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Stochastic Maximal System of Fuzzy Stochastic Delay Differential Equations with Continuous Coefficients

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ABSTRACT: This work aims to propose a new formulation of forward-backward fuzzy stochastic differential equations by taking the delay coefficients as continuous and imposing appropriate conditions to ensure the stability of the solution, with a discussion of the existence and uniqueness of the solution to this model of equations, as well as the achievement of the maximum solution.

Key Words: Stochastic differential equations, Forward-backward stochastic differential equations, Stochastic fuzzy differential equations, Delay differential Equations, Maximal solution.

Contents

1. Introduction

Suppose that (Ω, F, P) is a complete probability space with the natural filtration $\{F_t\}_{t\geq 0}$. Assume that $T > 0$, and $\{W(t),\ 0 \leq t \leq T\}$ is a d-dimensional Brownian motion defined on this space. Consider the following forward-backward stochastic differential equation (FBSDE):

$$\begin{cases} dY(t) = f(t, Y(t), \Theta(t))dt + [\acute{f}(t, Y(t)) - \Theta(t)]dW(t) \\ d\Psi(t) = g(t, \Psi(t), \Theta(t))dt - \Theta(t)dW(t) \\ Y(0) = \gamma\ ,\ \ \Psi(T) = J(Y(T)) \end{cases} \tag{1.1}$$

where the processes Y, Ψ and Θ are defined on $\mathbb{R}^a$, $\mathbb{R}^b$ and $\mathbb{R}^{a\times b}$, respectively with $E\int_0^T \|Y(t)\|^2 dt < \infty$. Also, the continuous function $f, g, \acute{f}$ and J are defined on $\mathbb{R}^a$, $\mathbb{R}^b$, $\mathbb{R}^{a\times b}$ and $\mathbb{R}^a$, respectively. By using Euclidean norm, we define $|Y| = tr(YY^{Trac})^{\frac{1}{2}}$, where $((Y^{Trac}))$ denotes the transpose. Therefore, the space $\mathbb{R}^{a\times b}$ is a Hilbert space.

Fabio [1] proposed a new model for backward stochastic differential equations under special conditions and demonstrated the existence and uniqueness of solution for this type of equations. He also expanded the work to include proposing a formula for the backward-forward stochastic differential equations and demonstrating that there is an existence and uniqueness of solution for this type of equations under Lipschitz's conditions. Jin et al. [2] presented a new numerical formulation for the forward-backward system of stochastic differential equations, then conducted a detailed study of the cumulative error. Based on this proposal, a comparison was conducted between several models to arrive at a decoupled forward stochastic differential equations. Peng et al. [3] presented a detailed study on the solvability of forward-backward stochastic differential equations with unstable coefficients. There is a group of researchers who proposed a new formula for stochastic differential equations by taking delay coefficients, and many studies were conducted on this model, including numerical convergence of the solution with the

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application of some numerical examples and other similar studies (see [4,5]). Fuzzy stochastic differential equations have attracted the attention of many researchers. A group of researchers presented studies on the existence and uniqueness of the solution to this type of equations, along with a group of numerical examples that illustrate the study (see [6,7]). In this work, the researcher deals with random integrals coupled with fuzzy Brownian motion, where this integral is built for fuzzy random functions and then expanded to include fuzzy random functions that are integrable [8]. Bernt and Agnes [9] studied in detail, through the principle of maximum risk reduction, a specific model of forward-backward stochastic differential equations with a set of applied examples. Falah [10], through proposing a model for forward-backward fuzzy stochastic differential equations, discussed the existence and uniqueness of the solution, with a study of the maximum solution by proposing a set of conditions on the continuous coefficients. In this work, a model of forward-backward fuzzy stochastic differential equations with continuous delay coefficients is proposed under certain conditions. The existence and uniqueness of the solution of this model are studied, and a detailed study of the maximum solution is presented.

2. Basic Notations and Hypothesis

Suppose that $\{W(t),\ 0 \le t \le T\}$ is a standard Brownian motion defined on the complete probability space (Ω, F, P). For any $t \in [0,T]$ and stochastic process $Y(t)$ with $\mathcal{N}$ is denoted the class of P-null set such that

$$F^Y_{s,t} = \sigma\{Y(r) - Y(s) :\ s \le r \le t\} \vee \mathcal{N}$$

is a σ-field generated by $\{Y(r) - Y(s)\},\ s \le r \le t$. Clearly, the σ-field $F(t), t \in [0,T]$ is neither increasing nor decreasing. Let us know the following spaces here:

i. Let $S^2(\Omega, F, P, \mathbb{R}^a)$ be the continuous space $\{F_r\}_{t\le r\le T}$-adapted process $Y : \Omega \times [t,T] \to \mathbb{R}^a$ such that $\|Y\|^2_{S^2} = E[\sup_{t\le r\le T} |Y(r)|^2] < \infty$.

ii. Let $Q^2(\Omega, F, P, \mathbb{R}^{a\times b})$ be the space of $\{F_r\}_{t\le r\le T}$-progressively measurable process $\Theta : \Omega \times [t,T] \to R^{a\times b}$ such that $\|\Theta\|^2_{Q^2} = E[\int_t^T |\Theta(r)|^2 dr] < \infty$.

iii. Let $G^2(\Omega, F, P, \mathbb{R}^a)$ be the space of $\{F_r\}_{t\le r\le T}$-measurable variable $Y : \Omega \times [0,T] \to \mathbb{R}^a$ such that $E|Y(t)|^2 < \infty, t \in [0,T]$.

iv. Let $O^2_{-T}(\mathbb{R}^a)$ be the space of measurable function $Y : [-T,0] \to \mathbb{R}^a$ such that $\sup_{-T\le t\le 0} |Y(t)|^2 < \infty$.

v. Let $O^2_{-T}(\mathbb{R}^{a\times b})$ be the space of measurable function $\Theta : [-T,0] \to \mathbb{R}^{a\times b}$ such that $\int_{-T}^0 |\Theta(t)|^2 dt < \infty$.

Let $MCS = MCS([0,T], S^2(\Omega, F, P, \mathbb{R}^a))$ be the class of all mean continuous second order such that the norm of stochastic process is defined as: $\|Y\|_{MCS} = \sup_{0\le t\le T}\ \|Y\|^2 = \sup_{0\le t\le T}(E(Y(t))^2)^{1/2}$. Therefore, we can define the norms $\|Y\|^2_{S^2} = E[\sup_{t\le r\le T} |Y(r)|^2]$ and $\|Y\|^2_{Q^2} = E\int_t^T |Y(r)|^2 dr,\ t \le r \le T$. Let us consider the forward-backward stochastic delay differential equation:

$$\begin{cases} dY(t) = f(r, Y(r), \Theta(r), Y_r, \Theta_r)dr + [\acute{f}(r, Y(r), Y_r) - \Theta(r)]dW(r), 0 \le r \le t, \\ Y = \gamma(\tau_0),\ -t \le \tau \le 0, \\ d\Psi(t) = g(r, \Psi(r), \Theta(r), \Psi_r, \Theta_r)dr - \Theta(r)dW(r),\ 0 \le r \le T, \\ \Psi_T = J(Y(T)),\ -T \le \tau \le 0, \end{cases} \tag{2.1}$$

where $f : \Omega \times [0,T] \times \mathbb{R}^a \times \mathbb{R}^{a\times b} \times O^2_{-T}(\mathbb{R}^a) \times O^2_{-T}(\mathbb{R}^{a\times b}) \to \mathbb{R}^a$, $g : \Omega \times [0,T] \times \mathbb{R}^b \times \mathbb{R}^{a\times b} \times O^2_{-T}(\mathbb{R}^b) \times O^2_{-T}(\mathbb{R}^{a\times b}) \to \mathbb{R}^b$, $\acute{f} : \Omega \times [0,T] \times \mathbb{R}^a \times O^2_{-T}(\mathbb{R}^a) \to \mathbb{R}^{a\times b}$ and $J : \Omega \times \mathbb{R}^a \to \mathbb{R}^a$ are product measurable depend on the past values such that $Y_r = (Y(r+\tau)),\ -T \le \tau \le 0, \Psi_r = (\Psi(r+\tau)),\ -T \le \tau \le 0$ and $\Theta_r = (\Theta(r+\tau)),\ -T \le \tau \le 0$. For a non-random, finitely valued measure τ supported on $[-T,0]$ and for $t \in [0,T], (Y^1, \Theta^1), (Y^2, \Theta^2) \in \mathbb{R}^a \times \mathbb{R}^{a\times b} and\ (Y^1_t, \Theta^1_t), (Y^2_t, \Theta^2_t) \in O^2_{-T}(\mathbb{R}^a) \times O^2_{-T}(\mathbb{R}^{a\times b})$, we impose the following hypothesis:

H1. $\forall\ (Y, \Psi, \Theta) \in \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^{a\times b}$ and for all constants $K_1, K_2, K_3, N_1, N_2, N_3 > 0$ such that:

i. $|f\left(t\ ,\ Y^1, \Theta^1\ ,\ Y_t^1\ , \Theta_t^1\right) - f\left(t\ ,\ Y^2, \Theta^2\ ,\ Y_t^2\ , \Theta_t^2\right)|^2 \leq K_1(|Y^1 - Y^2|^2 + \left|\Theta^1 - \Theta^2\right|^2)$ $+N_1(\int_{-T}^{0} |Y^1\left(t+\tau\right) - Y^2\left(t+\tau\right)|^2 \lambda d\tau + \int_{-T}^{0} \left|\Theta^1\left(t+\tau\right) - \Theta^2\left(t+\tau\right)\right|^2 \lambda d\tau)$.

ii. $|g\left(t\ ,\ \Psi^1, \Theta^1\ ,\ \Psi_t^1\ , \Theta_t^1\right) - g\left(t\ ,\ \Psi^2, \Theta^2\ ,\ \Psi_t^2\ , \Theta_t^2\right)|^2 \leq K_2(|\Psi^1 - \Psi^2|^2 + \left|\Theta^1 - \Theta^2\right|^2)$ $+N_2(\int_{-T}^{0} |\Psi^1\left(t+\tau\right) - \Psi^2\left(t+\tau\right)|^2 \lambda d\tau + \int_{-T}^{0} \left|\Theta^1\left(t+\tau\right) - \Theta^2\left(t+\tau\right)\right|^2 \lambda d\tau)$.

iii. $|\acute{f}\left(t\ ,\ Y^1,\ Y_t^1\right) - \acute{f}\left(t\ , Y^2,\ Y_t^2\right)|^2 \leq K_3(|Y^1 - Y^2|^2) + N_3(\int_{-T}^{0} |Y^1\left(t+\tau\right) - Y^2\left(t+\tau\right)|^2 \lambda d\tau)$.

H2. For all $K_4, K_5, K_6, K_7, K_8, N_4, N_5, N_6, N_7 > 0$ such that

i. $|f(t, Y(t), \Theta(t), Y_t, \Theta_t)|^2 \ \leq\ K_4(1 + |Y|^2 + |\Theta|^2) + N_4(\int_{-T}^{0} |Y\left(t+\tau\right)|^2 \lambda d\tau + \int_{-T}^{0} |\Theta(t+\tau)|^2 \lambda d\tau)$.

ii. $|g(t, \Psi(t), \Theta(t), \Psi_t, \Theta_t)|^2 \ \leq\ K_5(1 + |\Psi|^2 + |\Theta|^2) + N_5(\int_{-T}^{0} |\Psi\left(t+\tau\right)|^2 \lambda d\tau + \int_{-T}^{0} |\Theta(t+\tau)|^2 \lambda d\tau)$.

iii. $|\acute{f}(t, Y(t), Y_t)|^2 \leq\ K_6(1 + |Y|^2) + N_6(\int_{-T}^{0} |Y\left(t+\tau\right)|^2 \lambda d\tau)$,

iv. For all $-r \leq \tau < t \leq 0$ such that $E|\gamma(t) - \gamma(\tau)|^2 \leq K_7(t-\tau)$ and $|J\left(Y\right)|^2 \leq\ K_8\left(1 + |Y|^2\right) + N_7(\int_{-T}^{0} |Y\left(t+\tau\right)|^2 \lambda d\tau$.

H3. $E(\int_0^T |f(r\ , 0, 0, 0, 0)|^2)dr\ ) < \infty, E(\int_0^T |g(r, 0, 0, 0, 0)|^2)dr) < \infty, E(\int_0^T |\acute{f}(r\ , 0, 0)|^2)dr) < \infty$.
H4. $f(t, \cdot, \cdot, \cdot, \cdot) = 0, g(t\ , \cdot, \cdot, \cdot, \cdot) = 0$ and $\acute{f}(t\ , \cdot, \cdot) = 0$ for $t > 0$.
H5. For any $N, K > 0$ and $m \geq K$ such that

1. $|f(t, Y(t), \Theta(t), Y_t, \Theta_t)| \ \leq\ K(1 + |Y| + |\Theta|) + N(\int_{-T}^{0} |Y\left(t+\tau\right)| \lambda d\tau + \int_{-T}^{0} |\Theta\left(t+\tau\right)| \lambda d\tau)$.
2. $f(t, Y(t), \Theta(t), Y_t, \Theta_t) \leq f^{m+1}(t, Y(t), \Theta(t), Y_t, \Theta_t) \leq f^m(t, Y(t), \Theta(t), Y_t, \Theta_t)$.

3. Fuzzy Stochastic System

Assume that $\mathbb{A}(\mathbb{R}^a)$ is a family of all convex, compact and nonempty subsets $\mathbb{R}^a$ and $\mathbb{B}(\mathbb{R}^a)$ is a fuzzy set space of $\mathbb{R}^a$. Therefore, we define the set of functions $\mathbb{J} : \mathbb{R}^a \to [0, 1]$ such that $[\mathbb{J}]^u \in \mathbb{A}(\mathbb{R}^a)$ for every $u \in [0, 1]$ and $[\mathbb{J}]^u = \{x \in \mathbb{R}^a : \mathbb{J}\left(x\right) \geq u, u \in [0, 1]\} = [\mathbb{J}_L^u, \mathbb{J}_U^u]$, where $\mathbb{J}_L^u = \inf_{x \in \mathbb{R}^a}\{x \in [\mathbb{J}]^u\}$ and $\mathbb{J}_U^u = \sup_{x \in \mathbb{R}^a}\{x \in [\mathbb{J}]^u\}$ and $[\mathbb{J}]^0 = cl\{x \in \mathbb{R}^a : \mathbb{J}\left(x\right) > 0\}$.

Definition 3.1. *Suppose that (Ω, F, P) is a complete probability space, we define $X : \Omega \to \mathbb{B}(\mathbb{R}^a)$ is a fuzzy random variable, if for any $u \in\ [0, 1], [X]^a : \Omega \to \mathbb{A}(\mathbb{R}^a)$ is an F-measurable multifunction.*

Definition 3.2. *Suppose that (Ω, F, P) is a complete probability space, we define $X : [0, T] \times \Omega \to \mathbb{B}(\mathbb{R}^a)$ is a fuzzy stochastic process if $X(t, \cdot) = X(t) : \Omega \to \mathbb{B}(\mathbb{R}^a)$ is a fuzzy random variable.*

Definition 3.3. *Fuzzy stochastic process X is a_∞-continuous, if the mappings $X\left(\cdot, W\right) : [0, T] \to \mathbb{B}(\mathbb{R}^a)$ are a_∞-continuous functions.*

Here, we can consider the forward-backward fuzzy stochastic delay differential equation (FBFSDDE) as:

$$\begin{cases} dY(t) = f(r, Y(r), \Theta(r), Y_r, \Theta_r)dr + [\acute{f}(r, Y(r), Y_r) - \Theta(r)]dW(r) \\ d\Psi(t) = g(r, \Psi(r), \Theta(r), \Psi_r, \Theta_r)dr - \Theta(r)dW(r) \\ Y = \gamma(\tau_0),\ -t \leq \tau \leq 0,\ \Psi_T = J\left(Y\left(T\right)\right),\ -T \leq \tau \leq 0, \end{cases} \tag{3.1}$$

where the processes Y, Ψ and Θ are defined on $\mathbb{R}^a$, $\mathbb{R}^b$ and $\mathbb{R}^{a\times b}$, respectively and $\gamma(\tau_0)$ is a given F_1-measurable random variable with $E|\gamma|^2 < \infty$ as well as the functions $f : \Omega \times [0, T] \times \mathbb{R}^a \times \mathbb{R}^{a\times b} \times O^2_{-T}(\mathbb{R}^a) \times O^2_{-T}(\mathbb{R}^{a\times b}) \times \mathbb{B}(\mathbb{R}^a) \to \mathbb{B}(\mathbb{R}^a)$, $g : \Omega \times [0, T] \times \mathbb{R}^b \times \mathbb{R}^{a\times b} \times O^2_{-T}(\mathbb{R}^b) \times O^2_{-T}(\mathbb{R}^{a\times b}) \times \mathbb{B}(\mathbb{R}^b) \to \mathbb{B}(\mathbb{R}^b)$, $\acute{f} : \Omega \times [0, T] \times \mathbb{R}^a \times O^2_{-T}(\mathbb{R}^a) \times \mathbb{B}(\mathbb{R}^a) \to \mathbb{R}^{a\times b}$, $Y(0) : \Omega \to \mathbb{B}(\mathbb{R}^a)$ and $J : \Omega \times \mathbb{R}^a \times \mathbb{B}(\mathbb{R}^b) \to \mathbb{B}(\mathbb{R}^b)$. Now, let us present our numerical formula, let $0 = r_0 < r_1 < \cdots < r_n = T, n \geq 1$ be a partition on interval

$[0,T]$. Therefore, we define $\delta = \Delta r_{i+1} = r_{i+1} - r_i = \frac{T}{n}$, and $\Delta W_{r_{i+1}} = W_{r_{i+1}} - W_{r_i}$ and $\Delta r = max \Delta r_{i+1}$, for $i = 0, 1, \cdots, n-1,\ n \geq 1$. Her, we define FBFSDDE on the small interval $[r_i, r_{i+1}]$ as follows:

$$\begin{cases} Y(r_i) = \gamma + \int_{r_i}^{r_{i+1}} f(t, Y(t), \Theta(t), Y_t, \Theta_t)dt + \int_{r_i}^{r_{i+1}} \left[\acute{f}(t, Y(t), Y_t) - \Theta(t) \right] dW(t) \\ \Psi(r_i) = \Psi(r_{i+1}) + \int_{r_i}^{r_{i+1}} g(t, \Psi(t), \Theta(t), \Psi_t, \Theta_t)dt - \int_{r_i}^{r_{i+1}} \Theta(t) dW(t), \end{cases} \tag{3.2}$$

Let us consider approximation form as follows

$$\begin{cases} Y(r) = Y(0) + \int_0^r f(t, Y_i^n(t), \Theta_i^n(t), Y_{i\ t}^n, \Theta_{i\ t}^n)dt + \int_0^r \left[\acute{f}(t, Y_i^n(t), Y_{i\ t}^n) - \Theta_i^n(t) \right] dW(t) \\ \Psi(r) = \Psi(T) + \int_r^T g(t, \Psi_i^n(t), \Theta_i^n(t), \Psi_{i\ t}^n, \Theta_{i\ t}^n)dt - \int_r^T \Theta_i^n(t) dW(t), \end{cases} \tag{3.3}$$

where $t \in [0,T], i = 1, 2, \cdots, n$.

Lemma 3.4. *Under assumptions (H1-H4), the FBFSDDEs system (3.3) has an unique solution (Y^n, Ψ^n, Θ^n), and there exists a constant $C > 0$ such that*

$$\|Y^n\|_{S^2} + \|\Psi^n\|_{S^2} + \|\Theta^n\|_{S^2} \leq C.$$

Proof. First, we consider the general forward FSDDE

$$Y(r) = Y(0) + \int_0^r f(t, Y_i^n(t), \Theta_i^n(t), Y_{i\ t}^n, \Theta_{i\ t}^n)dt + \int_0^r \left[\acute{f}(t, Y_i^n(t), Y_{i\ t}^n) - \Theta_i^n(t) \right] dW(t), \tag{3.4}$$

where $r \in [0,T]$. Let us define the mapping
$H(\Phi^n, \Gamma^n) = (Y^n, \Theta^n)$. We define $(Y^{i,n}, \Theta^{i,n}) = H(\Phi^{i,n}, \Gamma^{i,n}),\ i = 1, 2$. Let $(\overline{Y^n}, \overline{\Theta^n}) = (Y^{1,n} - Y^{2,n}, \Theta^{1,n} - \Theta^{2,n})$. Applying Itô's formula to $|\overline{Y^n}(r)|^2, r \in [0,T]$, yields

$$E\left[|\overline{Y^n}(r)|^2\right] + E\left[\int_0^r |\overline{Y^n}(t)|^2 dt\right] + E\left[\int_0^r |\overline{\Theta^n}(t)|^2 dt\right] = 2E\left[\int_0^r \overline{Y^n}(t)\bar{f}(t)dt\right] E\left[\int_0^r |\bar{\acute{f}}(t)|^2 dt\right], \tag{3.5}$$

where

$$\bar{f}(t) = f(t, Y^{1,n}(t), \Theta^{1,n}(t), Y_t^{1,n}, \Theta_t^{1,n}) - f(t, Y^{2,n}(t), \Theta^{2,n}(t), Y_t^{2,n}, \Theta_t^{2,n}),$$
$$\bar{\acute{f}}(t) = \acute{f}(t, Y^{1,n}(t), Y_t^{1,n}) - \acute{f}(t, Y^{2,n}(t), Y_t^{2,n}).$$

From equation (3.5) and applying inequality $2xy \leq \frac{1}{c}x^2 + cy^2,\ c > 0$, we have

$$\begin{aligned} 2E\left[\int_0^r \overline{Y^n}(t)\bar{f}(t)dt\right] &\leq 2E\left[\int_0^r |\overline{Y^n}(t)||\bar{f}(t)|dt\right] \leq \frac{1}{c}E\left[\int_0^r |\overline{Y^n}(t)|^2 dt\right] + cE\left[\int_0^r |\bar{f}(t)|^2 dt\right] \\ &\leq \frac{1}{c}E\left[\int_0^r |\overline{Y^n}(t)|^2 dt\right] + cE\Big[\int_0^r \left[K_1(|\overline{\Phi^n}(t)|^2 + |\overline{\Gamma^n}(t)|^2)\right. \\ &+ N_1(\int_{-r}^0 |\overline{\Phi^n}(t+\tau)|^2 \lambda d\tau + \int_{-r}^0 |\overline{\Gamma^n}(t+\tau)|^2 \lambda d\tau)\Big] dt\Big] \\ &\leq \frac{1}{c}E\left[\int_0^r |\overline{Y^n}(t)|^2 dt\right] + cK_1 E\left[\int_0^r |\overline{\Phi^n}(t)|^2 dt\right] + cK_1 E\left[\int_0^r |\overline{\Gamma^n}(t)|^2 dt\right] \\ &+ cN_1 E\left[\int_0^r (\int_{-r}^0 |\overline{\Phi^n}(t+\tau)|^2 \lambda d\tau)dt\right] + cN_1 E\left[\int_0^r (\int_{-r}^0 |\overline{\Gamma^n}(t+\tau)|^2 \lambda d\tau)dt\right]. \end{aligned}$$

By changing of integration order argument, we have

$$\int_0^r (\int_{-r}^0 |\overline{\Phi^n}(t+\tau)|^2 \lambda d\tau)dt = \int_{-r}^0 (\int_0^r |\overline{\Phi^n}(t+\tau)|^2 dt)\lambda d\tau = \int_{-r}^0 (\int_\tau^{r+\tau} |\overline{\Phi^n}(t)|^2 dt)\lambda d\tau \leq \rho \int_0^r |\overline{\Phi^n}(t)|^2 dt,$$

and

$$\int_0^r (\int_{-r}^0 |\overline{\Gamma^n}(t+\tau)|^2 \lambda d\tau)dt = \int_{-r}^0 (\int_0^r |\overline{\Gamma^n}(t+\tau)|^2 dt)\lambda d\tau = \int_{-r}^0 (\int_\tau^{r+\tau} |\overline{\Gamma^n}(t)|^2 dt)\lambda d\tau \leq \rho \int_0^r |\overline{\Gamma^n}(t)|^2 dt,$$

where $\rho = \int_{-r}^{0} \lambda d\tau$. Consequently, we have

$$2E\left[\int_0^r \overline{Y^n}(t)\bar{f}(t)dt\right] \leq \frac{1}{c}E\left[\int_0^r \left|\overline{Y^n}(t)\right|^2 dt\right] + cK_1E\left[\int_0^r |\overline{\Phi^n}(t)|^2 dt\right]$$
$$+ cK_1E\left[\int_0^r |\overline{\Gamma^n}(t)|^2 dt\right] + cN_1\rho E\left[\int_0^r |\overline{\Phi^n}(t)|^2 dt\right] + cN_1\rho E\left[\int_0^r |\overline{\Gamma^n}(t)|^2 dt\right]$$
$$= \frac{1}{c}E\left[\int_0^r \left|\overline{Y^n}(t)\right|^2 dt\right] + (cK_1 + cN_1\rho)E\left[\int_0^r |\overline{\Phi^n}(t)|^2 dt\right]$$
$$+ (cK_1 + cN_1\rho)E\left[\int_0^r |\overline{\Gamma^n}(t)|^2 dt\right].$$

and

$$E\left[\int_0^r \left|\bar{\bar{f}}(t)\right|^2 dt\right] \leq E\left[\int_0^r [K_6(1 + |\overline{\Phi^n}(t)|^2) + N_6(\int_{-r}^0 \left|\overline{\Phi^n}\,(t+\tau)\right|^2 \lambda d\tau)dt\right]$$
$$\leq K_6E\int_0^r |\overline{\Phi^n}(t)|^2 dt + N_6E\left[\int_0^r (\int_{-r}^0 \left|\overline{\Phi^n}\,(t+\tau)\right|^2 \lambda d\tau)dt\right].$$

By the same changing of integration order argument above, we have

$$E\left[\int_0^r \left|\bar{\bar{f}}(t)\right|^2 dt\right] \leq K_6E\left[\int_0^r |\overline{Y^n}(t)|^2 dt\right] + N_6\rho E\left[\int_0^r |\overline{Y^n}(t)|^2 dt\right].$$

Substituting two parts above in (3.5), we have

$$E\left[\left|\overline{Y^n}(r)\right|^2\right] + E\left[\int_0^r \left|\overline{Y^n}(t)\right|^2 dt\right] + E\left[\int_0^r \left|\overline{\Theta^n}(t)\right|^2 dt\right] \leq \frac{1}{c}E\left[\int_0^r \left|\overline{Y^n}(t)\right|^2 dt\right]$$
$$+ (cK_1 + cN_1\rho)E\int_0^r |\overline{\Phi^n}(t)|^2 dt + (cK_1 + cN_1\rho)E\int_0^r |\overline{\Gamma^n}(t)|^2 dt$$
$$+ K_6E\int_0^r |\overline{Y^n}(t)|^2 dt + N_6\rho E\int_0^r |\overline{Y^n}(t)|^2 dt$$
$$= (\frac{1}{c} + K_6 + N_6\rho)E[\int_0^r \left|\overline{Y^n}(t)\right|^2 dt + (cK_1 + cN_1\rho)E\int_0^r |\overline{\Phi^n}(t)|^2 dt$$
$$+ (cK_1 + cN_1\rho)E\int_0^r |\overline{\Gamma^n}(t)|^2 dt.$$

Therefore, we have

$$(1 - \frac{1}{c} - K_6 - N_6\rho)E\left[\int_0^r \left|\overline{Y^n}(t)\right|^2 dt\right] + E\left[\int_0^r \left|\overline{\Theta^n}(t)\right|^2 dt\right]$$
$$\leq (cK_1 + cN_1\rho)E\left[\int_0^r |\overline{\Phi^n}(t)|^2 dt\right] + (cK_1 + cN_1\rho)E\left[\int_0^r |\overline{\Gamma^n}(t)|^2 dt\right].$$

From contraction mapping theorem, there is a unique fixed point $(Y^n, \Theta^n) \in S^2(\Omega, F, P, \mathbb{R}^a) \times Q^2(\Omega, F, P, \mathbb{R}^{a\times b})$ such that $H(Y^n, \Theta^n) = (Y^n, \Theta^n)$. There is a unique solution (Y^n, Θ^n) for the equation (3.4). Also, from a classical result for backward FSDDEs, then the existence and unique of solution (Ψ^n, Θ^n) are fulfilled.

Next step, we must prove that there exists a constant $C_1 > 0$ such that

$$\|Y^n\|_{S^2} + \|\Theta^n\|_{Q^2} \leq C_1.$$

By using Itô's formula to $|Y^n(t)|^2$, we have

$$|Y^n(r)|^2 + \int_0^r |Y^n(t)|^2 dt + \int_0^r |\Theta^n(t)|^2 dt = |\gamma|^2 + 2\int_0^r Y^n(t)f(t, Y^n(t), \Theta^n(t), {Y^n}_t, {\Theta^n}_t)dt$$
$$+ 2\int_0^r Y^n(t)\acute{f}(t, Y^n(t), {Y^n}_t)dW(t) + \int_0^r \left|\acute{f}(t, Y^n(t), {Y^n}_t)\right|^2 dt. \tag{3.6}$$

By applying inequality $2xy \leq \frac{1}{c}x^2 + cy^2,\ c > 0$, we have

$$
\begin{aligned}
2\int_0^r Y^n(t)f(t,Y^n(t),\Theta^n(t),{Y^n}_t,{\Theta^n}_t)dt &\leq \frac{1}{c}\int_0^r |Y^n(t)|^2 dt + c\int_0^r \left|f(t,Y^n(t),\Theta^n(t),{Y^n}_t,{\Theta^n}_t)\right|^2 dt \\
&\leq \frac{1}{c}\int_0^r |Y^n(t)|^2 dt + c\int_0^r \Big[K_4(1+|Y^n(t)|^2+|\Theta^n(t)|^2) \\
&+ N_4(\int_{-r}^0 |Y^n\,(t+\tau)\,|^2\lambda d\tau + \int_{-r}^0 |\Theta^n(t+\tau)\lambda d\tau))\Big] dt \\
&\leq \frac{1}{c}\int_0^r |Y^n(t)|^2 dt + cK_4\int_0^r |f(t,0,0,0,0)|^2 dt \\
&+ cK_4\int_0^r |Y^n(t)|^2 dt + cK_4\int_0^r |\Theta^n(t)|^2 dt \\
&+ cN_4\int_0^r\int_{-r}^0 |Y^n\,(t+\tau)\,|^2\lambda d\tau dt \\
&+ cN_4\int_0^r\int_{-r}^0 |\Theta^n\,(t+\tau)\,|^2\lambda d\tau dt.
\end{aligned}
$$

By the same changing of the integration order argument previously, we have

$$
\begin{aligned}
2\int_0^r Y^n(t)f(t,Y^n(t),\Theta^n(t),{Y^n}_t,{\Theta^n}_t)dt &\leq \frac{1}{c}\int_0^r |Y^n(t)|^2 dt + cK_4\int_0^r |f(t,0,0,0,0)|^2 dt \\
&+ cK_4\int_0^r |Y^n(t)|^2 dt + cK_4\int_0^r |\Theta^n(t)|^2 dt + \rho cN_4\int_0^r |Y^n(t)|^2 dt \\
&+ \rho cN_4\int_0^r |\Theta^n(t)|^2 dt = (\frac{1}{c} + cK_4 + \rho cN_4)\int_0^r |Y^n(t)|^2 dt \quad (3.7) \\
&+ (cK_4 + \rho cN_4)\int_0^r |\Theta^n(t)|^2 dt + cK_4\int_0^r |f(t,0,0,0,0)|^2 dt.
\end{aligned}
$$

and

$$
\begin{aligned}
2\int_0^r Y^n(t)\acute{f}(t,Y^n(t),{Y^n}_t)dW(t) &\leq \frac{1}{c}\int_0^r |Y^n(t)|^2 dt + c\int_0^r \left|\acute{f}(t,Y^n(t),{Y^n}_t)\right|^2 dt \\
&\leq \frac{1}{c}\int_0^r |Y^n(t)|^2 dt + c\int_0^r \Big[K_6(1+|Y^n|^2) + N_6(\int_{-r}^0 |Y^n\,(t+\tau)\,|^2\lambda d\tau)\Big] dt \\
&\leq \frac{1}{c}\int_0^r |Y^n(t)|^2 dt + cK_6\int_0^r \left|\acute{f}(t,0,0)\right|^2 dt + cK_6\int_0^r |Y^n(t)|^2 dt \\
&+ cN_6\int_0^r\int_{-r}^0 |Y^n\,(t+\tau)\,|^2\lambda d\tau dt \leq \frac{1}{c}\int_0^r |Y^n(t)|^2 dt + cK_6\int_0^r \left|\acute{f}(t,0,0)\right|^2 dt \\
&+ cK_6\int_0^r |Y^n(t)|^2 dt + cN_6\rho\int_0^r |Y^n(t)|^2 dt \\
&= (\frac{1}{c} + cK_6 + cN_6\rho)\int_0^r |Y^n(t)|^2 dt + cK_6\int_0^r \left|\acute{f}(t,0,0)\right|^2 dt,
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
\int_0^r \left|\acute{f}(t,Y^n(t),{Y^n}_t)\right|^2 dt &\leq \int_0^r \Big[K_6(1+|Y^n(t)|^2) + N_6(\int_{-r}^0 |Y^n\,(t+\tau)\,|^2\lambda d\tau)\Big] dt \\
&\leq K_6\int_0^r \left|\acute{f}(t,0,0)\right|^2 dt + K_6\int_0^r |Y^n(t)|^2 dt + N_6\rho\int_0^r |Y^n(t)|^2 dt \quad (3.9) \\
&= (K_6 + N_6\rho)\int_0^r |Y^n(t)|^2 dt + K_6\int_0^r \left|\acute{f}(t,0,0)\right|^2 dt
\end{aligned}
$$

Combining (3.7), (3.8) and (3.9) with (3.6), we have

$$
\begin{aligned}
|Y^n(r)|^2 + \int_0^r |Y^n(t)|^2 dt + \int_0^r |\Theta^n(t)|^2 dt \le |\gamma|^2 \\
&+ (\frac{1}{c} + cK_4 + \rho c N_4) \int_0^r |Y^n(t)|^2 dt \\
&+ (cK_4 + \rho c N_4) \int_0^r |\Theta^n(t)|^2 dt + cK_4 \int_0^r |f(t,0,0,0,0)|^2 dt \\
&+ (\frac{1}{c} + cK_6 + cN_6\rho) \int_0^r |Y^n(t)|^2 dt + cK_6 \int_0^r |\acute{f}(t,0,0)|^2 dt \\
&+ (K_6 + N_6\rho) \int_0^r |Y^n(t)|^2 dt + K_6 \int_0^r |\acute{f}(t,0,0)|^2 dt \\
&= |\gamma|^2 + (\frac{1}{c} + cK_4 + \rho c N_4 + \frac{1}{c} + cK_6 + cN_6\rho + K_6 + N_6\rho) \int_0^r |Y^n(t)|^2 dt \\
&+ (cK_4 + \rho c N_4) \int_0^r |\Theta^n(t)|^2 dt + cK_4 \int_0^r |f(t,0,0,0,0)|^2 dt + K_6 \int_0^r |\acute{f}(t,0,0)|^2 dt.
\end{aligned} \tag{3.10}
$$

By taking the expectation, we have

$$
\begin{aligned}
E|Y^n(r)|^2 + R_1 E\left[\int_0^r |Y^n(t)|^2 dt\right] + R_2 E\left[\int_0^r |\Theta^n(t)|^2 dt\right] \le E|\gamma|^2 \\
+ cK_4 E\left[\int_0^r |f(t,0,0,0,0)|^2 dt\right] + K_6 E[\int_0^r |\acute{f}(t,0,0)|^2 dt]
\end{aligned}
$$

where $R_1 = \frac{c-2-c^2K_4-\rho c^2 N_4 - c^2K_6 - \rho c^2 N_6 - cK_6 - \rho c N_6}{c}$ and $R_2 = 1 - cK_4 - \rho c N_4$. For choosing $R_1 > 0$ and $R_2 > 0$. Therefore, there exists a constant $D > 0$ depending on R_1 and R_2 such that

$$
\begin{aligned}
E|Y^n(r)|^2 + E\left[\int_0^r |Y^n(t)|^2 dt\right] + E\left[\int_0^r |\Theta^n(t)|^2 dt\right] \\
\le D[E|\gamma|^2 + E\left[\int_0^r |f(t,0,0,0,0)|^2 dt\right] + E\left[\int_0^r |\acute{f}(t,0,0)|^2 dt\right]].
\end{aligned}
$$

Applying the Burkholder-Davis-Gundy inequality and Young's inequality, then there exists a constant $C_1 > 0$ such that.

$$
E\left[\sup_{0\le t\le r} |Y^n(r)|^2 + \int_0^r |\Theta^n(t)|^2 dr\right] \le D\left[E|\gamma|^2 + E[\int_0^r |f(t,0,0,0,0)|^2 dt\right] + E\left[\int_0^r |\acute{f}(t,0,0)|^2 dt\right]] \le C_1,
$$

Thus,

$$
\|Y^n\|_{S^2} + \|\Theta^n\|_{Q^2} \le C_1. \tag{3.11}
$$

The next srep, we prove the backward FSDDE

$$
\Psi(r) = \Psi(T) + \int_r^T g(t, \Psi_i^n(t), \Theta_i^n(t), \Psi^n_{i\,t}, \Theta^n_{i\,t}) dt - \int_r^T \Theta_i^n(t) dW(t)
$$

has an unique solution (Ψ^n, Θ^n), and for $C_2 > 0$ we have

$$
\|\Psi^n\|_{S^2} + \|\Theta^n\|_{Q^2} \le C_2.
$$

From existence and uniqueness of the solution of the forward of FSDDE. The backward of FSDDE becomes a classical equation terminal condition $\Psi(T)$ and coefficient $g(t, \Psi^n(t), \Theta^n(t), \Psi^n{}_t, \Theta^n{}_t)$. Therefore, we deduce that the backward equation has a unique solution. By following the same technique of forward equation, we obtain

$$
\|\Psi^n\|_{S^2} + \|\Theta^n\|_{Q^2} \le C_2 \tag{3.12}
$$

where C_2 is a positive number. Let $C = C_1 + C_2$ and by combining (3.11) and (3.12), we have

$$\|Y^n\|_{S^2} + \|\Psi^n\|_{S^2} + \|\Theta^n\|_{Q^2} \leq C.$$

□

4. Maximal Solution

Definition 4.1. *Let $Y(r)$ be a solution of stochastic differential equation. $Y(r)$ is a maximal solution if every solution $X(r)$ such that $E(X^2(r)) < E(Y^2(r))$.*

Definition 4.2. *Let $X(r), Y(r) \in S^2([0,T], \mathbb{R}^a)$, $r \in [0,T]$, such that $\|X(r)\|^2 < \|Y(r)\|^2$. A function $h : S^2_{[0,T]}(\Omega, F, \mathbb{R}^a) \to S^2_{[0,T]}(\Omega, F, \mathbb{R}^a)$ is stochastically increasing if $\|h(r, X(r))\|^2 < \|h(r, Y(r))\|^2$.*

Definition 4.3. *Let $X(r), Y(r) \in S^2([0,T], \mathbb{R}^a)$, $r \in [0,T]$, such that $\|X(r)\|^2 < \|Y(r)\|^2$. A function $h : S^2_{[0,T]}(\Omega, F, \mathbb{R}^a) \to S^2_{[0,T]}(\Omega, F, \mathbb{R}^a)$ is stochastically decreasing if $\|h(r, X(r))\|^2 > \|h(r, Y(r))\|^2$.*

Lemma 4.4. *Assuming (Y^1, Θ^1) and (Y^2, Θ^2) are the solutions of the forward equation of FBFSDDE system (3.3). Then, if $f^1 \leq f^2$, it holds that also $Y^1 \leq Y^2$.*

Proof. From Itô's formula and applying inequality $2xy \leq \frac{1}{c}x^2 + cy^2,\ c > 0$, we have

$$\begin{aligned}
E(\overline{Y}(r))^2 + E\left[\int_0^r |\overline{\Theta}(t)|^2 dt\right] &= E(\overline{\gamma})^2 \\
&+ 2E\left[\int_0^r \overline{Y}(t)(f^1(t, Y^1(t), \Theta^1(t), Y^1{}_t, \Theta^1{}_t) - f^2(t, Y^2(t), \Theta^2(t), Y^2{}_t, \Theta^2{}_t))dt\right] \\
&+ E\left[\int_0^r |\acute{f}(t, Y^1(t), Y^1{}_t) - \acute{f}(t, Y^2(t), Y^2{}_t)|^2 dt\right] \leq \frac{1}{c}E\left[\int_0^r |\overline{Y}(t)|^2 dt\right] \\
&+ cE\left[\int_0^r |f^1(t, Y^1(t), \Theta^1(t), Y^1{}_t, \Theta^1{}_t) - f^2(t, Y^2(t), \Theta^2(t), Y^2{}_t, \Theta^2{}_t)|^2 dt\right] \\
&+ E\left[\int_0^r |\acute{f}(t, Y^1(t), Y^1{}_t) - \acute{f}(t, Y^2(t), Y^2{}_t)|^2 dt\right] \leq \frac{1}{c}E\left[\int_0^r |\overline{Y}(t)|^2 dt\right] \\
&+ cE\Bigg[\int_0^r |f^1(t, Y^1(t), \Theta^1(t), Y^1{}_t, \Theta^1{}_t) - f^1(t, Y^2(t), \Theta^2(t), Y^2{}_t, \Theta^2{}_t) \\
&+ f^1(t, Y^2(t), \Theta^2(t), Y^2{}_t, \Theta^2{}_t) - f^2(t, Y^2(t), \Theta^2(t), Y^2{}_t, \Theta^2{}_t)|^2 dt\Bigg] \\
&+ E\left[\int_0^r |\acute{f}(t, Y^1(t), Y^1{}_t) - \acute{f}(t, Y^2(t), Y^2{}_t)|^2 dt\right] \\
&\leq \frac{1}{c}E\left[\int_0^r |\overline{Y}(t)|^2 dt + cK_1 E[\int_0^r |\overline{Y}(t)|^2 dt\right] \\
&+ cK_1 E\left[\int_0^r |\overline{\Theta}(t)|^2 dt\right] + cN_1\rho E\left[\int_0^r |\overline{Y}(t)|^2 dt\right] \\
&+ N_1\rho E\left[\int_0^r |\overline{\Theta}(t)|^2 dt\right] + C_3 E\left[\int_0^v (|\overline{Y}(t)|^2 dt\right] \\
&+ R_3\mu E\left[\int_0^v |\overline{Y}(t)|^2) dt\right] \leq (\frac{1}{c} + cK_1 + cN_1\rho + K_3 + \rho R_3) E\left[\int_0^r |\overline{Y}(t)|^2 dt\right] \\
&+ (cK_1 + N_1\rho) E\left[\int_0^r |\overline{\Theta}(t)|^2 dt\right].
\end{aligned}$$

From Gronwall's inequality, $E(\overline{Y}(r))^2 = 0,\ r \in [0,T], ie., Y^1(r) \leq Y^2(r),\ r \in [0,T]$. Hence $Y^1 \leq Y^2$. □

Lemma 4.5. *Assuming* (Ψ^1, Θ^1) *and* (Ψ^2, Θ^2) *are the solutions of the backward equation of FBFSDDE system (3.3). Then, if* $g^1 \leq g^2$*, it holds that also* $\Psi^1 \leq \Psi^2$*.*

Proof. From Itô's formula and applying inequality $2xy \leq \frac{1}{c}x^2 + cy^2,\ c > 0$, we have

$$\begin{aligned}
E(\overline{\Psi}(r))^2 + E\left[\int_0^r |\overline{\Theta}(t)|^2 dt\right] &= 2E\left[\int_0^r \overline{\Psi}(t)(g^1(t,\Psi^1(t),\Theta^1(t),{\Psi^1}_t,{\Theta^1}_t) - g^2(t,\Psi^2(t),\Theta^2(t),{\Psi^2}_t,{\Theta^2}_t))dt\right] \\
&\leq \frac{1}{c}E\left[\int_0^r |\overline{\Psi}(t)|^2 dt\right] + cE\left[\int_0^r |g^1(t,\Psi^1(t),\Theta^1(t),{\Psi^1}_t,{\Theta^1}_t) - g^2(t,\Psi^2(t),\Theta^2(t),{\Psi^2}_t,{\Theta^2}_t)|^2 dt\right] \\
&\leq \frac{1}{c}E\left[\int_0^r |\overline{\Psi}(t)|^2 dt\right] + cE\Big[\int_0^r |g^1(t,\Psi^1(t),\Theta^1(t),{\Psi^1}_t,{\Theta^1}_t) - g^1(t,\Psi^2(t),\Theta^2(t),{\Psi^2}_t,{\Theta^2}_t) \\
&+ g^1(t,\Psi^2(t),\Theta^2(t),{\Psi^2}_t,{\Theta^2}_t) - g^2(t,\Psi^2(t),\Theta^2(t),{\Psi^2}_t,{\Theta^2}_t)|^2 dt\Big] \leq \frac{1}{c}E\left[\int_0^r |\overline{\Psi}(t)|^2 dt\right] \\
&+ cK_2E\left[\int_0^r |\overline{\Psi}(t)|^2 dt\right] + cK_2E\left[\int_0^r |\overline{\Theta}(t)|^2 dt\right] + cN_2\rho E\left[\int_0^r |\overline{\Psi}(t)|^2 dt\right] + cN_2\rho E\left[\int_0^r |\overline{\Theta}(t)|^2 dt\right] \\
&= (\frac{1}{c} + cK_2 + cN_2\rho)E\left[\int_0^r |\overline{\Psi}(t)|^2 dt\right] + (cK_2 + cN_2\rho)E\left[\int_0^r |\overline{\Theta}(t)|^2 dt\right].
\end{aligned}$$

From Gronwall's inequality, $E(\overline{\Psi}(r))^2 = 0,\ r \in [0,T]$, ie., $\Psi^1(r) \leq \Psi^2(r),\ r \in [0,T]$. Hence $\Psi^1 \leq \Psi^2$. □

Theorem 4.6. *Suppose that* $f, \acute{f}$ *and* g *are stochastically increasing functions. Under the assumptions (H1-H5), the FBFSDDEs system has a maximal solution* (Y, Ψ, Θ)*.*

Proof. We will prove that (Y^n, Ψ^n, Θ^n) is monotonic and its limit verifies system (3.1). In the beginning, we construct the start point of FBFSDDEs system. Consider the following two general forward equations of FBFSDDEs system:

$$\check{Y}^0(r) = \gamma + \int_0^r f^1(t, \check{Y}^0(t), \check{\Theta}^0(t), \check{Y}^0_t, \check{\Theta}^0_t)dt + \int_0^r \left[\acute{f}(t, \check{Y}^0(t), \check{Y}^0_t) - \check{\Theta}^0(t)\right] dW(t), \tag{4.1}$$

$$Y^0(r) = \gamma + \int_0^r f^2(t, Y^0(t), \Theta^0(t), {Y^0}_t, {\Theta^0}_t)dt + \int_0^r \left[\acute{f}(t, Y^0(t), {Y^0}_t) - \Theta^0(t)\right] dW(t). \tag{4.2}$$

From lemma 3.4, the Equations (4.1) and (4.2) have unique solutions $(\check{Y}^0, \check{\Theta}^0)$ and (Y^0, Θ^0), respectively, which satisfy $\|\check{Y}^0\|_{S^2} + \|\check{\Theta}^0\|_{Q^2} \leq C_1$ and $\|Y^0\|_{S^2} + \|\Theta^0\|_{Q^2} \leq C_1$. By lemma 4.4, we have $\check{Y}^0 \leq Y^0$. Also, we consider the following two general backward equations of FBFSDDEs system:

$$\check{\Psi}^0(r) = J\left(\check{Y}^0(T)\right) + \int_r^T g(t, \check{\Psi}^0(t), \check{\Theta}^0(t), \check{\Psi}^0_t, \check{\Theta}^0_t)dt - \int_r^T \check{\Theta}^0(t)dW(t), \tag{4.3}$$

$$\Psi^0(r) = J\left(Y^0(T)\right) + \int_r^T g(t, \Psi^0(t), \Theta^0(t), {\Psi^0}_t, {\Theta^0}_t)dt - \int_r^T \Theta^0(t)dW(t), \tag{4.4}$$

From lemma 3.4, the Equations (4.3) and (4.4) have unique solutions $(\check{\Psi}^0, \Theta^0)$ and (Ψ^0, Θ^0), respectively, which satisfy $\|\check{\Psi}^0\|_{S^2} + \|\check{\Theta}^0\|_{Q^2} \leq C_2$ and $\|\Psi^0\|_{S^2} + \|\Theta^0\|_{Q^2} \leq C_2$. By lemma 4.5, we have $\check{\Psi}^0 \leq \Psi^0$. Now, we will prove that the following backward of FBFSDDEs system has a solution (Ψ^1, Θ^1):

$$\Psi^1(r) = J\left(Y^0(T)\right) + \int_r^T g(t, \Psi^1(t), \Theta^1(t), {\Psi^1}_t, {\Theta^1}_t)dt - \int_r^T \Theta^1(t)dW(t),\ r \in [0,T]. \tag{4.5}$$

From Lemma 3.4, the following backward of FBFSDDEs system has an unique solution $(\Psi^{1,m}, \Theta^{1,m}),\ m \geq 1$.

$$\Psi^{1,m}(r) = J\left(Y^0(T)\right) + \int_r^T g^{1,m}(t, \Psi^{1,m}(t), \Theta^{1,m}(t), {\Psi^{1,m}}_t, {\Theta^{1,m}}_t)dt - \int_r^T \Theta^{1,m}(t)dW(t),\ r \in [0,T]. \tag{4.6}$$

From lemma 4.5, it follows that $\check{\Psi}^0(r) \leq \Psi^{1,m+1}(r) \leq \Psi^{1,m}(r) \leq \Psi^0(r),\ r \in [0,T]$. Apply Ito's formula to $|\Psi^{1,m}(r)|^2$, we have

$$|\Psi^{1,m}(r)|^2 + \int_r^T |\Theta^{1,m}(t)|^2 dt = |J\left(Y^0(T)\right)|^2 + 2\int_r^T \Psi^{1,m}(t) g^{1,m}(t, \Psi^{1,m}(t), \Theta^{1,m}(t), {\Psi^{1,m}}_t, {\Theta^{1,m}}_t) dt$$
$$- 2\int_r^T \Psi^{1,m}(t)\Theta^{1,m}(t) dW(t). \tag{4.7}$$

By taking expectations and applying inequality $2xy \leq \frac{1}{c}x^2 + cy^2, c > 0$, we have

$$\begin{aligned}
E\left[|\Psi^{1,m}(r)|^2\right] &+ E\left[\int_r^T |\Theta^{1,m}(t)|^2 dt\right] = E\left[|J\left(Y^0(T)\right)|^2\right] \\
&+ 2E\left[\int_r^T \Psi^{1,m}(t) g^{1,m}(t, \Psi^{1,m}(t), \Theta^{1,m}(t), {\Psi^{1,m}}_t, {\Theta^{1,m}}_t) dt\right] \leq E\left[|J\left(Y^0(T)\right)|^2\right] \\
&+ \frac{1}{c}E\left[\int_r^T |\Psi^{1,m}(t)|^2 dt\right] + cE\left[\int_r^T |g^{1,m}(t, \Psi^{1,m}(t), \Theta^{1,m}(t), {\Psi^{1,m}}_t, {\Theta^{1,m}}_t)|^2 dt\right] \\
&\leq E\left[|J\left(Y^0(T)\right)|^2\right] + \frac{1}{c}E\left[\int_r^T |\Psi^{1,m}(t)|^2 dt\right] + cK_5 E\left[\int_r^T (1 + |\Psi^{1,m}(t)|^2 + |\Theta^{1,m}(t)|^2) dt\right] \\
&+ cN_5 E\left[\int_r^T (\int_{-T}^0 |\Psi^{1,m}(t+\tau)|^2 \lambda d\tau + \int_{-T}^0 |\Theta^{1,m}(t+\tau)|^2 \lambda d\tau) dt\right] \\
&\leq D_1\left(1 + E\left[\int_r^T |\Psi^{1,m}(t)|^2 dt\right]\right) + (cK_5 + cN_5\rho) E\left[\int_r^T |\Theta^{1,m}(t)|^2 dt\right],
\end{aligned}$$

where D_1 depends on $K_5,\ N_5$ and C_2. Therefore, for any $r \in [0,T]$, we have

$$E\left[|\Psi^{1,m}(r)|^2\right] + (1 - cK_5 - cN_5\rho) E\left[\int_r^T |\Theta^{1,m}(t)|^2 dt\right] \leq D_1\left(1 + E\left[\int_r^T |\Psi^{1,m}(t)|^2 dt\right]\right).$$

According to Gronwall's inequality, we deduce

$$Sup_{r\in[0,T]} E\left[|\Psi^{1,m}(r)|^2\right] \leq D_1,\ E\left[\int_r^T |\Theta^{1,m}(t)|^2 dt\right] \leq D_1.$$

From (4.6), let us take the Sup and then take the expectation

$$\begin{aligned}
E\left[\sup_{r\in[0,T]} |\Psi^{1,m}(r)|^2\right] &\leq E\left|J\left(Y^0(T)\right)\right|^2 + 2E\int_0^T \Psi^{1,m}(t)\, g^{1,m}\left(t, \Psi^{1,m}(t), \Theta^{1,m}(t), {\Psi^{1,m}}_t, {\Theta^{1,m}}_t\right) dt \\
&+ \left(E\left[\sup_{r\in[0,T]} |\Psi^{1,m}(r)|^2\right]\right)^{\frac{1}{2}} \left(E\left[\sup_{r\in[0,T]} \left|\int_r^T \Theta^{1,m}(t) dW(t)\right|^2\right]\right)^{\frac{1}{2}} \\
&\leq K_8 E\left(1 + |Y^0(T)|^2\right) + N_7 E(\int_{-T}^0 |Y^0(t+\tau)|^2 \lambda d\tau + \frac{1}{c}E\left[\int_0^T |\Psi^{1,m}(t)|^2 dt\right] \\
&+ cE\left[\int_0^T \left|g^{1,m}\left(t, \Psi^{1,m}(t), \Theta^{1,m}(t), {\Psi^{1,m}}_t, {\Theta^{1,m}}_t\right)\right|^2 dt\right] \\
&+ \frac{1}{c}E\left[\sup_{r\in[0,T]} |\Psi^{1,m}(r)|^2\right] + cD_3 E\left[\int_0^T |\Theta^{1,m}(t)|^2 dt\right] \leq K_8 E\left(1 + |Y^0(T)|^2\right) \\
&+ N_7\rho + \frac{1}{c}E\left[\int_0^T |\Psi^{1,m}(t)|^2 dt\right] + cK_5 E\left[\int_0^T (1 + |\Psi^{1,m}(t)|^2 + |\Theta^{1,m}(t)|^2) dt\right] \\
&+ cN_5 E\left[\int_0^T (\int_{-T}^0 |\Psi^{1,m}(t+\tau)|^2 \lambda d\tau + \int_{-T}^0 |\Theta^{1,m}(t+\tau)|^2 \lambda d\tau) dt\right] \\
&+ \frac{1}{c}E\left[\sup_{r\in[0,T]} |\Psi^{1,m}(r)|^2\right] + cD_3 E\left[\int_0^T |\Theta^{1,m}(t)|^2 dt\right] \leq K_8 + K_8 E|Y^0(T)|^2
\end{aligned}$$

$$+ N_7\rho + \frac{1}{c}E\left[\int_0^T |\Psi^{1,m}(t)|^2 dt\right] + cK_5E\left[\int_0^T |Y^0(t)|^2 dt\right] + cK_5E\left[\int_0^T |\Psi^{1,m}(t)|^2 dt\right]$$
$$+ cK_5E\left[\int_0^T |\Theta^{1,m}(t)|^2 dt\right] + cK_5\rho E\left[\int_0^T |\Psi^{1,m}(t)|^2 dt\right] + cK_5\rho E\left[\int_0^T |\Theta^{1,m}(t)|^2 dt\right]$$
$$+ \frac{1}{c}E\left[\sup_{r\in[0,T]}|\Psi^{1,m}(r)|^2\right] + cD_3E\left[\int_0^T |\Theta^{1,m}(t)|^2 dt\right]$$
$$\leq \frac{1}{c}E\left[\sup_{r\in[0,T]}|\Psi^{1,m}(r)|^2\right] + (\frac{1}{c} + cK_5 + cK_5\rho)E\left[\int_0^T |\Psi^{1,m}(t)|^2 dt\right] + D_1.$$

Then,

$$E\left[\sup_{r\in[0,T]}|\Psi^{1,m}(r)|^2\right] \leq KE\left[\int_0^T |\Psi^{1,m}(t)|^2 dt\right] + D_1.$$

From (4.6), we have

$$E\left[\sup_{r\in[0,T]}|\Psi^{1,m}(r)|^2\right] \leq D_1.$$

From the monotone bounded convergence theorem, we deduce that $\lim_{m\to\infty} \Psi^{1,m} = \Psi^1$. From Fatou's theorem we deduce that $E\left[\sup_{r\in[0,T]} |\Psi^1(r)|^2\right] \leq D_1$. Using the theorems of dominated convergence and Dini's, we get $E\left[\sup_{r\in[0,T]} |\Psi^{1,m}(r) - \Psi^1(r)|^2\right] \to 0,\ m \to \infty$. Applying Ito's formula to $|\Psi^{1,j}(r) - \Psi^{1,m}(r)|^2$ and by Hölder's inequality, we have

$$E\left[\left|\Psi^{1,j}(0) - \Psi^{1,m}(0)\right|^2\right] + E\left[\int_0^T \left|\Theta^{1,j}(t) - \Theta^{1,m}(t)\right|^2 dt\right]$$
$$\leq 2(E\left[\int_0^T |\Psi^{1,j}(r) - \Psi^{1,m}(r)|^2 dt\right])^{\frac{1}{2}} (E\Big[\int_0^T |g^{1,j}(t, \Psi^{1,j}(t), \Theta^{1,j}(t), {\Psi^{1,j}}_t, {\Theta^{1,j}}_t)$$
$$- g^{1,m}(t, \Psi^{1,m}(t), \Theta^{1,m}(t), {\Psi^{1,m}}_t, {\Theta^{1,m}}_t)|^2 dt\Big])^{\frac{1}{2}}$$
$$\leq \frac{1}{c}E\left[\int_0^T |\Psi^{1,j}(r) - \Psi^{1,m}(r)|^2 dt\right] + cE\Big[\int_0^T |g^{1,j}(t, \Psi^{1,j}(t), \Theta^{1,j}(t), {\Psi^{1,j}}_t, {\Theta^{1,j}}_t)$$
$$- g^{1,m}(t, \Psi^{1,m}(t), \Theta^{1,m}(t), {\Psi^{1,m}}_t, {\Theta^{1,m}}_t)|^2 dt\Big] \leq \frac{1}{c}E\left[\int_0^T |\Psi^{1,j}(r) - \Psi^{1,m}(r)|^2 dt\right]$$
$$+ cK_2E\left[\int_0^T |\Psi^{1,j}(r) - \Psi^{1,m}(r)|^2 dt\right] + cK_2E\left[\int_0^T |\Theta^{1,j}(r) - \Theta^{1,m}(r)|^2 dt\right]$$
$$+ cN_2\rho E\left[\int_0^T |\Psi^{1,j}(r) - \Psi^{1,m}(r)|^2 dt\right] + cN_2\rho E\left[\int_0^T |\Theta^{1,j}(r) - \Theta^{1,m}(r)|^2 dt\right]$$
$$= (\frac{1}{c} + cK_2 + cN_2\rho)E\left[\int_0^T |\Psi^{1,j}(r) - \Psi^{1,m}(r)|^2 dt\right]$$
$$+ (cK_2 + cN_2\rho)E\left[\int_0^T |\Theta^{1,j}(r) - \Theta^{1,m}(r)|^2 dt\right].$$

Because $\{\Theta^{1,m}\}_{m\geq 1}$ is a Cauchy sequence in $Q^2(\Omega, F, P, \mathbb{R}^{a\times b})$, then

$$E\left[\int_0^T (|\Psi^{1,j}(r) - \Psi^{1,m}(r)|^2 + |\Theta^{1,j}(r) - \Theta^{1,m}(r)|^2)dt\right] \to 0,\ as\ j, m \to \infty$$

this means $\lim_{m\to\infty} \Theta^{1,m} = \Theta^1$. Hence, (Ψ^1, Θ^1) is a solution of backward of FBFSDDEs system. Using the same proof technique above it is possible to prove (Y^1, Θ^1) is a solution of forward of FBFSDDEs system. Next, it is obvious that the following backward of FBFSDDEs system has a unique solution

$(\Psi^{2,m}, \Theta^{2,m}),\ m \geq 1$:

$$\Psi^{2,m}(r) = J\left(Y^1(T)\right) + \int_r^T g^{2,m}(t, \Psi^{2,m}(t), \Theta^{2,m}(t), \Psi^{2,m}{}_t, \Theta^{2,m}{}_t)dt - \int_r^T \Theta^{2,m}(t)dW(t),\ r \in [0, T].$$

We kow $Y^1(T) \leq Y^0(T)$, we deduce $J\left(Y^1(T)\right) \leq J\left(Y^0(T)\right)$. Moreover, from Lemma 4.5, it follows that $\Psi^{2,m}(r) \leq \Psi^{1,m}(r)$, then $\Psi^2(r) \leq \Psi^1(r)$ as $m \to \infty$. Similarly, we prove that (Y^2, Θ^2) is the solution of forward of FBFSDDEs system:

$$Y^2(r) = \gamma + \int_0^r f(t, Y^2(t), \Theta^2(t), Y^2{}_t, \Theta^2{}_t)dt + \int_0^r \left[\acute{f}(t, Y^2(t), Y^2{}_t) - \Theta^2(t)\right] dW(t),$$

and $\check{Y}^0(r) \leq Y^2(r) \leq Y^1(r) \leq Y^0(r)$, for all $r \in [0, T]$. By the same procedure, we get the existence of a sequence (Y^m, Ψ^m, Θ^m) which satisfies (3.3). Therefore, for any $r \in [0, T]$ such that

$$\check{Y}^0(r) \leq \cdots \leq Y^{m+1}(r) \leq Y^m(r) \leq \cdots \leq Y^2(r) \leq Y^1(r) \leq Y^0(r),$$

$$\check{\Psi}^0(r) \leq \cdots \leq \Psi^{m+1}(r) \leq \Psi^m(r) \leq \cdots \leq \Psi^2(r) \leq \Psi^1(r) \leq \Psi^0(r).$$

Therefore, we deduce $Y^1 = \lim_{m\to\infty} Y^{1,m}$ and $\Psi^1 = \lim_{m\to\infty} \Psi^{1,m}$. Using the same proof technique above, we deduce $\Theta^1 = \lim_{m\to\infty} \Theta^{1,m}$. Hence, we conclude that $(\mathrm{Y},\ \Psi, \Theta) \in S^2(\Omega, F, P, \mathbb{R}^a) \times S^2(\Omega, F, P, \mathbb{R}^b) \times Q^2(\Omega, F, P, \mathbb{R}^{a\times b})$ is a solution of FBFSDDEs system (3.1).

Finally, Let us assume that $(\widetilde{Y},\ \widetilde{\Psi}, \widetilde{\Theta}) \in S^2(\Omega, F, P, \mathbb{R}^a) \times S^2(\Omega, F, P, \mathbb{R}^b) \times Q^2(\Omega, F, P, \mathbb{R}^{a\times b})$ is an arbitrary solution of the system (3.1). From lemma 4.4, we conclude $\check{Y}^0(r) \leq \widetilde{Y}^0(r) \leq Y^0(r), r \in [0, T]$. From $J\left(\widetilde{Y}(T)\right) \leq J\left(Y^0(T)\right)$, and then using lemma 4.5, we deduce immediately that $\check{\Psi}^0(r) \leq \widetilde{\Psi}^0(r) \leq \Psi^0(r), r \in [0, T]$. Therefore, it follows that $\Psi^1(r) \leq \Psi^0(r), r \in [0, T]$. Repeating the same procedure, we conclude

$$\check{Y}^0(r) \leq \widetilde{Y}^0(r) \leq \cdots \leq Y^m(r) \leq \cdots \leq Y^2(r) \leq Y^1(r) \leq Y^0(r),$$

$$\check{\Psi}^0(r) \leq \widetilde{\Psi}^0(r) \leq \cdots \leq \Psi^m(r) \leq \cdots \leq \Psi^2(r) \leq \Psi^1(r) \leq \Psi^0(r).$$

It implies that $\widetilde{Y} \leq Y$ and $\widetilde{\Psi} \leq \Psi$, that is the system of FBFSDDEs (3.1) has a maximal solution $(Y,\ \Psi, \Theta) \in S^2(\Omega, F, P, \mathbb{R}^a) \times S^2(\Omega, F, P, \mathbb{R}^b) \times Q^2(\Omega, F, P, \mathbb{R}^{a\times b})$. □

Theorem 4.7. *Under the assumptions (H1-H5), the FBFSDDEs system (3.1) has a unique solution (Y, Ψ, Θ).*

Proof. Let (Y^1, Ψ^1, Θ^1) and (Y^2, Ψ^2, Θ^2) be two solution of FBFSDDEs system (3.1). We set $(\overline{Y}, \overline{\Psi}, \overline{\Theta}) = (Y^1 - Y^2, \Psi^1 - \Psi^2, \Theta^1 - \Theta^2)$. Applying Itô's formula to $\overline{Y}, \overline{\Psi}$ and by using inequality $xy \leq \frac{1}{2c}x^2 + \frac{c}{2}y^2,\ c > 0$, we have

$$\begin{aligned}
E\left[\overline{Y^m}(T)J\left(\overline{Y^m}(T)\right)\right] \leq E\Bigg[\int_0^T & \overline{\Psi^m}(t)(f(t, Y^{2,m}(t), \Theta^{2,m}(t), Y^{2,m}{}_t, \Theta^{2,m}{}_t) \\
& - f(t, Y^{1,m}(t), \Theta^{1,m}(t), Y^{1,m}{}_t, \Theta^{1,m}{}_t))\ dt\Bigg] \\
& + E\Bigg[\int_0^T \overline{Y^m}(t)(g(t, \Psi^{2,m}(t), \Theta^{2,m}(t), \Psi^{2,m}{}_t, \Theta^{2,m}{}_t) \\
& - g(t, \Psi^{1,m}(t), \Theta^{1,m}(t), \Psi^{1,m}{}_t, \Theta^{1,m}{}_t))dt\Bigg] \\
& + E\left[\int_0^T |\Theta^{2,m}(t) - \Theta^{1,m}(t)||\acute{f}(t, Y^{2,m}(t), Y^{2,m}{}_t) - \acute{f}(t, Y^{1,m}(t), Y^{1,m}{}_t)|dt\right]
\end{aligned}$$

$$\leq \frac{1}{2c}E\left[\int_0^T |\overline{\Psi^m}(t)|^2 dt\right] + \frac{c}{2}E\Big[\int_0^T |f(t, Y^{2,m}(t), \Theta^{2,m}(t), Y^{2,m}{}_t, \Theta^{2,m}{}_t)$$

$$- f(t, Y^{1,m}(t), \Theta^{1,m}(t), Y^{1,m}{}_t, \Theta^{1,m}{}_t)|^2 dt\Big] + \frac{1}{2c}E\left[\int_0^T |\overline{Y^m}(t)|^2 dt\right]$$

$$+ \frac{c}{2}E\left[\int_0^T |g(t, \Psi^{2,m}(t), \Theta^{2,m}(t), \Psi^{2,m}{}_t, \Theta^{2,m}{}_t) - g(t, \Psi^{1,m}(t), \Theta^{1,m}(t), \Psi^{1,m}{}_t, \Theta^{1,m}{}_t)|^2 dt\right]$$

$$+ \frac{1}{2c}E\left[\int_0^T |\overline{\Theta^m}(t)|^2 dt\right] + \frac{c}{2}E\left[\int_0^T |\acute{f}(t, Y^{2,m}(t), Y^{2,m}{}_t) - \acute{f}(t, Y^{1,m}(t), Y^{1,m}{}_t)|^2 dt\right]$$

$$\leq (\frac{cK_1}{2} + \frac{cN_1\rho}{2} + \frac{1}{2c} + \frac{cK_3}{2} + \frac{cN_3\rho}{2})E\left[\int_0^T |\overline{Y^m}(t)|^2 dt\right] + (\frac{1}{2c} + \frac{cK_2}{2} + \frac{cN_2\rho}{2})E\left[\int_0^T |\overline{\Psi^m}(t)|^2 dt\right]$$

$$+ (\frac{cK_1}{2} + \frac{cN_1\rho}{2} + \frac{cK_2}{2} + \frac{cN_2\rho}{2} + \frac{1}{2c})E\left[\int_0^T |\overline{\Theta^m}(t)|^2 dt\right].$$

On the other hand, by Cauchy sequence, we have

$$E\left[\int_0^T |\overline{Y^m}(t)|^2 dt + \int_0^T |\overline{\Psi^m}(t)|^2 dt + \int_0^T |\overline{\Theta^m}(t)|^2 dt\right] \to 0, asm \to \infty.$$

Therefore, $E\left[\int_0^T |(Y^1(t), \Psi^1(t), \Theta^1(t)) - (Y^2(t), \Psi^2(t), \Theta^2(t))|^2 dt\right] = 0$, this means

$$(Y^1(t), \Psi^1(t), \Theta^1(t)) = (Y^2(t), \Psi^2(t), \Theta^2(t)).$$

□

5. Conclusion

This paper presents, under some suitable conditions, an ideal solution to overcome some of the risks of stocks in the stock market by proposing a fuzzy model for doubly stochastic differential equations in forward-backward forms. This solution is also used to find solutions to more complex problems.

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