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Reverse Hölder Inequality and Fibonacci numbers

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ABSTRACT: In this paper we present reverse Hölder-type inequalities with power sums. We apply these results to sums involving Fibonacci numbers.

Key Words: Fibonacci sequence, Fibonacci identity, power sum, Hölder inequality.

Contents

1. Introduction

The well-known classical Hölder inequality can be stated as follows.

Theorem 1.1 ([6]) *Let* $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n_+$.

(i) If $p > 1$ *and* $q = \frac{p}{p-1}$, *then*

$$\sum_{i=1}^{n} x_i y_i \le \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}}. \tag{1.1}$$

(ii) If $0 < p < 1$ *and* $q = \frac{p}{p-1}$, *then the reverse inequality holds in* (1.1).

The integral version of Hölder's inequality is the following.

Theorem 1.2 *Let* f, g *be positive continuous functions on* $[a, b]$. *If* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\int_a^b f(x)g(x)dx \le \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}}.$$

The Hölder inequality has been extensively studied by numerous researchers, and as a result, many of its generalizations and refinements have been obtained so far. See, for example [5,6,11] and the references therein.
In [11], the author gave the following integral reverses of Hölder's inequality:

Theorem 1.3 ([11]) *Let* f *and* g *be positive functions satisfying*

$$0 < m \le f(x)g(x), \quad \forall x \in [a, b].$$

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\frac{\displaystyle\int_a^b f^p(x)dx \int_a^b g^q(x)dx}{\left(\displaystyle\int_a^b f^{pq}(x)dx\right)^{1/q} \left(\displaystyle\int_a^b g^{pq}(x)\, dx\right)^{1/p}} \le \frac{1}{m} \int_a^b f(x)g(x)dx. \tag{1.2}$$

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Theorem 1.4 ([11]) *Let f and g be positive functions satisfying*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M, \quad \forall x \in [a,b].$$

Let $p > 0, q > 0$. Then

$$\left(\int_a^b f^p(x)dx\right)^{1/p}\left(\int_a^b g^q(x)dx\right)^{1/q} \leq \frac{M}{m}\left(\int_a^b (f(x)g(x))^{p/2}dx\right)^{1/p}\left(\int_a^b (f(x)g(x))^{q/2}dx\right)^{1/q}. \quad (1.3)$$

Recently, in [3] authors obtained a chain of inequalities for power sums using Hölder's and Cauchy's inequalities and their conversions. Motivated by their approach, in this paper we provide the discrete versions of Theorems 1.3 and 1.4, which we then use to derive a series of inequalities for power sums.

We use the following notation: for $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ we denote

$$S_n^{[\alpha]}(\mathbf{x}) = \sum_{i=1}^n x_i^\alpha. \quad (1.4)$$

We will also use the following result:

Proposition 1.1 ([5]) *If $\alpha > \beta > 0$ then*

$$\left(S_n^{[\alpha]}(\mathbf{x})\right)^{1/\alpha} \leq \left(S_n^{[\beta]}(\mathbf{x})\right)^{1/\beta}. \quad (1.5)$$

2. Reverse Hölder inequalities

Theorem 2.1 *Let $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}_+^n$ and*

$$0 < m \leq x_i y_i, \quad i = 1, \ldots, n.$$

Let $q > p > 0$. Then

$$\frac{\left(\sum_{i=1}^n x_i^p\right)\left(\sum_{i=1}^n y_i^q\right)}{\left(\sum_{i=1}^n x_i^{\frac{pq}{q-p}}\right)^{1-\frac{p}{q}}\left(\sum_{i=1}^n y_i^{\frac{q^2}{p}}\right)^{\frac{p}{q}}} \leq \frac{1}{m}\sum_{i=1}^n x_i y_i. \quad (2.1)$$

Proof:

$$\begin{aligned}
\sum_{i=1}^n x_i^p &= \sum_{i=1}^n x_i^{\frac{p}{q}} y_i^{\frac{p}{q}} x_i^{p-\frac{p}{q}} y_i^{-\frac{p}{q}} \leq \left(\sum_{i=1}^n x_i y_i\right)^{\frac{p}{q}}\left(\sum_{i=1}^n x_i^{\frac{q}{q-p}(p-\frac{p}{q})} y_i^{-\frac{p}{q-p}}\right)^{\frac{q-p}{q}} \\
&\leq \left(\sum_{i=1}^n x_i y_i\right)^{\frac{p}{q}}\left(\sum_{i=1}^n x_i^{\frac{q}{q-p}(p-\frac{p}{q})}\left(\frac{x_i}{m}\right)^{\frac{p}{q-p}}\right)^{\frac{q-p}{q}} \qquad (2.2) \\
&= \frac{1}{m^{\frac{p}{q}}}\left(\sum_{i=1}^n x_i y_i\right)^{\frac{p}{q}}\left(\sum_{i=1}^n x_i^{\frac{pq}{q-p}}\right)^{\frac{q-p}{q}}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{i=1}^n y_i^q &= \sum_{i=1}^n x_i^{\frac{q-p}{q}} y_i^{\frac{q-p}{q}} y_i^{q-\frac{q-p}{q}} x_i^{-\frac{q-p}{q}} \leq \left(\sum_{i=1}^n x_i y_i\right)^{\frac{q-p}{q}}\left(\sum_{i=1}^n y_i^{\frac{q}{p}(q-\frac{q-p}{q})} x_i^{-\frac{q-p}{q}}\right)^{\frac{p}{q}} \\
&\leq \left(\sum_{i=1}^n x_i y_i\right)^{\frac{q-p}{q}}\left(\sum_{i=1}^n y_i^{\frac{q}{p}(q-\frac{q-p}{q})}\left(\frac{y_i}{m}\right)^{\frac{q-p}{p}}\right)^{\frac{p}{q}} \qquad (2.3) \\
&= \frac{1}{m^{\frac{q-p}{q}}}\left(\sum_{i=1}^n x_i y_i\right)^{\frac{q-p}{q}}\left(\sum_{i=1}^n y_i^{\frac{q^2}{p}}\right)^{\frac{p}{q}}
\end{aligned}$$

By multiplying (2.2) and (2.3), we obtain the desired result (2.1). Hence, the proof is complete. □

Remark 2.1 *An analogous inequality holds for integrals.*

The following lemmas provide the discrete versions of inequalities (1.2) and (1.3).

Lemma 2.1 *Let* $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}_+^n$ *and*

$$0 < m \le x_i y_i, \quad i = 1, \ldots, n.$$

Let $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\frac{\left(\sum_{i=1}^n x_i^p\right)\left(\sum_{i=1}^n y_i^q\right)}{\left(\sum_{i=1}^n x_i^{pq}\right)^{1/q}\left(\sum_{i=1}^n y_i^{pq}\right)^{1/p}} \le \frac{1}{m}\sum_{i=1}^n x_i y_i. \tag{2.4}$$

Proof: Similar to the proof of Theorem 2.1. First, we see that

$$\begin{aligned}\sum_{i=1}^n x_i^p &= \sum_{i=1}^n x_i^{\frac{1}{p}} y_i^{\frac{1}{p}} x_i^{p-\frac{1}{p}} y_i^{-\frac{1}{p}} \le \left(\sum_{i=1}^n x_i y_i\right)^{\frac{1}{p}} \left(\sum_{i=1}^n x_i^{q(p-\frac{1}{p})} y_i^{-\frac{q}{p}}\right)^{\frac{1}{q}} \\ &\le \left(\sum_{i=1}^n x_i y_i\right)^{\frac{1}{p}} \left(\sum_{i=1}^n x_i^{q(p-\frac{1}{p})} \left(\frac{x_i}{m}\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \\ &= \frac{1}{m^{\frac{1}{p}}} \left(\sum_{i=1}^n x_i y_i\right)^{\frac{1}{p}} \left(\sum_{i=1}^n x_i^{pq}\right)^{\frac{1}{q}}\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\sum_{i=1}^n y_i^q &= \sum_{i=1}^n x_i^{\frac{1}{q}} y_i^{\frac{1}{q}} y_i^{q-\frac{1}{q}} x_i^{-\frac{1}{q}} \le \left(\sum_{i=1}^n x_i y_i\right)^{\frac{1}{q}} \left(\sum_{i=1}^n y_i^{p(q-\frac{1}{q})} x_i^{-\frac{p}{q}}\right)^{\frac{1}{p}} \\ &\le \left(\sum_{i=1}^n x_i y_i\right)^{\frac{1}{q}} \left(\sum_{i=1}^n y_i^{p(q-\frac{1}{q})} \left(\frac{y_i}{m}\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \\ &= \frac{1}{m^{\frac{1}{q}}} \left(\sum_{i=1}^n x_i y_i\right)^{\frac{1}{q}} \left(\sum_{i=1}^n y_i^{pq}\right)^{\frac{1}{p}}.\end{aligned} \tag{2.6}$$

By multiplying (2.5) and (2.6), we obtain (2.4). □

Lemma 2.2 *Let* $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}_+^n$ *and*

$$0 < m \le \frac{x_i}{y_i} \le M, \quad i = 1, \ldots, n. \tag{2.7}$$

Let $p > 0, q > 0$. *Then*

$$\left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}} \le \frac{M}{m} \left(\sum_{i=1}^n (x_i y_i)^{\frac{p}{2}}\right)^{\frac{1}{p}} \left(\sum_{i=1}^n (x_i y_i)^{\frac{q}{2}}\right)^{\frac{1}{q}}. \tag{2.8}$$

Proof: From the assumption (2.7), we have

$$m+1 \leq \frac{x_i+y_i}{y_i} \leq M+1, \quad i=1,\ldots,n \tag{2.9}$$

and

$$\frac{M+1}{M} \leq \frac{x_i+y_i}{x_i} \leq \frac{m+1}{m}, \quad i=1,\ldots,n. \tag{2.10}$$

The above inequalities imply

$$\begin{aligned}
x_i &\leq \frac{M}{M+1}(x_i+y_i), \quad y_i \leq \frac{1}{m+1}(x_i+y_i) \\
x_i+y_i &\leq \frac{m+1}{m}x_i, \qquad x_i+y_i \leq (M+1)y_i \\
(x_i+y_i)^2 &\leq \frac{(m+1)(M+1)}{m}x_iy_i.
\end{aligned}$$

Now, from the last inequality, it follows that

$$\left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{\frac{1}{p}} \leq \left(\frac{(m+1)(M+1)}{m}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n}(x_iy_i)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

$$\left(\sum_{i=1}^{n}(x_i+y_i)^q\right)^{\frac{1}{q}} \leq \left(\frac{(m+1)(M+1)}{m}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n}(x_iy_i)^{\frac{q}{2}}\right)^{\frac{1}{q}}.$$

Therefore, we have

$$\begin{aligned}
\left(\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n}y_i^q\right)^{\frac{1}{q}} &\leq \frac{M}{(M+1)(m+1)} \left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n}(x_i+y_i)^q\right)^{\frac{1}{q}} \\
&\leq \frac{M}{m}\left(\sum_{i=1}^{n}(x_iy_i)^{\frac{p}{2}}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n}(x_iy_i)^{\frac{q}{2}}\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.11}$$

This completes the proof. □

In the sequel, we derive a series of inequalities with power sums using the reverse Hölder inequalities from Theorem 2.1 and Lemmas 2.1 and 2.2.

Theorem 2.2 *Let $q > p > 0$ and $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}_+^n$. If u and v are real numbers such that $\alpha = \frac{u}{p}+\frac{v}{q}$ and $m = \min_i \{x_i^\alpha\}$ then*

$$\frac{S_n^{[u]}(\mathbf{x})\ S_n^{[v]}(\mathbf{x})}{\left(S_n^{\left[\frac{uq}{q-p}\right]}(\mathbf{x})\right)^{1-\frac{p}{q}} \left(S_n^{\left[\frac{vq}{p}\right]}(\mathbf{x})\right)^{p/q}} \leq \frac{1}{m}S_n^{[\alpha]}(\mathbf{x}). \tag{2.12}$$

Proof: We take the substitutions $x_i \to x_i^{u/p}$ and $y_i \to x_i^{v/q}$ in Theorem 2.1, whereby the inequality (2.1) is transformed into (2.12). Also, note that the condition $0 < m \leq x_iy_i$ is satisfied for $m = \min_i \{x_i^\alpha\}$ so inequality (2.12) holds. □

Theorem 2.3 *Let $p > 1, q = \frac{p}{p-1}$, $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}_+^n$ and $m = \min_i \{x_i^\alpha\}$.*

(i) Let $x_i \geq 1,\ i = 1, \ldots, n$. If u and v are real numbers such that $\alpha = \frac{u}{p} + \frac{v}{q}$ and $0 < \alpha < \beta$ then

$$\begin{aligned} \frac{S_n^{[u]}(\mathbf{x})\ S_n^{[v]}(\mathbf{x})}{\left(S_n^{[uq]}(\mathbf{x})\right)^{1/q}\left(S_n^{[vp]}(\mathbf{x})\right)^{1/p}} &\leq \frac{1}{m} S_n^{[\alpha]}(\mathbf{x}) \leq \frac{1}{m \cdot n^{\frac{\alpha}{\beta}-1}}\left(S_n^{[\beta]}(\mathbf{x})\right)^{\alpha/\beta} \\ &\leq \frac{1}{m} S_n^{[\beta]}(\mathbf{x}) \leq \frac{1}{m}\left(S_n^{[\alpha]}(\mathbf{x})\right)^{\beta/\alpha}. \end{aligned} \tag{2.13}$$

(ii) If u and v are real numbers such that $\alpha = \frac{u}{p} + \frac{v}{q}$ and $\alpha > \beta > 0$ then

$$\frac{S_n^{[u]}(\mathbf{x})\ S_n^{[v]}(\mathbf{x})}{\left(S_n^{[uq]}(\mathbf{x})\right)^{1/q}\left(S_n^{[vp]}(\mathbf{x})\right)^{1/p}} \leq \frac{1}{m} S_n^{[\alpha]}(\mathbf{x}) \leq \frac{1}{m}\left(S_n^{[\beta]}(\mathbf{x})\right)^{\alpha/\beta}. \tag{2.14}$$

Proof:

(i) Taking substitutions $x_i \to x_i^{u/p}$ and $y_i \to x_i^{v/q}$ in Lemma 2.1 we observe that the condition $0 < m \leq x_i y_i$ is satisfied for $m = \min_i \{x_i^\alpha\}$, and the inequality (2.4) is transformed into

$$\frac{\left(\sum_{i=1}^n x_i^u\right)\left(\sum_{i=1}^n x_i^v\right)}{\left(\sum_{i=1}^n x_i^{uq}\right)^{1/q}\left(\sum_{i=1}^n x_i^{vp}\right)^{1/p}} \leq \frac{1}{m}\sum_{i=1}^n x_i^\alpha. \tag{2.15}$$

To obtain the desired inequalities, we first apply the reverse Jensen's inequality to the function $x \mapsto x^{\alpha/\beta}$, where $\alpha < \beta$, on the right-hand side of (2.15). Then, we use the monotonicity of the exponential function $x \mapsto b^x$, where $b = \frac{1}{n}\sum_{i=1}^n x_i^\beta \geq 1$, and finally apply inequality (1.5):

$$\begin{aligned} \frac{\left(\sum_{i=1}^n x_i^u\right)\left(\sum_{i=1}^n x_i^v\right)}{\left(\sum_{i=1}^n x_i^{uq}\right)^{1/q}\left(\sum_{i=1}^n x_i^{vp}\right)^{1/p}} &\leq \frac{1}{m}\sum_{i=1}^n x_i^\alpha = \frac{1}{m}\sum_{i=1}^n \left(x_i^\beta\right)^{\alpha/\beta} \leq \frac{n}{m}\left(\frac{1}{n}\sum_{i=1}^n x_i^\beta\right)^{\alpha/\beta} \\ &\leq \frac{1}{m}\sum_{i=1}^n x_i^\beta \leq \left(\sum_{i=1}^n x_i^\alpha\right)^{\frac{\beta}{\alpha}}. \end{aligned} \tag{2.16}$$

(ii) Similar to the proof of (i), we apply Lemma 2.1 with substitutions $x_i \to x_i^{u/p}$ and $y_i \to x_i^{v/q}$, and then the moment inequality (1.5).

□

Theorem 2.4 *Let u and v be real numbers such that $\alpha = \frac{u}{p} + \frac{v}{q}$. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, and let $m = \min_i \left\{x_i^{\frac{u}{p}-\frac{v}{q}}\right\}$, $M = \max_i \left\{x_i^{\frac{u}{p}-\frac{v}{q}}\right\}$.*

(i) If $0 < p, q < 1$ then we have

$$\left(S_n^{[u]}(\mathbf{x})\right)^{\frac{1}{p}}\left(S_n^{[v]}(\mathbf{x})\right)^{\frac{1}{q}} \leq \frac{M}{m}\left(S_n^{[\alpha p/2]}(\mathbf{x})\right)^{\frac{1}{p}}\left(S_n^{[\alpha q/2]}(\mathbf{x})\right)^{\frac{1}{q}} \leq \frac{M}{m} n^{\frac{1}{p}+\frac{1}{q}-1}\left(S_n^{[\alpha]}(\mathbf{x})\right). \tag{2.17}$$

(ii) Let $x_i \geq 1,\ i = 1, \ldots, n$. If $p, q \geq 1$ and $\alpha \geq 0$ then we have

$$\left(S_n^{[u]}(\mathbf{x})\right)^{\frac{1}{p}} \left(S_n^{[v]}(\mathbf{x})\right)^{\frac{1}{q}} \leq \frac{M}{m} \left(S_n^{[\alpha p/2]}(\mathbf{x})\right)^{\frac{1}{p}} \left(S_n^{[\alpha q/2]}(\mathbf{x})\right)^{\frac{1}{q}} \leq \frac{Mn^2}{m} S_n^{[\alpha p]}(\mathbf{x})\, S_n^{[\alpha q]}(\mathbf{x}). \tag{2.18}$$

Proof:

(i) By substituting $x_i \to x_i^{u/p}$ and $y_i \to x_i^{v/q}$ into Lemma 2.2, we observe that the condition $0 < m \leq \frac{x_i}{y_i} \leq M$ is satisfied for $m = \min_i \left\{x_i^{\frac{u}{p} - \frac{v}{q}}\right\}$ and $M = \max_i \left\{x_i^{\frac{u}{p} - \frac{v}{q}}\right\}$. With this substitution, inequality (2.8) becomes:

$$\left(\sum_{i=1}^n x_i^u\right)^{\frac{1}{p}} \left(\sum_{i=1}^n x_i^v\right)^{\frac{1}{q}} \leq \frac{M}{m} \left(\sum_{i=1}^n x_i^{\frac{\alpha p}{2}}\right)^{\frac{1}{p}} \left(\sum_{i=1}^n x_i^{\frac{\alpha q}{2}}\right)^{\frac{1}{q}}. \tag{2.19}$$

By applying Jensen's inequality to the functions $x \mapsto x^{p/2}$ and $x \mapsto x^{q/2}$ we obtain:

$$\sum_{i=1}^n (x_i^\alpha)^{\frac{p}{2}} \leq n^{1-\frac{p}{2}} \left(\sum_{i=1}^n x_i^\alpha\right)^{\frac{p}{2}} \tag{2.20}$$

$$\sum_{i=1}^n (x_i^\alpha)^{\frac{q}{2}} \leq n^{1-\frac{q}{2}} \left(\sum_{i=1}^n x_i^\alpha\right)^{\frac{q}{2}} \tag{2.21}$$

Using the monotonicity of the functions $x \mapsto x^{1/p}$ and $x \mapsto x^{1/q}$ we derive:

$$\left(\sum_{i=1}^n (x_i^\alpha)^{\frac{p}{2}}\right)^{\frac{1}{p}} \leq n^{\frac{1}{p}-\frac{1}{2}} \left(\left(\sum_{i=1}^n x_i^\alpha\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} = n^{\frac{1}{p}-\frac{1}{2}} \left(\sum_{i=1}^n x_i^\alpha\right)^{\frac{1}{2}} \tag{2.22}$$

$$\left(\sum_{i=1}^n (x_i^\alpha)^{\frac{q}{2}}\right)^{\frac{1}{q}} \leq n^{\frac{1}{q}-\frac{1}{2}} \left(\left(\sum_{i=1}^n x_i^\alpha\right)^{\frac{q}{2}}\right)^{\frac{1}{q}} = n^{\frac{1}{q}-\frac{1}{2}} \left(\sum_{i=1}^n x_i^\alpha\right)^{\frac{1}{2}} \tag{2.23}$$

Combining (2.19), (2.22) and (2.23) we obtain:

$$\left(\sum_{i=1}^n x_i^u\right)^{\frac{1}{p}} \left(\sum_{i=1}^n x_i^v\right)^{\frac{1}{q}} \leq \frac{M}{m} \left(\sum_{i=1}^n (x_i^\alpha)^{\frac{p}{2}}\right)^{\frac{1}{p}} \left(\sum_{i=1}^n (x_i^\alpha)^{\frac{q}{2}}\right)^{\frac{1}{q}} \leq \frac{M}{m} n^{\frac{1}{p}+\frac{1}{q}-1} \sum_{i=1}^n x_i^\alpha \tag{2.24}$$

Hence, (2.17) holds.

(ii)

$$\left(\sum_{i=1}^n x_i^u\right)^{\frac{1}{p}} \left(\sum_{i=1}^n x_i^v\right)^{\frac{1}{q}} \leq \frac{M}{m} \left(\sum_{i=1}^n x_i^{\frac{\alpha p}{2}}\right)^{\frac{1}{p}} \left(\sum_{i=1}^n x_i^{\frac{\alpha q}{2}}\right)^{\frac{1}{q}} \tag{2.25}$$

$$\leq \frac{M}{m} \left(\sum_{i=1}^n x_i^{\frac{\alpha p}{2}}\right)^{2} \left(\sum_{i=1}^n (x_i)^{\frac{\alpha q}{2}}\right)^{2} \tag{2.26}$$

$$\leq \frac{Mn^2}{m} \sum_{i=1}^n x_i^{\alpha p} \sum_{i=1}^n x_i^{\alpha q} \tag{2.27}$$

First, we use (2.8) with substitutions $x_i \to x_i^{u/p}$, $y_i \to x_i^{v/q}$ to obtain (2.25). Then, in (2.26) we use the monotonicity of the exponential function $x \mapsto b^x,\ b = \sum_{i=1}^n x_i^{\alpha p/2} \geq 1$, for $\alpha \geq 0$. In (2.27) we use Jensen's inequality for the function $x \mapsto x^2$.

$\square$

3. Applications

In this section, we apply the obtained results to the Fibonacci sequence to estimate the power sums of Fibonacci numbers.

The classical Fibonacci numbers are defined by the linear recurrence relation

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-2} + F_{n-1}, \quad n = 2, 3, \ldots.$$

In the papers [1], [2], and [9], the following identity is given:

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}. \tag{3.1}$$

Using our notation, where $x_i = F_i$ for $i = 1, \ldots, n$, identity (3.1) can be rewritten as:

$$S_n^{[2]}(\mathbf{x}) = F_n F_{n+1}. \tag{3.2}$$

By applying Theorem 2.3 with $x_i = F_i$, $\beta = 2$, and using the identity (3.2), along with the observation that the condition $x_i \geq 1$ holds for all i, we obtain the following theorem.

Theorem 3.1 *Let $p > 1$ and $q = \frac{p}{p-1}$.*

(i) If u and v are real numbers such that $\alpha = \frac{u}{p} + \frac{v}{q}$ and $0 < \alpha < 2$ then

$$\frac{\left(\sum_{i=1}^{n} F_i^u\right)\left(\sum_{i=1}^{n} F_i^v\right)}{\left(\sum_{i=1}^{n} F_i^{uq}\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n} F_i^{vp}\right)^{\frac{1}{p}}} \leq \sum_{i=1}^{n} F_i^{\alpha} \leq \frac{1}{n^{\frac{\alpha}{2}-1}} \left(F_n F_{n+1}\right)^{\alpha/2} \leq F_n F_{n+1} \leq \left(\sum_{i=1}^{n} F_i^{\alpha}\right)^{2/\alpha}. \tag{3.3}$$

(ii) If u and v are real numbers such that $\alpha = \frac{u}{p} + \frac{v}{q}$ and $\alpha > 2$ then

$$\frac{\left(\sum_{i=1}^{n} F_i^u\right)\left(\sum_{i=1}^{n} F_i^v\right)}{\left(\sum_{i=1}^{n} F_i^{uq}\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n} F_i^{vp}\right)^{\frac{1}{p}}} \leq \sum_{i=1}^{n} F_i^{\alpha} \leq \left(F_n F_{n+1}\right)^{\alpha/2}. \tag{3.4}$$

Furthermore, by applying Theorems 2.2 and 2.4 with $x_i = F_i$ and $\alpha = 2$, we obtain the following results.

Theorem 3.2 *Let $q > p > 0$. If u and v are real numbers such that $\frac{u}{p} + \frac{v}{q} = 2$ then*

$$\frac{\left(\sum_{i=1}^{n} F_i^u\right)\left(\sum_{i=1}^{n} F_i^v\right)}{\left(\sum_{i=1}^{n} F_i^{\frac{uq}{q-p}}\right)^{1-\frac{p}{q}}\left(\sum_{i=1}^{n} F_i^{\frac{vq}{p}}\right)^{p/q}} \leq F_n F_{n+1}. \tag{3.5}$$

Theorem 3.3 *Let u and v be real numbers such that $\frac{u}{p} + \frac{v}{q} = 2$, and let $m = \min_i \left\{F_1^{\frac{u}{p}-\frac{v}{q}}, F_n^{\frac{u}{p}-\frac{v}{q}}\right\}$ and $M = \max_i \left\{F_1^{\frac{u}{p}-\frac{v}{q}}, F_n^{\frac{u}{p}-\frac{v}{q}}\right\}$.*

(i) *If* $0 < p, q < 1$ *then*

$$\left(\sum_{i=1}^{n} F_i^u\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} F_i^v\right)^{\frac{1}{q}} \leq \frac{M}{m} \left(\sum_{i=1}^{n} F_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} F_i^q\right)^{\frac{1}{q}} \leq \frac{M}{m} n^{\frac{1}{p}+\frac{1}{q}-1} F_n F_{n+1}. \tag{3.6}$$

(ii) *If* $p, q \geq 1$ *then*

$$\left(\sum_{i=1}^{n} F_i^u\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} F_i^v\right)^{\frac{1}{q}} \leq \frac{M}{m} \left(\sum_{i=1}^{n} F_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} F_i^q\right)^{\frac{1}{q}} \leq \frac{Mn^2}{m} \left(\sum_{i=1}^{n} F_i^{2p}\right) \left(\sum_{i=1}^{n} F_i^{2q}\right) \leq \frac{Mn^4}{m} F_n^{2(p+q)}. \tag{3.7}$$

In [3], the authors studied power sums of Fibonacci numbers in cases where the exponents are distinct. For example, in the case when $\alpha = -1$ and $x_i = F_i F_{i+2}$ the following identity holds (see [4])

$$S_n^{[-1]}(\mathbf{x}) = 1 - \frac{1}{F_{n+1} F_{n+2}}. \tag{3.8}$$

In the case when $\alpha = 1$ and $x_i = F_i F_{i+2}$ we have

$$\begin{aligned} S_n^{[1]}(\mathbf{x}) &= \sum_{i=1}^{n} F_i F_{i+2} = \sum_{i=1}^{n} F_i (F_i + F_{i+1}) = \sum_{i=1}^{n} F_i^2 + \sum_{i=1}^{n} F_i F_{i+1} \\ &= F_n F_{n+1} + F_{n+1}^2 - \frac{1 + (-1)^n}{2} = F_{n+1} F_{n+2} - \frac{1 + (-1)^n}{2}. \end{aligned} \tag{3.9}$$

By using (3.8) and (3.9) we obtain the following theorem.

Theorem 3.4 *Let* $p > 1$*. Then the following statements hold:*

(i) *If* $\beta > 1$*, then*

$$\begin{aligned} &\frac{\sum_{i=1}^{n} (F_i F_{i+2})^{2p-1} \left(1 - \frac{1}{F_{n+1} F_{n+2}}\right)}{\left(\sum_{i=1}^{n} (F_i F_{i+2})^{\frac{2p^2-p}{p-1}}\right)^{\frac{p-1}{p}} \left(\sum_{i=1}^{n} (F_i F_{i+2})^{-p}\right)^{\frac{1}{p}}} \leq \frac{1}{2} \left(F_{n+1} F_{n+2} - \frac{1 + (-1)^n}{2}\right) \\ &\leq \frac{1}{2 \cdot n^{\frac{1}{\beta}-1}} \left(\sum_{i=1}^{n} (F_i F_{i+2})^{\beta}\right)^{\frac{1}{\beta}} \leq \frac{1}{2} \sum_{i=1}^{n} (F_i F_{i+2})^{\beta} \leq \frac{1}{2} \left(F_{n+1} F_{n+2} - \frac{1 + (-1)^n}{2}\right)^{\beta}. \end{aligned} \tag{3.10}$$

(ii) *If* $\beta < 1$*, then*

$$\begin{aligned} &\frac{\sum_{i=1}^{n} (F_i F_{i+2})^{2p-1} \left(1 - \frac{1}{F_{n+1} F_{n+2}}\right)}{\left(\sum_{i=1}^{n} (F_i F_{i+2})^{\frac{2p^2-p}{p-1}}\right)^{\frac{p-1}{p}} \left(\sum_{i=1}^{n} (F_i F_{i+2})^{-p}\right)^{\frac{1}{p}}} \leq \frac{1}{2} \left(F_{n+1} F_{n+2} - \frac{1 + (-1)^n}{2}\right) \\ &\leq \frac{1}{2} \left(\sum_{i=1}^{n} (F_i F_{i+2})^{\beta}\right)^{\frac{1}{\beta}}. \end{aligned} \tag{3.11}$$

Proof: Take $x_i = F_i F_{i+2}$ for $i = 1, \ldots, n$, $\alpha = 1$, and $v = -1$ in Theorem 2.3 and use identities (3.8) and (3.9). $\square$

4. Conclusion

In this paper, we obtained new inequalities for power sums using reverse Hölder inequalities. We have shown that, by applying Theorems 2.2, 2.3 and 2.4, along with identity (3.2) related to Fibonacci numbers, a series of inequalities can be derived. Additionally, we present identities found in [4], [7], [8] and [10] that can be used in a similar way:

$$
\begin{aligned}
&\text{for } i = 1, \dots, n,\\
&\quad x_i = F_i,\ \beta = 1,\ S_n^{[1]}(\mathbf{x}) = F_{n+2} - 2,\\
&\quad x_i = F_{2i-1},\ \beta = 1,\ S_n^{[1]}(\mathbf{x}) = F_{2n},\\
&\quad x_i = F_{2i},\ \beta = 1,\ S_n^{[1]}(\mathbf{x}) = F_{2n+1} - 1,\\
&\quad x_i = iF_i,\ \beta = 1\ \ S_n^{[1]}(\mathbf{x}) = F_{n+2} - F_{n+3} + 2,\\
&\quad x_i = F_iF_{3i},\ \beta = 1,\ S_n^{[1]}(\mathbf{x}) = F_nF_{n+1}F_{2n+1},\\
&\quad x_i = F_iF_{i+1},\ \beta = 1,\ \ S_n^{[1]}(\mathbf{x}) = F_{n+1}^2 - \frac{1+(-1)^n}{2}.\\
&\quad x_i = F_{4i-2},\ \ \beta = 1,\ S_n^{[1]}(\mathbf{x}) = F_{2n}^2,\\
&\quad x_i = \binom{n}{i}F_i,\ \ \beta = 1,\ S_n^{[1]}(\mathbf{x}) = F_{2n},\\
&\quad x_i = F_i\ \ \beta = 6,\ \ S_n^{[6]}(\mathbf{x}) = \frac{1}{4}(F_n^5F_{n+3} + F_{2n}),\\
&\quad x_i = F_i,\ \ \beta = 3,\ \ S_n^{[3]}(\mathbf{x}) = \frac{1}{10}(F_{3n+2} - (-1)^n 6F_{n-1} + 5),
\end{aligned}
$$

$$
\text{for } i = 1, \dots, 2n+1,\ x_i = F_i\sqrt{\binom{2n+1}{i}},\ \beta = 2,\ S_{2n+1}^{[2]}(\mathbf{x}) = 5^nF_{2n+1}.
$$

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