\| \leq& \|R_n(0)\| \left[\frac{2(1-\eta)}{(2-\eta)M(\eta)}m_5 + \frac{2\eta}{(2-\eta)M(\eta)}m_5 t\right]^n.
\end{aligned} \tag{4.16}$$

Therefore, a solution for the model considered exists and is continuous. Now, to show that equation (4.7) is a solution to model (3.2), we do the following:

$$\begin{aligned}
S(t) - S(0) &= S_n(t) - A_n(t),\\
V(t) - V(0) &= V_n(t) - B_n(t),\\
I_w(t) - I_w(0) &= I_{w_n}(t) - C_n(t),\\
I_R(t) - I_R(0) &= I_{R_n}(t) - D_n(t),\\
R(t) - R(0) &= R_n(t) - E_n(t).
\end{aligned} \tag{4.17}$$

Here, we have

$$\begin{aligned}
\|A_n(t)\| =& \left\| \frac{2(1-\eta)}{(2-\eta)M(\eta)} \left[\omega_1(t,S) - \omega_1(t,S_{n-1})\right] \right.\\
&\left. + \frac{2\eta}{(2-\eta)M(\eta)} \textstyle\int_0^t \left[\omega_1(\zeta,S) - \omega_1(\zeta,S_{n-1})\right] d\zeta \right\|,\\
\leqslant& \frac{2(1-\eta)}{(2-\eta)M(\eta)} \|\omega_1(t,S) - \omega_1(t,S_{n-1})\|\\
&+ \frac{2\eta}{(2-\eta)M(\eta)} \int_0^t \|\omega_1(\zeta,S) - \omega_1(\zeta,S_{n-1})\| d\zeta,\\
\leqslant& \frac{2(1-\eta)}{(2-\eta)M(\eta)} m_1 \|S - S_{n-1}\| + \frac{2\eta}{(2-\eta)M(\eta)} m_1 \|S - S_{n-1}\| t.
\end{aligned} \tag{4.18}$$

Using this process recursively, we get

$$\|A_n(t)\| \leq \left(\frac{2(1-\eta)}{(2-\eta)M(\eta)} + \frac{2\eta}{(2-\eta)M(\eta)} t\right)^{n+1} m_1^{n+1} k_1, \tag{4.19}$$

and

$$\lim_{n\to\infty} \|A_n(t)\| \to 0$$

Similarly, we obtain $\lim_{n\to\infty} \|B_n(t)\| \to 0$, $\lim_{n\to\infty} \|C_n(t)\| \to 0$, $\lim_{n\to\infty} \|D_n(t)\| \to 0$, $\lim_{n\to\infty} \|E_n(t)\| \to 0$. This completes the proof of the existence theorem.

Next, we prove the uniqueness of the solution of the fractional model (S, V, I_w, I_R, R) given in equation (3.2). We assume that the influenza model (S, V, I_w, I_R, R) has a different decision system $(S^*, V^*, I_w^*, I_R^*, R^*)$, then

$$\begin{aligned}
S(t) - S^*(t) =& \frac{2(1-\eta)}{(2-\eta)M(\eta)} \left[\omega_1(t,S) - \omega_1(t,S^*)\right]\\
&+ \frac{2\eta}{(2-\eta)M(\eta)} \textstyle\int_0^t \left[\omega_1(\zeta,S) - \omega_1(\zeta,S^*)\right] d\zeta,
\end{aligned} \tag{4.20}$$

taking the norms

$$\begin{aligned}\|S(t)-S^*(t)\| \leqslant & \frac{2(1-\eta)}{(2-\eta)M(\eta)}\|\omega_1(t,S)-\omega_1(t,S^*)\| \\ & +\frac{2\eta}{(2-\eta)M(\eta)}\int_0^t \|\omega_1(\zeta,S)-\omega_1(\zeta,S^*)\|d\zeta,\end{aligned} \tag{4.21}$$

applying the results given in (4.5, 4.6) on equation (4.21) we get

$$\begin{aligned}\|S(t)-S^*(t)\| \leqslant & \frac{2(1-\eta)}{(2-\eta)M(\eta)}m_1\|S(t)-S^*(t)\| \\ & +\frac{2\eta}{(2-\eta)M(\eta)}m_1 t\|S(t)-S^*(t)\|,\end{aligned} \tag{4.22}$$

$$\implies \|S(t)-S^*(t)\|\left(1-\frac{2(1-\eta)}{(2-\eta)M(\eta)}m_1-\frac{2\eta}{(2-\eta)M(\eta)}m_1 t\right) \leq 0. \tag{4.23}$$

□

Theorem 4.3 *The fractional* $(S,\ V,\ I_w,\ I_R,\ R)$ *model (3.2) has a unique solution if*

$$\left(1-\frac{2(1-\eta)}{(2-\eta)M(\eta)}m_1-\frac{2\eta}{(2-\eta)M(\eta)}m_1 t\right) > 0 \tag{4.24}$$

Proof: Using equation (4.23), we get

$$\|S(t)-S^*(t)\|\left(1-\frac{2(1-\eta)}{(2-\eta)M(\eta)}m_1-\frac{2\eta}{(2-\eta)M(\eta)}m_1 t\right) \leq 0 \tag{4.25}$$

Using properties of norms, equation (4.24) and (4.25) becomes

$$\|S(t)-S^*(t)\| = 0,$$

thus

$$S(t) = S^*(t), \tag{4.26}$$

similarly, we can prove that

$$\begin{aligned} V(t) =& V^*(t), \\ I_w(t) =& I_w^*(t), \\ I_R(t) =& I_R^*(t), \\ R(t) =& R^*(t). \end{aligned} \tag{4.27}$$

Therefore, we obtain a unique solution of fractional model. □

5. HATM for fractional $(S,\ V,\ I_w,\ I_R,\ R)$ Influenza model

In this section, we use HATM [27] to solve the fractional model $(S,\ V,\ I_w,\ I_R,\ R)$ (3.2). First, we apply the Laplace transform to the fractional influenza model (3.2), which gives us:

$$\begin{aligned}
\frac{s\mathcal{L}[S]-S(0)}{s+\eta(1-s)} =&\mathcal{L}[\Lambda+\Theta R-(\gamma+\mu+\mu_1+\mu_2)S(t)],\\
\frac{s\mathcal{L}[V]-V(0)}{s+\eta(1-s)} =&\mathcal{L}[\gamma S(t)-((1-\beta)\mu_1+(1-\beta)\mu_2+\mu)V(t)],\\
\frac{s\mathcal{L}[I_w]-I_w(0)}{s+\eta(1-s)} =&\mathcal{L}[\mu_1 S(t)+(1-\beta)\mu_1 V(t)-(b+\mu+a_w+\delta_w)I_w(t)],\\
\frac{s\mathcal{L}[I_R]-I_R(0)}{s+\eta(1-s)} =&\mathcal{L}[\mu_2 S(t)+(1-\beta)\mu_2 V(t)+bI_w(t)-(\mu+\delta_r+a_r)I_R(t)],\\
\frac{s\mathcal{L}[R]-R(0)}{s+\eta(1-s)} =&\mathcal{L}[\delta_w I_w(t)+\delta_r I_R(t)-(\Theta+\mu)R(t)].
\end{aligned} \tag{5.1}$$

On simplifying it gives,

$$\begin{aligned}
\mathcal{L}[S]-\frac{c_1}{s}-\frac{[s+\eta(1-s)]\Lambda}{s^2}-\frac{s+\eta(1-s)}{s}\mathcal{L}[\Theta R(t)-(\gamma+\mu+\mu_1+\mu_2)S(t)] =&0,\\
\mathcal{L}[V]-\frac{c_2}{s}-\frac{s+\eta(1-s)}{s}\mathcal{L}[\gamma S(t)-((1-\beta)\mu_1+(1-\beta)\mu_2+\mu)V(t)] =&0,\\
\mathcal{L}[I_w]-\frac{c_3}{s}-\frac{s+\eta(1-s)}{s}\mathcal{L}[\mu_1 S(t)+(1-\beta)\mu_1 V(t)-(b+\mu+a_w+\delta_w)I_w(t)] =&0,\\
\mathcal{L}[I_R]-\frac{c_4}{s}-\frac{s+\eta(1-s)}{s}\mathcal{L}[\mu_2 S(t)+(1-\beta)\mu_2 V(t)+bI_w(t)-(\mu+\delta_r+a_r)I_R(t)] =&0,\\
\mathcal{L}[R]-\frac{c_5}{s}-\frac{s+\eta(1-s)}{s}\mathcal{L}[\delta_w I_w(t)+\delta_r I_R(t)-(\Theta+\mu)R(t)] =&0.
\end{aligned} \tag{5.2}$$

The nonlinear operators are defined as follows:

$$\begin{aligned}
N_1[\Psi_1(t;Z)] =& \mathcal{L}[\Psi_1(t;Z)]-\frac{c_1}{s}-\frac{[s+\eta(1-s)]\Lambda}{s^2}\\
&-\frac{s+\eta(1-s)}{s}\mathcal{L}[\Theta\Psi_5(t;Z)-(\gamma+\mu+\mu_1+\mu_2)\Psi_1(t;Z)]=0,\\
N_2[\Psi_2(t;Z)] =& \mathcal{L}[\Psi_2(t;Z)]-\frac{c_2}{s}\\
&-\tfrac{s+\eta(1-s)}{s}\mathcal{L}[\gamma(\Psi_1(t;Z)-((1-\beta)\mu_1+(1-\beta)\mu_2+\mu)\Psi_2(t;Z)]=0,\\
N_3[\Psi_3(t;Z)] =& \mathcal{L}[\Psi_3(t;Z)]-\tfrac{c_3}{s}-\tfrac{s+\eta(1-s)}{s}\mathcal{L}[\mu_1\Psi_1(t;Z)\\
&+(1-\beta)\mu_1\Psi_2(t;Z)-(b+\mu+a_w+\delta_w)\Psi_3(t;Z)]=0,\\
N_4[\Psi_4(t;Z)] =& \mathcal{L}[\Psi_4(t;Z)]-\tfrac{c_4}{s}-\tfrac{s+\eta(1-s)}{s}\mathcal{L}[\mu_2\Psi_1(t;Z)\\
&+(1-\beta)\mu_2\Psi_2(t;Z)+b\Psi_3(t;Z)-(\mu+\delta_r+a_r)\Psi_4(t;Z)]=0,\\
N_5[\Psi_5(t;Z)] =& \mathcal{L}[\Psi_5(t;Z)]-\tfrac{c_5}{s}-\tfrac{s+\eta(1-s)}{s}\\
&\mathcal{L}[\delta_w\Psi_3(t;Z)+\delta_r\Psi_4(t;Z)-(\Theta+\mu)\Psi_5(t;Z)]=0.
\end{aligned}$$

Putting the values of (μ_1 and μ_2) from equation(3.1), we get

$$
\begin{aligned}
N_1[\Psi_1(t;Z)] &= \mathcal{L}[\Psi_1(t;Z)] - \frac{c_1}{s} - \frac{[s+\eta(1-s)]\Lambda}{s^2} - \frac{s+\eta(1-s)}{s}\mathcal{L}\left[\Theta\Psi_5(t;Z)\right.\\
&\quad -(\gamma+\mu)\Psi_1(t;Z) - \sigma_w\Psi_3(t;Z)\Psi_1(t;Z) - \sigma_r(1+b^2)\Psi_4(t;Z)\Psi_1(t;Z)] = 0,\\
N_2[\Psi_2(t;Z)] &= \mathcal{L}[\Psi_2(t;Z)] - \frac{c_2}{s} - \frac{s+\eta(1-s)}{s}\mathcal{L}[\gamma(\Psi_1(t;Z) - ((1-\beta)\sigma_w\\
&\quad \Psi_3(t;Z)\Psi_2(t;Z) + (1-\beta)\sigma_r(1+b^2)\Psi_4(t;Z)\Psi_2(t;Z) + \mu\Psi_2(t;Z))] = 0,\\
N_3[\Psi_3(t;Z)] &= \mathcal{L}[\Psi_3(t;Z)] - \frac{c_3}{s} - \frac{s+\eta(1-s)}{s}\mathcal{L}[\sigma_w\Psi_3(t;Z)\Psi_1(t;Z)\\
&\quad +(1-\beta)\sigma_w\Psi_3(t;Z)\Psi_2(t;Z) - (b+\mu+a_w+\delta_w)\Psi_3(t;Z)] = 0,\\
N_4[\Psi_4(t;Z)] &= \mathcal{L}[\Psi_4(t;Z)] - \frac{c_4}{s} - \frac{s+\eta(1-s)}{s}\mathcal{L}[\sigma_r(1+b^2)\Psi_4(t;Z)\Psi_1(t;Z)\\
&\quad +(1-\beta)\sigma_r(1+b^2)\Psi_4(t;Z)\Psi_2(t;Z) + b\Psi_3(t;Z) - (\mu+\delta_r+a_r)\Psi_4(t;Z)] = 0,\\
N_5[\Psi_5(t;Z)] &= \mathcal{L}[\Psi_5(t;Z)] - \frac{c_5}{s} - \frac{s+\eta(1-s)}{s}\\
&\quad \mathcal{L}[\delta_w\Psi_3(t;Z) + \delta_r\Psi_4(t;Z) - (\Theta+\mu)\Psi_5(t;Z) = 0,
\end{aligned}
\tag{5.3}
$$

and thus we have,

$$
\begin{aligned}
R_{1,j}(\vec{S}_{(j-1)}) &= \mathcal{L}[S_{(j-1)}] - \left(\frac{c_1}{s} + \frac{[s+\eta(1-s)]\Lambda}{s^2}\right)(1-\chi_j)\\
&\quad -\frac{s+\eta(1-s)}{s}\mathcal{L}\left[\Theta R_{(j-1)} - (\gamma+\mu)S_{(j-1)}\right.\\
&\quad \left.-\sigma_w(\Sigma_{d=0}^{(j-1)} I_{wd}S_{(j-1)-d}) - \sigma_r(1+b^2)\left(\Sigma_{d=0}^{(j-1)} I_{Rd}S_{(j-1)-d}\right)\right],\\
R_{2,j}(\vec{V}_{(j-1)}) &= \mathcal{L}[V_{(j-1)}] - (\frac{c_2}{s})(1-\chi_j) - \frac{s+\eta(1-s)}{s}\\
&\quad \mathcal{L}\left[\gamma S_{(j-1)} - ((1-\beta)\sigma_w\left(\Sigma_{d=0}^{(j-1)} I_{wd}V_{(j-1)-d}\right)\right.\\
&\quad \left.+(1-\beta)\sigma_r(1+b^2)\left(\Sigma_{d=0}^{(j-1)} I_{Rd}V_{(j-1)-d}\right) + \mu V_{(j-1)})\right],\\
R_{3,j}(\overrightarrow{I_w}_{(j-1)}) &= \mathcal{L}[I_{w_{(j-1)}}] - (\frac{c_3}{s})(1-\chi_j) - \frac{s+\eta(1-s)}{s}\\
&\quad \mathcal{L}\left[\sigma_w(\Sigma_{d=0}^{(j-1)} I_{wd}S_{(j-1)-d}) + (1-\beta)\sigma_w\left(\Sigma_{d=0}^{(j-1)} I_{wd}V_{(j-1)-d}\right)\right.\\
&\quad \left.-(b+\mu+a_w+\delta_w)\, I_{w_{(j-1)}}\right],\\
R_{4,j}(\overrightarrow{I_R}_{(j-1)}) &= \mathcal{L}[I_{R_{(j-1)}}] - (\frac{c_4}{s})(1-\chi_j) - \frac{s+\eta(1-s)}{s}\\
&\mathcal{L}\left[\sigma_r(1+b^2)\left(\Sigma_{d=0}^{(j-1)} I_{Rd}S_{(j-1)-d}\right) + (1-\beta)\sigma_r(1+b^2)\right.\\
&\quad \left.\left(\Sigma_{d=0}^{(j-1)} I_{Rd}V_{(j-1)-d}\right) + bI_{w_{((j-1))}} - (\mu+\delta_r+a_r)\, I_{R_{((j-1))}}\right],\\
R_{5,j}(\vec{R}_{(j-1)}) &= \mathcal{L}[R_{(j-1)}] - (\frac{c_5}{s})(1-\chi_j) - \frac{s+\eta(1-s)}{s}\\
&\quad \mathcal{L}\left[\delta_w I_{w_{(j-1)}} + \delta_r\delta_w I_{R_{(j-1)}} - (\Theta+\mu)R_{(j-1)}\right].
\end{aligned}
\tag{5.4}
$$

Further the deformation equations of j^{th} order are expressed as

$$\begin{aligned}\mathcal{L}[S_j(t)-\chi_j S_{(j-1)}(t)] =& hR_{1,j}(\overrightarrow{S}_{(j-1)}),\\ \mathcal{L}[V_j(t)-\chi_j V_{(j-1)}(t)] =& hR_{2,j}(\overrightarrow{V}_{(j-1)}),\\ \mathcal{L}[I_{w_j}(t)-\chi_j I_{w_{(j-1)}}(t)] =& hR_{3,j}(\overrightarrow{I_{w}}_{(j-1)}),\\ \mathcal{L}[I_{R_j}(t)-\chi_j I_{R_{(j-1)}}(t)] =& hR_{4,j}(\overrightarrow{I_{R}}_{(j-1)}),\\ \mathcal{L}[R_j(t)-\chi_j R_{(j-1)}(t)] =& hR_{5,j}(\overrightarrow{R}_{(j-1)}).\end{aligned} \tag{5.5}$$

Applying the inverse laplace transform to equation (5.5), we get

$$\begin{aligned}S_j(t) =& \chi_j S_{(j-1)}(t) + h\mathcal{L}^{-1}[R_{1,j}(\overrightarrow{S}_{(j-1)})],\\ V_j(t) =& \chi_j V_{(j-1)}(t) + h\mathcal{L}^{-1}[R_{2,j}(\overrightarrow{V}_{(j-1)})],\\ I_{w_j}(t) =& \chi_j I_{w_{(j-1)}}(t) + h\mathcal{L}^{-1}[R_{3,j}(\overrightarrow{I_{w}}_{(j-1)})],\\ I_{R_j}(t) =& \chi_j I_{R_{(j-1)}}(t) + h\mathcal{L}^{-1}[R_{4,j}(\overrightarrow{I_{R}}_{(j-1)})],\\ R_j(t) =& \chi_j R_{(j-1)}(t) + h\mathcal{L}^{-1}[R_{5,j}(\overrightarrow{R}_{(j-1)})].\end{aligned} \tag{5.6}$$

Using the initial conditions $S_0(t) = c_1 + [1 + \eta(t-1)]\Lambda$, $V_0(t) = c_2$, $I_{w_0} = c_3$, $I_{R_0} = c_4$, $R_0(t) = c_5$, and solving equation (5.6) for j= 1, 2, 3, ... , we get the values of $S(t),\ V(t),\ I_w(t),\ I_R(t),\ R(t)$ for $j \geq 1$. Thus the solution of fractional model can be obtained as

$$\begin{aligned}S(t) =& S_0 + S_1 + S_2 + ...\\ V(t) =& V_0 + V_1 + V_2 + ...\\ I_w(t) =& I_{w_0} + I_{w_1} + I_{w_2} + ...\\ I_R(t) =& I_{R_0} + I_{R_1} + I_{R_2} + ...\\ R(t) =& R_0 + R_1 + R_2 + ...\end{aligned} \tag{5.7}$$

6. Results and Discussions

In this section, numerical simulations has been carried out. Numerical results of the model (3.2) are calculated using HATM.

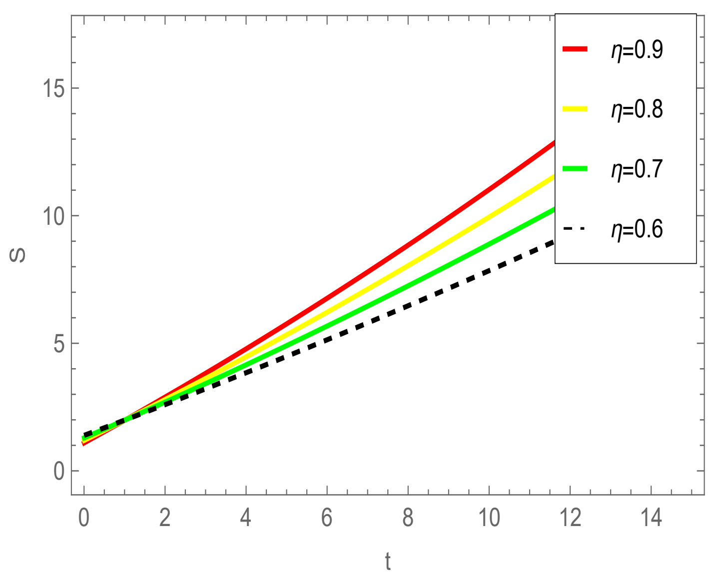

Figure 1: The dynamics of susceptible population for fractional values of $\eta = 0.6, 0.7, 0.8, 0.9$.

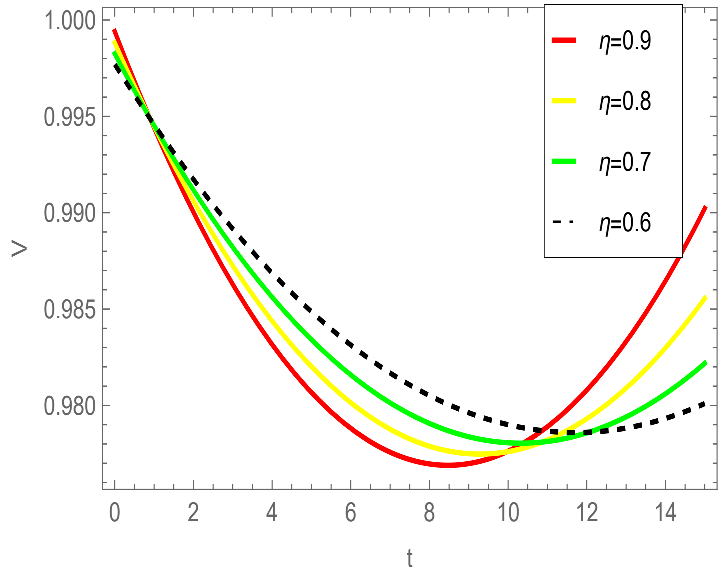

Figure 2: The dynamics of Vaccinated population for fractional values of $\eta = 0.6, 0.7, 0.8, 0.9$.

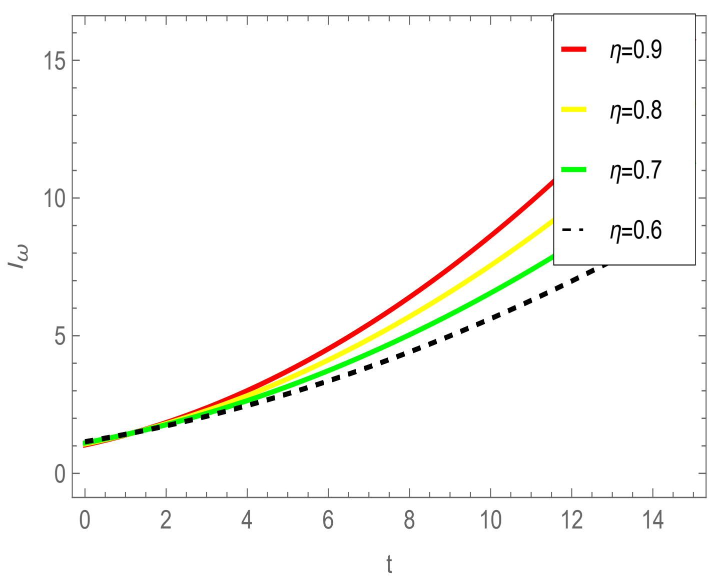

Figure 3: The dynamics of Infected with Wild-type strain population for fractional values of $\eta = 0.6, 0.7, 0.8, 0.9$.

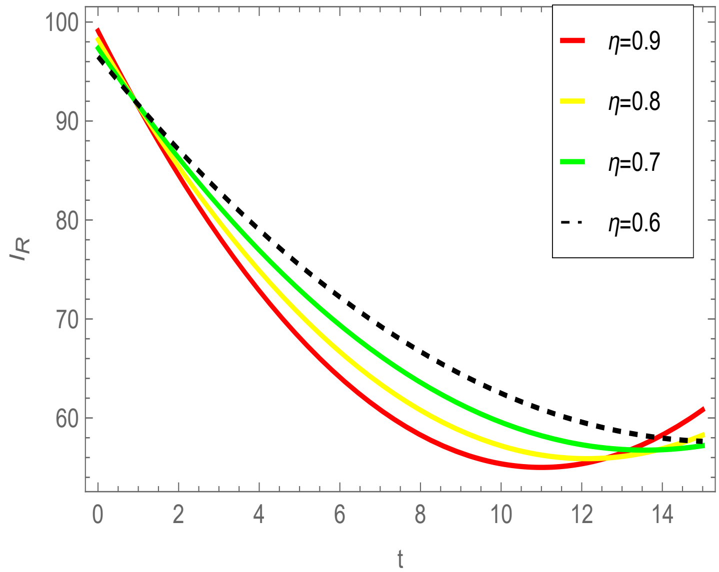

Figure 4: The dynamics of Infected with resistant strain population for fractional values of $\eta = 0.6, 0.7, 0.8, 0.9$.

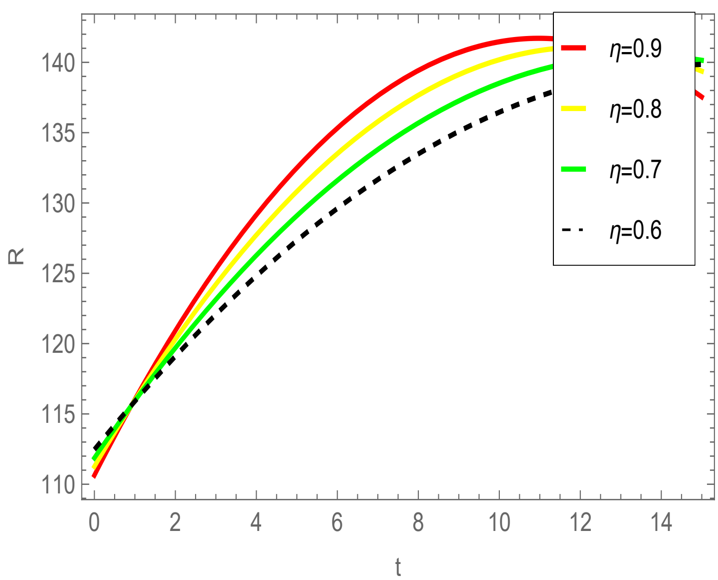

Figure 5: The dynamics of Recovered population for fractional values of $\eta = 0.6, 0.7, 0.8, 0.9$.

To calculate the numerical results for the model(3.2), we have taken the values of various parameters using [4] $\sigma_w = 0.00102$ day^{-1}, $\sigma_r = 0.00026$ day^{-1}, $\beta = 0.77$, $\gamma = 0.00027375$ day^{-1}, b = 0.118, $\delta_w = 0.1998$ day^{-1}, $\delta_r = 0.0714$ day^{-1}, $\Theta = 0.00833$ day^{-1}, $a_w = 0.01$, $\frac{1}{\mu}$ =70 $\times$ 365 days, $\Lambda = 0.0381$, $a_r = 0.021$ and the initial conditions are taken as $S_0(t) = 1$, $V_0(t) = 1$, $I_{w_0}(t) = 1$, $I_{R_0}(t) = 100$, $R_0(t) = 110$.

Figures 1-5 show the investigation for the susceptible, vaccinated, infected with wild-type strain, infected with resistant strain and recovered population, respectively, for different values of η. In Fig. 1, we can observe an increase in the susceptible population when time increases with different fractional order η. It indicates that effective interventions have reduced the infection rate, resulting in fewer people becoming infected, leading to an increase in the number of susceptible individuals.

Figure 2 shows that as time increases, the number of vaccinations for the susceptible population also increases. Over time, there is greater availability of the vaccine which allows more people in the susceptible population to get vaccinated.

In Figure 3 it can be observed that as time lapses, the population infected with wild-type strain increase but for fractional values of η this rise occurs at a slower rate. In Figure 4, it can be revealed that the number of infected people with resistant strain decreases with time and also the profile is shifting to the higher side with increasing fractional order.

From Figure 5 we can observe that there is a significant and rapid increase in the recovered population of different fractional order η. This graph allows for a more flexible and realistic representation of the recovery process, where the rate of recovery does not follow simple linear patterns but instead depends on the fractional order, accounting for complex factors like memory effects, delayed recovery in the population.

As a result, the figures represent dynamics and potential outcomes of the fractional order model and are more useful rather than the integral model.

7. Conclusion

In this paper, we studied a fractional $(S,\ V,\ I_w,\ I_R,\ R)$ model of the transmission dynamics of the influenza virus in the aspect of drug resistance. The graphical representations show the change in fractional order in susceptible, vaccinated, infected with wild strain, infected with resistant strain and recovered populations. From the results, we conclude that the fractional-order models are more convincing and satisfactory in predicting how the dynamics of influenza will change rather than the integer-order model.

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