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Existence of multiple nontrivial solutions for $2n$-th order discrete boundary value problem

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ABSTRACT: We established the existence of multiple solutions for a class of nonlinear $2n$-th-order discrete boundary value problems by using variational methods and critical point theory. One application is included to illustrate our theoretical finding.

Key Words: Discrete boundary value problems, $2n$-th order, variational methods, critical point theory.

Contents

1. Introduction

Let $n \geq 1$ be a positive integer. In this work we study the following nonlinear $2n$-th order boundary value problem

$$\begin{cases} \sum_{k=0}^{n}(-1)^k\Delta^{2k}w(t-k) = g(t,w(t)), & t\in[1,N]_{\mathbb{Z}}, \\ \Delta^i w(-(n-1)) = \Delta^i w(N-(n-1)), & i\in[0,2n-1]_{\mathbb{Z}}, \end{cases} \tag{1.1}$$

where $N \geq n$ is an integer, $[1,N]_{\mathbb{Z}}$ denotes the discrete interval $\{1,2,...,N\}$, Δ is the forward difference operator defined by $\Delta u(t) = u(t+1)-u(t)$, $\Delta^0 u(t) = u(t)$, $\Delta^i u(t) = \Delta^{i-1}(\Delta u(t))$ for $i = 1,2,3,...,2n$ and $g : [1,N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function in the second variable, i.e., for any fixed $t \in [1,N]_{\mathbb{Z}}$ a function $g(t,.)$ is continuous.
A solution of (1.1) is a function $w : [-(n-1), N+n]_{\mathbb{Z}} \longrightarrow \mathbb{R}$ which satisfies both equations of (1.1).

Let λ_0, λ_1,..., λ_{N-1} be the eigenvalues of the linear boundary value problem corresponding to the problem (1.1)

$$\begin{cases} \sum_{k=0}^{n}(-1)^k\Delta^{2k}w(t-k) = \lambda w(t), & t\in[1,N]_{\mathbb{Z}}, \\ \Delta^i w(-(n-1)) = \Delta^i w(N-(n-1)), & i\in[0,2n-1]_{\mathbb{Z}}. \end{cases} \tag{1.2}$$

From [4], it can be concluded that the issue (1.2) has precisely N real eigenvalues λ_j, $j \in [0,N-1]_{\mathbb{Z}}$ which satisfiy

$$\begin{cases} \lambda_j = \mu_0 + 2\sum_{l=1}^{n}\mu_l cos(\frac{2\pi lj}{N}),\ j\in[0,N-1]_{\mathbb{Z}}, \\ \lambda_j = \lambda_{N-j},\ j\in[1,N-1]_{\mathbb{Z}}, \end{cases}$$

with $\mu_l = (-1)^l\sum_{j=l}^{n} C_{2j}^{j+l}$ for any $l \in [0,n]_{\mathbb{Z}}$.

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Since $\lambda_j = \lambda_{N-j}$, for any $j \in [1, N-1]_{\mathbb{Z}}$ the problem (1.2) has $q+1$ different eigenvalues, where $q = \frac{N-1}{2}$ when N is odd, or $q = \dfrac{N}{2}$ when N is even. Consequently, this notation can be used to write these eigenvalues in the following way

$$0 < \lambda_0 < \lambda_1 < ... < \lambda_q.$$

In the sequel φ_j, for any $j \in [0, N-1]_{\mathbb{Z}}$, denotes the correspondent orthonormal eigenfunction of λ_j. Further, the symbol span$\{\varphi_1, \varphi_2 ..., \varphi_j\}$ stands for the $\mathbb{R}$-linear subspace of E_N generated by $\varphi_1, \varphi_2, ..., \varphi_j$, where E_N is defined in (2.1).

We want to focus on the fact that the problem (1.1) can be seen as discrete counterpart of the following $2n$-th order differential equation

$$\begin{cases} \sum\limits_{k=0}^{n} (-1)^k \dfrac{d^{2k} w(t)}{dt^{2k}} = g(t, w(t)), \quad t \in]0,1[\,, \\ \qquad w^{(i)}(0) = w^{(i)}(1), \quad i \in [0, 2n-1]_{\mathbb{Z}}. \end{cases}$$

It is widely recognized that effective mathematical modeling of significant problems across various research fields, such as computer science, neural networks, biological systems, and population dynamics, is based on nonlinear difference equations, and the study leads to enormous findings. Many existing results of nontrivial solutions for differential equations have been obtained in recent years due to the fast development of studying the boundary value problems for differential equations, where various methods and techniques have been used, for example, fixed point theorems methods, coincidence degree theory, and topological degree theory. For more details, see [1, 2, 4, 5, 6, 7, 11, 14]. Critical point theory and variational methods are powerful tools to investigate the existence of solutions to various problems in differential equations [8, 9, 10, 12, 13, 15 − 25] .

This paper is concerned with proving the existence and multiplicity of critical point theory and variational techniques solutions to discrete nonlinear 2n-th order problems (1.1). The rest of this paper is organized as follows: Section 2 contains some preliminary lemmas, whereas the main results will be proved in section 3. Section 4 includes an illustrative application.

The following theorems are the key findings of this paper.

Theorem 1.1 *Assume that*

(G_1) *there exists* γ *with* $\gamma < \sum\limits_{l=0}^{n} C_{2l}^{l} + 2\sum\limits_{l=1}^{n}\sum\limits_{k=l}^{n} (-1)^l C_{2k}^{k+l}$ *such that*

$$\limsup_{|x|\to\infty} \frac{2G(t,x)}{x^2} \leq \gamma \quad \textit{for every } t \in [1, N]_{\mathbb{Z}},$$

where $G(t,x) = \int_0^x g(t,s)ds$ *for* $(t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}$*;*

(G_2) *there exists an integer* $p \in [1, q]_{\mathbb{Z}}$ *such that*

$$\liminf_{x\to 0} \frac{2G(t,x)}{x^2} > \sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n} (-1)^l C_{2k}^{k+l} \cos(\frac{2\pi l p}{N}), \quad \textit{for every } t \in [1, N]_{\mathbb{Z}},$$

where $q = \frac{N-1}{2}$ *when* N *is odd, or* $q = \frac{N}{2}$ *when* N *is even.*

Then, the problem (1.1) *has at least one nontrivial solution.*

Theorem 1.2 *Assume that* (G_1) *and* (G_3) *hold, where*

(G_3) *there exists an integer* $p \in [1, q]_{\mathbb{Z}}$ *such that*

$$\liminf_{x\to 0} \frac{2G(t,x)}{x^2} > \sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l}\cos\left(\frac{2\pi lp}{N}\right)$$

and

$$\limsup_{x\to 0} \frac{2G(t,x)}{x^2} < \sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l}\cos\left(\frac{2\pi l(p+1)}{N}\right),$$

for every $t \in [1, N]_{\mathbb{Z}}$.

Then, the problem (1.1) *has at least two nontrivial solutions.*

Theorem 1.3 *Assume that* (G_1), (G_4) *and* (G_5) *hold, where*

(G_4) $\liminf_{x\to 0} \frac{2G(t,x)}{x^2} > \sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l}\cos\left(\frac{2\pi lq}{N}\right)$, *for every* $t \in [1, N]_{\mathbb{Z}}$;

(G_5) $g(t,x)$ *is odd in* x, *i.e.,* $g(t,-x) = -g(t,x)$ *for* $(t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}$.

Then the problem (1.1) *has at least* $2q$ *nontrivial solutions.*

2. Preliminary lemmas

In the present paper, we define a vector space E_N by

$$E_N = \{w : [-(n-1), N+n]_{\mathbb{Z}} \longrightarrow \mathbb{R} \mid \Delta^i w(-(n-1)) = \Delta^i w(N-(n-1)),\ \ i = 0,1,2,3,...,2n-1\}, \quad (2.1)$$

E_N can be equipped with inner product $\langle .,. \rangle$ and norm $\|.\|$ as follows:

$$\langle w, v\rangle = \sum_{t=1}^{N} w(t)v(t), \qquad \forall w, v \in E_N$$

$$\|w\| = \left(\sum_{t=1}^{N} |w(t)|^2\right)^{1/2}, \quad \forall w \in E_N.$$

We also put, for every $w \in E_N$

$$\|w\|_\infty = \max_{t\in[1,N]_{\mathbb{Z}}} |w(t)|. \quad (2.2)$$

Remark 2.1 *It is easy to see that, for any* $w \in E_N$, *we have*

$$\begin{aligned}
w(-(n-1)) &= w(N-(n-1))\\
w(-(n-1)+1) &= w(N-(n-1)+1)\\
w(-(n-1)+2) &= w(N-(n-1)+2)\\
\vdots \quad & \quad \vdots\\
w(0) &= w(N)\\
w(1) &= w(N+1)\\
\vdots \quad & \quad \vdots\\
w(n) &= w(N+n).
\end{aligned} \quad (2.3)$$

Clearly, $(E_N, \|.\|)$ is an N dimensional reflexive Banach space. Since it is isomorphic to the finite dimensional space $\mathbb{R}^N$. When we say that the vector $w = (w(1), ..., w(N)) \in \mathbb{R}^N$, we understand that w can be extended to a vector in E_N so that (2.3) holds, that is, w can be extended to the vector

$$(w(N-(n-1)), w(N-(n-1)+1), ..., w(N), w(1), w(2), ..., w(N), w(1), ..., w(n)) \in E_N$$

and when we write $E_N = \mathbb{R}^N$, we mean the elements in $\mathbb{R}^N$ have been extended in the above sense.

For $w \in E_N$, let the functional Ψ be denoted by

$$\Psi(w) = \frac{1}{2}\sum_{t=1}^{N}\sum_{k=0}^{n}|\Delta^k w(t-k)|^2 - \sum_{t=1}^{N} G(t, w(t)), \tag{2.4}$$

Then, it is easy to see that $\Psi \in C^1(E_N, \mathbb{R})$ and its derivative $\Psi'(w)$ at $w \in E_N$ is given by

$$\Psi'(w).v = \sum_{t=1}^{N}\left[\sum_{k=0}^{n}\Delta^k w(t-k)\Delta^k v(t-k) - g(t, w(t))v(t)\right] \text{ for any } v \in E_N. \tag{2.5}$$

By [4, Lemma 2.3], Ψ' can be written as

$$\Psi'(w).v = \sum_{t=1}^{N}\left[\sum_{k=0}^{n}(-1)^k\Delta^{2k} w(t-k) - g(t, w(t))\right] v(t), \text{ for any } v \in E_N.$$

Thus, finding solutions of (1.1) is equivalent to finding critical point of the functional Ψ.

From [4], Ψ can be rewritten as

$$\Psi(w) = \frac{1}{2}\left\langle \sum_{k=0}^{n} A_k w, w\right\rangle - \sum_{t=1}^{N} G(t, w(t)), \tag{2.6}$$

where

$$\sum_{k=0}^{n} A_k = \begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n & a_{n+1} & a_{n+2} & \cdots & a_{N-(n+1)} & a_{N-n} & a_{N-(n-1)} & \cdots & a_{N-2} & a_{N-1} \\
a_{N-1} & a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} & a_n & a_{n+1} & \cdots & \vdots & \vdots & a_{N-n} & \cdots & a_{N-3} & a_{N-2} \\
a_{N-2} & a_{N-1} & a_0 & \cdots & a_{n-3} & a_{n-2} & a_{n-1} & a_n & \cdots & \vdots & \vdots & \vdots & \cdots & a_{N-4} & a_{N-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
a_3 & a_4 & a_5 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & a_0 & a_1 & a_2 \\
a_2 & a_3 & a_4 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & a_{N-1} & a_0 & a_1 \\
a_1 & a_2 & a_3 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & a_{N-2} & a_{N-1} & a_0
\end{pmatrix}_{N\times N},$$

with

$$\begin{aligned} a_l &= (-1)^l \sum_{j=l}^{n} C_{2j}^{j+l}, && l \in [0,n]_{\mathbb{Z}}, \\ a_l &= 0, && l \in [n+1, N-(n+1)]_{\mathbb{Z}}, \\ a_l &= (-1)^{N-l} \sum_{j=N-l}^{n} C_{2j}^{j+N-l}, && l \in [N-n, N-1]_{\mathbb{Z}}. \end{aligned}$$

Remark 2.2 *The eigenvalues of the problem* $(1,2)$ *are exactly the eigenvalues of the matrix* $\sum_{k=0}^{n} A_k$*, then for every* $w \in E_N$*,*

$$\left[\sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l}\right]\|w\|^2 \leq \left\langle \sum_{k=0}^{n} A_k w, w \right\rangle \leq \left[\sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l}\cos\left(\frac{2\pi l q}{N}\right)\right]\|w\|^2.$$

The space of continuously Fréchet-differentiable functionals from E into $\mathbb{R}$ is denoted by $C^1(E,\mathbb{R})$.

Definition 2.1 Let E be a real Banach space, and $\Psi \in C^1(E,\mathbb{R})$. Ψ is said to satisfy the Palais-Smale (PS) condition if any sequence $(y_t) \subset E$ for which $(\Psi(y_t))$ is bounded and $\Psi'(y_t) \longrightarrow 0$ as $t \to \infty$, possesses a convergent subsequence. The sequence (y_t) is called a (PS) sequence.

Lemma 2.1 (*see* [3]) *Let* E *be a Banach space and* $E = E_1 \bigoplus E_2$ *where* E_2 *is a finite dimensional subspace of* E*. Suppose that* $\Psi \in C^1(E,\mathbb{R})$ *satisfies the* (PS) *condition,* $\Psi(0) = 0$ *and for some* $r > 0$

$$\Psi(y) \geq 0, \quad \text{for } y \in E_1,\ \|y\| \leq r,$$

$$\Psi(y) \leq 0, \quad \text{for } y \in E_2,\ \|y\| \leq r.$$

Assume also that Ψ *is bounded below and* $\inf_{u\in E} \Psi(y) < 0$*. Then* Ψ *possesses at least two nontrivial critical points.*

Lemma 2.2 (*see* [5]) *Let* E *be a Banach space and let* $\Phi \in C^1(E;\mathbb{R})$ *be an even, bounded from below functional that satisfies the Palais–Smale (PS) condition. Suppose that* $\Phi(0) = 0$ *and there is a subset* $K \subset E$ *such that* K *is homeomorphic to* S^{r-1} *via an odd mapping, where* S^{r-1} *is the* $r-1$ *dimensional unit sphere and* $\sup_{u\in K} \Phi(u) < 0$*. Then* Φ *has at least* r *disjoint pairs of nontrivial critical points.*

3. Proof of the main results

Proof of Theorem 1.1. From (G_1) there exixts a constant $\sigma > 0$ such that

$$\frac{2G(t,x)}{x^2} \leq \gamma + \varepsilon, \quad \text{for every } (t,|x|) \in [1,N]_{\mathbb{Z}} \times]\sigma, +\infty[,$$

where $\varepsilon > 0$ stisfying

$$\varepsilon < \sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l} - \gamma. \tag{3.1}$$

Thus,

$$G(t,x) \leq \frac{1}{2}(\gamma+\varepsilon)x^2, \quad \text{for every } (t,|x|) \in [1,N]_{\mathbb{Z}} \times]\sigma, +\infty[, \tag{3.2}$$

On the other hand, by the continuity of $x \longrightarrow G(t,x)$, there exists ξ_t such that

$$|G(t,x)| \leqslant \xi_t, \quad \text{for every } (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,\sigma]. \tag{3.3}$$

For any $w \in E_N$, let $S_1 = \{t \in [1, N]_{\mathbb{Z}} : \ |w(t)| > \sigma\}$ and $S_2 = \{t \in [1, N]_{\mathbb{Z}} : \ |w(t)| \leq \sigma\}$. Using (2.6), (3.2) and (3.3), we have

$$\begin{aligned}\Psi(w) &= \frac{1}{2}\left\langle \sum_{k=0}^{n} A_k w, w\right\rangle - \sum_{t=1}^{N} G(t, w(t)) \\ &\geqslant \frac{1}{2}\left[\sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^{l} C_{2k}^{k+l}\right]\|w\|^2 - \sum_{t\in S_1} G(t, w(t)) - \sum_{t\in S_2} G(t, w(t)) \\ &\geqslant \frac{1}{2}\left[\sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^{l} C_{2k}^{k+l} - \gamma - \varepsilon\right]\|w\|^2 - \sum_{t=1}^{N}\zeta_t,\end{aligned}$$

for any $w \in E_N$.
Then, in view of (3.1), $\Psi(w) \longrightarrow +\infty$ as $\|w\| \to +\infty$ which means that Ψ is coercive and bounded from below, hence there is a minimum point of Ψ at some $w_0 \in E_N$, i.e., $\Psi(w_0) = \inf_{w\in E_N} \Psi(w)$, which is a critical point of Ψ and it is a solution of the problem (1.1).
Now, we prove that w_0 is nontrivial. Using (G_2), there exists $\delta > 0$ such that

$$G(t, x) > \frac{1}{2}\left[\sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^{l} C_{2k}^{k+l}\cos\left(\frac{2\pi l p}{N}\right)\right] x^2 \quad \text{for } (t, |x|) \in [1, N]_{\mathbb{Z}} \times [0, \delta],$$

On the other hand, from the Cauchy Schwartz inequality, one has

$$|w(t)| \leqslant \sum_{t=1}^{N} |w(t)| \leqslant \sqrt{N}\|w\| \quad \text{for every } t \in [1, N]_{\mathbb{Z}}.$$

Therefore, we get

$$\|w\|_\infty = \max_{t\in[1,N]_{\mathbb{Z}}} |w(t)| \leqslant \sqrt{N}\|w\|, \quad \forall w \in E_N.$$

Hence, for every $w \in \bar{B}(0, \rho_1) \bigcap E_1$, where $\rho_1 > 0$ and $E_1 = \text{span}\{\varphi_1, ..., \varphi_p\}$ we have

$$\|\omega\|_\infty \leqslant \sqrt{N}\rho_1.$$

Consequently, if we take $\rho_1 < \dfrac{\delta}{\sqrt{N}}$, we obtain

$$G(t, w(t)) > \frac{1}{2}\left[\sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^{l} C_{2k}^{k+l}\cos\left(\frac{2\pi l p}{N}\right)\right] w^2(t) \quad \text{for every } t \in [1, N]_{\mathbb{Z}}. \tag{3.4}$$

Moreover, if $w \in E_1$, there exist $a_1, a_2, ..., a_p \in \mathbb{R}$, such that $w = \sum_{j=1}^{p} a_j\varphi_j$.
Hence, we obtain

$$\begin{aligned}\left\langle \sum_{k=0}^{n} A_k w, w\right\rangle &= \sum_{j=1}^{p}\left[\sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^{l} C_{2k}^{k+l}\cos\left(\frac{2\pi l j}{N}\right)\right] a_j^2 \\ &\leq \left[\sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^{l} C_{2k}^{k+l}\cos\left(\frac{2\pi l p}{N}\right)\right]\|w\|^2.\end{aligned} \tag{3.5}$$

Using (2.6), (3.4) and (3.5), we have

$$\Psi(w) \leq \frac{1}{2}\left[\sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^{l} C_{2k}^{k+l}\cos\left(\frac{2\pi l p}{N}\right)\right]\left(\|w\|^2 - \|w\|^2\right) = 0, \text{ for every } w \in \overline{B}(0, \rho_1) \cap E_1.$$

Now, if $\Psi(w)=0$, every $w\in E_1$ with $\|w\|\leq\rho_1$ is a solution of the problem (1.1). If $\Psi(w)<0$, for any $w\in\overline{B}(0,\rho_1)\cap E_1$, then $\inf_{w\in E_N}\Psi(w)<0$, therefore $\Psi(w_0)<0$. The proof of Theorem is complete.

Proof of Theorem 1.2.

We will use Lemma 2.1 to prove this result.

From the proof of Theorem 1.1, Ψ is coercive hence every (PS) sequence (u_n) is bounded. In view of the fact that the dimension of E_N is finite, Ψ satisfies (PS) conditions.

Moreover, $\Psi(w)\leq 0$, for every $w\in\overline{B}(0,\rho_1)\cap E_1$ for some $0<\rho_1<\dfrac{\delta}{\sqrt{N}}$, where $E_1=\operatorname{span}\{\varphi_1,...,\varphi_p\}$.

Now, we prove that there exists $\rho_2>0$ such that

$$\Psi(w)\geq 0,\ \text{ for every } w\in\overline{B}(0,\rho_2)\cap E_2 \text{ where } E_2=\operatorname{span}\{\varphi_{p+1},...,\varphi_q\}.$$

Let $w\in E_2$, then there exist b_{p+1}, b_{p+2},..., $b_q\in\mathbb{R}$ such that $w=\sum\limits_{j=p+1}^{q}b_j\varphi_j$.

It is easy to see that

$$\begin{aligned}\left\langle\sum_{k=0}^{n}A_k w,w\right\rangle &=\sum_{j=p+1}^{q}\left[\sum_{l=0}^{n}C_{2l}^{l}+2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l}\cos\left(\frac{2\pi lj}{N}\right)\right]b_j^2\\ &\geq\left[\sum_{l=0}^{n}C_{2l}^{l}+2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l}\cos\left(\frac{2\pi l(p+1)}{N}\right)\right]\|w\|^2.\end{aligned}$$

Therefore, we obtain

$$\Psi(w)\geq\frac{1}{2}\left[\sum_{l=0}^{n}C_{2l}^{l}+2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l}\cos\left(\frac{2\pi l(p+1)}{N}\right)\right]\|\, w\|^2-\sum_{t=1}^{N}G(t,w(t)).$$

Further, from (G_3), there exists $\delta>0$ such that

$$G(t,x)<\frac{1}{2}\left[\sum_{l=0}^{n}C_{2l}^{l}+2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l}\cos\left(\frac{2\pi l(p+1)}{N}\right)\right]x^2,\quad\text{for every }\ (t,|x|)\in[1,N]_{\mathbb{Z}}\times[0,\delta]\,.$$

This, taking $\rho_2<\dfrac{\delta}{\sqrt{N}}$, for every $w\in\overline{B}(0,\rho_2)\cap E_2$, we have

$$\Psi(w)\geq\frac{1}{2}\left[\sum_{l=0}^{n}C_{2l}^{l}+2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l}\cos\left(\frac{2\pi l(p+1)}{N}\right)\right]\left(\|w\|^2-\|w\|^2\right)=0.$$

Thus, by choosing $\rho=\min\{\rho_1,\rho_2\}$, we deduce that

$$\Psi(w)\leq 0,\ \text{ for } w\in\overline{B}(0,\rho)\cap E_1,$$

and

$$\Psi(w)\geq 0,\ \text{ for } w\in\overline{B}(0,\rho)\cap E_2.$$

If $\inf_{w\in E_N}\Psi(w)\geq 0$, then $\Psi(w)=\inf_{w\in E_N}\Psi(w)=0$ for every $w\in\overline{B}(0,\rho)\cap E_1$, which implies that all $w\in E_1$ with $\|w\|\leq\rho$ are solutions of the problem (1.1).

If $\inf_{w\in E_N}\Psi(w)<0$, then all conditions of Lemma 2.1 are satisfied. Then, the problem (1.1) has at least two nontrivial solutions.

Proof of Theorem 1.3. From (G_4), there exists $\mu>0$ such that

$$G(t,x)\geq\frac{1}{2}(F_0-\varepsilon)x^2\quad\text{for }\ (t,|x|)\in[1,N]_{\mathbb{Z}}\times[0,\mu]\,,\tag{3.6}$$

where $F_0 = \liminf_{x\to 0} \frac{2G(t,x)}{x^2}$ and $0 < \varepsilon < F_0 - \left[\sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l}\cos\left(\frac{2\pi l q}{N}\right)\right]$.

Put

$$K = \left\{ w \in E_N \mid \|w\| = \frac{\mu}{\sqrt{N}} \right\}.$$

It is clear to see that $|w(t)| \le \mu$, for any $w \in K$.
Let Ψ be defined in (2.6). Then, for any $w \in K$

$$\begin{aligned}
\Psi(w) &\le \frac{1}{2}\left[\sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l}\cos\left(\frac{2\pi l q}{N}\right)\right]\|w\|^2 - \frac{1}{2}(F_0-\varepsilon)\|w\|^2 \\
&= \frac{1}{2}\frac{\mu^2}{N}\left[\sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l}\cos\left(\frac{2\pi l q}{N}\right) - F_0 + \varepsilon\right] < 0.
\end{aligned}$$

Thus, we obtain $\sup_{w\in K}\Psi(w) < 0$. On the other hand, from the proof of theorem 1.1, Ψ is bounded from below and satisfies the (PS) conditions.
Let S^{q-1} be the unit sphere in $\mathbb{R}^q$ and define

$$T : K \longrightarrow S^{q-1} \quad \text{by} \quad T(w) = \frac{\sqrt{N}}{\mu} w.$$

Then, T is an odd homeomorphism between K and S^{q-1}. Hence, all conditions of Lemma 2.2 are satisfied, so Ψ has at least $2q$ nontrivial critical points, which are nontrivial solutions of problem (1.1).

4. Application

Let us consider the function

$$g(t,x) = \frac{\lambda_p + \lambda_{p+1}}{2}\frac{x}{1+x^4} - tx^3, \quad (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}.$$

where $\lambda_j = \sum_{l=0}^{n} C_{2l}^{l} + 2\sum_{l=1}^{n}\sum_{k=l}^{n}(-1)^l C_{2k}^{k+l}\cos\left(\frac{2\pi l j}{N}\right)$ for $j \in \{p, p+1\}$.
The corresponding primitive is given by

$$G(t,x) = \int_0^x g(t,s)\,ds = \frac{\lambda_p + \lambda_{p+1}}{4}\cdot\arctan(x^2) - \frac{t}{4}x^4.$$

It is obvious that

$$\limsup_{|x|\to\infty} \frac{2G(t,x)}{x^2} = -\infty.$$

On the other hand, we expand $\arctan(x^2)$ near zero

$$\arctan(x^2) = x^2 - \frac{1}{3}x^6 + o(x^6).$$

Then,

$$\lim_{x\to 0}\frac{2G(t,x)}{x^2} = \frac{\lambda_p + \lambda_{p+1}}{2}.$$

Therefore, (G_1) and (G_3) are satisfied, then the problem (1.1) possesses at least two nontrivial solutions.

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