



Semiregularization and Semiregular Properties in Fuzzy Multiset Topological Spaces

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ABSTRACT: This paper aims to investigate semiregular spaces and its properties within the context of fuzzy M -topological spaces. To achieve this, we examine the characteristics of regular open sets in fuzzy M -topological spaces. Our research reveals that FM -semiregularization is not inherited in subspaces. Furthermore, we define various separation axioms using quasi-coincidence and analyze whether these axioms or related fuzzy M -topological concepts possess FM -semiregular properties or not.

Keywords: Fuzzy multiset, quasi-coincidence, fuzzy multi semiregular space, separation axiom, FM -semiregular property, FM -extremally disconnected space.

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1. Introduction

Classical set theory, a core idea in mathematics, concerns the organization of well-defined and distinct objects into groups. However, in certain situations it is essential to repeat elements and classical set theory does not adequately address these cases. In order to circumvent this type of issues, Yager [29] introduced multiset theory, Girish and John [5,6,7] developed a topological structure based on it. Different notions of M -topology were proposed in these research articles [4,11,12,16,24,25,26,18,19,28,2].

In contrast, to tackle vagueness and uncertainty issues, Zadeh [30] introduced fuzzy set theory. By utilizing the notion of fuzzy set, Chang [1] proposed a topological framework. Thereafter, a variety of separation axioms have been studied based on Chang's definition of fuzzy topological space and to define this variety of separation axioms, sometimes fuzzy points [3] are used or the concept of quasi-coincidence [27] is used.

The concept of the M -set has been explored in [29,14,15] in a fuzzy environment to handle situations where elements can be repeated and have varying degrees of membership. It has broad applications in various fields, including data mining, decision theory, machine learning, pattern recognition, logic and artificial intelligence etc. in [13,22,14,10,23]. Subsequently, Moshahary [17] has then assessed a topological structure in it. Moreover, Hoque et al. [8] have conducted a thorough discussion of the concepts of closure, interior operator and connectedness in fuzzy multiset setting. The idea of generalized closed set, various forms of generalized connectedness and different separation axioms has been examined in [20] and [21].

Bourbaki [34] showed that for every topological space, there exists a corresponding semi-regularization topology that is coarser than the original topology. Later, Mršević [33] demonstrated how various properties of topological spaces are preserved or reflected in their associated semi-regularization topologies. In the context of fuzzy topology, the concept of semi-regularization has been explored by Mukherjee et

2020 *Mathematics Subject Classification*: 54A05, 54A10, 54D10, 54D70, 54G05.

Submitted May 08, 2025. Published April 01, 2026

al. [32]. Contemplating all those factors, we attempt to investigate semiregularization and associated characteristics in an exclusive environment—a fuzzy multiset topological space.

The layout of the treatise is as follows: Section 2 presents the necessary preliminaries for this study. Section 3 provides the foundation of the paper by defining regular open set in fuzzy multi context and exploring their various attributes in relation to the basis structure. Section 4 explores the concept of semiregular space and the semiregular property within the same frame work. Along with, the specific fuzzy M -topological spaces are identified that inherently possesses FM -semiregular properties.

2. Preliminaries

Some of the basic definitions and important results that are required for this purpose are provided sequentially in this section.

Definition 2.1 [6] *Let X be the universal set. Then a count function C_M which associates each member of X to a natural number, defines a multiset (M -set) M over X .*

In a multiset M , $C_M(u)$ stands for the number of appearances of $u \in X$ in M . Zero is considered as a count of those elements which are outside of M .

Definition 2.2 [14] *A fuzzy multiset F is defined as $F = \{(\mu_F^1(y), \mu_F^2(y), \dots, \mu_F^p(y))/y : y \in X\}$, where the sequence of membership values $\mu_F^1(y) \geq \mu_F^2(y) \geq \dots \geq \mu_F^p(y)$ is of decreasing in nature for $y \in X$.*

In order to understand, a fuzzy M -set $F = \{1/(y, 0.1), 2/(y, 0.7), 3/(z, 0.3)\}$ can be written as $\{(0.7, 0.7, 0.1)/y, (0.3, 0.3, 0.3)/z\}$ or $\{(y, 0.7), (y, 0.7), (y, 0.1), (z, 0.3), (z, 0.3), (z, 0.3)\}$.

Consider two fuzzy M -sets F_1, F_2 from X . Then for all $i = 1, 2, \dots, p$ and for all $y \in X$ if

- (i) $\mu_{F_1}^i(y) \leq \mu_{F_2}^i(y)$, then $F_1 \subseteq F_2$,
- (ii) $\mu_{F_1}^i(y) = \mu_{F_2}^i(y)$, then $F_1 = F_2$,
- (iii) $U = F_1 \cup F_2$, then $\mu_U^i(y) = \max\{\mu_{F_1}^i(y), \mu_{F_2}^i(y)\}$,
- (iv) $I = F_1 \cap F_2$, then $\mu_I^i(y) = \min\{\mu_{F_1}^i(y), \mu_{F_2}^i(y)\}$,
- (v) the complement of F is F_1^c then $\mu_{F_1^c}^i(y) = (1 - \mu_{F_1}^i)(y)$.

Definition 2.3 [8] *Let a fuzzy M -set is drawn from the universal set X . Then $\tilde{0}_{FM}^X = \{(0, 0, \dots, 0)/y : y \in X\}$ defines a null fuzzy M -set and $\tilde{1}_{FM}^X = \{(1, 1, \dots, 1)/y : y \in X\}$ defines a whole fuzzy M -set.*

Definition 2.4 [8] *Let δ be any set of fuzzy M -sets of X . If the three requirements listed below are met, δ is considered as a fuzzy multiset topology (briefly, fuzzy M -topology):*

- (i) $\tilde{0}_{FM}^X, \tilde{1}_{FM}^X \in \delta$,
- (ii) $\cup_{i \in \Lambda} F_i \in \delta$ if $F_i \in \delta$,
- (iii) $F \cap G \in \delta$ if $F, G \in \delta$.

The elements of δ are fuzzy open M -sets and their complements are called fuzzy closed M -sets.

Definition 2.5 [17] *Let δ be arbitrary fuzzy M -topology on X and $F \subseteq X$. Then*

- (i) interior of F is
 $\delta\text{-int}(F) = \cup\{G : G \in \delta, G \subseteq F\}$,
- (ii) closure of F
 $\delta\text{-cl}(F) = \cap\{G : G \in \delta^c, F \subseteq G\}$.

The study [21] delves into several fundamental aspects of fuzzy M -topology namely fuzzy M -point including their behavior in subspaces and their role in various separation axioms. The following Definitions 2.6 to 2.9 and Theorem 2.1 have been added from [21] to enhance the subsequent work.

Definition 2.6 A fuzzy M -point can also be expressed as fuzzy M -set $F \in X$ s.t.

$$C_F(x, \beta_i) = \begin{cases} m_i & \text{if } x \in X; \\ 0 & \text{if } y \neq x, y \in X; \end{cases}$$

where $0 < \beta_i < 1$ for $i = 1, 2, \dots, p$ and $\beta_1, \beta_2, \dots, \beta_p$ are various membership values of x .

A fuzzy M -point is written as $x_{\beta_1, \beta_2, \dots, \beta_p}^{m_1, m_2, \dots, m_p}$ where $(x, \beta_1), (x, \beta_2), \dots, (x, \beta_p)$ repeats m_1, m_2, \dots, m_p times respectively.

x is known here the support of the fuzzy M -point.

Definition 2.7 A fuzzy M -point $x_{\beta_1, \beta_2, \dots, \beta_p}^{m_1, m_2, \dots, m_p} \in F$ iff $m_i \leq \omega \forall i = 1, 2, \dots, p$ and $\max\{\beta_i\} \leq \mu_F^k(x)$, where ω indicates the maximum multiplicity of a fuzzy element $x \in F$.

Definition 2.8 If supports of two fuzzy M -points x_α^n and y_β^m are different then the fuzzy M -points are called distinct.

Definition 2.9 A fuzzy M -set F is called a neighborhood (briefly, nbd) of any fuzzy M -point x_β^n if \exists a fuzzy open M -set O s.t. $x_\beta^n \in O \subset F$.

Theorem 2.1 Let δ be any fuzzy M -topology on X . Consider any subspace (Y, δ_Y) of X and $F \subset Y$ such that $\mu_Y^i(x) = \mu_X^i(x)$ for all x in the support of Y , whenever $\mu_{(F \cap Y)}^i > 0$ where $i = 1, 2, \dots, p$. Then

- (i) $F \in \delta_Y^c$ iff $F = Y \cap G$ with $G \in \delta^c$;
- (ii) $cl_Y(F) = Y \cap cl_X(F)$, where the closure of F in X and Y is indicated by the symbols $cl_X(F)$ and $cl_Y(F)$ respectively.

Definition 2.10 [31] Let (X, δ) be a fuzzy topological space, then $F \subset X$ is called a fuzzy regular open set if $F = \text{int}(cl(F))$.

Definition 2.11 [32] Let (X, δ) be a fuzzy topological space. Consider the set of all fuzzy regularly open sets in X . Then it is easy to see that it forms a base for some fuzzy topology on X . We call this topology the fuzzy semiregularization topology of δ , to be denoted by δ_s .

3. Fuzzy Regular Open M -Set: Towards Characterizations and Basis for Fuzzy M -Topology

This section mainly deals with some key concepts of fuzzy M -topology like basis, sub-basis and the prospect of regular open sets along with some characteristics of it.

Definition 3.1 Let δ be any fuzzy M -topology on X . A subfamily \mathcal{L} of fuzzy sub- M -sets of X is called a basis for fuzzy M -topology δ if for each $F \in \delta$, there exists a subfamily $\{L_\alpha\} \subseteq \mathcal{L}$ s.t. $\forall x \in X$,

$$\mu_F^j(x) = \sup\{\mu_{L_\alpha}^j(x) : L_\alpha \in \mathcal{L}, x \in X\}, j = 1, 2, 3, \dots, p.$$

Definition 3.2 A subfamily \mathcal{S} of δ is a sub-basis of fuzzy M -topology δ iff the basis for δ is formed by the family of finite intersection of members of \mathcal{S} .

Remark 3.1 Consider \mathcal{L} is a basis for δ , then a fuzzy M -topology δ produced by \mathcal{L} can be described as: a fuzzy sub- M -set F of X is said to be a fuzzy open M -set in X if for each fuzzy M -point $x_\alpha^m \in F$, there is atleast one basis element $G \in \mathcal{L}$ s.t. $x_\alpha^m \in G$ and $G \subseteq F$.

Definition 3.3 The fuzzy M -topology produced by sub-basis \mathcal{S} is defined to be the collection δ of all unions of finite intersections of elements of \mathcal{S} .

Definition 3.4 [9] A fuzzy M -point $x_{\alpha_1, \alpha_2, \dots, \alpha_p}^{m_1, m_2, \dots, m_p}$ is said to be quasi-coincident with a fuzzy M -set F , denoted by $x_{\alpha_1, \alpha_2, \dots, \alpha_p}^{m_1, m_2, \dots, m_p} qF$ iff there exists $x \in X$ s.t. $\max\{\alpha_i\} > \mu_{F^c}^j(x)$ for atleast one $j = 1, 2, \dots, p$.

If two fuzzy M -sets F and G are quasi-coincident at $x \in X$, then both $\mu_F^j(x)$ and $\mu_G^j(x)$ are non-zero at $x \in X$ and hence F and G are intersect at $x \in X$.

Definition 3.5 [9] A fuzzy M -set F is quasi-coincident with a fuzzy M -set G i.e., FqG iff $\exists x \in X$ s.t. $\mu_F^j(x) > \mu_{G^c}^j(x)$ for atleast one $j = 1, 2, \dots, p$; otherwise these are not quasi-coincident, denoted by $F \not qG$.

Definition 3.6 [9] A fuzzy M -set F is called Q -neighborhood (briefly, Q -nbd) of a fuzzy M -point x_β^n iff \exists a fuzzy open M -set G s.t. $x_\beta^n qG \subset F$.

Result 1 [9] For two fuzzy M -sets F and G , $F \not qG^c$ iff $F \subseteq G$.

Lemma 3.1 Let $\{F_\beta\}$ be a family of fuzzy M -sets in a fuzzy M -topological space (X, δ) . Then a fuzzy M -point $x_{\alpha_1, \alpha_2, \dots, \alpha_p}^{m_1, m_2, \dots, m_p}$ is quasi-coincident with $\bigcup F_\beta$ iff there exists some F_β s.t. $x_{\alpha_1, \alpha_2, \dots, \alpha_p}^{m_1, m_2, \dots, m_p} qF_\beta$.

Proof: Let us assume that $x_{\alpha_1, \alpha_2, \dots, \alpha_p}^{m_1, m_2, \dots, m_p}$ is a fuzzy M -point which is quasi-coincident with $\bigcup F_\beta$. That is,

$$\begin{aligned} & x_{\alpha_1, \alpha_2, \dots, \alpha_p}^{m_1, m_2, \dots, m_p} qG \text{ where } G = \bigcup F_\beta; \\ \iff & \text{there exists } x \in X \text{ s.t. } \sup\{\alpha_i\} > 1 - \mu_G^j(x), \text{ for atleast one } j = 1, 2, \dots, p; \\ \iff & \text{there exists } F_\beta \text{ s.t. } \sup\{\alpha_i\} > 1 - \mu_{F_\beta}^j(x), \text{ for atleast one } j = 1, 2, \dots, p; \\ \iff & \text{there exists } F_\beta \text{ s.t. } x_{\alpha_1, \alpha_2, \dots, \alpha_p}^{m_1, m_2, \dots, m_p} qF_\beta. \end{aligned}$$

□

Theorem 3.1 A subfamily \mathcal{L} of a fuzzy M -topology δ for X forms a basis for δ iff for each fuzzy M -point $x_\alpha^m (= x_{\alpha_1, \alpha_2, \dots, \alpha_p}^{m_1, m_2, \dots, m_p})$ of X and for each open Q -nbd O of x_α^m , \exists atleast a member $F \in \mathcal{L}$ s.t. $x_\alpha^m qF \subset O$.

Proof: Let \mathcal{L} be any basis for a fuzzy M -topology δ on X . Again, let us consider O be an open Q -nbd for a fuzzy M -point $x_\alpha^m \in X$. Then $x_\alpha^m qO$. Since \mathcal{L} is a basis so, $O = \bigcup F$ where $F \in \mathcal{L}$. Thus, $x_\alpha^m q(\bigcup F)$, which indicates that \exists some $F \in \mathcal{L}$ s.t. $x_\alpha^m qF \subset O$.

Conversely, let \mathcal{L} is not a basis for fuzzy M -topology δ . Then $\exists G \in \delta$ s.t.

$$G \neq S = \bigcup \{F \in \mathcal{L} : F \subset G\}$$

and hence \exists a fuzzy M -point $x_\alpha^m (= x_{\alpha_1, \alpha_2, \dots, \alpha_p}^{m_1, m_2, \dots, m_p})$ s.t.

$$\mu_S^j(x) < \mu_G^j(x) \text{ for atleast one } j = 1, 2, \dots, p.$$

Now, consider $j = p$ for which $\mu_S^p(x) < \mu_G^p(x)$. Then $1 - \mu_S^p(x) = r > 0$ and we obtain a fuzzy M -point x_r^p .

Since $\mu_G^p(x) + r > \mu_S^p(x) + r = 1$, so $x_r^p qG$.

Again, we have

$$\begin{aligned} & \mu_F^j(x) \leq \mu_S^j(x), \text{ for } j = 1, 2, \dots, p; \\ \implies & \mu_F^j(x) + r \leq \mu_S^j(x) + r, \text{ for } j = 1, 2, \dots, p; \\ & = 1 \\ \implies & r \leq 1 - \mu_F^j(x), \text{ for } j = 1, 2, \dots, p; \\ \implies & x_r^p \in F^c, \text{ which is a contradiction, since } F \in \mathcal{L}. \end{aligned}$$

Hence the proof is completed and hence the claim is established. □

Lemma 3.2 For fuzzy open M -sets F and G in a fuzzy M -topological space (X, δ) ,
 $F \not\leq G \implies \delta\text{-cl}(F) \not\leq G$ and $F \not\leq \delta\text{-cl}(G)$.

Proof: The proof is straightforward and is therefore omitted. \square

Definition 3.7 Let (X, δ) be a fuzzy M -topological space and F be any fuzzy M -set taken from X . Then the fuzzy M -set F is called a fuzzy regular open M -set (in short, FM -regular open set) of X if $F = \delta\text{-int}(\delta\text{-cl}(F))$.

Fuzzy regular closed M -set (in short, FM -regular closed set) is the complement of an FM -regular open set.

The set of all FM -regular open sets and FM -regular closed sets are denoted by $\mathcal{RO}_{FM}(X, \delta)$ and $\mathcal{RC}_{FM}(X, \delta)$ respectively.

Remark 3.2 Every FM -regular open (closed) set is a fuzzy open (closed) M -set; however, this may not always happen in the reverse case as seen in the example 3.1. Furthermore, two FM -regular open (closed) sets cannot be united (intersected) to form a FM -regular open (closed) set which is also clear from the same example.

Example 3.1 Let us take $X = \{x, y\}$ and $\delta = \{\tilde{0}_{FM}^X, \tilde{1}_{FM}^X, K, L, M\}$ where

$K = \{1/(x, 1), 1/(x, 0.4)\}$, $L = \{1/(y, 0.6)\}$ and $M = \{1/(x, 1), 1/(x, 0.4), 1/(y, 0.6)\}$, which shows that δ is a fuzzy M -topology on X .

Here, FM -regular open sets are $\tilde{0}_{FM}^X$, $\tilde{1}_{FM}^X$, K and L , whereas $K \cup L = M = \{1/(x, 1), 1/(x, 0.4), 1/(y, 0.6)\}$ is not an FM -regular open set, though M is a fuzzy open M -set.

Theorem 3.2 The intersection (union) of two FM -regular open sets (FM -regular closed sets) is an FM -regular open set (FM -regular closed set).

Proof: Let F_1 and F_2 be two FM -regular open sets. Then $F_1 \cap F_2$ being a fuzzy open M -set, $F_1 \cap F_2 \subseteq \delta\text{-int}(\delta\text{-cl}(F_1 \cap F_2))$.

Again, $\delta\text{-int}(\delta\text{-cl}(F_1 \cap F_2)) \subseteq \delta\text{-int}(\delta\text{-cl}(F_1)) = F_1$ and $\delta\text{-int}(\delta\text{-cl}(F_1 \cap F_2)) \subseteq \delta\text{-int}(\delta\text{-cl}(F_2)) = F_2$.

So, $\delta\text{-int}(\delta\text{-cl}(F_1 \cap F_2)) \subseteq F_1 \cap F_2$.

Then, $F_1 \cap F_2 = \delta\text{-int}(\delta\text{-cl}(F_1 \cap F_2))$.

Therefore, the intersection of FM -regular open sets is an FM -regular open set.

Along with similar arguments, one can easily prove that the union of two FM -regular closed sets is an FM -regular closed set. \square

Theorem 3.3 In any fuzzy M -topological space (X, δ) ,

- (i) $\delta\text{-int}(F)$ is an FM -regular open set if F be a member of δ^c ;
- (ii) $\delta\text{-cl}(F)$ is an FM -regular closed set if F be a member of δ .

Proof: (i) Considering a fuzzy closed M -set F , it is obtained that

$$\begin{aligned} \delta\text{-int}(F) &\subseteq F \\ \implies \delta\text{-cl}(\delta\text{-int}(F)) &\subseteq \delta\text{-cl}(F) = F \\ \implies \delta\text{-int}(\delta\text{-cl}(\delta\text{-int}(F))) &\subseteq \delta\text{-int}(F). \end{aligned} \tag{3.1}$$

Also,

$$\begin{aligned} \delta\text{-int}(F) &\subseteq \delta\text{-cl}(\delta\text{-int}(F)) \\ \implies \delta\text{-int}(F) &\subseteq \delta\text{-int}(\delta\text{-cl}(\delta\text{-int}(F))). \end{aligned} \tag{3.2}$$

From equation (3.1) and (3.2), it is obvious that $\delta\text{-int}(F) = \delta\text{-int}(\delta\text{-cl}(\delta\text{-int}(F)))$ i.e., $\delta\text{-int}(F)$ is an FM -regular open set.

Along with similar arguments, one can prove (ii). \square

Remark 3.3 Example 3.1 makes it evident that $\mathcal{RO}_{FM}(X, \delta)$ does not form a fuzzy M -topology. But it may form a basis for some fuzzy M -topology.

4. Fuzzy Multi Semiregular Spaces and Semiregular Properties

This section, in particular, is the primary contribution of the study wherein we examine the fuzzy semiregularization M -topology that arises from the set of fuzzy regular open M -sets. Also, FM -semiregular space, FM -semiregularization, FM -semiregular property and several types of separation axioms are thoroughly defined and certain connections are made between those.

Definition 4.1 Let (X, δ) be a fuzzy M -topological space. The collection of all FM -regular open sets $\mathcal{RO}_{FM}(X, \delta)$ forms a basis for some smaller fuzzy M -topology which is called a fuzzy semiregularization M -topology of δ and is denoted by δ_s^{FM} . Then the pair (X, δ_s^{FM}) is called an FM -semiregularization of (X, δ) .

In general, $\delta_s^{FM} \subset \delta$.

Definition 4.2 Let (X, δ) be a fuzzy M -topological space and δ_s^{FM} be a fuzzy semiregularization M -topology of δ . If $\delta = \delta_s^{FM}$ then the space is called a fuzzy multi semiregular space (briefly, FM -semiregular space).

Theorem 4.1 Let (X, δ) be a fuzzy M -topological space and (X, δ_s^{FM}) be FM -semiregularization of (X, δ) . Then for any fuzzy open M -set $F \in X$, we have following relations:

- (i) $\delta_s^{FM}\text{-cl}(F) = \delta\text{-cl}(F)$.
- (ii) $\delta_s^{FM}\text{-int}(\delta_s^{FM}\text{-cl}(F)) = \delta\text{-int}(\delta\text{-cl}(F))$.

Proof: (i) Let $F \subseteq X$. Then $\delta\text{-cl}(F)$ is an FM -regular closed set and so its complement is an FM -regular open set. Thus, $\delta\text{-cl}(F)$ is a fuzzy closed M -set in δ_s^{FM} . Hence, $\delta_s^{FM}\text{-cl}(F) \subseteq \delta\text{-cl}(F)$. Conversely, we know that $\delta_s^{FM} \subseteq \delta$, so $\delta\text{-cl}(F) \subseteq \delta_s^{FM}\text{-cl}(F)$. Thus, $\delta_s^{FM}\text{-cl}(F) = \delta\text{-cl}(F)$.

(ii) Since $\delta_s^{FM} \subseteq \delta$, we obtain $\delta_s^{FM}\text{-int}(\delta_s^{FM}\text{-cl}(F)) \subseteq \delta\text{-int}(\delta\text{-cl}(F))$. In contrast, from (i) we get, $\delta\text{-int}(\delta\text{-cl}(F)) \subseteq \delta_s^{FM}\text{-cl}(F)$. Since $\delta\text{-int}(\delta\text{-cl}(F)) \in \delta_s^{FM}$, so $\delta\text{-int}(\delta\text{-cl}(F)) \subseteq \delta_s^{FM}\text{-int}(\delta_s^{FM}\text{-cl}(F))$. Thus, $\delta\text{-int}(\delta\text{-cl}(F)) = \delta_s^{FM}\text{-int}(\delta_s^{FM}\text{-cl}(F))$. \square

Note 1 For a fuzzy M -topological space (X, δ) ,

- (i) Clearly, $\mathcal{RO}_{FM}(X, \delta) = \mathcal{RO}_{FM}(X, \delta_s^{FM})$. Hence, FM -semiregularization of (X, δ) and (X, δ_s^{FM}) are equal.
- (ii) The process of semiregularizing a fuzzy M -topology is nothing but an idempotent operation i.e., $(\delta_s^{FM})_s^{FM} = (\delta_s^{FM})$.

Theorem 4.2 For a fuzzy M -topological space (X, δ) , the assertions below are identical to each other:

- (i) (X, δ) is an FM -semiregular space.
- (ii) For each $F \in \delta$ and each fuzzy M -point x_α^m s.t. $x_\alpha^m q F$, $\exists G \in \delta$ s.t. $x_\alpha^m q G \subseteq \delta\text{-int}(\delta\text{-cl}(G)) \subseteq F$.
- (iii) For each $K \in \delta^c$ and each fuzzy M -point $x_\alpha^m \notin K$, \exists atleast one fuzzy regular closed M -set L s.t. $K \subseteq L$ and $x_\alpha^m \notin L$.
- (iv) For each fuzzy M -set N and each $P \in \delta$ s.t. $N q P$, \exists atleast one FM -regular open set R s.t. $N q R$ and $R \subseteq P$.

Proof: (i) \Rightarrow (ii) Let (X, δ) be any FM -semiregular space and $F \in \delta$.

Then $\delta = \delta_s^{FM}$ and $F = \bigcup_{n \in \mathbb{N}} B_n$, where each B_n are fuzzy regular open M -set i.e., $B_n = \delta\text{-int}(\delta\text{-cl}(B_n))$. So, $B_n \subseteq F$, for each $n \in \mathbb{N}$.

Now, let x_α^m be a fuzzy M -point s.t. $x_\alpha^m q F$. Then,

$$\begin{aligned} & x_\alpha^m q F \\ \Rightarrow & \text{there exists a fuzzy regular open } M\text{-set } B_n = G(\text{say}) \text{ s.t. } x_\alpha^m q G \subseteq F \\ \Rightarrow & \text{there exists } G \in \delta \text{ s.t. } x_\alpha^m q G \subseteq \delta\text{-int}(\delta\text{-cl}(G)) \subseteq F. \end{aligned}$$

(ii) \Rightarrow (iii) Let for each $F \in \delta$ and each fuzzy M -point x_α^m s.t. $x_\alpha^m q F$, there exists $G \in \delta$ s.t. $x_\alpha^m q G \subseteq \delta\text{-int}(\delta\text{-cl}(G)) \subseteq F$. Then we have

$$\begin{aligned} & x_\alpha^m q F \\ \Rightarrow & \max\{\alpha\} > \mu_{F^c}^j(x), \text{ for atleast one } j = 1, 2, \dots, k, \\ \Rightarrow & x_\alpha^m \notin F^c, \end{aligned}$$

and

$$\begin{aligned} & x_\alpha^m q G \subseteq \delta\text{-int}(\delta\text{-cl}(G)) \subseteq F \\ \Rightarrow & x_\alpha^m \notin G^c \text{ and } G \text{ is a fuzzy regular open } M\text{-set s.t. } G \subseteq F \\ \Rightarrow & x_\alpha^m \notin L \text{ and } K \subseteq L \text{ where } G^c = L. \end{aligned}$$

Consequently, for every $K \in \delta^c$ and every fuzzy M -point $x_\alpha^m \notin K$, there is an FM -regular closed set L s.t. $K \subseteq L$ and $x_\alpha^m \notin L$.

(iii) \Rightarrow (iv) Let K be any element of δ^c and n/u be any fuzzy M -point which is not included in K then \exists an FM -regular closed set L s.t. $K \subseteq L$ and $n/u \notin L$.

Eventually, for each $P = K^c \in \delta$ and fuzzy M -point n/u s.t. $n/u q P$, \exists atleast one FM -regular open set $R = L^c$ s.t. $n/u q R \subseteq P$.

Since, this holds for each fuzzy M -point $n/u \in M$, so for each fuzzy M -set N and each $P \in \delta$ s.t. $N q P$, there exists an FM -regular open set R s.t. $N q R \subseteq P$.

(iv) \Rightarrow (i) Let $P \in \delta$. Then from (iv) we have, for each fuzzy M -set N and each $P \in \delta$ s.t. $N q P$, there is a fuzzy regular open M -set R s.t. $N q R \subseteq P$. This gives that

$$\begin{aligned} & \bigcup R = P \\ \Rightarrow & P \in \delta_s, \text{ since } R \text{ is an } FM\text{-regular open set} \\ \Rightarrow & P \in \delta_s. \end{aligned}$$

As a result, $\delta \subseteq \delta_s$ which shows that the space is an FM -semiregular space. \square

Theorem 4.3 Let δ be any fuzzy M -topology on X and (X, τ) be any FM -semiregular space s.t. $\delta_s^{FM} \subset \tau \subset \delta$. Then $\delta_s^{FM} = \tau$.

Proof: Let us consider an FM -regular open set F in (X, τ) . Since $\delta_s^{FM} \subset \tau \subset \delta$ and $\delta\text{-cl}(F) = \delta_s^{FM}\text{-cl}(F)$, so we obtain $\delta_s^{FM}\text{-cl}(F) = \tau\text{-cl}(F)$.

Again, $\delta_s^{FM}\text{-int}(\delta_s^{FM}\text{-cl}(F)) = \delta\text{-int}(\delta\text{-cl}(F))$, which implies $\delta\text{-int}(\delta\text{-cl}(F)) = \tau\text{-int}(\tau\text{-cl}(F)) = F$. Hence, $F \in \delta_s^{FM}$.

In due course, we have $\tau = \delta_s^{FM}$. \square

Note 2 It is noted that if δ be any fuzzy M -topology on X , then the space (X, δ_s^{FM}) represents the largest FM -semiregular space among all those FM -semiregular spaces which are coarser than (X, δ) .

In general, the property of FM -semiregularization is not necessarily inherited by its subspaces. This is illustrated by the following example:

Example 4.1 Let $X = \{x, y\}$ and $\delta = \{\tilde{0}_{FM}^X, \tilde{1}_{FM}^X, \{1/(x, 1), 1/(x, 0.4)\}, \{1/(y, 0.6)\}, \{1/(x, 1), 1/(x, 0.4), 1/(y, 0.6)\}\}$. Then δ forms a fuzzy M -topology on X .

Considering any arbitrary fuzzy sub- M -set $Y = \{1/(x, 0.7), 1/(x, 0.6), 1/(y, 0.3)\}$ of X . Then $(\delta_Y)_s^{FM} \neq (\delta_s^{FM})_Y$.

However, for certain fuzzy sub- M -sets, the FM -semiregularization may be well preserved in the subspace which is provided below.

Theorem 4.4 Let (X, δ) be a fuzzy M -topological space and $Y \subset X$. Then the following hold if Y is either a fuzzy dense M -set or a fuzzy open M -set:

$$(i) RO_{FM}(Y, \delta_Y) = \{Y \cap R : R \in RO_{FM}(X, \delta)\}.$$

$$(ii) (\delta_Y)_s^{FM} = (\delta_s^{FM})_Y.$$

Proof:

(i) **Case-I:** Let Y be a fuzzy open M -set in (X, δ) and $Z \in RO_{FM}(Y, \delta_Y)$. Then

$$\begin{aligned} \delta_Y\text{-cl}(Z) &= \delta\text{-cl}(Z) \cap Y \\ \implies \delta_Y\text{-int}(\delta_Y\text{-cl}(Z)) &= \delta\text{-int}(\delta\text{-cl}(Z)) \cap Y \\ &= R \cap Y, \end{aligned}$$

$$\text{where } R = \delta\text{-int}(\delta\text{-cl}(Z)) \in RO_{FM}(X, \delta).$$

Conversely, let $Y \in RO_{FM}(X, \delta)$ and $Z = Y \cap R$. Then

$$\begin{aligned} \delta_Y\text{-int}(\delta_Y\text{-cl}(Z)) &= \delta\text{-int}(\delta\text{-cl}(Y \cap R) \cap Y) \\ &= \delta\text{-int}(\delta\text{-cl}(R) \cap Y) \\ &= Y \cap R \\ &= Z. \end{aligned}$$

Hence $Z \in RO_{FM}(Y, \delta_Y)$.

Case-II: Suppose Y be a fuzzy dense M -set of (X, δ) and $Z \subseteq Y$.

We have to show that, $\delta_Y\text{-int}(\delta_Y\text{-cl}(Z)) = \delta\text{-int}(\delta\text{-cl}(Z)) \cap Y$.

If $x_\alpha^m \in \delta_Y\text{-int}(\delta_Y\text{-cl}(Z))$, then there is atleast one fuzzy open M -set D s.t. $x_\alpha^m \in D \cap Y \subset \delta\text{-cl}(Z) \cap Y$.

If $D \not\subseteq \delta\text{-cl}(Z)$, then $D \cap Z^c$ is a non-empty fuzzy open M -set of X and Y being a fuzzy dense M -set in X , there exists $y_\beta^m \in Y$ s.t. $y_\beta^m \in D \cap (\delta\text{-cl}(Z))^c$. Thus $y_\beta^m \in D \cap Y$ and $y_\beta^m \notin \delta\text{-cl}(Z) \cap Y$ because $y_\beta^m \in (\delta\text{-cl}(Z))^c$. But this is a contradiction. Hence, $x_\alpha^m \in D \subset \delta\text{-cl}(Z)$ and therefore $x_\alpha^m \in \delta\text{-int}(\delta\text{-cl}(Z))$ and thus $\delta_Y\text{-int}(\delta_Y\text{-cl}(Z)) \subset \delta\text{-int}(\delta\text{-cl}(Z)) \cap Y$.

Conversely, if $x_\alpha^m \in \delta\text{-int}(\delta\text{-cl}(Z)) \cap Y$, then there is atleast one fuzzy open M -set D s.t. $x_\alpha^m \in D \subset \delta\text{-cl}(Z)$.

So, $x_\alpha^m \in D \cap Y \subset \delta\text{-cl}(Z) \cap Y$ and thus

$$x_\alpha^m \in \delta_Y\text{-int}(\delta\text{-cl}(Z) \cap Y) = \delta_Y\text{-int}(\delta_Y\text{-cl}(Z)).$$

Hence, $\delta\text{-int}(\delta\text{-cl}(Z) \cap Y) \subset \delta_Y\text{-int}(\delta_Y\text{-cl}(Z))$.

Therefore we obtain, $\delta_Y\text{-int}(\delta_Y\text{-cl}(Z)) = \delta\text{-int}(\delta\text{-cl}(Z)) \cap Y$.

(ii) This proof immediately comes from (i). □

In order to connect with FM -semiregular properties, we have defined a variety of spaces below.

Definition 4.3 Consider a fuzzy M -topology δ on X . If

- (i) for two distinct fuzzy M -points $x_\alpha^m, y_\beta^n \in X$, either x_α^m possesses an open nbd that is not quasi-coincident with y_β^n or, y_β^n possesses an open nbd that is not quasi-coincident with x_α^m , then X is called $FM\text{-}\mathfrak{S}_0$ space;
- (ii) for two distinct fuzzy M -points $x_\alpha^m, y_\beta^n \in X$, x_α^m possesses an open nbd that is not quasi-coincident with y_β^n and y_β^n possesses an open nbd that is not quasi-coincident with x_α^m , then X is called $FM\text{-}\mathfrak{S}_1$ space;
- (iii) any two distinct fuzzy M -points $x_\alpha^m, y_\beta^n \in X$ have two open nbds that are not quasi-coincident, then the space is called $FM\text{-}\mathfrak{S}_2$ space;
- (iv) any two distinct fuzzy M -points $x_\alpha^m, y_\beta^n \in X$ have fuzzy multi open nbds F and G respectively such that $\delta\text{-cl}(F)$ and $\delta\text{-cl}(G)$ are not quasi-coincident, then the space is called $FM\text{-}\mathfrak{S}_{T_2\frac{1}{2}}$ space.

Definition 4.4 Let δ be a fuzzy M -topology on X . Then the space (X, δ) is called an expansion over a fuzzy M -topological space (X, δ') iff $\delta' \subset \delta$.

Definition 4.5 Let (X, δ) be a fuzzy M -topological space and δ_s^{FM} be a fuzzy semiregularization M -topology on δ . A property P is called an FM -semiregular property, whenever (X, δ) shares the property P iff (X, δ_s^{FM}) shares the same property.

Note 3 Let δ and δ' be two fuzzy M -topologies on X with $\delta \subset \delta'$. Then every element of δ is also an element of δ' . Hence by Definition 4.3, $FM\text{-}\mathfrak{S}_0$, $FM\text{-}\mathfrak{S}_1$, $FM\text{-}\mathfrak{S}_2$ and $FM\text{-}\mathfrak{S}_{T_2\frac{1}{2}}$ are expansive properties.

Remark 4.1 From the above note, it can be said that a fuzzy M -topological space (X, δ) is $FM\text{-}\mathfrak{S}_0$ (resp. $FM\text{-}\mathfrak{S}_1$) if (X, δ_s^{FM}) is $FM\text{-}\mathfrak{S}_0$ (resp. $FM\text{-}\mathfrak{S}_1$). However, this may not always happen in the reverse case.

Example 4.2 Considering $X = \{x, y\}$ with $\delta = \{\tilde{0}_{FM}^X, \tilde{1}_{FM}^X, \{2/(y, 1)\}\}$, the space obviously forms an $FM\text{-}\mathfrak{S}_0$ space. But $\delta_s^{FM} = \{\tilde{0}_{FM}^X, \tilde{1}_{FM}^X\}$, which is not an $FM\text{-}\mathfrak{S}_0$ space.

Example 4.3 Considering an infinite space X and a fuzzy M -topology δ on it whose fuzzy open M -sets A are of the form

$\{(\mu_A^i)/x : x \in X; \mu_A^i(x) < 1 \text{ for finite number of element and } \mu_A^i(x) = 1 \text{ for infinite number of element where } i = 1, 2, \dots, n\}$. Then it is found that (X, δ_s^{FM}) fails to be an $FM\text{-}\mathfrak{S}_1$ space whereas the original space is an $FM\text{-}\mathfrak{S}_1$ space.

From the Remark 4.1, we can say that $FM\text{-}\mathfrak{S}_0$ (resp. $FM\text{-}\mathfrak{S}_1$) is not an FM -semiregular property.

Theorem 4.5 $FM\text{-}\mathfrak{S}_2$ property is an FM -semiregular property.

Proof: Let (X, δ) be a fuzzy M -topological space and δ_s^{FM} be a fuzzy semiregularization M -topology of δ .

If (X, δ_s^{FM}) is an $FM\text{-}\mathfrak{S}_2$ space, then (X, δ) is also an $FM\text{-}\mathfrak{S}_2$ space which is clear from Note 3.

For the other part, consider an $FM\text{-}\mathfrak{S}_2$ space (X, δ) and x_α^m, y_β^n be two distinct fuzzy M -points in X . Then x_α^m, y_β^n have open nbd F, G respectively in (X, δ) s.t. $F \not\subseteq G$. Then by Lemma 3.2, we have $\delta\text{-cl}(F)$ is not quasi-coincident with G and so $\delta\text{-int}(\delta\text{-cl}(F))$ fails to be quasi-coincident with G .

We found that $\delta\text{-int}(\delta\text{-cl}(F))$ is not quasi-coincident with $\delta\text{-cl}(G)$ from Lemma 3.2.

Let us take, $\delta\text{-int}(\delta\text{-cl}(F)) = F_1$ and $\delta\text{-int}(\delta\text{-cl}(G)) = G_1$. Then F_1 and G_1 are δ_s^{FM} -open nbd of x_α^m, y_β^n respectively s.t. F_1 is not quasi-coincident with G_1 . This proves that, (X, δ_s^{FM}) is an $FM\text{-}\mathfrak{S}_2$ -space. \square

Theorem 4.6 *The property of being $FM\text{-}\mathfrak{S}_{T_2\frac{1}{2}}$ is an FM -semiregular property.*

Proof: Let (X, δ) be a fuzzy M -topological space and δ_s^{FM} be a fuzzy semiregularization M -topology of δ . It is clear from Note 3 that, if (X, δ_s^{FM}) is an $FM\text{-}\mathfrak{S}_{T_2\frac{1}{2}}$ space then (X, δ) is also an $FM\text{-}\mathfrak{S}_{T_2\frac{1}{2}}$ space.

For the other part, let (X, δ) be an $FM\text{-}\mathfrak{S}_{T_2\frac{1}{2}}$ space and x_α^m, y_β^n be two distinct fuzzy M -points in X . Then there exist two open nbds F, G of x_α^m and y_β^n respectively s.t. $\delta\text{-cl}(F)$ is not quasi-coincident with $\delta\text{-cl}(G)$. Consider $\delta\text{-int}(\delta\text{-cl}(F)) = F_s$ and $\delta\text{-int}(\delta\text{-cl}(G)) = G_s$. Then F_s is not quasi-coincident with G_s . Since $\delta_s^{FM}\text{-cl}(F_s) = \delta\text{-cl}(F_s)$ and $\delta_s^{FM}\text{-cl}(G_s) = \delta\text{-cl}(G_s)$ and hence we must obtain that $\delta_s^{FM}\text{-cl}(F_s)$ is not quasi-coincident with $\delta_s^{FM}\text{-cl}(G_s)$, where F_s and G_s are open nbds of x_α^m and y_β^n respectively in (X, δ_s^{FM}) . Thus, (X, δ_s^{FM}) is an $FM\text{-}\mathfrak{S}_{T_2\frac{1}{2}}$ space. \square

Definition 4.6 *A fuzzy M -topological space (X, δ) is called FM -extremally disconnected iff $\delta\text{-cl}(F) \in \delta$ for every $F \in \delta$.*

Theorem 4.7 *FM -extremally disconnectedness is an FM -semiregular property.*

Proof: Let δ be any fuzzy M -topology on X s.t. the space is an FM -extremally disconnected space and consider $F \in \delta_s^{FM}$. Again, assume that $\delta_s^{FM}\text{-cl}(F) = \delta\text{-cl}(F) = G$ (say). Then from FM -extremally disconnectedness of (X, δ) , we have $G \in \delta$ and so $\delta_s^{FM}\text{-cl}(F) = \delta\text{-int}(\delta\text{-cl}(F)) = \delta\text{-int}(\delta\text{-cl}(G)) \in \delta_s^{FM}$.

Hence (X, δ_s^{FM}) is FM -extremally disconnected space.

Conversely, let (X, δ_s^{FM}) is FM -extremally disconnected space. For any $F \in \delta$, we have $\delta_s^{FM}\text{-cl}(F) = \delta\text{-cl}(F)$ and $\delta\text{-int}(\delta\text{-cl}(F)) \in \delta_s^{FM}$.

Let, $\delta\text{-int}(\delta\text{-cl}(F)) = G$. Then $\delta\text{-cl}(F) = \delta\text{-cl}(G) = \delta_s^{FM}\text{-cl}(G) \in \delta_s^{FM} \subset \delta$. Thus, (X, δ) is FM -extremally disconnected space. \square

Let us denote $clo(X, \delta)$ be the collection of all fuzzy M -sets of (X, δ) which are both fuzzy closed as well as open M -sets.

Proposition 4.1 *For any fuzzy M -topological space (X, δ) , $clo(X, \delta) = clo(X, \delta_s^{FM})$.*

Proof: Let $F \in clo(X, \delta)$. Then F is an FM -regular open M -set and hence $F \in \delta_s^{FM}$. Also, $F^c \in \delta_s^{FM}$. Thus $F \in clo(X, \delta_s^{FM})$.

On the other hand, since $\delta_s^{FM} \subset \delta$, we have $clo(X, \delta_s^{FM}) \subset clo(X, \delta)$. \square

Corollary 4.1 *Let δ be any fuzzy M -topology on X , then (X, δ) is connected iff (X, δ_s^{FM}) is connected.*

Definition 4.7 *Let δ be any fuzzy M -topology on X . If for every fuzzy M -point $x_\alpha^m \in X$ and every open Q -nbd F of x_α^m , there is $G \in \delta$ s.t. $x_\alpha^m q G \subseteq \delta\text{-cl}(G) \subseteq F$, then X is called FM -regular space.*

In general, the expansion of topologies does not sustain the higher separation features. This explains the significant differences in the FM -regularity between (X, δ) and its FM -semiregularization (X, δ_s^{FM}) .

Definition 4.8 *Let δ be any fuzzy M -topology on X . If for every fuzzy M -point $x_\alpha^m \in X$ and for every FM -regular open Q -nbd F of x_α^m , there is a FM -regular open Q -nbd G of x_α^m s.t. $\delta\text{-cl}(G) \subseteq F$ then the space is known as FM -almost regular.*

Theorem 4.8 *Every space which is FM-regular, it must be FM-almost regular.*

Proof: Considering fuzzy M -point x_α^m from the space (X, δ) , which is FM-regular and F be any FM-regular open Q -nbd of x_α^m . Then there is an FM-regular open set G s.t. $x_\alpha^m q G \subseteq F$. Now, G being an FM-regular open set, it is fuzzy open M -set. By FM-regularity of X , there is $G \in \delta$ s.t. $x_\alpha^m q G \subseteq \delta\text{-cl}(G) \subseteq F$. Therefore, X is an FM-almost regular space. \square

Example 4.4 *Let $X = \{x, y\}$ and a fuzzy M -topology on X be $\delta = \{\tilde{0}_{FM}^X, \tilde{1}_{FM}^X, K, L, M, N\}$ where $K = \{1/(x, 0.5), 1/(y, 1)\}$, $L = \{1/(x, 1), 1/(y, 0.3), 1/(y, 0.8)\}$, $M = \{1/(x, 1), 1/(y, 1), 1/(y, 0.3)\}$ and $N = \{1/(x, 0.5), 1/(y, 0.8)\}$.*

Though (X, δ) is an FM-almost regular space but fails to be an FM-regular space, since for the fuzzy M -point $y_{0.7}^1$ and open Q -nbd L of $y_{0.7}^1$, there is $N \in \delta$ s.t. $y_{0.7}^1 q N \subseteq \text{cl}(N)$ but $\text{cl}(N) = \{1/(x, 0.5), 1/(y, 1)\} \not\subseteq L$.

Nevertheless, considering the semiregularization of a given space, we obtain an equivalent condition between FM-regular and FM-almost regular space, which is given below:

Theorem 4.9 *Let δ be any fuzzy M -topology on X . Then for an FM-almost regular space (X, δ) , (X, δ_s^{FM}) must be FM-regular as a necessary and sufficient condition.*

Proof: The proof is straightforward so it is omitted. \square

Corollary 4.2 *Let δ be any fuzzy M -topology on X . Then for (X, δ) , (X, δ_s^{FM}) is an FM-almost regular space iff it is FM-regular.*

Note 4 *From Theorem 4.9 and Corollary 4.2, we conclude that FM-almost regularity and FM-regularity are FM-semiregular properties.*

5. Conclusions

This article presented the notion of fuzzy multi regular open sets providing comprehensive characterizations and demonstrating that their collection gives rise to a fuzzy semiregularization M -topology. Furthermore, we have introduced FM-semiregular space and highlighted specific fuzzy M -topological spaces that naturally exhibit FM-semiregular properties, offering new insights into their topological behavior. Moreover, different kinds of topological space can be unified by the FM-semiregular property. By defining various spaces within fuzzy M -topology, one can investigate their structure to determine whether they exhibit FM-semiregular properties or not. The concept of FM-semiregular space can be extended by introducing the notion of S -closed space in the said environment. FM-semiregular spaces can be applied to network topology and can be used to identify clusters and patterns in high-dimensional data.

References

1. Chang, C.L., Fuzzy topological spaces, *J Math Anal Appl*, 24(1), 182–190, 1968.
2. El-Sheikh, S.A., Omer, R. A-K., Raafat, M., Separation axioms on multiset topological spaces, *J. New Theory*, 7, 11–21, 2015.
3. Fora, A.A, Separation axioms for fuzzy spaces, *Fuzzy sets and systems*, 33, 59–75, 1989.
4. Girish, K.P., John, S.J., Relations and functions in multiset context, *Information Sciences*, 179(6), 758–768, 2009.
5. Girish, K.P., John, S.J., Rough multiset and its multiset topology, *Transactions on Rough Sets XIV*, Springer, Berlin, Heidelberg, 62–80, 2011.
6. Girish, K.P., John, S.J., On multiset topologies, *Theory Appl Math Comput Sci.*, 2 37–52, 2012.
7. Girish, K.P., John, S.J., Multiset topologies induced by multiset relations, *Information Sciences*, 188,298–313, 2012.
8. Hoque, M.M., Bhattacharya, B., Tripathy, B.C., A novel approach in the light of fuzzy multiset topology, *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.*, 93, 675–683, 2023.

9. Hoque, M.M., Bhattacharya, B., Tripathy, B.C., Mixed fuzzy multiset topology induced by quasi-coincidence, Communicated.
10. Kim, K.S., Miyamoto, S., Application of fuzzy multisets to fuzzy database systems. *In: Soft Computing in Intelligent Systems and Information Processing*. Proceedings of the 1996 Asian Fuzzy Systems Symposium. IEEE, (1996) 115–120, DOI:10.1109/AFSS.1996.583573
11. Kumar, P.R., John, S.J., Samanta, S., On redundancy, separation and connectedness in multiset topological spaces, *AIMS Mathematics*, 5(3), 2484–2499, 2020.
12. Kumar, P.R., John, S.J., Samanta, S., Compactness and subspaces M-topologies, *Soft computing*, 26, 4115–4122, 2022.
13. Li, B., Fuzzy bags and applications, *Fuzzy Sets Syst*, 34(1), 61–71, 1990.
14. Miyamoto, S., Fuzzy multisets and their generalizations, *Workshop on membrane computing*, Springer, Berlin, Heidelberg, 225–235, 2000.
15. Miyamoto, S., Remarks on basics of fuzzy sets and fuzzy multisets, *Fuzzy Sets and Systems*, 156, 427–431, 2005.
16. Mahalakshmi, P.M., Thangavelu, P., M-Connectedness in M-topology, *Int. J. Pure Appl. Math.*, 106, 21–25, 2016.
17. Moshahary, J., Topological structure on fuzzy multisets, *Int J Sci Res Manag*, 5, 6461–6465, 2017.
18. Ray, G.C., Dey, S., Mixed multiset topological space and separation axioms, *Indian Journal of Pure and Applied Mathematics*, 53(1), 92–99, 2022.
19. Ray, S., Bhattacharya, B., A study on countability in the context of multiset topological spaces, *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.*, 94(4), 457–461, 2024.
20. Ray, S., Bhattacharya, B., A study on connectedness via generalized fuzzy closed multiset, *Proceedings of the Fifth International Conference on Emerging Trends in Mathematical Sciences & Computing (IEMSC-24)*, 94(4), 469–478, 2024.
21. Ray, S., Hoque, M.M., Bhattacharya, B., Tripathy, B.C., Separation Axioms in Fuzzy Multiset Topological Spaces, *Knowledge and information systems*, 67, 5043–5055, 2025.
22. Riesgo, Á., Alonso, P., Diaz, I., Montes, S., Basic operations for fuzzy multisets, *Int J Approx Reason*, 101, 107–118, 2018.
23. Rocacher, D., On fuzzy bags and their application to flexible querying, *Fuzzy Sets Syst*, 140(1), 93–110, 2003.
24. Shravan, K., Tripathy, B.C., Generalised closed sets in multiset topological space, *Proyecciones J. Math.*, 37, 223–237, 2018.
25. Shravan, K., Tripathy, B.C., Multiset mixed topological space, *Soft computing*, 23, 9801–9805, 2019.
26. Shravan, K., Tripathy, B.C., Metrizable in multiset topological spaces, *Bulletin of the Transilvania University of Brasov*, 13(62) (2), 683–696, 2020.
27. Sinha, S.P., Separation axioms for fuzzy topological spaces, *Fuzzy sets and systems*, 45, 261–270, 1992.
28. Thankachan, B., John, S.J., Kumar P, R., Exploring maximal and minimal open subsets in m-topological spaces: A comprehensive analysis, *Results in Nonlinear Analysis*, 7(1), 176–190, 2024.
29. Yager, R.R., On the theory of bags, *International Journal of General System*, 13(1), 23–37, 1986.
30. Zadeh, L.A., Fuzzy sets, *Inform Control*, 8, 338–353, 1965.
31. Azad, K.K., On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, *Journal of Mathematical Analysis and Applications*, 82, 14–32, 1981.
32. Mukherjee, M. N., Ghosh, B., Fuzzy semiregularization topologies and fuzzy submaximal spaces, *Fuzzy Sets and Systems*, 44, 283–294, 1991.
33. Mrsevic, M., Reilly, I.L., Vamanamurthy, M.K., On semi-regularization topologies, *J. Austral. Math. Soc. (Series A)*, 38, 40–54, 1985.
34. Bourbaki, N., (1966) General topology, *Addison-Wesley, Mass.*

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