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Infinitely Many Solutions for a Elliptic System Involving Critical Sobolev Growth, Hardy Potential and Concave-Convex Nonlinearity

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ABSTRACT: In this paper, we will prove the existence of two disjoint and infinite sets of solutions for the following elliptic system with critical Sobolev exponents and Hardy potential

$$\begin{cases} -\Delta u - t\frac{u}{|x|^2} = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} + \frac{2p}{p+q}|u|^{p-2}u|v|^{q} & \text{in } \Omega,\\ -\Delta v - t\frac{v}{|x|^2} = \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + \frac{2q}{p+q}|u|^{p}|v|^{q-2}v & \text{in } \Omega,\\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smoothly bounded domain containing the origin, $N \geq 7$, $\alpha + \beta = 2^*$, $\bar{t} = \frac{(N-2)^2}{4}$, $t \in [0, \bar{t} - 4)$, $2^* - \sqrt{1 - \frac{t}{\bar{t}}} < p + q < 2$ and $2^* := \frac{2N}{N-2}$ denotes the critical Sobolev exponent.

Keywords: Laplacian, hardy potential, critical Sobolev exponent, infinitely many solutions, Pohozaev identity.

Contents

1. Introduction

This paper is concerned with the existence of infinitely many solutions for the following elliptic problem:

$$\begin{cases} -\Delta u - t\frac{u}{|x|^2} = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} + \frac{2p}{p+q}|u|^{p-2}u|v|^{q} & \text{in } \Omega,\\ -\Delta v - t\frac{v}{|x|^2} = \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + \frac{2q}{p+q}|u|^{p}|v|^{q-2}v & \text{in } \Omega,\\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p + q < 2$, $\alpha, \beta > 1$ satisfy $\alpha + \beta = 2^*$, ν is an outward normal to the boundary $\partial\Omega$ and $2^* = \frac{2N}{N-2}$, $N \geq 3$, is a critical Sobolev exponent. We assume that Ω is a smooth open bounded domain in $\mathbb{R}^N$ with $0 \in \Omega$.
The corresponding energy functional to (1.1) is

$$\begin{aligned} I(u,v) =& \frac{1}{2}\int_{\Omega}\left(|\nabla u|^2 + |\nabla v|^2 - t\frac{|u|^2 + |v|^2}{|x|^2}\right)dx \\ &- \frac{2}{\alpha+\beta}\int_{\Omega}|u|^{\alpha}|v|^{\beta}dx - \frac{2}{p+q}\int_{\Omega}|u|^{p}|v|^{q}dx. \end{aligned} \tag{1.2}$$

Problem (1.1) is related to the well-known Hardy inequality :

$$\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}dx \leq \frac{1}{\bar{t}}\int_{\mathbb{R}^N}|\nabla u|^2 dx, \quad \forall u \in C_0^{\infty}\left(\mathbb{R}^N\right).$$

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When $t = 0$ and $u = v$ in problem (1.1), it reduces to the following scalar semi-linear elliptic problem with concave-convex nonlinearities.

$$\begin{cases} -\Delta u = |u|^{p-2}u + a|u|^{q-2}u, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In the celebrated paper [2] Ambrosetti, Brezis and Cerami showed that

(1) For all $0 < q < 2 < p$, there exists $\Lambda > 0$ such that, for all $a \in (0, \Lambda)$, problem (1.3) has a minimal solution u_a with negative energy.

(2) Let $1 < q < 2 < p \leq 2^*$. Then, for all $a \in (0, \Lambda)$, problem (1.3) has a second solution $v_a > u_a$.

(3) Let $1 < q < 2 < p \leq 2^*$. Then there exists $a^* > 0$ such that for all $a \in (0, a^*)$, problem (1.3) has infinitely many negative energy solutions

(4) Let $1 < q < 2 < p < 2^*$. Then, for all $a \in (0, a^*)$, problem (1.3) has also infinitely many positive energy solutions.

After this work, the concave-convex problem has been extensively studied and some important and interesting result has been obtained, see for instance [6,7,9,15,16,20]. In the last section of [2], authors proposed one open problem : whether problem (1.3) has infinitely many solutions with positive energy, when, $p = 2^*$, for $a > 0$ small enough. On the other hand Devillanova and Solimini [18] considered the following problem

$$\begin{cases} -\Delta u = \mu u + |u|^{2^*-2}u & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $\mu > 0$. They established the existence of infinitely many solutions if $N \geq 7$. Their crucial idea is to show the strong convergence of approximating solutions of (1.4). The main ingredient used to achieve this goal is to obtain some estimates for approximating solutions of (1.4) in a carefully defined safe region, and then a local Pohozaev identity is used to obtain the result. Using the similar approaches in [18], Pigong Han in [19] showed hat if $N > \frac{2(q+1)}{q-1}$, then problem (1.3) admits an infinite sets of solutions with positive energy. Others affirmative answers to above open problem may be found in [11,14]. As far as we know, there is no easy answer for this question concerning problem (1.1). The main result of this paper is as follows

Theorem 1. *If we assume that* $N \geq 7$, $t \in [0, \bar{t} - 4)$ *and* $2^* - \sqrt{1 - \frac{t}{\bar{t}}} < p + q < 2$ *then*

(i) There exists a sequence of solutions $(u_k, v_k)_k$ *of* (1.1) *such that* $I(u_k, v_k) < 0$ *and* $I(u_k, v_k) \to 0$ *as* $k \to +\infty$.

(ii) There exists a sequence of solutions $(\tilde{u}_k, \tilde{v}_k)_k$ *of* (1.1) *such that* $I(\tilde{u}_k, \tilde{v}_k) > 0$ *and* $I(\tilde{u}_k, \tilde{v}_k) \to +\infty$ *as* $k \to +\infty$.

When $p = 2$, $q = 0$ and $u = v$ the method introduced in [18] was also used by Daomin Cao and Shusen Yan in [13] to establish that if $N \geq 7$, $a > 0$ and $\mu \in [0, \bar{\mu} - 4)$, then the problem (1.1) has infinitely many solutions. For more similar results, we refer the reader to [8,12,22]. Our main argument use the same strategy as that used in [13]. But some difficulty arise in applying the Moser iteration since we don't have a reverse Hölder inequality when $1 < p + q < 2$. To overcome this difficulty, we give here an argument, which works for the subcritical term $|u|^{p+q}$ with $1 < p + q < 2^*$ (see the argument used in the proof of Proposition 5 below). This paper is organized as follows. Section 2 is devoted to the strong convergence of approximating solutions of (1.1) in $H_0^1(\Omega)$. By applying the Fountain Theorem and its dual form [5,25], we prove Theorem 1 in Section 3.

To conclude this introduction, we explain some notations used in what follows. Denote the norms of the spaces $H_0^1(\Omega) \times H_0^1(\Omega)$, $L^p(\Omega)(1 \leq p < \infty)$ by $\|(u, v)\| := \left(\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx\right)^{\frac{1}{2}}$, $\|u\|_p := \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}$ respectively. By symbol C we denote a generic constant whose value may change from line to line.

2. Strong Convergence of Approximating Solutions in $H_0^1(\Omega)$.

We consider the following perturbed problem:

$$\begin{cases} -\Delta u - t\frac{u}{|x|^2} = \frac{2(\alpha-\epsilon)}{\alpha+\beta-2\epsilon}|u|^{\alpha-2-\epsilon}u|v|^{\beta-\epsilon} + \frac{2p}{p+q}|u|^{p-2}u|v|^q \quad \text{in } \Omega, \\ -\Delta v - t\frac{v}{|x|^2} = \frac{2(\beta-\epsilon)}{\alpha+\beta-2\epsilon}|u|^{\alpha-\epsilon}|v|^{\beta-2-\epsilon}v + \frac{2q}{p+q}|u|^p|v|^{q-2}v \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \Omega. \end{cases} \tag{2.1}$$

where $\epsilon > 0$ is a small constant such that $0 < \varepsilon < \min\{\alpha-1, \beta-1\}$. A pair of functions $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is said to be a weak solution of problem (2.1) if it holds for any $\varphi = (\varphi_1, \varphi_2) \in H^1(\Omega) \times H^1(\Omega)$

$$\begin{aligned} &\int_\Omega \left(\nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2 - t\frac{u\varphi_1}{|x|^2} - t\frac{v\varphi_2}{|x|^2}\right) dx \\ &- \frac{2(\alpha-\epsilon)}{\alpha+\beta-2\epsilon}\int_\Omega |u|^{\alpha-2-\epsilon}u|v|^{\beta-\epsilon}\varphi_1 dx - \frac{2(\beta-\epsilon)}{\alpha+\beta-2\epsilon}\int_\Omega |u|^{\alpha-\epsilon}|v|^{\beta-2-\epsilon}v\varphi_2 dx \\ &- \frac{2p}{p+q}\int_\Omega |u|^{p-2}|v|^q u\varphi_1 dx - \frac{2q}{p+q}\int_\Omega |u|^p|v|^{q-2}v\varphi_2 dx = 0. \end{aligned}$$

The corresponding energy functional of problem (2.1) is defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$\begin{aligned} I^\epsilon(u,v) =& \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 - t\frac{|u|^2+|v|^2}{|x|^2}\right) dx \\ &- \frac{2}{\alpha+\beta-2\epsilon}\int_\Omega |u|^{\alpha-\epsilon}|v|^{\beta-\epsilon}dx - \frac{2}{p+q}\int_\Omega |u|^p|v|^q dx. \end{aligned} \tag{2.2}$$

We first introduce some notations and terminologies which will be used in the sequel. Let (u,v) be a solution of problem (2.1). Set $\tilde{u} := |u|$, $\tilde{v} = |v|$ (extended by zero out of Ω) we have $\tilde{u}, \tilde{v} \in H_0^1(\mathbb{R}^N)$. Then for any $\varphi \in H^1(\mathbb{R}^N)$ with $\varphi \geq 0$

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla\tilde{u}\nabla\varphi = \int_\Omega \nabla|u|.\nabla\varphi dx \\ &= \int_{\partial\Omega} \varphi\frac{\partial|u|}{\partial n}ds - \int_\Omega |u|^{-1}u\,\mathrm{div}(\nabla u)\varphi dx \\ &\leq \int_\Omega u|u|^{-1}\left(t\frac{u}{|x|^2} + \frac{2(\alpha-\epsilon)}{\alpha+\beta-2\epsilon}|u|^{\alpha-2-\epsilon}u|v|^{\beta-\epsilon} + \frac{2p}{p+q}|u|^{p-2}u|v|^q\right)\varphi dx \\ &= \int_{\mathbb{R}^N}\left(t\frac{\tilde{u}}{|x|^2} + \frac{2(\alpha-\epsilon)}{\alpha+\beta-2\epsilon}\tilde{u}^{\alpha-1-\epsilon}\tilde{v}^{\beta-\epsilon} + \frac{2p}{p+q}\tilde{u}^{p-1}\tilde{v}^q\right)\varphi dx \\ &\leq \int_{\mathbb{R}^N}\left(t\frac{\tilde{u}+\tilde{v}}{|x|^2} + \frac{2(\alpha-\epsilon)}{2^*-2\epsilon}\left(\tilde{u}+\tilde{v}\right)^{2^*-1-2\epsilon} + \frac{2p}{p+q}\left(\tilde{u}+\tilde{v}\right)^{p+p-1}\right)\varphi dx, \end{aligned}$$

which implies in the sense of distribution

$$-\Delta\tilde{u} - t\frac{\tilde{u}+\tilde{v}}{|x|^2} \leq \frac{2(\alpha-\epsilon)}{2^*-2\epsilon}\left(\tilde{u}+\tilde{v}\right)^{2^*-1-2\epsilon} + \frac{2p}{p+q}\left(\tilde{u}+\tilde{v}\right)^{p+p-1}.$$

Similarly,

$$-\Delta\tilde{v} - t\frac{\tilde{u}+\tilde{v}}{|x|^2} \leq \frac{2(\beta-\epsilon)}{2^*-2\epsilon}\left(\tilde{u}+\tilde{v}\right)^{2^*-1-2\epsilon} + \frac{2q}{p+q}\left(\tilde{u}+\tilde{v}\right)^{p+p-1}.$$

It follows that

$$\begin{aligned} -\Delta(\tilde{u}+\tilde{v}) - t\frac{\tilde{u}+\tilde{v}}{|x|^2} &\leq 2(\tilde{u}+\tilde{v})^{2^*-1} + 2(\tilde{u}+\tilde{v})^{p+q-1} \\ &\leq 3(\tilde{u}+\tilde{v})^{2^*-1} + A \quad \text{in } \Omega, \end{aligned} \tag{2.3}$$

where A is a positive constant. So in next section we can only consider the estimates of solutions to (2.3) in $H_0^1\left(\mathbb{R}^N\right)$.

Definition 1. *Let $\{(u_n, v_n)\}$ be a given sequence.*

- *(i) We shall say that $\{(u_n, v_n)\}$ is a controlled sequence if $u_n + v_n$ is a solution to problem (2.3).*
- *(ii) We shall say that $\{(u_n, v_n)\}$ is a balanced sequence if each $\{(u_n, v_n)\}$ solves (2.1) with $\epsilon = \epsilon_n > 0$ and $\epsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$.*

Remark 1. *Every balanced sequence $\{(u_n, v_n)\}$ is a controlled sequence. In addition, we also may assume that controlled sequence $\{(u_n, v_n)\}$ is nonnegative in Ω, that is, $u_n, v_n \geq 0$ in Ω, since we could replace them with their absolute values respectively.*

For any $\sigma > 0$ and $x \in \mathbb{R}^N$, we define

$$\rho_{x,\sigma}(u,v) := \sigma^{\frac{N}{2^*}} \left(u(\sigma(\cdot - x)), v(\sigma(\cdot - x))\right), u, v \in H_0^1(\Omega).$$

From Appendix in [13], we have the following decomposition of approximating solutions.

Proposition 2. *Suppose $N \geq 3$. Let (u_n, v_n) be a solution of (2.1) with $\epsilon = \epsilon_n \to 0$, satisfying $\|(u_n, v_n)\| \leq C$ for some constant C. Then*

(i) u_n can be decomposed as

$$(u_n, v_n) = (u_0, v_0) + \sum_{j=1}^{m} \rho_{0,\sigma_{n,j}}(U_j, V_j) + \sum_{j=m+1}^{k} \rho_{x_{n,j},\sigma_{n,j}}(U_j, V_j) + (u_\infty, v_\infty), \tag{2.4}$$

where $(u_\infty, v_\infty) \to 0$ in $H^1(\Omega) \times H^1(\Omega)$, (u_0, v_0) is the weak limit of a suitable subsequence (u_n, v_n) . For $j = m+1, \cdots, k$, $x_{n,j} \in \Omega$, $\sigma_{n,j} d(x_{n,j}, \partial\Omega) \to +\infty$, $\sigma_{n,j}|x_{n,j}| \to +\infty$, as $n \to \infty$, and (U_j, V_j) is a solution of

$$\begin{cases} -\Delta u = b_j \left(\frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}\right), & \text{in } \mathbb{R}^N, \\ -\Delta v = b_j \left(\frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v\right), & \text{in } \mathbb{R}^N, \\ (u,v) \in \left(D^{1,2}(\mathbb{R}^N)\right)^2, \end{cases}$$

for some $b_j \in (0,1]$. While for $j = 1, \cdots, m$, $\sigma_{n,j} \to +\infty$ as $n \to \infty$, and (U_j, V_j) is a solution of

$$\begin{cases} -\Delta u - \frac{t}{|x|^2}u = b_j \left(\frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}\right), & \text{in } \mathbb{R}^N, \\ -\Delta v - \frac{t}{|x|^2}v = b_j \left(\frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v\right), & \text{in } \mathbb{R}^N, \\ (u,v) \in \left(D^{1,2}(\mathbb{R}^N)\right)^2, \end{cases}$$

for some $b_j \in (0,1]$.

(ii) Set $x_{n,i} = 0$ for $i = 1, \ldots, m$. For $i, j = 1, \ldots, k$, if $i \neq j$, then, as $n \to \infty$,

$$\frac{\sigma_{n,j}}{\sigma_{n,i}} + \frac{\sigma_{n,i}}{\sigma_{n,j}} + \sigma_{n,j}\sigma_{n,i}|x_{n,i} - x_{n,j}|^2 \to \infty.$$

We recall that $D^{1,2}(\mathbb{R}^N) := \left\{u \in L^{2^*}(\mathbb{R}^N) / |\nabla u| \in L^2(\mathbb{R}^N)\right\}$ equipped with the following norm

$$u \longmapsto \|\nabla u\|_{L^2(\mathbb{R}^N)}.$$

We call $\{(u_n, v_n)\}$ a concentrating sequence if the limit in (2.4) holds in the H^1 -strong topology. Among all the bubbles in (2.4), we can choose one with slowest concentration rate, denoted by σ_n, which concentrates in $x_n = x_{n,i}$ in the slowest way. We may always choose a constant $\bar{C} > 0$ such that the region

$$\mathcal{A}_n^1 := \left(B_{(\bar{C}+5)\sigma_n^{-\frac{1}{2}}}(x_n) \backslash B_{\bar{C}\sigma_n^{-\frac{1}{2}}}(x_n)\right) \cap \Omega,$$

does not contain any concentration point of (u_n, v_n) for every n. We call this region a safe region. We consider two thinner subsets as follows

$$\mathcal{A}_n^2 := \left(B_{(\bar{C}+4)\sigma_n^{-\frac{1}{2}}}(x_n) \backslash B_{(\bar{C}+1)\sigma_n^{-\frac{1}{2}}}(x_n)\right) \cap \Omega,$$

and

$$\mathcal{A}_n^3 := \left(B_{(\bar{C}+3)\sigma_n^{-\frac{1}{2}}}(x_n) \backslash B_{(\bar{C}+2)\sigma_n^{-\frac{1}{2}}}(x_n)\right) \cap \Omega.$$

For any $p_2 < 2^* < p_1$, $\alpha > 0$ and $\sigma > 0$, we consider the following relation:

$$\begin{cases} \|u_1\|_{p_1} \leq \alpha, \\ \|u_2\|_{p_2} \leq \alpha \sigma^{\frac{N}{2^*} - \frac{N}{p_2}}. \end{cases} \tag{2.5}$$

Define $\|u\|_{p_1,p_2,\sigma} = \inf\{\alpha > 0 :$ there are u_1 and u_2, such that (2.5) holds and $|u| \leq u_1 + u_2$. From (2.5) we have the following two results related to the controlled concentrating sequences.

Proposition 3. *Let $(u_n, v_n)_{n\in\mathbb{N}}$ be a controlled sequence. For any p_1, $p_2 \in \left(\frac{2^*}{2}, \frac{2^*\sqrt{\bar{t}}}{\sqrt{\bar{t}}-\sqrt{\bar{t}-t}}\right)$, $p_2 < 2^* < p_1$, there is a constant C, depending on p_1, p_2, N and t, such that*

$$\| |u_n| + |v_n| \|_{p_1,p_2,\sigma_n} \leq C.$$

Here σ_n is the smallest concentration rate of the bubbles in (u_n, v_n).

Lemma 4. *Let $\{(u_n, v_n)\}$ be a controlled concentrating sequence. Then for any $p_1 \in \left(2^*, \frac{2^*\sqrt{\bar{t}}}{\sqrt{\bar{t}}-\sqrt{\bar{t}-t}}\right)$, there exists a constant $C > 0$, independent of n, such that*

$$\frac{1}{r^{N-1}} \int_{\partial B_r(y)} w_n \leq C\sigma_n^{\frac{N}{2p_1}}, \quad \forall y \in \mathbb{R}^N,$$

for all $r \in \left[\bar{C}\sigma_n^{-\frac{1}{2}}, (\bar{C}+5)\sigma_n^{-\frac{1}{2}}\right]$, where $w_n(x) = |u_n(x)| + |v_n(x)|$.

We set

$$\Phi_n(x) := w_n\left(\sigma_n^{-1/2}x\right), \; x \in \Omega_n,$$

where $\Omega_n := \left\{x : \sigma_n^{-1/2}x \in \Omega\right\}$.
Using the inequality (2.3), it is easy to check that Φ_n (extended by zero out of Ω) satisfies

$$-\Delta\Phi_n \leq \sigma_n^{-1}\left(\frac{t}{|x|^2}\Phi_n + 3\Phi_n^{2^*-1} + A\right) \text{ in } \mathbb{R}^N. \tag{2.6}$$

From (2.6) we have

$$\int_{\mathbb{R}^N} \nabla\Phi_n \nabla\varphi \mathrm{d}x \leq \sigma_n^{-1} \int_{\mathbb{R}^N} f(\Phi_n)\varphi \mathrm{d}x, \; \forall\varphi \in C_0^\infty\left(\mathbb{R}^N\right), \phi \geq 0, \tag{2.7}$$

where

$$f(\lambda) := \frac{t}{|x|^2}\lambda + 3\lambda^{2^*-1} + A, \quad \lambda \geq 0.$$

We have the following estimate:

Proposition 5. *Let $\{(u_n, v_n)\}$ be a controlled concentrating sequence, there exists a constant $C > 0$, independent of n, such that*

$$\int_{\mathcal{A}_n^2} |w_n|^{2\beta^2} dx \leq C\sigma_n^{-\frac{N}{2}+\frac{N\beta^2}{p_1}},$$

where $\beta := \frac{2^*}{2}$.

Proof. For fixed $y \in \mathcal{A}_n^2$ and $0 < r < R \leq 1$, we set $z_n := \sigma_n^{1/2} y$ and let $\chi \in \mathcal{C}_0^\infty(B(z_n, R))$ be a cut-off function with $0 \leq \chi \leq 1$, $\chi = 1$ on $B(z_n, r)$ and $|\nabla \chi| \leq \frac{1}{R-r}$. For every $\beta > 1$ and $M > 0$, we define

$$\varphi = \chi^2 \Phi_n \widetilde{\Phi}_n^{2(\beta-1)}$$

where $\widetilde{\Phi}_n = \min\{\Phi_n, M\}$. In what follows C denotes several positive constants independent of n, r, R, β and M.
Note that $\nabla \Phi_n \nabla \widetilde{\Phi}_n = |\nabla \widetilde{\Phi}_n|^2$ and $\nabla \Phi_n \nabla \widetilde{\Phi}_n \Phi_n \widetilde{\Phi}_n^{2(\beta-1)-1} = |\nabla \widetilde{\Phi}_n|^2 \widetilde{\Phi}_n^{2(\beta-1)}$.

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla \Phi_n \nabla \varphi dx \\
&= \int_{\mathbb{R}^N} \nabla \Phi_n . \left[2\chi \nabla \chi \Phi_n \widetilde{\Phi}_n^{2(\beta-1)} + \chi^2 \widetilde{\Phi}_n^{2(\beta-1)} \nabla \Phi_n + 2(\beta-1)\chi^2 \Phi_n \widetilde{\Phi}_n^{2(\beta-1)-1} \nabla \widetilde{\Phi}_n \right] dx \\
&= \int_{\mathbb{R}^N} \chi^2 |\nabla \Phi_n|^2 \widetilde{\Phi}_n^{2(\beta-1)} dx + 2(\beta-1) \int_{\mathbb{R}^N} \chi^2 |\nabla \widetilde{\Phi}_n|^2 \widetilde{\Phi}_n^{2(\beta-1)} dx \\
&\quad + 2 \int_{\mathbb{R}^N} \chi \Phi_n \widetilde{\Phi}_n^{2(\beta-1)} \nabla \chi \nabla \Phi_n dx \\
&\leq \sigma_n^{-1} \int_{\mathbb{R}^N} \left(\frac{t}{|x|^2} \Phi_n + 3\Phi_n^{2^*-1} + A \right) \chi^2 \Phi_n \widetilde{\Phi}_n^{2(\beta-1)} dx.
\end{aligned}$$

By the Young's inequality, we have

$$|(\chi \nabla \Phi_n) \cdot (\Phi_n \nabla \chi)| \leq |\nabla \chi|^2 \Phi_n^2 + \frac{1}{4} \chi^2 |\nabla \Phi_n|^2.$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \chi^2 |\nabla \Phi_n|^2 \widetilde{\Phi}_n^{2(\beta-1)} dx + 2(\beta-1) \int_{\mathbb{R}^N} \chi^2 |\nabla \widetilde{\Phi}_n|^2 \widetilde{\Phi}_n^{2(\beta-1)} dx \\
&\quad \leq \sigma_n^{-1} \int_{\mathbb{R}^N} \left(\frac{t}{|x|^2} \Phi_n + 3\Phi_n^{2^*-1} + A \right) \chi^2 \Phi_n \widetilde{\Phi}_n^{2(\beta-1)} dx \\
&\quad + 2 \int_{\mathbb{R}^N} \widetilde{\Phi}_n^{2(\beta-1)} \left[|\nabla \chi|^2 \Phi_n^2 + \frac{1}{4} \chi^2 |\nabla \Phi_n|^2 \right] dx,
\end{aligned}$$

and it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \chi^2 |\nabla \Phi_n|^2 \widetilde{\Phi}_n^{2(\beta-1)} dx + 4(\beta-1) \int_{\mathbb{R}^N} \chi^2 |\nabla \widetilde{\Phi}_n|^2 \widetilde{\Phi}_n^{2(\beta-1)} dx \\
&\quad \leq 2\sigma_n^{-1} \int_{\mathbb{R}^N} \left(\frac{t}{|x|^2} \Phi_n + 3\Phi_n^{2^*-1} + A \right) \chi^2 \Phi_n \widetilde{\Phi}_n^{2(\beta-1)} dx \\
&\quad + 4 \int_{\mathbb{R}^N} \widetilde{\Phi}_n^{2(\beta-1)} |\nabla \chi|^2 \Phi_n^2 dx.
\end{aligned} \tag{2.8}$$

Now we take $\beta = \frac{2^*}{2} > 1$. Consider $\Psi_M := \chi \Phi_n \widetilde{\Phi}_n^{\beta-1}$, we have

$$\begin{aligned}
\nabla \Psi_M &= \nabla \chi \Phi_n \widetilde{\Phi}_n^{(\beta-1)} + \chi \nabla \Phi_n \widetilde{\Phi}_n^{(\beta-1)} + (\beta-1)\chi \Phi_n \widetilde{\Phi}_n^{(\beta-2)} \nabla \widetilde{\Phi}_n \\
&= \chi \nabla \Phi_n \widetilde{\Phi}_n^{(\beta-1)} + (\beta-1)\chi \widetilde{\Phi}_n^{(\beta-1)} \nabla \widetilde{\Phi}_n + \nabla \chi \Phi_n \widetilde{\Phi}_n^{(\beta-1)}
\end{aligned}$$

since $\beta > 1$,

$$\begin{aligned}
|\nabla \Psi_M|^2 &\leq C \left[\chi^2 |\nabla \Phi_n|^2 \widetilde{\Phi}_n^{2(\beta-1)} + (\beta-1)^2 \chi^2 \widetilde{\Phi}_n^{2(\beta-1)} |\nabla \widetilde{\Phi}_n|^2 + \widetilde{\Phi}_n^{2(\beta-1)} |\nabla \chi|^2 {\Phi_n}^2 \right] \\
&\leq C\beta \left[\chi^2 |\nabla \Phi_n|^2 \widetilde{\Phi}_n^{2(\beta-1)} + 4(\beta-1) \chi^2 \widetilde{\Phi}_n^{2(\beta-1)} |\nabla \widetilde{\Phi}_n|^2 + \widetilde{\Phi}_n^{2(\beta-1)} |\nabla \chi|^2 {\Phi_n}^2 \right].
\end{aligned} \tag{2.9}$$

Formula (2.8) and (2.9) yields that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla \Psi_M|^2 dx\\
&\leq C\beta \left[\int_{\mathbb{R}^N} \chi^2 |\nabla \Phi_n|^2 \widetilde{\Phi}_n^{2(\beta-1)} dx + 4(\beta-1)\int_{\mathbb{R}^N} \chi^2 |\nabla \widetilde{\Phi}_n|^2 \widetilde{\Phi}_n^{2(\beta-1)} dx\right.\\
&\quad \left. + \int_{\mathbb{R}^N} \widetilde{\Phi}_n^{2(\beta-1)} |\nabla \chi|^2 {\Phi_n}^2 \right]\\
\leq & C\beta \left[\sigma_n^{-1} \int_{\mathbb{R}^N} \left(\frac{t}{|x|^2}\Phi_n + 3\Phi_n^{2^*-1} + A\right) \chi^2 \Phi_n \widetilde{\Phi}_n^{2(\beta-1)} dx + \int_{\mathbb{R}^N} \widetilde{\Phi}_n^{2(\beta-1)} |\nabla \chi|^2 \Phi_n^2 dx\right]\\
&\leq C\beta \left[\sigma_n^{-1} \int_{\mathbb{R}^N} \frac{t}{|x|^2} \Phi_n^2 \chi^2 \widetilde{\Phi}_n^{2(\beta-1)} dx + \sigma_n^{-1} \int_{\mathbb{R}^N} \Phi_n^{2^*} \chi^2 \widetilde{\Phi}_n^{2(\beta-1)} dx\right.\\
&\quad \left. + A\sigma_n^{-1} \int_{\mathbb{R}^N} \chi^2 \Phi_n \widetilde{\Phi}_n^{2(\beta-1)} dx + \int_{\mathbb{R}^N} \widetilde{\Phi}_n^{2(\beta-1)} |\nabla \chi|^2 \Phi_n^2 dx\right].
\end{aligned} \tag{2.10}$$

By the Hardy embedding theorem,

$$\int_{\mathbb{R}^N} \frac{t}{|x|^2} \Phi_n^2 \chi^2 \widetilde{\Phi}_n^{2(\beta-1)} dx = t\int_{\mathbb{R}^N} \frac{(\Phi_n \chi \widetilde{\Phi}_n^{(\beta-1)})^2}{|x|^2} dx \leq \frac{t}{\bar{t}} \int_{\mathbb{R}^N} |\nabla \Psi_M|^2 dx.$$

Formula (2.10) yields that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla \Psi_M|^2 dx \leq \sigma_n^{-1} \int_{\mathbb{R}^N} \Phi_n^{2^*} \chi^2 \widetilde{\Phi}_n^{2(\beta-1)} dx + A\sigma_n^{-1} \int_{\mathbb{R}^N} \chi^2 \Phi_n \widetilde{\Phi}_n^{2(\beta-1)} dx\\
+ \int_{\mathbb{R}^N} \widetilde{\Phi}_n^{2(\beta-1)} |\nabla \chi|^2 \Phi_n^2 dx.
\end{aligned}$$

By the Sobolev embedding theorem

$$\begin{aligned}
\left(\int_{\mathbb{R}^N} (\Psi_M)^{2^*} dx\right)^{\frac{2}{2^*}} \leq \sigma_n^{-1} \int_{\mathbb{R}^N} \Phi_n^{2^*} \chi^2 \widetilde{\Phi}_n^{2(\beta-1)} dx + A\sigma_n^{-1} \int_{\mathbb{R}^N} \chi^2 \Phi_n \widetilde{\Phi}_n^{2(\beta-1)} dx\\
+ \int_{\mathbb{R}^N} \widetilde{\Phi}_n^{2(\beta-1)} |\nabla \chi|^2 \Phi_n^2 dx.
\end{aligned} \tag{2.11}$$

By the Hölder inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Phi_n^{2^*} \chi^2 \widetilde{\Phi}_n^{2(\beta-1)} dx = \int_{\mathbb{R}^N} \Phi_n^{2^*-2} \chi^2 \Phi_n^2 \widetilde{\Phi}_n^{2(\beta-1)} dx\\
&\leq \left[\int_{B(z_n,R)} \Phi_n^{2^*} dx\right]^{\frac{2^*-2}{2^*}} \left[\int_{\mathbb{R}^N} \left(\chi \Phi_n \widetilde{\Phi}_n^{\beta-1}\right)^{2^*} dx\right]^{\frac{2}{2^*}}\\
&\leq \left[\int_{B(z_n,R)} \Phi_n^{2^*} dx\right]^{\frac{2^*-2}{2^*}} \left[\int_{\mathbb{R}^N} \left(\chi \Phi_n \widetilde{\Phi}_n^{\beta-1}\right)^{2^*} dx\right]^{\frac{2}{2^*}}\\
&= \left[\int_{B(z_n,R)} \Phi_n^{2^*} dx\right]^{\frac{2^*-2}{2^*}} \left[\int_{\mathbb{R}^N} (\Psi_M)^{2^*} dx\right]^{\frac{2}{2^*}}.
\end{aligned}$$

Since $\widetilde{\Phi}_n \leq \Phi_n$, we have

$$\int_{\mathbb{R}^N} \widetilde{\Phi}_n^{2(\beta-1)} |\nabla \chi|^2 \Phi_n^2 dx \leq \int_{\mathbb{R}^N} \Phi_n^{2^*} |\nabla \chi|^2 dx,$$

and

$$\int_{\mathbb{R}^N} \chi^2 \Phi_n \widetilde{\Phi}_n^{2(\beta-1)} dx \leq \int_{\mathbb{R}^N} \chi^2 \Phi_n^{2^*-1} dx \leq \int_{B(z_n,R)} \Phi_n^{2^*-1} dx.$$

So

$$\begin{aligned}&\left(\int_{\mathbb{R}^N} (\Psi_M)^{2^*} dx\right)^{\frac{2}{2^*}}\\ &\leq C\sigma_n^{-1}\left[\int_{B(z_n,R)} \Phi_n^{2^*} dx\right]^{\frac{2}{N}} \left(\int_{\mathbb{R}^N} (\Psi_M)^{2^*} dx\right)^{\frac{2}{2^*}} + C\sigma_n^{-1}\int_{\mathbb{R}^N} \chi^2\Phi_n^{2^*-1}dx\\ &\quad + C\int_{\mathbb{R}^N} \Phi_n^{2^*}|\nabla\chi|^2 dx.\end{aligned}$$

Since $B\left(y, \sigma_n^{-1/2}\right) \subset \mathcal{A}_n^1$, and $\mathcal{A}_n^1$ does not contain any concentration point of Φ_n, we can deduce that

$$\begin{aligned}\sigma_n^{-1}\left[\int_{B(z_n,R)} \Phi_n^{2^*} dx\right]^{\frac{2}{N}} &\leq \sigma_n^{-1}\left[\int_{B(z_n,1)} \Phi_n^{2^*} dx\right]^{\frac{2}{N}}\\ &= \left[\int_{B\left(y,\sigma_n^{-1/2}\right)} |w_n|^{2^*} dx\right]^{\frac{2}{N}} \to 0\end{aligned}$$

as $n \to +\infty$. It follows that

$$\left(\int_{\mathbb{R}^N} (\Psi_M)^{2^*} dx\right)^{\frac{2}{2^*}} \leq C\int_{\mathbb{R}^N} \chi^2\Phi_n^{2^*-1}dx + C\int_{\mathbb{R}^N} \Phi_n^{2^*}|\nabla\chi|^2.$$

Let M go to infinity, we obtain that

$$\left(\int_{B(z_n,r)} \Phi_n^{2^*\beta}dx\right)^{\frac{2}{2^*}} \leq C\int_{\mathbb{R}^N} \chi^2\Phi_n^{2^*-1}dx + C\int_{\mathbb{R}^N} \Phi_n^{2^*}|\nabla\chi|^2 dx. \tag{2.12}$$

It then follows from (2.12) and Young inequality that

$$\begin{aligned}\left(\int_{B(z_n,r)} \Phi_n^{2^*\beta}dx\right)^{\frac{1}{\beta 2^*}} &\leq \frac{C}{(R-r)^{\frac{1}{\beta}}}\left[\int_{B(z_n,R)} \Phi_n^{2^*} dx\right]^{\frac{1}{2\beta}} + C\left[\int_{B(z_n,R)} \Phi_n^{2^*-1}dx\right]^{\frac{1}{2\beta}}\\ &\leq C\left[\frac{1}{(R-r)^{\frac{1}{\beta}}}+1\right]\left[\int_{B(z_n,R)} \Phi_n^{2^*} dx\right]^{\frac{1}{2^*}} + C < \infty.\end{aligned} \tag{2.13}$$

Using the interpolation inequality and Young's inequality we then have

$$\begin{aligned}\left(\int_{B(z_n,r)} \Phi_n^{2^*\beta}dx\right)^{\frac{1}{2^*\beta}} &\leq C\left[\frac{1}{(R-r)^{\frac{1}{\beta}}}+1\right]\|\Phi_n\|_{L^1(B(z_n,R))}^k\\ &\quad\times \|\Phi_n\|_{L^{2^*\beta}(B(z_n,R))}^{1-k} + C\\ &\leq \frac{1}{2}\|\Phi_n\|_{L^{2^*\beta}(B(z_n,R))}\\ &\quad + C\left[\frac{1}{(R-r)^{\frac{1}{\beta}}}+1\right]\|\Phi_n\|_{L^1(B(z_n,R))} + C,\end{aligned} \tag{2.14}$$

where the number k is given by $\frac{1}{2^*} = k + \frac{1-k}{\beta 2^*}$.

Using iteration argument, we set $r_i = R - \frac{R-r}{i+1}$, $i \in \mathbb{N}$, we obtain

$$\begin{aligned}\left(\int_{B(z_n,r_0)} \Phi_n^{2^*\beta} dx\right)^{\frac{1}{2^*\beta}} \leq & \frac{1}{2^i} \left\|\Phi_n\right\|_{L^{2^*\beta}(B(z_n,r_i))} \\ & + C\sum_{j=1}^{i} \frac{1}{2^{j-1}} \left[\frac{1}{(r_j - r_{j-1})^{\frac{1}{t}}} + 1\right] \left\|\Phi_n\right\|_{L^1(B(z_n,r_j))} \\ & + C\sum_{j=1}^{i} \frac{1}{2^{j-1}}.\end{aligned} \tag{2.15}$$

It is easy to see that

$$\sum_{i=1}^{+\infty} \frac{1}{2^{i-1}} \left[\frac{1}{(r_i - r_{i-1})^{\frac{1}{t}}} + 1\right] < \infty \quad \textit{and} \quad \sum_{j=1}^{+\infty} \frac{1}{2^{j-1}} < \infty.$$

Letting i go to infinity in (2.15), we obtain then that

$$\left(\int_{B(z_n,r)} \Phi_n^{2^*\beta} dx\right)^{\frac{1}{2^*\beta}} \leq C \left\|\Phi_n\right\|_{L^1(B(z_n,1))} + C. \tag{2.16}$$

Letting $r \to \frac{1}{2}$ in (2.16), we infer that

$$\left\|\Phi_n\right\|_{L^{2\beta^2}\left(B\left(z_n,\frac{1}{2}\right)\right)} \leq C \left\|\Phi_n\right\|_{L^1(B(z_n,1))} + C. \tag{2.17}$$

On the other hand, we have from $B\left(y, \sigma_n^{-1/2}\right) \subset \mathcal{A}_n^1$ and Lemma 4

$$\begin{aligned}\int_{B(z_n,1)} \Phi_n dx = & \sigma_n^{\frac{N}{2}} \int_{B\left(y,\sigma_n^{-1/2}\right)} w_n dx \\ & \leq \sigma_n^{\frac{N}{2}} \int_{\mathcal{A}_n^1} w_n dx = \sigma_n^{\frac{N}{2}} \int_{\bar{C}\sigma_n^{\frac{-1}{2}}}^{(\bar{C}+5)\sigma_n^{\frac{-1}{2}}} \left[\int_{\partial B_r(y)} w_n d\sigma\right] dr \\ & \leq C\sigma_n^{\frac{N}{2}} \sigma_n^{\frac{N}{2p_1}} \int_{\bar{C}\sigma_n^{\frac{-1}{2}}}^{(\bar{C}+5)\sigma_n^{\frac{-1}{2}}} r^{N-1} dr \leq C\sigma_n^{\frac{N}{2p_1}}.\end{aligned}$$

Combining this with (2.17) and using the definition of v_n, we obtain then the desired result. □

As a consequence of the previous lemma we have the following estimates which play a crucial role in the proof of Proposition 7 below.

Lemma 6. *Let $(u_n, v_n)_{n\in\mathbb{N}}$ be a controlled sequence. For any $s \leq 2^*$ there exists a positive constant C such that for any n*

(i) $\displaystyle\int_{\mathcal{A}_n^2} |w_n|^s dx \leq C\sigma_n^{-\frac{N}{2}+\frac{Ns}{2p_1}}$.

(ii) $\displaystyle\int_{\mathcal{A}_n^3} |\nabla w_n|^2 dx \leq C\sigma_n^{\frac{2-N}{2}+\frac{N}{p_1}}$.

Proof. (i) By Hölder's inequality and Proposition 5 we obtain for any $s \leq 2^*$,

$$\begin{aligned}\int_{\mathcal{A}_n^2} |w_n|^s dx &\leq C \left(\int_{\mathcal{A}_n^2} |w_n|^{2\beta^2} \right)^{\frac{s}{2\beta^2}} \sigma_n^{-\frac{N}{2}\left(1-\frac{s}{2\beta^2}\right)} \\ &\leq C \sigma_n^{(-\frac{N}{2}+\frac{N\beta^2}{p_1})\frac{s}{2\beta^2}} \sigma_n^{-\frac{N}{2}+\frac{Ns}{4\beta^2}} \\ &\leq C \sigma_n^{-\frac{N}{2}+\frac{Ns}{2p_1}}.\end{aligned}$$

(ii) For each $n \in \mathbb{N}$, consider a function $\phi_n \in C_0^2\left(\mathcal{A}_n^2\right)$ such that $\phi_n = 1$ in $\mathcal{A}_n^3$, $0 \leq \phi_n \leq 1$ and

$$|\nabla \phi_n| \leq C \sigma_n^{1/2}.$$

We have

$$\int_\Omega \nabla w_n \nabla\left(\phi_n^2 w_n\right) \leq C \int_\Omega \left(\frac{t}{|x|^2}|w_n| + |w_n|^{2^*-1} + A \right) \phi_n^2 |w_n|. \tag{2.18}$$

On the other hand, it follows from (i) that

$$\begin{aligned}\int_{A_n^3} \frac{\phi_n^2 |w_n|^2}{|x|^2} dx &\leq \left(\int_{A_n^3} \phi_n^2 |w_n|^{2^*} dx \right)^{\frac{2}{2^*}} \left(\int_{A_n^3} \frac{\phi_n^2}{|x|^{\frac{22^*}{2^*-2}}} dx \right)^{1-\frac{2}{2^*}} \\ &\leq C \sigma_n^{(-\frac{N}{2}+\frac{N2^*}{2p_1})\frac{2}{2^*}} \sigma_n^{-\frac{1}{2}\left(N-\frac{22^*}{2^*-2}\right)\left(1-\frac{2}{2^*}\right)} = C\sigma_n^{\frac{2-N}{2}+\frac{N}{p_1}}.\end{aligned} \tag{2.19}$$

Sinse $p_1 > 2^*$, we see $\frac{-N}{2} + \frac{2^*N}{2p_1} < \frac{2-N}{2} + \frac{N}{p_1}$ and $\frac{-N}{2} + \frac{N}{2p_1} < \frac{2-N}{2} + \frac{N}{p_1}$. Thus, from (i) and (2.18), we obtain that

$$\int_{A_n^3} |\nabla w_n|^2 \leq C\sigma_n^{\frac{2-N}{2}+\frac{N}{p_1}} + C\sigma_n^{\frac{-N}{2}+\frac{2^*N}{2p_1}} + C\sigma_n^{\frac{-N}{2}+\frac{N}{2p_1}} \leq C\sigma_n^{\frac{2-N}{2}+\frac{N}{p_1}}.$$

The proof of Lemma 6 is complet. □

Proposition 7. *Let $(u_n, v_n)_{n\in\mathbb{N}}$ be a balanced sequence, satisfying $\| (u_n, v_n) \| \leq C$ for some constant independent of n, the sequence $(u_n, v_n)_{n\in\mathbb{N}}$ converges strongly in $H_0^1(\Omega)$.*

Proof. Take a $t_n \in [\bar{C}+2, \bar{C}+3]$, satisfying

$$\begin{aligned}&\int_{\partial B_{t_n\sigma_n^{-1/2}(x_n)}} \left(\sigma_n^{-1}|u_n|^{2^*-\epsilon_n}|v_n|^{2^*-\epsilon_n} + |u_n|^p|v_n|^q + \sigma_n^{-1}\left(|\nabla u_n|^2 + |\nabla v_n|^2\right)\right) \\ &+ \int_{\partial B_{t_n\sigma_n^{-1/2}(x_n)}} t\sigma_n^{-1}\frac{u_n^2+v_n^2}{|x|^2} \\ &\leq C\sigma_n^{1/2}\int_{A_n^3}\left(\sigma_n^{-1}|u_n|^{2^*-\epsilon_n}|v_n|^{2^*-\epsilon_n} + |u_n|^p|v_n|^q + \sigma_n^{-1}\left(|\nabla u_n|^2 + |\nabla v_n|^2\right)\right) \\ &+ \int_{A_n^3} t\sigma_n^{-1}\frac{u_n^2+v_n^2}{|x|^2}.\end{aligned} \tag{2.20}$$

Using Lemma 6, (2.19) and (2.20), we obtain

$$\begin{aligned}&\int_{\partial B_{t_n\sigma_n^{-1/2}(x_n)}} \left(\sigma_n^{-1}|u_n|^{2^*-\epsilon_n}|v_n|^{2^*-\epsilon_n} + |u_n|^p|v_n|^q + \sigma_n^{-1}\left(|\nabla u_n|^2 + |\nabla v_n|^2\right)\right) \\ &+ \int_{\partial B_{t_n\sigma_n^{-1/2}(x_n)}} t\sigma_n^{-1}\frac{u_n^2+v_n^2}{|x|^2} \leq C\sigma_n^{\frac{1}{2}-\frac{N}{2}+\frac{N}{p_1}}.\end{aligned} \tag{2.21}$$

We have three different cases:

(i) $B_{t_n\sigma_n^{-\frac{1}{2}}}(x_n) \cap \left(\mathbb{R}^N \backslash \Omega\right) \neq \emptyset$;

(ii) $B_{t_n\sigma_n^{-\frac{1}{2}}}(x_n) \subset \Omega$ and $0 \notin \overline{B_{t_n\sigma_n^{-\frac{1}{2}}}(x_n)}$;

(iii) $B_{t_n\sigma_n^{-\frac{1}{2}}}(x_n) \subset \Omega$ and $0 \in \overline{B_{t_n\sigma_n^{-\frac{1}{2}}}(x_n)}$.

We have the following local Pohozaev identity for u_n and v_n on $B_n = B_{t_n\sigma_n^{-\frac{1}{2}}}(x_n) \bigcap \Omega$:

$$\begin{aligned}
&2N\left(\frac{1}{2^*-2\epsilon_n}-\frac{1}{2^*}\right)\int_{B_n}|u_n|^{\alpha-\epsilon_n}|v_n|^{\beta-\epsilon_n}dx + t\int_{B_n}\frac{x\cdot x_0\left(|u_n|^2+|v_n|^2\right)}{|x|^4}dx\\
&+2N\left(\frac{1}{p+q}-\frac{1}{2^*}\right)\int_{B_n}|u_n|^p|v_n|^q dx\\
&=\frac{2}{2^*-2\epsilon_n}\int_{\partial B_n}|u_n|^{\alpha-\epsilon_n}|v_n|^{\beta-\epsilon_n}(x-x_0)\cdot\nu d\sigma\\
&+\frac{2}{p+q}\int_{\partial B_n}|u_n|^p|v_n|^q(x-x_0)\cdot\nu d\sigma\\
&+\int_{\partial B_n}\left(\left(\nabla u_n\cdot(x-x_0)\right)\left(\nabla u_n\cdot\nu\right)+\left(\nabla v_n\cdot(x-x_0)\right)\left(\nabla v_n\cdot\nu\right)\right)d\sigma\\
&+\frac{t}{2}\int_{\partial B_n}\frac{|u_n|^2+|v_n|^2}{|x|^2}(x-x_0)\cdot\nu d\sigma-\frac{1}{2}\int_{\partial B_n}\left(|\nabla u_n|^2+|\nabla v_n|^2\right)(x-x_0)\cdot\nu d\sigma\\
&+\frac{N}{2^*}\int_{\partial B_n}\left((\nabla u_n\cdot\nu)u_n+(\nabla v_n\cdot\nu)v_n\right)d\sigma
\end{aligned}\tag{2.22}$$

where ν is the outward normal to ∂B_n. The point x_0 in (2.22) is chosen as follows. In case (i), we take $x_0 \in \mathbb{R}^N\backslash\Omega$ with $|x_0-x_n| \leq 2t_n\sigma_n^{-\frac{1}{2}}$ and $\nu\cdot(x-x_0) \leq 0$ in $\partial\Omega\cap B_n$. With this x_0, we can check $x\cdot x_0 \geq 0$ in B_n. In case (ii), we take a point $x_0 = x_n$. Then $x\cdot x_0 \geq 0$ in B_n. In case (iii), we take $x_0 = 0$. Thus, in any cases we have $x\cdot x_0 \geq 0$ in B_n.
In fact, in case (i) and case (ii), $u_n, v_n \in C^2\left(\bar{B}_n\right)$. So, (2.22) is the usual local Pohozaev identity. Now, we prove that (2.22) holds as well in case (iii). Since $\int_\Omega\left(|\nabla u_n|^2+|\nabla v_n|^2\right) \leq C$, we can choose $\theta_j \to 0$ as $j \to +\infty$, such that

$$\begin{aligned}
&\theta_j\int_{\partial B_{\theta_j}(0)}\left(|\nabla u_n|^2+|\nabla v_n|^2\right)+\theta_j\int_{\partial B_{\theta_j}(0)}|u_n|^{\alpha-\epsilon_n}|v_n|^{\beta-\epsilon_n}\\
&+\theta_j\int_{\partial B_{\theta_j}(0)}t\frac{|u_n|^2+|v_n|^2}{|x|^2}+\theta_j\int_{\partial B_{\theta_j}(0)}|u_n|^p|v_n|^q \to 0.
\end{aligned}\tag{2.23}$$

Let $B_{n,\theta} = B_n \backslash B_\theta(0)$. Then, $u_n, v_n \in C^2\left(\overline{B_{n,\theta_j}}\right)$. So, we have

$$\begin{aligned}
&2N\left(\frac{1}{2^*-2\epsilon_n}-\frac{1}{2^*}\right)\int_{B_{n,\theta_j}}|u_n|^{\alpha-\epsilon_n}|v_n|^{\beta-\epsilon_n}dx+t\int_{B_{n,\theta_j}}\frac{x\cdot x_0\left(|u_n|^2+|v_n|^2\right)}{|x|^4}dx\\
&+2N\left(\frac{1}{p+q}-\frac{1}{2^*}\right)\int_{B_{n,\theta_j}}|u_n|^p|v_n|^q dx\\
&=\frac{2}{2^*-2\epsilon_n}\int_{\partial B_{n,\theta_j}}|u_n|^{\alpha-\epsilon_n}|v_n|^{\beta-\epsilon_n}(x-x_0)\cdot\nu d\sigma\\
&+\frac{2}{p+q}\int_{\partial B_{n,\theta_j}}|u_n|^p|v_n|^q(x-x_0)\cdot\nu d\sigma \qquad (2.24)\\
&+\int_{\partial B_{n,\theta_j}}\left(\left(\nabla u_n\cdot(x-x_0)\right)\left(\nabla u_n\cdot\nu\right)+\left(\nabla v_n\cdot(x-x_0)\right)\left(\nabla v_n\cdot\nu\right)\right)d\sigma\\
&+\frac{t}{2}\int_{\partial B_{n,\theta_j}}\frac{|u_n|^2+|v_n|^2}{|x|^2}(x-x_0)\cdot\nu d\sigma-\frac{1}{2}\int_{\partial B_{n,\theta_j}}\left(|\nabla u_n|^2+|\nabla v_n|^2\right)(x-x_0)\cdot\nu d\sigma\\
&+\frac{N}{2^*}\int_{\partial B_{n,\theta_j}}((\nabla u_n\cdot\nu)u_n+(\nabla v_n\cdot\nu)v_n)d\sigma.
\end{aligned}$$

Using (2.23), together with Lemma B.1 in [13], we find

$$\begin{aligned}
&\Big|\int_{\partial B_{\theta_j}(0)}\left(\nabla u_n\cdot\nu u_n+\nabla v_n\cdot\nu v_n\right)\ \mathrm{d}\sigma\Big|\\
&\leq\left(\int_{\partial B_{\theta_j}(0)}\left(|\nabla u_n|^2+\nabla v_n|^2\right)\ \mathrm{d}\sigma\right)^{1/2}\theta_j^{\frac{N-1}{2}-\sqrt{\bar{t}}+\sqrt{\tilde{t}-t}}=o(1)\theta_j^{\sqrt{t}-t}=o(1),
\end{aligned} \qquad (2.25)$$

and

$$\begin{aligned}
&\frac{2}{2^*-2\epsilon_n}\int_{\partial B_{\theta_j}(0)}|u_n|^{\alpha-\epsilon_n}|v_n|^{\beta-\epsilon_n}(x-x_0)\cdot\nu d\sigma\\
&+\frac{2}{p+q}\int_{\partial B_{\theta_j}(0)}|u_n|^p|v_n|^q(x-x_0)\cdot\nu d\sigma\\
&+\int_{\partial B_{\theta_j}(0)}\left(\left(\nabla u_n\cdot(x-x_0)\right)\left(\nabla u_n\cdot\nu\right)+\left(\nabla v_n\cdot(x-x_0)\right)\left(\nabla v_n\cdot\nu\right)\right)d\sigma\\
&+\frac{t}{2}\int_{\partial B_{\theta_j}(0)}\frac{|u_n|^2+|v_n|^2}{|x|^2}(x-x_0)\cdot\nu d\sigma-\frac{1}{2}\int_{\partial B_{\theta_j}(0)}\left(|\nabla u_n|^2+|\nabla v_n|^2\right)(x-x_0)\cdot\nu d\sigma \qquad (2.26)\\
&+\frac{N}{2^*}\int_{\partial B_{\theta_j}(0)}((\nabla u_n\cdot\nu)u_n+(\nabla v_n\cdot\nu)v_n)d\sigma\\
&=O\Bigg(\theta_j\int_{\partial B_{\theta_j}(0)}|\nabla u_n|^2+|\nabla v_n|^2+\theta_j\int_{\partial B_{\theta_j}(0)}|u_n|^{\alpha-\epsilon_n}|v_n|^{\beta-\epsilon_n}\\
&\qquad+\theta_j\int_{\partial B_{\theta_j}(0)}t\frac{|u_n|^2+|v_n|^2}{|x|^2}+\theta_j\int_{\partial B_{\theta_j}(0)}|u_n|^p|v_n|^q\Bigg)=o(1).
\end{aligned}$$

So, letting $j\to+\infty$ in (2.24), in view of (2.25) and (2.26), we obtain (2.22).
Since $2^*-2\epsilon_n<2^*$, the first term in the left hand side of (2.22) is non-negative. By the choice of x_0, the

second term in the left hand side of (2.22) is non-negative as well. We thus obtain from (2.22) that

$$
\begin{aligned}
&2N\left(\frac{1}{p+q}-\frac{1}{2^*}\right)\int_{B_n}|u_n|^p|v_n|^q dx \le \frac{2}{2^*-2\epsilon_n}\int_{\partial B_n}|u_n|^{\alpha-\epsilon_n}|v_n|^{\beta-\epsilon_n}(x-x_0)\cdot\nu d\sigma\\
&+\frac{2}{p+q}\int_{\partial B_n}|u_n|^p|v_n|^q(x-x_0)\cdot\nu d\sigma\\
&+\int_{\partial B_n}\left(\left(\nabla u_n\cdot(x-x_0)\right)\left(\nabla u_n\cdot\nu\right)+\left(\nabla v_n\cdot(x-x_0)\right)\left(\nabla v_n\cdot\nu\right)\right)d\sigma\\
&+\frac{t}{2}\int_{\partial B_n}\frac{|u_n|^2+|v_n|^2}{|x|^2}(x-x_0)\cdot\nu d\sigma-\frac{1}{2}\int_{\partial B_n}\left(|\nabla u_n|^2+|\nabla v_n|^2\right)(x-x_0)\cdot\nu d\sigma\\
&+\frac{N}{2^*}\int_{\partial B_n}((\nabla u_n\cdot\nu)u_n+(\nabla v_n\cdot\nu)v_n)d\sigma.
\end{aligned}
\tag{2.27}
$$

Now we decompose ∂B_n into $\partial B_n=\partial_i B_n\cup\partial_e B_n$ where $\partial_i B_n=\partial B_n\cap\Omega$ and $\partial_e B_n=\partial B_n\cap\partial\Omega$. Noting $u_n=0$ and $v_n=0$ on $\partial\Omega$, we find

$$
\begin{aligned}
&\frac{2}{2^*-2\epsilon_n}\int_{\partial_e B_n}|u_n|^{\alpha-\epsilon_n}|v_n|^{\beta-\epsilon_n}(x-x_0)\cdot\nu d\sigma\\
&+\frac{2}{p+q}\int_{\partial_e B_n}|u_n|^p|v_n|^q(x-x_0)\cdot\nu d\sigma\\
&+\int_{\partial_e B_n}\left(\left(\nabla u_n\cdot(x-x_0)\right)\left(\nabla u_n\cdot\nu\right)+\left(\nabla v_n\cdot(x-x_0)\right)\left(\nabla v_n\cdot\nu\right)\right)d\sigma\\
&+\frac{t}{2}\int_{\partial_e B_n}\frac{|u_n|^2+|v_n|^2}{|x|^2}(x-x_0)\cdot\nu d\sigma-\frac{1}{2}\int_{\partial_e B_n}\left(|\nabla u_n|^2+|\nabla v_n|^2\right)(x-x_0)\cdot\nu d\sigma\\
&+\frac{N}{2^*}\int_{\partial_e B_n}((\nabla u_n\cdot\nu)u_n+(\nabla v_n\cdot\nu)v_n)d\sigma\\
=&\frac{1}{2}\int_{\partial_e B_n}\left(|\nabla u_n|^2+|\nabla v_n|^2\right)(x-x_0)\cdot\nu d\sigma\le 0.
\end{aligned}
\tag{2.28}
$$

So, we can rewrite (2.27) as

$$
\begin{aligned}
&2N\left(\frac{1}{p+q}-\frac{1}{2^*}\right)\int_{B_n}|u_n|^p|v_n|^q dx\\
&\le\frac{2}{\alpha+\beta-2\epsilon_n}\int_{\partial_i B_n}|u_n|^{\alpha-\epsilon_n}|v_n|^{\beta-\epsilon_n}(x-x_0)\cdot\nu d\sigma\\
&+\frac{2}{p+q}\int_{\partial_i B_n}|u_n|^p|v_n|^q(x-x_0)\cdot\nu d\sigma\\
&+\int_{\partial_i B_n}\left(\left(\nabla u_n\cdot(x-x_0)\right)\left(\nabla u_n\cdot\nu\right)+\left(\nabla v_n\cdot(x-x_0)\right)\left(\nabla v_n\cdot\nu\right)\right)d\sigma\\
&+\frac{t}{2}\int_{\partial_i B_n}\frac{|u_n|^2+|v_n|^2}{|x|^2}(x-x_0)\cdot\nu d\sigma-\frac{1}{2}\int_{\partial_i B_n}\left(|\nabla u_n|^2+|\nabla v_n|^2\right)(x-x_0)\cdot\nu d\sigma\\
&+\frac{N}{2^*}\int_{\partial_i B_n}((\nabla u_n\cdot\nu)u_n+(\nabla v_n\cdot\nu)v_n)d\sigma.
\end{aligned}
\tag{2.29}
$$

Using (2.21), noting that $|x-x_0|\le C\sigma_n^{-\frac{1}{2}}$ for $x\in\partial_i B_n$, we see RHS of (2.29)

$$
\begin{aligned}
&\le C\sigma_n^{-\frac{1}{2}}\int_{\partial_i B_n}|u_n|^{\alpha-\epsilon_n}|v_n|^{\beta-\epsilon_n}+|u_n|^p|v_n|^q+\frac{t}{2}\frac{|u_n|^2+|v_n|^2}{|x|^2}+|\nabla u_n|^2+|\nabla v_n|^2 d\sigma\\
&\qquad+C\int_{\partial_i B_n}|\nabla u_n||u_n|+|\nabla v_n||v_n|\mathrm{d}\sigma\le C\sigma_n^{-\frac{N-2}{2}+\frac{N}{p_1}}.
\end{aligned}
\tag{2.30}
$$

On the other hand, let $B'_n = B_{\sigma_n^{-1}}(x_n) \cap \Omega$. We decompose u_n as $u_n = u_0 + u_{n,1} + u_{n,2}$, with $u_{n,1} = \sum_{j=1}^{m} \rho_{0,\sigma_{n,j}}(U_j) + \sum_{j=m+1}^{k} \rho_{x_{n,j},\sigma_{n,j}}(U_j)$ and $u_{n,2} = u_\infty$ which converges strongly to 0 in $D^{1,2}(\mathbb{R}^N)$. Similary we decompose v_n as $v_n = v_0 + v_{n,1} + v_{n,2}$, with $v_{n,1} = \sum_{j=1}^{m} \rho_{0,\sigma_{n,j}}(V_j) + \sum_{j=m+1}^{k} \rho_{x_{n,j},\sigma_{n,j}}(V_j)$ and $v_{n,2} = v_\infty$ which converges strongly to 0 in $D^{1,2}(\mathbb{R}^N)$.

It is well known that the inequality holds:

$$|a+b|^p|c+d|^q \geq \frac{1}{4}|a|^p|c|^q - C\left(|a|^p|d|^q + |b|^p|c|^q + |b|^p|d|^q\right) \quad \forall a,b,c,d \in \mathbb{R}$$

where $p, q > 0$ and $C > 0$. Then we deduce that for n large enough,

$$\begin{aligned}\int_{B_n} |u_n|^p|v_n|^q dx &\geq \int_{B'_n} |u_n^2 + (u_n^1 + u_n^0)|^p|v_n^2 + (v_n^1 + v_n^0)|^q dx \\ &\geq \frac{1}{4}\int_{B'_n} |u_{n,1}|^p|v_{n,1}|^q dx \\ &\quad - C\int_{B'_n} [|u_{n,1}|^p|v_0|^q + |u_{n,1}|^p|v_{n,2}|^q + |u_0|^p|v_{n,1}|^q + |u_0|^p|v_0|^q \\ &\quad + |u_0|^p|v_{n,2}|^q + |u_{n,2}|^p|v_{n,1}|^q + |u_{n,2}|^p|v_0|^q + |u_{n,2}|^p|v_{n,2}|^q]\, dx.\end{aligned} \tag{2.31}$$

Now we estimate each term in the right-hand side of (2.31). Therefore, after a direct calculation, we have

$$\int_{B'_n} |u_{n,1}|^p|v_{n,1}|^q dx \geq C\sigma_n^{\frac{(N-2)(p+q)}{2}-N}, \tag{2.32}$$

$$\int_{B'_n} |u_0|^p|v_0|^q \leq C\sigma_n^{\frac{(N-2)(p+q)}{2}-N},$$

$$\int_{B'_n} |u_0|^p|v_{n,1}|^q \leq \left(\int_{B_n} |v_{n,1}|^{2^*}\right)^{\frac{q}{2^*}} |B'_n|^{1-\frac{q}{2^*}} = C\sigma_n^{\frac{qN}{2^*}-N} \leq C\sigma_n^{\frac{(N-2)(p+q)}{2}-N},$$

$$\int_{B'_n} |u_0|^p|v_{n,2}|^q \leq \left(\int_{B'_n} |v_{n,2}|^{2^*}\right)^{\frac{q}{2^*}} |B'_n|^{1-\frac{q}{2^*}} = C\sigma_n^{\frac{qN}{2^*}-N} \leq \sigma_n^{\frac{(N-2)(p+q)}{2}-N},$$

$$\begin{aligned}\int_{B'_n} |u_{n,1}|^p|v_{n,2}|^q &\leq \left(\int_{B'_n} |u_{n,1}|^{2^*}\right)^{\frac{p}{2^*}} \left(\int_{B'_n} |v_{n,2}|^{2^*}\right)^{\frac{q}{2^*}} |B'_n|^{1-\frac{p+q}{2^*}} \\ &\leq C\sigma_n^{\frac{(N-2)(p+q)}{2}-N},\end{aligned}$$

and similarly,

$$\int_{B'_n} [|u_{n,1}|^p|v_0|^q + |u_{n,2}|^p|v_{n,1}|^q + |u_{n,2}|^p|v_0|^q + |u_{n,2}|^p|v_{n,2}|^q] \leq C\sigma_n^{\frac{(N-2)(p+q)}{2}-N}.$$

Inserting the above estimates into (2.31), we conclude that there exists $c' > 0$ such

$$\int_{B_n} |u_n|^p|v_n|^q \geq c'\sigma_n^{\frac{(N-2)(p+q)}{2}-N}. \tag{2.33}$$

From (2.30) and (2.33), we find

$$\sigma_n^{\frac{(N-2)(p+q)}{2}-N} \leq C\sigma_n^{-\frac{N}{2^*}+\frac{N}{p_1}}, \tag{2.34}$$

where $p_1 > 2^*$ is any constant, satisfying $p_1 < \frac{2^*\sqrt{\bar{t}}}{\sqrt{\bar{t}}-\sqrt{\bar{t}-t}}$.

Choose $p_1 = \frac{2N}{(N-2)(p+q+1)-2N} + \delta$ with $p_1 < \frac{2^*\sqrt{\bar{t}}}{\sqrt{\bar{t}}-\sqrt{\bar{t}-t}}$, where $\delta > 0$ is a small constant. This can be achieved if $t < \bar{t} - 4$. With this p_1, we know $\frac{(2-N)(p+q)}{2} + N < \frac{N}{2^*} - \frac{N}{p_1}$. So, we obtain a contradiction from (2.34). □

3. The Proof of Theorem 1.

In this section, we first introduce some notations (see [21,24,25]). Denote the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ by $0<\lambda_1\le\lambda_2\le\cdots$, and the corresponding eigenfunctions by $e_1(x),e_2(x),\ldots$. Then $\{e_i(x)\}_{i=1}^{\infty}$ consist of an orthogonal basis in $H_0^1(\Omega)$.

It is well known that the nontrivial solutions of problem (2.1) are the corresponding nonzero critical points of the following energy functional defined on $H_0^1(\Omega)\times H_0^1(\Omega)$:

$$\begin{aligned} I^{\epsilon}(u,v) &= \frac{1}{2}\int_\Omega\left(|\nabla u|^2+|\nabla v|^2-t\frac{|u|^2+|v|^2}{|x|^2}\right)dx \\ &\quad -\frac{2}{\alpha+\beta-2\epsilon}\int_\Omega |u|^{\alpha-\epsilon}|v|^{\beta-\epsilon}dx-\frac{2}{p+q}\int_\Omega |u|^p|v|^q dx. \end{aligned}$$

Set $\tilde{Y}_k:=\oplus_{j=1}^{k}e_j,\ \tilde{Z}_k:=\overline{\oplus_{j=k}^{\infty}e_j},\ Y_k:=\tilde{Y}_k\times\tilde{Y}_k,\ Z_k:=\tilde{Z}_k\times\tilde{Z}_k,$

$$B_k:=\{(u,v)\in Y_k\ /\ \|(u,v)\|\le\rho_k\},$$

$$N_k:=\{(u,v)\in Z_k\ /\ \|(u,v)\|=r_k\},$$

where $\rho_k>r_k>0$.
Define

$$\Gamma_k:=\left\{\gamma\in C\left(B_k,H_0^1(\Omega)\times H_0^1(\Omega)\right)/\gamma\mid_{\partial B_k}=id\right\},$$

$$c_k:=\inf_{\gamma\in\Gamma_k}\max_{(u,v)\in B_k}I(\gamma(u,v)),\ c_k^n:=\inf_{\gamma\in\Gamma_k}\max_{(u,v)\in B_k}I^{\epsilon_n}(\gamma(u,v)),$$

$$b_k^n:=\inf_{(u,v)\in N_k}I^{\epsilon_n}(u,v),$$

where $\epsilon_n>0$ and $\epsilon_n\to0$ as $n\to\infty$.
For simplicity, we set $p_n=2^*-2\epsilon_n\in(2,2^*)$.

The proof of (i) of Theorem 1.

Using the arguments similar to those of Theorem 3.20 in [25], it is not difficult to verify that for every $k\ge k_0$, there exist $\rho_k>r_k>0$ such that $\rho_k\to0$ as $k\to+\infty$ and for all $n\in\mathbb{N}$

$$a_k^n:=\inf_{\substack{(u,v)\in Z_k\\ \|(u,v)\|=\rho_k}}I^{\epsilon_n}(u,v)\ge0,\ \ b_k^n:=\max_{\substack{(u,v)\in Y_k\\ \|(u,v)\|=r_k}}I^{\epsilon_n}(u,v)<0,$$

$$b_k:=\max_{\substack{(u,v)\in Y_k\\ \|(u,v)\|=r_k}}I(u,v)<0,\ \ d_k^n:=\inf_{\substack{(u,v)\in Z_k\\ \|(u,v)\|\le\rho_k}}I^{\epsilon_n}(u,v)\to0$$

as $k\to+\infty$, and $(\epsilon_n)_n$ is a decrcasing sequence with $\epsilon_n>0$ and $\epsilon_n\to0$ as $n\to\infty$. So by Theorem 3.18 in [25] (Dual fountain theorem), the functional I^{ϵ_n} has a sequence of critical points, denoted by $(\tilde{u}_k^n,\tilde{v}_k^n)$, moreover

$$I^{\epsilon_n}(\tilde{u}_k^n,\tilde{v}_k^n)=c_k^n\in[d_k^n,b_k^n].$$

Since c_k^n is negative, it is easy to see that

$$\left(\frac{1}{2}-\frac{1}{2^*-2\epsilon_n}\right)\|(\tilde{u}_k^n,\tilde{v}_k^n)\|^2<C\left(\frac{1}{p+q}-\frac{1}{2^*-2\epsilon_n}\right)\int_\Omega|\tilde{u}_k^n|^p|\tilde{v}_k^n|^q dx.$$

By using Young inequality and Sobolev's embedding, we deduce that $(\tilde{u}_k^n,\tilde{v}_k^n)$ is bounded in $H_0^1(\Omega)\times H_0^1(\Omega)$. Applying Proposition 7 we can find a subsequence of $(\tilde{u}_k^n,\tilde{v}_k^n)$, which strongly converges to a solution $(\tilde{u}_k,\tilde{v}_k)$ of (1.1) at level c_k, with $c_k:=\lim\limits_{n+\infty}c_k^n$. We claim first that for any $k\ge k_0$, $c_k<0$. In fact, since ∂B_k is compact and the functionals $(I^{\epsilon_n})_n$ are equicontinuous, then $b_k^n\to b_k$. It follows that $c_k\le b_k<0$. Secondly, we claim that $\lim\limits_{k\to+\infty}c_k=0$. In fact, for any $k\ge k_0$ there exists $n_k>k$ such that

$$|c_k^{n_k}-c_k|<\frac{1}{k}.$$

Let $\delta \in (0, \delta_0)$ be a fixed number, where

$$\delta_0 := \inf_{u \in H_0^1(\Omega), \|u\|_2 = 1} \int_\Omega |\nabla u|^2 dx > 0.$$

Define

$$\alpha_k := \inf_{u \in \tilde{Z}_k, \|u\|_{p_{n_k}} = 1} \int_\Omega \left(|\nabla u|^2 - \delta |u|^2 \right) dx, \tag{3.1}$$

where $p_{n_k} = 2^* - 2\epsilon_{n_k} \in (2, 2^*)$. We will show that, up to a subsequence, $\alpha_k \to +\infty$ as $k \to \infty$. Since $p_{n_k} < 2^*$, then the scalar α_k can be achieved by a function $v_k \in \tilde{Z}_k$, which satisfies

$$-\Delta v_k = \alpha_k |v_k|^{p_{n_k} - 2} v_k + \delta v_k \text{ in } \Omega, \text{ and } v_k = 0 \text{ on } \partial\Omega.$$

By using Sobolev's embedding, we have that

$$\alpha_k = \int_\Omega \left(|\nabla v_k|^2 - \delta |v_k|^2 \right) dx \geq \left(1 - \frac{\delta}{\delta_0} \right) \int_\Omega |\nabla v_k|^2 dx.$$

If $\alpha_k \not\to \infty$ as $k \to \infty$, then $\displaystyle\int_\Omega |\nabla v_k|^2 dx \leq C$. From Proposition 7, we conclude that $(v_k)_k$ converges strongly in $H_0^1(\Omega)$. Since $v_k \in \tilde{Z}_k$, up to a subsequence, we may assume that

$$v_k \to 0 \text{ in } H_0^1(\Omega).$$

By using Hölder inequality, it follows that $\displaystyle\lim_{k \to \infty} \int_\Omega |v_k|^{p_{n_k}} dx = 0$, which is a contradiction due to $\displaystyle\int_\Omega |v_k|^{p_{n_k}} dx = 1$. Thus $\alpha_k \to \infty$ as $k \to \infty$. On the other hand, if we suppose $(u, v) \in Z_k$, $\|(u, v)\| \leq \rho_k$ and k large enough, then

$$\begin{aligned}
I^{\epsilon_k}(u, v) &= \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 - t \frac{|u|^2 + |v|^2}{|x|^2} \right) dx \\
&\quad - \frac{2}{\alpha + \beta - 2\epsilon_k} \int_\Omega |u|^{\alpha - \epsilon_k} |v|^{\beta - \epsilon_k} dx - \frac{2}{p + q} \int_\Omega |u|^p |v|^q dx \\
&\geq \frac{1}{2} \|(u, v)\|^2 - \frac{t}{2\bar{t}} \|(u, v)\|^2 - C \alpha_k^{-\frac{p_{n_k}}{2}} \|(u, v)\|^{p_{n_k}} - C \|(u, v)\|^{p+q} \\
&\geq \frac{1}{4} \|(u, v)\|^2 - C \alpha_k^{-\frac{p_{n_k}}{2}} \|(u, v)\|^{p_{n_k}} - C \|(u, v)\|^{p+q} \\
&\geq h(\|(u, v)\|) - C \alpha_k^{-\frac{p_{n_k}}{2}} \|(u, v)\|^{p_{n_k}},
\end{aligned}$$

where $h(t) = \frac{1}{4} t^2 - C t^{p+q}$. Since the function h is decreasing on $[0, \rho_k]$ for k large enough, it follows that

$$I^{\epsilon_k}(u_k^{n_k}, v_k^{n_k}) \geq \frac{\rho_k^2}{4} - C \alpha_k^{-\frac{p_{n_k}}{2}} \rho_k^{p_{n_k}} - C \rho_k^{p+q}.$$

Hence we get

$$c_k^{n_k} \geq d_k^{n_k} \geq \frac{\rho_k^2}{4} - C \alpha_k^{-\frac{p_{n_k}}{2}} \rho_k^{p_{n_k}} - C \rho_k^{p+q}.$$

Since $\rho_k \to 0$, $\alpha_k \to +\infty$ as $k \to \infty$, it follows that

$$\lim_{k \to +\infty} c_k = \lim_{k \to +\infty} c_k^{n_k} = 0.$$

The conclusion of (i) of Theorem 1 is now obvious.

The proof of (ii) of Theorem 1.

Using the arguments of Theorem 3.7 in [25], we can prove that for every k, there exist $\varrho_k > \tau_k > 0$ such that $\varrho_k \to +\infty$ as $k \to +\infty$ and

$$\mathrm{a}_k^n := \max_{\substack{(u,v)\in Y_k\\ \|(u,v)\|=\varrho_k}} I^{\epsilon_n}(u,v) \le 0,\ \mathrm{b}_k^n := \inf_{\substack{(u,v)\in Z_k\\ \|(u,v)\|=r_k}} I^{\epsilon_n}(u,v) \to \infty \text{ as } k \to +\infty,$$

where

$$\mathrm{B}_k := \{(u,v) \in Y_k : \|(u,v)\| \le \varrho_k\},\ \mathrm{N}_k := \{(u,v) \in Z_k : \|(u,v)\| = \tau_k\},$$

and $(\epsilon_n)_n$ is a decreasing sequence with $\epsilon_n > 0$ and $\epsilon_n \to 0$ as $n \to \infty$. So by Theorem 3.6 in [25] (Fountain theorem), we conclude that I^{ϵ_n} has a sequence of critical points, denoted by $(u_k^n, v_k^n)_n$. Moreover, $\mathfrak{c}_k^n = I^{\epsilon_n}(u_k^n, v_k^n)$, where

$$\mathfrak{c}_k^n := \inf_{\gamma\in\Gamma_k} \max_{(u,v)\in\mathrm{B}_k} I^{\epsilon_n}(\gamma(u,v)),$$

and

$$\Gamma_k := \left\{\gamma \in \mathcal{C}\left(\mathrm{B}_k, H_0^1(\Omega) \times H_0^1(\Omega)\right) : \gamma_{|\partial\mathrm{B}_k} = id\right\}.$$

We claim that for any $k \in \mathbb{N}$,

$$\mathfrak{c}_k^n \to \mathfrak{c}_k := \inf_{\gamma\in\Gamma_k} \max_{(u,v)\in B_k} I(\gamma(u,v)) \text{ as } n \to +\infty.$$

For any $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$,

$$I(u,v) = I^{\epsilon_n}(u,v) + G_n(u,v),$$

where

$$G_n(u,v) = \frac{2}{\alpha+\beta-2\epsilon_n}\int_\Omega |u|^{\alpha-\epsilon_n}|v|^{\beta-\epsilon_n}dx - \frac{2}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^\beta dx$$

Since B_k is compact and the functionals $\pm G_n$ are equicontinuous on B_k, we derive that $\lim\limits_{n\to\infty} \sup\limits_{(u,v)\in B_k} \pm G_n(u,v) = 0$. Note that $id \in \Gamma_k$, for any $k \in \mathbb{N}^*$, we deduce that

$$\begin{aligned}
\mathfrak{c}_k &= \inf_{\gamma\in\Gamma_k} \sup_{(u,v)\in B_k} I(\gamma(u,v)) \\
&\le \underline{\lim}_{n\to\infty} \inf_{\gamma\in\Gamma_k} \sup_{(u,v)\in B_k} I^{\epsilon_n}(\gamma(u,v)) + \overline{\lim}_{n\to\infty} \sup_{(u,v)\in B_k} (G_n(u,v)) \qquad (3.2)\\
&= \underline{\lim}_{n\to\infty} \mathfrak{c}_k^n.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\overline{\lim}_{n\to\infty} \mathfrak{c}_k^n &= \overline{\lim}_{n\to\infty} \inf_{\gamma\in\Gamma_k} \sup_{(u,v)\in B_k} I^{\epsilon_n}(\gamma(u,v)) \\
&\le \overline{\lim}_{n\to\infty} \inf_{\gamma\in\Gamma_k} \sup_{(u,v)\in B_k} I(\gamma(u,v)) + \overline{\lim}_{n\to\infty} \sup_{(u,v)\in B_k} (-G_n(u,v)) \qquad (3.3)\\
&= \mathfrak{c}_k.
\end{aligned}$$

Combining (3.2) with (3.3), we infer that

$$\lim_{n\to\infty} \mathfrak{c}_k^n = \mathfrak{c}_k. \qquad (3.4)$$

We have $I^{\epsilon_n}(u_k^n, v_k^n) = \mathfrak{c}_k^n$ and $I^{\epsilon_n\prime}(u_k^n, v_k^n)(u_k^n, v_k^n) = 0$. From this we obtain

$$\left(\frac{1}{2} - \frac{1}{p_n}\right)\|(u_k^n, v_k^n)\|^2 - C\left(\frac{1}{p+q} - \frac{1}{p_n}\right)\int_\Omega |u_k^n|^p|v_k^n|^q dx < \mathfrak{c}_k^n.$$

Since $(\mathfrak{c}_k^n)_n$ is bounded, then for all n

$$\left(\frac{1}{2}-\frac{1}{p_n}\right)\|(u_k^n,v_k^n)\|^2 < C\left(\frac{1}{p+q}-\frac{1}{p_n}\right)\int_\Omega |u_k^n|^p|v_k^n|^q dx + C.$$

By Young inequality, Sobolev's embedding and the fact that $p+q<2$, we get that $(u_k^n,v_k^n)_n$ is bounded in $H_0^1(\Omega)\times H_0^1(\Omega)$. Applying Proposition 7 we can find a subsequence of $(u_k^n,v_k^n)_n$, still denoted by $(u_k^n,v_k^n)_n$, such that

$$(u_k^n,v_k^n)\to(u_k,v_k) \text{ strongly in } H_0^1(\Omega)\times H_0^1(\Omega),$$

for some $(u_k,v_k)\in H_0^1(\Omega)\times H_0^1(\Omega)$ and $I(u_k,v_k)=\mathfrak{c}_k$. Therefore, (u_k,v_k) is solution of (1.1). Now we claim that

$$\lim_{k\to\infty}\mathfrak{c}_k=+\infty.$$

Using arguments as in the proof of (i), we see that $(u_k^n,v_k^n)_n$ is bounded in $H_0^1(\Omega)\times H_0^1(\Omega)$ and then we can find a subsequence of $(u_k^n,v_k^n)_n$ which strongly converges to a solution (u_k,v_k) of (1.1) at level $\mathfrak{c}_k$. It follows that for every $k\in\mathbb{N}$, there exists $n_k>k$ such that

$$|\mathfrak{c}_k^{n_k}-\mathfrak{c}_k|<\frac{1}{k}.$$

By using Young inequality, we obtain

$$I^{\epsilon_n}(u,v)\geq\frac{1}{4}\|(u,v)\|^2-C\alpha_k^{-\frac{p_{n_k}}{2}}\|(u,v)\|^{p_{n_k}}-C,$$

where α_k is given in (3.1). Choosing $\tau_k=\left(\frac{\alpha_k^{\frac{p_{n_k}}{2}}}{2Cp_{n_k}}\right)^{\frac{1}{p_{n_k}-2}}$. If $(u,v)\in Z_k$ and $\|(u,v)\|=\tau_k$, we obtain that

$$I^{\epsilon_n}(u,v)\geq\frac{1}{4}\left(1-\frac{2}{p_{n_k}}\right)\left(\frac{\alpha_k^{\frac{p_{n_k}}{2}}}{2Cp_{n_k}}\right)^{\frac{2}{p_{n_k}-2}}-C.$$

Since we have that $\alpha_k\to\infty$ as $k\to\infty$, then $b_k^{n_k}\to\infty$ as $k\to\infty$. By Theorem 3.5 in [25], we have that $\mathfrak{c}_k^{n_k}\geq\mathrm{b}_k^{n_k}$ and so $\lim\limits_{k\to\infty}\mathfrak{c}_k=\lim\limits_{k\to\infty}\mathfrak{c}_k^{n_k}=+\infty$. The conclusion of (ii) is now obvious.

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