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Introduction to the Stieltjes extension of Dunford and Pettis-type integrals on time scales

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ABSTRACT: Riemann-Stieltjes type integrals on time scales are explored in this paper. Theoretical definitions of the Riemann-Stieltjes-Dunford, Riemann-Stieltjes-Pettis and Riemann-Stieltjes-Gelfand integrals are presented. Scalarly Riemann-Stieltjes integral, weak Riemann-Stieltjes integral and weak* scalarly Riemann-Stieltjes integral definitions are put forward; and a few relations between these integrals are established. Uniform convergence of a sequence of functions which are Riemann-Stieltjes-Dunford, Riemann-Stieltjes-Pettis and Riemann-Stieltjes-Gelfand integrable are also formulated.

Key Words: Banach space, Riemann-Stieltjes integral, Pettis integral, Dunford integral, Gelfand integral, Δ-integral, ∇-integral.

Contents

1. Introduction and Preliminaries

S. Hilger, in 1988, as part of his doctoral degree [8] introduced a concept which in motivation and theory unified discrete and continuous analysis. He called this theory the measure chain calculus (m.c. calculus) which later came to be known as time scale calculus. Transcripts of his dissertation were published in 1990 as [9] [also view [10]]. As theoretical framework, Hilger presented three axioms [refer [9] for more insight] and concluded that any set, say $\mathbb{T}$, that satisfied these three axioms forms a measure chain and called set $\mathbb{T}$ a time scale. Any non-empty closed subset of $\mathbb{R}$ is a time scale, a direct excerpt from Hilger's paper concludes this, "... any closed subset of $\mathbb{R}$ bears the structure of a measure chain in a natural manner." [9].

Hilger defined two operators- forward jump operator and backward jump operator. The forward jump operator, denoted by σ, is defined as a mapping $\sigma : \mathbb{T} \to \mathbb{T}$ such that $\sigma(t) = \inf \{r \in \mathbb{T} : r > t\}$. While the backward jump operator is denoted by ρ and is defined as a mapping $\rho : \mathbb{T} \to \mathbb{T}$ such that $\rho(t) = \sup \{r \in \mathbb{T} : r < t\}$. Using the notion of the forward jump operator Hilger formulated a derivative called the delta derivative (Δ-derivative). A little over a decade later, another notion of derivative was

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formulated by F. M. Atici et al. [2] using the backward jump operator called the nabla derivative (∇-derivative). This nabla derivative was previously hinted in the works of C. D. Ahlbrandt et al. [1] where they introduced a notion of derivative called the alpha derivative which consisted both the delta and nabla derivatives as special cases.

In literature several integration notions are discussed including the Riemann and Riemann-Stieltjes integrations.
The intervals on which these integrals are defined, i.e. intervals on time scale $\mathbb{T}$ are defined as- assuming $x \leq y$ (refer [4])-
$[x,y]_{\mathbb{T}} = \{t \in \mathbb{T} : x \leq t \leq y\}$; $(x,y)_{\mathbb{T}} = \{t \in \mathbb{T} : x < t < y\}$;
$[x,y)_{\mathbb{T}} = \{t \in \mathbb{T} : x \leq t < y\}$; $(x,y]_{\mathbb{T}} = \{t \in \mathbb{T} : x < t \leq y\}$.

The Riemann integral for real-valued functions on time scales was formulated by S. Sailer [4,6] using the concept of Darboux sum definition; and by G. Sh. Guseinov et al. [6,7] using the concept of Riemann sum definition. The latter also proved that the two different approaches of the Riemann integral for real-valued functions on time scales are in essence equal [6].

Below we give the definition of Riemann Δ-integral for real-valued functions; for the nabla definition the reader is referred to [7].

Given $[x,y]_{\mathbb{T}}$ be a closed interval of $\mathbb{T}$; let $\mathcal{P}$ represent the collection of all possible partitions of $[x,y]_{\mathbb{T}}$. Consider partition $\mathcal{Q} = \{x = t_0 < t_1 < \ldots < t_i = y\} \in \mathcal{P}$ with $t_0, t_1, \ldots, t_i$ being the finite points of division. Subintervals are taken to be of the form $[t_{z-1}, t_z)_{\mathbb{T}}$, for $1 \leq z \leq i$, which we will call the Δ-subinterval. From each of these Δ-subintervals we choose $\vartheta_z \in [t_{z-1}, t_z)_{\mathbb{T}}$ arbitrarily and call it the Δ-tag. A point-interval collection defined as $\breve{\mathcal{Q}} = \{(\vartheta_z, [t_{z-1}, t_z)_{\mathbb{T}})\}_{z=1}^{i}$ is considered, which we call the Δ-tagged partition. Mesh of partition $\mathcal{Q}$ is defined as: mesh$-(\mathcal{Q}) = \max_{1\leq z\leq i} [t_z - t_{z-1}] > 0$. For some $\delta > 0$, $\mathcal{Q}_\delta$ will represent a partition of $[x,y]_{\mathbb{T}}$ with mesh δ satisfying the property: for each $z = 1, 2, \ldots, i$ we have either $t_z - t_{z-1} \leq \delta$ or $t_z - t_{z-1} > \delta \wedge \rho(t_z) = t_{z-1}$ (here $\wedge$ stands for "and"). Henceforth, $\breve{\mathcal{Q}}_\delta$ will mean a Δ-tagged partition with mesh δ satisfying the above property.

Definition 1.1 *[7] A function $k : [x,y]_{\mathbb{T}} \to \mathbb{R}$ is Riemann Δ-integrable if there exists an $\overline{I} \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any Δ-tagged partition $\breve{\mathcal{Q}}_\delta$ we have, $\left|\sum_{z=1}^{i} k(\vartheta_z)(t_z - t_{z-1}) - \overline{I}\right| < \varepsilon$. Here $\overline{I} = \overline{R}\int_x^y k(t)\Delta t$, where $\overline{R}\int_x^y k(t)\Delta t$ is called the Riemann Δ-integral.*

The Riemann-Stieltjes integral for real-valued functions on time scales was formulated by S. Sailer [4,11] using the concept of Darboux sum definition; and re-investigated by D. Mozyrska et al. [11]. The Riemann sum definition of Riemann-Stieltjes integral is given in [12] by the same authors. Equivalence of the above two approaches is proved in [14].

Below we give the definition of Riemann-Stieltjes Δ-integral and Riemann-Stieltjes ∇-integral for real-valued functions.

Let $\breve{\mathcal{Q}}_\delta$ be a Δ-tagged partition with mesh δ. Considering ψ to be a real-valued monotone increasing function on $[x,y]_{\mathbb{T}}$, and define $\psi(\mathcal{Q}) = \{\psi(x) = \psi(t_0) < \ldots < \psi(t_i) = \psi(y)\}$; $\psi(t_z) - \psi(t_{z-1})$ will be positive.

Definition 1.2 *[12] Let function $k : [x,y]_{\mathbb{T}} \to \mathbb{R}$ and let $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x,y]_{\mathbb{T}}$ is said to be Riemann-Stieltjes Δ-integrable if there exists an $\overline{I} \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any Δ-tagged partition $\breve{\mathcal{Q}}_\delta$ we have, $\left|\sum_{z=1}^{i} k(\vartheta_z)[\psi(t_z) - \psi(t_{z-1})] - \overline{I}\right| < \varepsilon$. Here $\overline{I} = \overline{RS}\int_x^y k(t)\Delta\psi(t)$, where $\overline{RS}\int_x^y k(t)\Delta\psi(t)$ denotes the Riemann-Stieltjes Δ-integral.*

For the sake of clarity we will let $\mathcal{R} \in \mathcal{P}$ denote the partition for the ∇-integral.

Consider partition $\mathcal{R} = \{x = t_0 < t_1 < \ldots < t_i = y\} \in \mathcal{P}$ with $t_0, t_1, \ldots, t_i$ being the finite points of division. Subintervals are taken to be of the form $(t_{z-1}, t_z]_{\mathbb{T}}$, for $1 \leq z \leq i$, which we will call the ∇-subinterval. From each of these ∇-subintervals we choose $\xi_z \in (t_{z-1}, t_z]_{\mathbb{T}}$ arbitrarily and call it the ∇-tag. A point-interval collection defined as $\breve{\mathcal{R}} = \{(\xi_z, (t_{z-1}, t_z]_{\mathbb{T}})\}_{z=1}^{i}$ is considered, which we call the ∇-tagged partition. Mesh of partition $\mathcal{R}$ is defined as: mesh$-(\mathcal{R}) = \max_{1\leq z\leq i}\left[t_z - t_{z-1}\right] > 0$. For some $\delta > 0$, $\mathcal{R}_\delta$ will represent a partition of $[x,y]_{\mathbb{T}}$ with mesh δ satisfying the property: for each $z = 1,2,\ldots,i$ we have either $t_z - t_{z-1} \leq \delta$ or $t_z - t_{z-1} > \delta \wedge t_z = \sigma(t_{z-1})$. Henceforth, $\breve{\mathcal{R}}_\delta$ will mean a ∇-tagged partition with mesh δ satisfying the above property. Considering ψ to be a real-valued monotone increasing function on $[x,y]_{\mathbb{T}}$, and define $\psi(\mathcal{R}) = \{\psi(x) = \psi(t_0) < \ldots < \psi(t_i) = \psi(y)\}$; $\psi(t_z) - \psi(t_{z-1})$ will be positive.

Definition 1.3 *[12] Let function $k : [x,y]_{\mathbb{T}} \to \mathbb{R}$ and let $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x,y]_{\mathbb{T}}$ is said to be Riemann-Stieltjes ∇-integrable if there exists an $\underline{I} \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any ∇-tagged partition $\breve{\mathcal{R}}_\delta$ we have, $\left|\sum_{z=1}^{i} k(\xi_z)\left[\psi(t_z) - \psi(t_{z-1})\right] - \underline{I}\right| < \varepsilon$. Here $\underline{I} = \underline{RS}\int_x^y k(t)\nabla\psi(t)$, where $\underline{RS}\int_x^y k(t)\nabla\psi(t)$ denotes the Riemann-Stieltjes ∇-integral.*

The Riemann integral for Banach-valued functions on time scales was formulated by B. Aulbach et al. [3]. The Riemann-type integrals such as the Riemann-Dunford and Riemann-Pettis integrals including scalarly Riemann integral were introduced in [13]. The weak Riemann integral was defined by M. Cichoń [5].

Throughout this paper, $\mathfrak{X}$ will denote a Banach space and $\mathfrak{X}^*$ its dual.

Below we give the definition of Riemann Δ-integral for functions whose values lie in the Banach space (Banach-valued functions), weak Riemann Δ-integral, scalarly Riemann Δ-integral, Riemann-Dunford Δ-integral and Riemann-Pettis Δ-integral; for the nabla definition the reader is referred to [13].

Definition 1.4 *[3] A function $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ is Riemann Δ-integrable if there exists an $\overline{I} \in \mathfrak{X}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any Δ-tagged partition $\breve{\mathcal{Q}}_\delta$ we have, $\left\|\sum_{z=1}^{i}(t_z - t_{z-1}) \cdot k(\vartheta_z) - \overline{I}\right\| < \varepsilon$. Here $\overline{I} = (\overline{R})\int_x^y k(t)\Delta t$, where $(\overline{R})\int_x^y k(t)\Delta t$ is called the Riemann Δ-integral.*

Definition 1.5 *[5] Weak Riemann Δ-integral: A function $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ is weak Riemann Δ-integrable on $[x,y]_{\mathbb{T}}$ if there exists an $\overline{I} \in \mathfrak{X}$ such that for every $\varepsilon > 0$ and every functional $\mathfrak{m}^* \in \mathfrak{X}^*$, there exists $\delta > 0$ hence for any Δ-tagged partition $\breve{\mathcal{Q}}_\delta$ we have, $\left|\mathfrak{m}^*\left(\sum_{z=1}^{i}\left(t_z - t_{z-1}\right) \cdot k(\vartheta_z) - \overline{I}\right)\right| < \varepsilon$. Here $\overline{I} = (\overline{wR})\int_x^y k(t)\Delta t$, where $(\overline{wR})\int_x^y k(t)\Delta t$ denotes the weak Riemann Δ-integral.*

Definition 1.6 *[13] Scalarly Riemann Δ-integral: A function $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ is scalarly Riemann Δ-integrable on $[x,y]_{\mathbb{T}}$ if there exists an $\overline{I} \in \mathbb{R}$ such that for every $\varepsilon > 0$ and every functional $\mathfrak{m}^* \in \mathfrak{X}^*$, there exists $\delta > 0$ such that for any Δ-tagged partition $\breve{\mathcal{Q}}_\delta$ we have, $\left|\sum_{z=1}^{i}\left(t_z - t_{z-1}\right) \cdot \mathfrak{m}^*(k(\vartheta_z)) - \overline{I}\right| < \varepsilon$. Here $\overline{I} = (\overline{sR})\int_x^y k(t)\Delta t$, where $(\overline{sR})\int_x^y k(t)\Delta t$ denotes the scalarly Riemann Δ-integral.*

Definition 1.7 *[13] Riemann-Dunford Δ-integral: A function $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ is Riemann-Dunford Δ-integrable on $[x,y]_{\mathbb{T}}$ if for each functional $\mathfrak{m}^* \in \mathfrak{X}^*$, the function $\mathfrak{m}^*(f) : [x,y]_{\mathbb{T}} \to \mathbb{R}$ is Riemann Δ-integrable on $[x,y]_{\mathbb{T}}$, and there exists an element $\mathfrak{m}^{**} \in \mathfrak{X}^{**}$ such that*

$$\mathfrak{m}^{**}(\mathfrak{m}^*) = \overline{R}\int_x^y \mathfrak{m}^*\big(k(t)\big)\Delta t$$

for all $\mathfrak{m}^* \in \mathfrak{X}^*$, then k is said to be Riemann-Dunford Δ-integrable on $[x,y]_{\mathbb{T}}$. The element $\mathfrak{m}^{**}$ is called the Riemann-Dunford Δ-integral of k over $[x,y]_{\mathbb{T}}$ and is denoted by $(\overline{RD})\int_x^y k(t)\Delta t$.

Definition 1.8 *[13] Riemann-Pettis Δ-integral: A function $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ is said to be Riemann-Pettis Δ-integrable on $[x,y]_{\mathbb{T}}$ if for each functional $\mathfrak{m}^* \in \mathfrak{X}^*$, the function $\mathfrak{m}^*(k) : [x,y]_{\mathbb{T}} \to \mathbb{R}$ is Riemann Δ-integrable on $[x,y]_{\mathbb{T}}$, and there exists an element $\mathfrak{m} \in \mathfrak{X}$ such that*

$$\mathfrak{m}(\mathfrak{m}^*) = \overline{R}\int_x^y \mathfrak{m}^*\big(k(t)\big)\Delta t$$

for all $\mathfrak{m}^* \in \mathfrak{X}^*$, then k is said to be Riemann-Pettis Δ-integrable on $[x,y]_{\mathbb{T}}$. The element $\mathfrak{m}$ is called the Riemann-Pettis Δ-integral of k over $[x,y]_{\mathbb{T}}$ and is denoted by $(\overline{RP})\int_x^y k(t)\Delta t$.

In the following sections we give our definition of the Riemann-Stieltjes integral for Banach-valued functions on time scales and explore the Riemann-Stieltjes-type integrals.

2. Banach-valued Riemann-Stieltjes integration

Theoretical definitions of the Riemann-Stieltjes Δ-integral and Riemann-Stieltjes ∇-integral for functions whose values lie in the Banach space on time scales are presented in this section.

2.1. Banach-valued Riemann-Stieltjes Δ-integral

Considering $\breve{\mathcal{Q}}_\delta$ to be a Δ-tagged partition with mesh δ; and let ψ be a real-valued monotone increasing function on $[x, y]_{\mathbb{T}}$.

The Riemann-Stieltjes Δ-sum, $(\overline{\mathrm{RS}})(k; \breve{\mathcal{Q}}_\delta; \psi)$, of the Banach-valued function k evaluated at the Δ-tags is formulated as,

$$(\overline{\mathrm{RS}})(k; \breve{\mathcal{Q}}_\delta; \psi) := \sum_{z=1}^{i} \Big[\psi(t_z) - \psi(t_{z-1})\Big] \cdot k(\vartheta_z).$$

Definition 2.1 *Riemann-Stieltjes Δ-integral: Let function $k : [x, y]_{\mathbb{T}} \to \mathfrak{X}$ and let $\psi : [x, y]_{\mathbb{T}} \to \mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x, y]_{\mathbb{T}}$ is said to be Riemann-Stieltjes Δ-integrable if there exists an $\overline{I} \in \mathfrak{X}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any Δ-tagged partition $\breve{\mathcal{Q}}_\delta$ we have,*

$$\Big\|(\overline{\mathrm{RS}})(k; \breve{\mathcal{Q}}_\delta; \psi) - \overline{I}\Big\| < \varepsilon.$$

Here $\overline{I} = (\overline{\mathrm{RS}}) \int_x^y k(t)\Delta\psi(t)$, where $(\overline{\mathrm{RS}}) \int_x^y k(t)\Delta\psi(t)$ denotes the Riemann-Stieltjes Δ-integral.

The set of all Banach-valued Riemann-Stieltjes Δ-integrable functions on $[x, y]_{\mathbb{T}}$ will be denoted by $(\mathrm{RS})_\Delta\big\{[x, y]_{\mathbb{T}}, \mathfrak{X}, \psi\big\}$.

2.2. Banach-valued Riemann-Stieltjes ∇-integral

Considering $\breve{\mathcal{R}}_\delta$ to be a ∇-tagged partition with mesh δ; and let ψ be a real-valued monotone increasing function on $[x, y]_{\mathbb{T}}$.

The Riemann-Stieltjes ∇-sum, $(\underline{\mathrm{RS}})(k; \breve{\mathcal{R}}_\delta; \psi)$, of the Banach-valued function k evaluated at the ∇-tags is formulated as,

$$(\underline{\mathrm{RS}})(k; \breve{\mathcal{R}}_\delta; \psi) := \sum_{z=1}^{i} \Big[\psi(t_z) - \psi(t_{z-1})\Big] \cdot k(\xi_z).$$

Definition 2.2 *Riemann-Stieltjes ∇-integral: Let function $k : [x, y]_{\mathbb{T}} \to \mathfrak{X}$ and let $\psi : [x, y]_{\mathbb{T}} \to \mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x, y]_{\mathbb{T}}$ is said to be Riemann-Stieltjes ∇-integrable if there exists an $\underline{I} \in \mathfrak{X}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any ∇-tagged partition $\breve{\mathcal{R}}_\delta$ we have,*

$$\Big\|(\underline{\mathrm{RS}})(k; \breve{\mathcal{R}}_\delta; \psi) - \underline{I}\Big\| < \varepsilon.$$

Here $\underline{I} = (\underline{\mathrm{RS}}) \int_x^y k(t)\nabla\psi(t)$, where $(\underline{\mathrm{RS}}) \int_x^y k(t)\nabla\psi(t)$ denotes the Riemann-Stieltjes ∇-integral.

The set of all Banach-valued Riemann-Stieltjes ∇-integrable functions on $[x, y]_{\mathbb{T}}$ will be denoted by $(\mathrm{RS})_\nabla\big\{[x, y]_{\mathbb{T}}, \mathfrak{X}, \psi\big\}$.

3. Banach-valued Riemann-Stieltjes type integrals

Theoretical definitions of the Riemann-Stieltjes-Dunford integral; Riemann-Stieltjes-Pettis integral and Riemann-Stieltjes-Gelfand integral for functions whose values lie in the Banach space on time scales are presented in this section.

3.1. Riemann-Stieltjes-Dunford integral

Definition 3.1 *Riemann-Stieltjes-Dunford Δ-integral: Let function $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ and let $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x,y]_{\mathbb{T}}$ is said to be Riemann-Stieltjes-Dunford Δ-integrable on $[x,y]_{\mathbb{T}}$ if for each functional $m^* \in \mathfrak{X}^*$, the function $m^*(k) : [x,y]_{\mathbb{T}} \to \mathbb{R}$ is Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$, and there exists an element $m^{**} \in \mathfrak{X}^{**}$ such that*

$$m^{**}(m^*) = \overline{\mathrm{RS}} \int_x^y m^*\big(k(t)\big)\Delta\psi(t)$$

for all $m^* \in \mathfrak{X}^*$, then k is said to be Riemann-Stieltjes-Dunford Δ-integrable on $[x,y]_{\mathbb{T}}$. The element m^{**} is called the Riemann-Stieltjes-Dunford Δ-integral of k over $[x,y]_{\mathbb{T}}$ and is denoted by $(\overline{\mathrm{RSD}}) \int_x^y k(t)\Delta\psi(t)$.

The set of all Banach-valued Riemann-Stieltjes-Dunford Δ-integrable functions on $[x,y]_{\mathbb{T}}$ will be denoted by $(\mathrm{RSD})_\Delta\big\{[x,y]_{\mathbb{T}}, \mathfrak{X}, \psi\big\}$.

Corollary 3.1 *If k is Riemann-Stieltjes-Dunford Δ-integrable on $[x,y]_{\mathbb{T}}$ then, for each $m^* \in \mathfrak{X}^*$ the function $m^*(k)$ is Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$.*

We provide the $\varepsilon - \delta$ definition of the Riemann-Stieltjes-Dunford Δ-integral.

Definition 3.2 *Riemann-Stieltjes-Dunford Δ-integral: Let function $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ and let $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x,y]_{\mathbb{T}}$ is said to be Riemann-Stieltjes-Dunford Δ-integrable on $[x,y]_{\mathbb{T}}$ if there exists an element $m^{**} \in \mathfrak{X}^{**}$ such that for any $\varepsilon > 0$ and every functional $m^* \in \mathfrak{X}^*$ there exists $\delta > 0$ hence for any Δ-tagged partition $\breve{\mathcal{Q}}_\delta$ we have,*

$$\left| m^*\left(\sum_{z=1}^{i} \Big[\psi(t_z) - \psi(t_{z-1})\Big] \cdot k(\vartheta_z) \right) - m^{**}(m^*) \right| < \varepsilon.$$

Here $m^{**} = (\overline{\mathrm{RSD}}) \int_x^y k(t)\Delta\psi(t)$, where $(\overline{\mathrm{RSD}}) \int_x^y k(t)\Delta\psi(t)$ is called the Riemann-Stieltjes-Dunford Δ-integral.

Below we define the scalarly Riemann-Stieltjes Δ-integral (Definition 3.3) and establish its relationship with the Riemann-Stieltjes-Dunford Δ-integral (Definition 3.1).

Definition 3.3 *Scalarly Riemann-Stieltjes Δ-integral: Let function $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ and let $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x,y]_{\mathbb{T}}$ is said to be scalarly Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$ if there exists an $\overline{I} \in \mathbb{R}$ such that for every $\varepsilon > 0$ and every functional $m^* \in \mathfrak{X}^*$ there exists $\delta > 0$ hence for any Δ-tagged partition $\breve{\mathcal{Q}}_\delta$ we have,*

$$\left| \sum_{z=1}^{i} \Big[\psi(t_z) - \psi(t_{z-1})\Big] \cdot m^*(k(\vartheta_z)) - \overline{I} \right| < \varepsilon.$$

Here $\overline{I} = (s\overline{\mathrm{RS}}) \int_x^y k(t)\Delta\psi(t)$, where $(s\overline{\mathrm{RS}}) \int_x^y k(t)\Delta\psi(t)$ is called the scalarly Riemann-Stieltjes Δ-integral.

Theorem 3.1 *If k is Riemann-Stieltjes-Dunford Δ-integrable on $[x,y]_{\mathbb{T}}$ then, k is scalarly Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$.*

Proof: Given k is Riemann-Stieltjes-Dunford Δ-integrable on $[x,y]_{\mathbb{T}}$ implies by Definition 3.1 that for every functional $m^* \in \mathfrak{X}^*$, the function $m^*(k) : [x,y]_{\mathbb{T}} \to \mathbb{R}$ is Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$ (Corollary 3.1), hence by Definition 3.3 it implies that k is scalarly Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$. □

Remark 3.1 If k is Riemann-Stieltjes-Dunford Δ-integrable on $[x,y]_{\mathbb{T}}$ then, k is integrable on every Δ-subinterval of $[x,y]_{\mathbb{T}}$.

The criterion of integrability for Riemann-Stieltjes-Dunford Δ-integral is as follows-

Theorem 3.2 *Let $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ be Riemann-Stieltjes-Dunford Δ-integrable then, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any Δ-tagged partitions $\breve{\mathcal{Q}}_1, \breve{\mathcal{Q}}_2 \in \mathcal{P}$ both having mesh δ, we have*

$$\left| m^*\left(\sum_{z=1}^{i} \Big[\psi(t_z^1) - \psi(t_{z-1}^1)\Big] \cdot k(\vartheta_z^1) \right) - m^*\left(\sum_{z=1}^{s} \Big[\psi(t_z^2) - \psi(t_{z-1}^2)\Big] \cdot k(\vartheta_z^2) \right) \right| < \varepsilon \tag{3.1}$$

for all $m^* \in \mathfrak{X}^*$ (superscript 1 denotes elements from $\breve{\mathcal{Q}}_1$ and superscript 2 denotes elements from $\breve{\mathcal{Q}}_2$).

Proof: Given $k \in (\mathrm{RSD})_\Delta\big\{[x,y]_{\mathbb{T}}, \mathfrak{X}, \psi\big\}$ implies for any $\varepsilon > 0$ there exists $\delta > 0$ such that for Δ-tagged partition $\breve{\mathcal{Q}}_1$ and $\breve{\mathcal{Q}}_2$ of $[x,y]_{\mathbb{T}}$ both with mesh δ, we have

$$\left| m^*\left(\textstyle\sum_{z=1}^{i} \Big[\psi(t_z^1) - \psi(t_{z-1}^1)\Big] \cdot k(\vartheta_z^1) \right) - m^{**}(m^*) \right| < \tfrac{\varepsilon}{2}$$

and,

$$\left| m^*\left(\textstyle\sum_{z=1}^{s} \Big[\psi(t_z^2) - \psi(t_{z-1}^2)\Big] \cdot k(\vartheta_z^2) \right) - m^{**}(m^*) \right| < \tfrac{\varepsilon}{2}.$$

Therefore we have,

$$\begin{aligned}
&\left| m^*\left(\sum_{z=1}^{i} \Big[\psi(t_z^1) - \psi(t_{z-1}^1)\Big] \cdot k(\vartheta_z^1) \right) - m^*\left\{ \sum_{z=1}^{s} \Big[\psi(t_z^2) - \psi(t_{z-1}^2)\Big] \cdot k(\vartheta_z^2) \right\} \right| \\
&\leq \left| m^*\left(\sum_{z=1}^{i} \Big[\psi(t_z^1) - \psi(t_{z-1}^1)\Big] \cdot k(\vartheta_z^1) \right) - m^{**}(m^*) \right| \\
&+ \left| m^{**}(m^*) - m^*\left(\sum_{z=1}^{s} \Big[\psi(t_z^2) - \psi(t_{z-1}^2)\Big] \cdot k(\vartheta_z^2) \right) \right| < \varepsilon.
\end{aligned}$$

Hence, if k is given to be Riemann-Stieltjes-Dunford Δ-integrable then Eq. 3.1 holds. □

We proceed to define the ∇-integral, statements and proofs of theorems are omitted due to its similarity with the Δ-integral.

Definition 3.4 *Riemann-Stieltjes-Dunford ∇-integral: Let function $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ and let $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x,y]_{\mathbb{T}}$ is said to be Riemann-Stieltjes-Dunford ∇-integrable on $[x,y]_{\mathbb{T}}$ if for each functional $m^* \in \mathfrak{X}^*$, the function $m^*(k) : [x,y]_{\mathbb{T}} \to \mathbb{R}$ is Riemann-Stieltjes ∇-integrable on $[x,y]_{\mathbb{T}}$, and there exists an element $m^{**} \in \mathfrak{X}^{**}$ such that*

$$m^{**}(m^*) = \underline{\mathrm{RS}} \int_x^y m^*\big(k(t)\big) \nabla\psi(t)$$

for all $m^* \in \mathfrak{X}^*$, then k is said to be Riemann-Stieltjes-Dunford ∇-integrable on $[x,y]_{\mathbb{T}}$. The element m^{**} is called the Riemann-Stieltjes-Dunford ∇-integral of k over $[x,y]_{\mathbb{T}}$ and is denoted by $(\underline{\mathrm{RSD}}) \int_x^y k(t)\nabla\psi(t)$.

The set of all Banach valued Riemann-Stieltjes-Dunford ∇-integrable functions on $[x,y]_{\mathbb{T}}$ will be denoted by $(\mathrm{RSD})_\nabla\big\{[x,y]_{\mathbb{T}}, \mathfrak{X}, \psi\big\}$.

Definition 3.3 gives the scalarly Riemann-Stieltjes Δ-integral, ∇-integral is given below.

Definition 3.5 *Scalarly Riemann-Stieltjes ∇-integral: Let function $k : [x, y]_{\mathbb{T}} \to \mathfrak{X}$ and let $\psi : [x, y]_{\mathbb{T}} \to \mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x, y]_{\mathbb{T}}$ is said to be scalarly Riemann-Stieltjes ∇-integrable on $[x, y]_{\mathbb{T}}$ if there exists an $\underline{I} \in \mathbb{R}$ such that for every $\varepsilon > 0$ and every functional $m^* \in \mathfrak{X}^*$ there exists $\delta > 0$ such that for any ∇-tagged partition $\breve{\mathcal{R}}_\delta$ we have,*

$$\left| \sum_{z=1}^{i} \Big[\psi(t_z) - \psi(t_{z-1})\Big] \cdot m^*(k(\xi_z)) - \underline{I} \right| < \varepsilon.$$

Here $\underline{I} = (s\underline{\mathrm{RS}}) \int_x^y k(t)\nabla\psi(t)$.

3.2. Riemann-Stieltjes-Pettis integral

Definition 3.6 *Riemann-Stieltjes-Pettis Δ-integral: Let function $k : [x, y]_{\mathbb{T}} \to \mathfrak{X}$ and let $\psi : [x, y]_{\mathbb{T}} \to \mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x, y]_{\mathbb{T}}$ is said to be Riemann-Stieltjes-Pettis Δ-integrable on $[x, y]_{\mathbb{T}}$ if for each functional $m^* \in \mathfrak{X}^*$, the function $m^*(k) : [x, y]_{\mathbb{T}} \to \mathbb{R}$ is Riemann-Stieltjes Δ-integrable on $[x, y]_{\mathbb{T}}$, and there exists an element $m \in \mathfrak{X}$ such that*

$$m(m^*) = \overline{\mathrm{RS}} \int_x^y m^*\big(k(t)\big)\Delta\psi(t)$$

for all $m^* \in \mathfrak{X}^*$, then k is said to be Riemann-Stieltjes-Pettis Δ-integrable on $[x, y]_{\mathbb{T}}$. The element m is called the Riemann-Stieltjes-Pettis Δ-integral of k over $[x, y]_{\mathbb{T}}$, and is denoted by $(\overline{\mathrm{RSP}}) \int_x^y k(t)\Delta\psi(t)$.

The set of all Banach valued Riemann-Stieltjes-Pettis Δ-integrable functions on $[x, y]_{\mathbb{T}}$ will be denoted by $(\mathrm{RSP})_\Delta\big\{[x, y]_{\mathbb{T}}, \mathfrak{X}, \psi\big\}$.

Corollary 3.2 *If k is Riemann-Stieltjes-Pettis Δ-integrable on $[x, y]_{\mathbb{T}}$ then, for each $m^* \in \mathfrak{X}^*$ the function $m^*(k)$ is Riemann-Stieltjes Δ-integrable on $[x, y]_{\mathbb{T}}$.*

We provide the $\varepsilon - \delta$ definition of the Riemann-Pettis Δ-integral.

Definition 3.7 *Riemann-Stieltjes-Pettis Δ-integral: Let function $k : [x, y]_{\mathbb{T}} \to \mathfrak{X}$ and let $\psi : [x, y]_{\mathbb{T}} \to \mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x, y]_{\mathbb{T}}$ is said to be Riemann-Stieltjes-Pettis Δ-integrable on $[x, y]_{\mathbb{T}}$ if there exists an element $m \in \mathfrak{X}$ such that for any $\varepsilon > 0$ and every functional $m^* \in \mathfrak{X}^*$ there exists $\delta > 0$ such that for any Δ-tagged partition $\breve{\mathcal{Q}}_\delta$ we have,*

$$\left| m^*\left(\sum_{z=1}^{i} \Big[\psi(t_z) - \psi(t_{z-1})\Big] \cdot k(\vartheta_z) \right) - m(m^*) \right| < \varepsilon.$$

Here $m = (\overline{\mathrm{RSP}}) \int_x^y k(t)\Delta\psi(t)$, where $(\overline{\mathrm{RSP}}) \int_x^y k(t)\Delta\psi(t)$ is called the Riemann-Stieltjes-Pettis Δ-integral.

Theorem 3.3 *Let $k : [x, y]_{\mathbb{T}} \to \mathfrak{X}$ be Riemann-Stieltjes-Pettis Δ-integrable then the value of $m(m^*)$, for $m \in \mathfrak{X}$ and for all $m^* \in \mathfrak{X}^*$, is unique.*

Proof: Let us assume that function k has two values $(m)'(m^*)$, $(m)''(m^*)$ for $(m)'$, $(m)'' \in \mathfrak{X}$ both satisfying the definition and let $\varepsilon > 0$.
There exists $\delta'_{\frac{\varepsilon}{2}} > 0$ such that for any Δ-tagged partition $\breve{\mathcal{Q}}_{\delta'_{\frac{\varepsilon}{2}}}$ we have,

$$\left| m^*\Big(\textstyle\sum_{z=1}^{i} \Big[\psi(t'_z) - \psi(t'_{z-1})\Big] \cdot k(\vartheta_z) \Big) - (m)'(m^*) \right| < \tfrac{\varepsilon}{2}.$$

Also, there exists $\delta''_{\frac{\varepsilon}{2}} > 0$ such that for any Δ-tagged partition $\breve{\mathcal{Q}}_{\delta''_{\frac{\varepsilon}{2}}}$ we have,

$$\left|m^*\Big(\sum_{z=1}^{i}\Big[\psi(t''_z)-\psi(t''_{z-1})\Big]\cdot k(\vartheta_z)\Big)-(m)''(m^*)\right|<\tfrac{\varepsilon}{2}.$$

Let $\delta_\epsilon=\min\left\{\delta'_{\frac{\varepsilon}{2}},\delta''_{\frac{\varepsilon}{2}}\right\}>0$ and let $\breve{\mathcal{Q}}_{\delta_\varepsilon}$ be the corresponding Δ-tagged partition. Since mesh of $\breve{\mathcal{Q}}_{\delta_\epsilon}$ is lesser or equal to the mesh of $\breve{\mathcal{Q}}_{\delta'_{\frac{\varepsilon}{2}}}$ and $\breve{\mathcal{Q}}_{\delta''_{\frac{\varepsilon}{2}}}$ thus we have by definition-

$$\left|m^*\Big(\sum_{z=1}^{i}\Big[\psi(t_z)-\psi(t_{z-1})\Big]\cdot k(\vartheta_z)\Big)-(m)'(m^*)\right|<\tfrac{\varepsilon}{2}$$

and,

$$\left|m^*\Big(\sum_{z=1}^{i}\Big[\psi(t_z)-\psi(t_{z-1})\Big]\cdot k(\vartheta_z)\Big)-(m)''(m^*)\right|<\tfrac{\varepsilon}{2},$$

whence it follows from triangle inequality that,

$$\begin{aligned}\Big|(m)'(m^*)-(m)''(m^*)\Big| &= \Bigg|(m)'(m^*)-m^*\Bigg(\sum_{z=1}^{i}\Big[\psi(t_z)-\psi(t_{z-1})\Big]\cdot k(\vartheta_z)\Bigg)+m^* \\ &\quad \Bigg(\sum_{z=1}^{i}\Big[\psi(t_z)-\psi(t_{z-1})\Big]\cdot k(\vartheta_z)\Bigg)-(m^{**})''(m^*)\Bigg| \\ &\le \Bigg|(m)'(m^*)-m^*\Bigg(\sum_{z=1}^{i}\Big[\psi(t_z)-\psi(t_{z-1})\Big]\cdot k(\vartheta_z)\Bigg)\Bigg|+\Bigg|m^* \\ &\quad \Bigg(\sum_{z=1}^{i}\Big[\psi(t_z)-\psi(t_{z-1})\Big]\cdot k(\vartheta_z)\Bigg)-(m)''(m^*)\Bigg|<\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, hence proved. □

We state a theorem which gives the relationship between our definition of the Riemann-Stieltjes-Pettis Δ-integral (Definition 3.6) and the definition of the scalarly Riemann-Stieltjes Δ-integral (Definition 3.3).

Theorem 3.4 *If k is Riemann-Stieltjes-Pettis Δ-integrable on $[x,y]_{\mathbb{T}}$ then, the function k is scalarly Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$.*

Proof: Given k is Riemann-Stieltjes-Pettis Δ-integrable on $[x,y]_{\mathbb{T}}$ implies by Definition 3.6 that for every functional $m^*\in\mathfrak{X}^*$, the function $m^*(k):[x,y]_{\mathbb{T}}\to\mathbb{R}$ is Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$ (Corollary 3.2), hence by Definition 3.3 it implies that k is scalarly Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$. □

Below we define the weak Riemann-Stieltjes Δ-integral (Definition 3.8) and establish its relationship with the Riemann-Stieltjes-Pettis Δ-integral (Definition 3.6).

Definition 3.8 *Weak Riemann-Stieltjes Δ-integral: Let function $k:[x,y]_{\mathbb{T}}\to\mathfrak{X}$ and let $\psi:[x,y]_{\mathbb{T}}\to\mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x,y]_{\mathbb{T}}$ is said to be weak Riemann-Stieltjes Δ-integrable if there exists an $\overline{I}\in\mathfrak{X}$ such that for every $\varepsilon>0$ and every functional $m^*\in\mathfrak{X}^*$ there exists $\delta>0$ such that for any Δ-tagged partition $\breve{\mathcal{Q}}_\delta$ we have,*

$$\left\|m^*\Bigg(\sum_{z=1}^{i}\Big[\psi(t_z)-\psi(t_{z-1})\Big]\cdot k(\vartheta_z)-\overline{I}\Bigg)\right\|<\varepsilon.$$

Here $\overline{I}=(w\overline{\mathrm{RS}})\int_x^y k(t)\Delta\psi(t)$, where $(w\overline{\mathrm{RS}})\int_x^y k(t)\Delta\psi(t)$ is called weak Riemann-Stieltjes Δ-integral.

Theorem 3.5 *If k is Riemann-Stieltjes-Pettis Δ-integrable on $[x,y]_{\mathbb{T}}$ then, k is weak Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$.*

Proof: Given k is Riemann-Stieltjes-Pettis Δ-integrable on $[x,y]_{\mathbb{T}}$ implies by Definition 3.6 that for every functional $m^* \in \mathfrak{X}^*$, the function m^*k is Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$, and that there exists an element $m \in \mathfrak{X}$ such that

$$m(m^*) = \overline{\text{RS}} \int_x^y m^*(k(t))\Delta\psi(t),$$

for all $m^* \in \mathfrak{X}^*$.

By the weak Riemann-Stieltjes Δ-integral definition (Definition 3.8) and from Theorem 3.3 implying the uniqueness of $m(m^*)$, for $m \in \mathfrak{X}$ and for all $m^* \in \mathfrak{X}^*$, we conclude the theorem. □

Remark 3.2 If k is Riemann-Stieltjes-Pettis Δ-integrable on $[x,y]_{\mathbb{T}}$ then, k is integrable on every Δ-subinterval of $[x,y]_{\mathbb{T}}$.

The criterion of integrability for Riemann-Stieltjes-Pettis Δ-integral on time scales is as follows-

Theorem 3.6 *Let $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ be Riemann-Stieltjes-Pettis Δ-integrable, then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any Δ-tagged partitions $\breve{\mathcal{Q}}_1, \breve{\mathcal{Q}}_2 \in \mathcal{P}$ having mesh δ, we have*

$$\left| m^*\left(\sum_{z=1}^{i} \Big[\psi(t_z^1) - \psi(t_{z-1}^1)\Big] \cdot k(\vartheta_z^1) \right) - m^*\left(\sum_{z=1}^{s} \Big[\psi(t_z^2) - \psi(t_{z-1}^2)\Big] \cdot k(\vartheta_z^2) \right) \right| < \varepsilon \tag{3.2}$$

for all $m^* \in \mathfrak{X}^*$ (superscript 1 denotes elements from $\breve{\mathcal{Q}}_1$ and superscript 2 denotes elements from $\breve{\mathcal{Q}}_2$).

Proof: Given $k \in (\text{RSP})_{\Delta}\big([x,y]_{\mathbb{T}}, \mathfrak{X}, \psi\big)$, implies for any $\varepsilon > 0$ there exists $\delta > 0$ such that for Δ-tagged partition $\breve{\mathcal{Q}}_1$ and $\breve{\mathcal{Q}}_2$ of $[x,y]_{\mathbb{T}}$ with mesh δ, we have

$$\left| m^*\left(\textstyle\sum_{z=1}^{i} \Big[\psi(t_z^1) - \psi(t_{z-1}^1)\Big] \cdot k(\vartheta_z^1) \right) - m(m^*) \right| < \tfrac{\varepsilon}{2}$$

and,

$$\left| m^*\left(\textstyle\sum_{z=1}^{s} \Big[\psi(t_z^2) - \psi(t_{z-1}^2)\Big] \cdot k(\vartheta_z^2) \right) - m(m^*) \right| < \tfrac{\varepsilon}{2}.$$

Therefore we have,

$$\begin{aligned} &\left| m^*\left(\sum_{z=1}^{i} \Big[\psi(t_z^1) - \psi(t_{z-1}^1)\Big] \cdot k(\vartheta_z^1) \right) - m^*\left(\sum_{z=1}^{s} \Big[\psi(t_z^2) - \psi(t_{z-1}^2)\Big] \cdot k(\vartheta_z^2) \right) \right| \\ &\quad \leq \left| m^*\left(\sum_{z=1}^{i} \Big[\psi(t_z^1) - \psi(t_{z-1}^1)\Big] \cdot k(\vartheta_z^1) \right) - m(m^*) \right| \\ &\quad + \left| m(m^*) - m^*\left(\sum_{z=1}^{s} \Big[\psi(t_z^2) - \psi(t_{z-1}^2)\Big] \cdot k(\vartheta_z^2) \right) \right| < \varepsilon. \end{aligned}$$

Hence, if k is given to be Riemann-Stieltjes-Pettis Δ-integrable then Eq. 3.2 holds. □

We proceed to define the ∇-integral, statements and proofs of theorems are omitted due to its similarity with the Δ-integral.

Definition 3.9 *Riemann-Stieltjes-Pettis ∇-integral: Let function $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ and let $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x,y]_{\mathbb{T}}$ is said to be Riemann-Stieltjes-Pettis ∇-integrable on $[x,y]_{\mathbb{T}}$ if for each functional $m^* \in \mathfrak{X}^*$, the function $m^*(k) : [x,y]_{\mathbb{T}} \to \mathbb{R}$ is Riemann-Stieltjes ∇-integrable on $[x,y]_{\mathbb{T}}$, and there exists an element $m \in \mathfrak{X}$ such that*

$$m(m^*) = \underline{\text{RS}} \int_x^y m^*\big(k(t)\big)\nabla\psi(t)$$

for all $m^* \in \mathfrak{X}^*$, then k is said to be Riemann-Stieltjes-Pettis ∇-integrable on $[x,y]_{\mathbb{T}}$. The element m is called the Riemann-Stieltjes-Pettis ∇-integral of k over $[x,y]_{\mathbb{T}}$, and is denoted by $(\underline{\text{RSP}}) \int_x^y k(t)\nabla\psi(t)$.

The set of all Banach valued Riemann-Stieltjes-Pettis ∇-integrable functions on $[x,y]_{\mathbb{T}}$ will be denoted by $(\mathrm{RSP})_{\nabla}\big([x,y]_{\mathbb{T}},\mathfrak{X},\psi\big)$.

Definition 3.8 gives the weak Riemann-Stieltjes Δ-integral, ∇-integral is given below.

Definition 3.10 *Weak Riemann-Stieltjes ∇-integral: Let function $k:[x,y]_{\mathbb{T}}\to\mathfrak{X}$ and let $\psi:[x,y]_{\mathbb{T}}\to\mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x,y]_{\mathbb{T}}$ is said to be weak Riemann-Stieltjes ∇-integrable if there exists an $\underline{I}\in\mathfrak{X}$ such that for every $\varepsilon>0$ and every functional $m^*\in\mathfrak{X}^*$ there exists $\delta>0$ such that for any ∇-tagged partition $\breve{\mathcal{R}}_\delta$ we have,*

$$\left\| m^*\bigg(\sum_{z=1}^{i}\Big[\psi(t_z)-\psi(t_{z-1})\Big]\cdot k(\xi_z)-\underline{I}\bigg)\right\|<\varepsilon.$$

Here $\underline{I}=(w\underline{\mathrm{RS}})\int_x^y k(t)\nabla\psi(t)$, where $(w\underline{\mathrm{RS}})\int_x^y k(t)\nabla\psi(t)$ is called weak Riemann-Stieltjes ∇-integral.

3.3. Riemann-Stieltjes-Gelfand integral

Definition 3.11 *Riemann-Stieltjes-Gelfand Δ-integral: Let function $k:[x,y]_{\mathbb{T}}\to\mathfrak{X}^*$ and let $\psi:[x,y]_{\mathbb{T}}\to\mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x,y]_{\mathbb{T}}$ is said to be Riemann-Stieltjes-Gelfand Δ-integrable on $[x,y]_{\mathbb{T}}$ if for each $m\in\mathfrak{X}$, the function $k(m):[x,y]_{\mathbb{T}}\to\mathbb{R}$ is Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$, and there exists an element $m^*\in\mathfrak{X}^*$ such that*

$$m^*(m)=\overline{\mathrm{RS}}\int_x^y m\big(k(t)\big)\Delta\psi(t)$$

for all $m\in\mathfrak{X}$, then k is said to be Riemann-Stieltjes-Gelfand Δ-integrable on $[x,y]_{\mathbb{T}}$. The element m^* is called the Riemann-Stieltjes-Gelfand Δ-integral of k over $[x,y]_{\mathbb{T}}$ and is denoted by $(\overline{\mathrm{RSG}})\int_x^y k(t)\Delta\psi(t)$.

The set of all Banach valued Riemann-Stieltjes-Gelfand Δ-integrable functions on $[x,y]_{\mathbb{T}}$ will be denoted by $(\mathrm{RSG})_{\Delta}\big\{[x,y]_{\mathbb{T}},\mathfrak{X},\psi\big\}$.

Corollary 3.3 *If k is Riemann-Stieltjes-Gelfand Δ-integrable on $[x,y]_{\mathbb{T}}$ then, for each $m\in\mathfrak{X}$ the function $k(m)$ is Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$.*

We provide the $\varepsilon-\delta$ definition of the Riemann-Stieltjes-Gelfand Δ-integral.

Definition 3.12 *Riemann-Stieltjes-Gelfand Δ-integral: Let function $k:[x,y]_{\mathbb{T}}\to\mathfrak{X}^*$ and let $\psi:[x,y]_{\mathbb{T}}\to\mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x,y]_{\mathbb{T}}$ is said to be Riemann-Stieltjes-Gelfand Δ-integrable on $[x,y]_{\mathbb{T}}$ if there exists an element $m^*\in\mathfrak{X}^*$ such that for any $\varepsilon>0$ and every $m\in\mathfrak{X}$ there exists $\delta>0$ hence for any Δ-tagged partition $\breve{\mathcal{Q}}_\delta$ we have,*

$$\left| m\bigg(\sum_{z=1}^{i}k(\vartheta_z).\Big[\psi(t_z)-\psi(t_{z-1})\Big]\bigg)-m^*(m)\right|<\varepsilon.$$

Here $m^*=(\overline{\mathrm{RSG}})\int_x^y k(t)\Delta\psi(t)$, where $(\overline{\mathrm{RSG}})\int_x^y k(t)\Delta\psi(t)$ is called the Riemann-Stieltjes-Gelfand Δ-integral.

Below we define the weak* scalarly Riemann-Stieltjes Δ-integral (Definition 3.13) and establish its relationship with the Riemann-Stieltjes-Gelfand Δ-integral (Definition 3.11).

Definition 3.13 *Weak* scalarly Riemann-Stieltjes Δ-integral: Let function $k:[x,y]_{\mathbb{T}}\to\mathfrak{X}^*$ and let $\psi:[x,y]_{\mathbb{T}}\to\mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x,y]_{\mathbb{T}}$ is said to be* *weak* scalarly Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$ if there exists an $\overline{I} \in \mathbb{R}$ such that for every $\varepsilon > 0$ and every $m \in \mathfrak{X}$ there exists $\delta > 0$ hence for any Δ-tagged partition $\breve{\mathcal{Q}}_\delta$ we have,*

$$\left| \sum_{z=1}^{i} \Big[\psi(t_z) - \psi(t_{z-1})\Big] \cdot m(k(\vartheta_z)) - \overline{I} \right| < \varepsilon.$$

Here $\overline{I} = (ws\overline{\mathrm{RS}}) \int_x^y k(t)\Delta\psi(t)$, where $(ws\overline{\mathrm{RS}}) \int_x^y k(t)\Delta\psi(t)$ is called the weak scalarly Riemann-Stieltjes Δ-integral.*

Theorem 3.7 *If k is Riemann-Stieltjes-Gelfand Δ-integrable on $[x,y]_{\mathbb{T}}$ then, k is weak* scalarly Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$.*

Proof: Given k is Riemann-Stieltjes-Gelfand Δ-integrable on $[x,y]_{\mathbb{T}}$ implies by Definition 3.11 that for every $m \in \mathfrak{X}$, the function $m(k) : [x,y]_{\mathbb{T}} \to \mathbb{R}$ is Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$ (Corollary 3.3), hence by Definition 3.13 it implies that k is weak* scalarly Riemann-Stieltjes Δ-integrable on $[x,y]_{\mathbb{T}}$. □

Remark 3.3 If k is Riemann-Stieltjes-Gelfand Δ-integrable on $[x,y]_{\mathbb{T}}$ then, k is integrable on every Δ-subinterval of $[x,y]_{\mathbb{T}}$.

The criterion of integrability for Riemann-Stieltjes-Gelfand Δ-integral is as follows-

Theorem 3.8 *Let $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}^*$ be Riemann-Stieltjes-Gelfand Δ-integrable then, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any Δ-tagged partitions $\breve{\mathcal{Q}}_1, \breve{\mathcal{Q}}_2 \in \mathcal{P}$ both having mesh δ, we have*

$$\left| m\Big(\sum_{z=1}^{i} k(\vartheta_z^1).\Big[\psi(t_z^1) - \psi(t_{z-1}^1)\Big]\Big) - m\Big(\sum_{z=1}^{s} k(\vartheta_z^2).\Big[\psi(t_z^2) - \psi(t_{z-1}^2)\Big]\Big) \right| < \varepsilon \tag{3.3}$$

for all $m \in \mathfrak{X}$ (superscript 1 denotes elements from $\breve{\mathcal{Q}}_1$ and superscript 2 denotes elements from $\breve{\mathcal{Q}}_2$).

Proof: Given $k \in (\mathrm{RSG})_\Delta\big\{[x,y]_{\mathbb{T}}, \mathfrak{X}, \psi\big\}$ implies for any $\varepsilon > 0$ there exists $\delta > 0$ such that for Δ-tagged partition $\breve{\mathcal{Q}}_1$ and $\breve{\mathcal{Q}}_2$ of $[x,y]_{\mathbb{T}}$ both with mesh δ, we have

$$\left| m\Big(\textstyle\sum_{z=1}^{i} k(\vartheta_z^1).\Big[\psi(t_z^1) - \psi(t_{z-1}^1)\Big]\Big) - m^*(m) \right| < \tfrac{\varepsilon}{2}$$

and,

$$\left| m\Big(\textstyle\sum_{z=1}^{s} k(\vartheta_z^2).\Big[\psi(t_z^2) - \psi(t_{z-1}^2)\Big]\Big) - m^*(m) \right| < \tfrac{\varepsilon}{2}.$$

Therefore we have,

$$\begin{aligned}
&\left| m\Big(\sum_{z=1}^{i} k(\vartheta_z^1).\Big[\psi(t_z^1) - \psi(t_{z-1}^1)\Big]\Big) - m\Big\{ \sum_{z=1}^{s} k(\vartheta_z^2).\Big[\psi(t_z^2) - \psi(t_{z-1}^2)\Big]\Big\} \right| \\
&\leq \left| m\Big(\sum_{z=1}^{i} k(\vartheta_z^1).\Big[\psi(t_z^1) - \psi(t_{z-1}^1)\Big]\Big) - m^*(m) \right| \\
&+ \left| m^*(m) - m\Big(\sum_{z=1}^{s} k(\vartheta_z^2).\Big[\psi(t_z^2) - \psi(t_{z-1}^2)\Big]\Big) \right| < \varepsilon.
\end{aligned}$$

Hence, if k is given to be Riemann-Stieltjes-Gelfand Δ-integrable then Eq. 3.3 holds. □

We proceed to define the ∇-integral, statements and proofs of theorems are omitted due to its similarity with the Δ-integral.

Definition 3.14 *Riemann-Stieltjes-Gelfand ∇-integral: Let function $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}^*$ and let $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x,y]_{\mathbb{T}}$ is said to be Riemann-Stieltjes-Gelfand ∇-integrable on $[x,y]_{\mathbb{T}}$ if for each $m \in \mathfrak{X}$, the function $k(m) : [x,y]_{\mathbb{T}} \to \mathbb{R}$ is Riemann-Stieltjes ∇-integrable on $[x,y]_{\mathbb{T}}$, and there exists an element $m^* \in \mathfrak{X}^*$ such that*

$$m^*(m) = \underline{\mathrm{RS}} \int_x^y m\big(k(t)\big) \nabla\psi(t)$$

for all $m \in \mathfrak{X}$, then k is said to be Riemann-Stieltjes-Gelfand ∇-integrable on $[x,y]_{\mathbb{T}}$. The element m^* is called the Riemann-Stieltjes-Gelfand ∇-integral of k over $[x,y]_{\mathbb{T}}$ and is denoted by $(\underline{\mathrm{RSG}}) \int_x^y k(t)\nabla\psi(t)$.

The set of all Banach valued Riemann-Stieltjes-Gelfand ∇-integrable functions on $[x,y]_{\mathbb{T}}$ will be denoted by $(\mathrm{RSG})_\nabla\big\{[x,y]_{\mathbb{T}}, \mathfrak{X}, \psi\big\}$.

Definition 3.13 gives the weak* scalarly Riemann-Stieltjes Δ-integral, ∇-integral is given below.

Definition 3.15 *Weak* scalarly Riemann-Stieltjes ∇-integral: Let function $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}^*$ and let $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$ be a monotone increasing function. Function k with respect to ψ on $[x,y]_{\mathbb{T}}$ is said to be weak* scalarly Riemann-Stieltjes ∇-integrable on $[x,y]_{\mathbb{T}}$ if there exists an $\underline{I} \in \mathbb{R}$ such that for every $\varepsilon > 0$ and every $m \in \mathfrak{X}$ there exists $\delta > 0$ hence for any ∇-tagged partition $\breve{\mathcal{R}}_\delta$ we have,*

$$\left| \sum_{z=1}^{i} \Big[\psi(t_z) - \psi(t_{z-1})\Big] \cdot m(k(\xi_z)) - \underline{I} \right| < \varepsilon.$$

Here $\underline{I} = (ws\underline{\mathrm{RS}}) \int_x^y k(t)\nabla\psi(t)$, where $(ws\underline{\mathrm{RS}}) \int_x^y k(t)\nabla\psi(t)$ is called the weak scalarly Riemann-Stieltjes ∇-integral.*

4. Convergence theorems

The convergence theorem involving the notion of uniform convergence for Riemann-Stieltjes-Dunford integral; Riemann-Stieltjes-Pettis integral and Riemann-Stieltjes-Gelfand integral on time scales are formulated in this section.

We first introduce the concept of RSD-equiintegrable; RSP-equiintegrable and RSG-equiintegrable on time scales, and using these definitions of equiintegrability we obtain the convergence results.

4.1. RSD-equiintegrable convergence theorem

Definition 4.1 RSD *Δ-equiintegrable: A collection $(\mathrm{RSD})_\Delta$ of functions is called Riemann-Stieltjes-Dunford Δ-equiintegrable on $[x,y]_{\mathbb{T}}$ if for every $k \in (\mathrm{RSD})_\Delta$, $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ with respect to a monotone increasing function $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$, there exists an element $m^{**} \in \mathfrak{X}^{**}$ such that for any $\varepsilon > 0$ and every functional $m^* \in \mathfrak{X}^*$ there exists $\delta > 0$ hence for any Δ-tagged partition $\breve{\mathcal{Q}}_\delta$ we have,*

$$\left| m^*\left(\sum_{z=1}^{i} \Big[\psi(t_z) - \psi(t_{z-1})\Big] \cdot k(\vartheta_z) \right) - m^{**}(m^*) \right| < \varepsilon,$$

for each $k \in (\mathrm{RSD})_\Delta$.

Definition 4.2 RSD *∇-equiintegrable: A collection $(\mathrm{RSD})_\nabla$ of functions is called Riemann-Stieltjes-Dunford ∇-equiintegrable on $[x,y]_{\mathbb{T}}$ if for every $k \in (\mathrm{RSD})_\nabla$, $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ with respect to be a monotone increasing function $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$, there exists an element $m^{**} \in \mathfrak{X}^{**}$ such that for any $\varepsilon > 0$ and every functional $m^* \in \mathfrak{X}^*$ there exists $\delta > 0$ hence for any ∇-tagged partition $\breve{\mathcal{R}}_\delta$ we have,*

$$\left| m^*\left(\sum_{z=1}^{i} \Big[\psi(t_z) - \psi(t_{z-1})\Big] \cdot k(\xi_z) \right) - m^{**}(m^*) \right| < \varepsilon,$$

for each $k \in (\mathrm{RSD})_\nabla$.

Using the concept of RSD Δ-equiintegrable we formulate the convergence theorem for Riemann-Stieltjes-Dunford Δ-integrals. The case of the ∇-integral can be obtained in a similar manner using the above ∇-integral definition (Definition 3.4 and Definition 4.2) hence not explicitly stated here.

Theorem 4.1 *Let $\{k_r\}$ be a sequence of functions that are* RSD Δ*-equiintegrable on $[x,y]_{\mathbb{T}}$ that converges uniformly to k on $[x,y]_{\mathbb{T}}$ then, k is also Riemann-Stieltjes-Dunford Δ-integrable on $[x,y]_{\mathbb{T}}$ and*

$$\overline{\mathrm{RS}}\int_x^y m^*\big(k(t)\big)\Delta\psi(t) = \lim_{r\to\infty}\overline{\mathrm{RS}}\int_x^y m^*\big(k_r(t)\big)\Delta\psi(t).$$

Proof: Let $\{k_r\}$ be a sequence of functions that are RSD Δ-equiintegrable on $[x,y]_{\mathbb{T}}$ implies for every functional $m^*\in\mathfrak{X}^*$ there exists an $m_r^{**}\in\mathfrak{X}^{**}$ for each k_r such that $\{m^*(k_r)\}$ is Riemann-Stieltjes Δ-integrable and $m_r^{**}(m^*) = \overline{\mathrm{RS}}\int_x^y m^*\big(k_r(t)\big)\Delta\psi(t)$ for each r. Given $\{k_r\}$ converges uniformly to k on $[x,y]_{\mathbb{T}}$ implies $\{m^*(k_r)\}$ also converges uniformly to $m^*(k)$, thus for every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that for all $r\geq N$ and all $t\in[x,y)_{\mathbb{T}}$ we have $\Big|m^*\big(k_r(t)\big)-m^*\big(k(t)\big)\Big|<\varepsilon$.
Consequently if $r,p\geq N$, then

$$\Big|m^*\big(k_r(t)\big)-m^*\big(k_p(t)\big)\Big| < \Big|m^*\big(k_r(t)\big)-m^*\big(k(t)\big)\Big| + \Big|m^*\big(k(t)\big)-m^*\big(k_p(t)\big)\Big| < 2\varepsilon$$

for all $t\in[x,y)_{\mathbb{T}}$.
Hence $\Big|m^*\big(k_r(t)\big)-m^*\big(k_p(t)\big)\Big| \leq \sup_{t\in[x,y)_{\mathbb{T}}}\Big|m^*\big(k_r(t)\big)-m^*\big(k_p(t)\big)\Big| < 2\varepsilon$, implying $-2\varepsilon < m^*\big(k_r(t)\big) - m^*\big(k_p(t)\big) < 2\varepsilon$ for all $t\in[x,y)_{\mathbb{T}}$. Thus,

$$-2\varepsilon(y-x) < \overline{\mathrm{RS}}\int_x^y m^*\big(k_r(t)\big)\Delta\psi(t) - \overline{\mathrm{RS}}\int_x^y m^*\big(k_p(t)\big)\Delta\psi(t) < 2\varepsilon(y-x).$$

Since $\varepsilon>0$ is arbitrary, hence the sequence $\Big(\overline{\mathrm{RS}}\int_x^y m^*\big(k_r(t)\big)\Delta\psi(t)\Big)$ is a Cauchy sequence and therefore converges to some number say $m_r^{**}(m^*)$, given $m_r^{**}\in\mathfrak{X}^{**}$ and every functional $m^*\in\mathfrak{X}^*$.

We will now show that $k\in(\mathrm{RSD})_\Delta\{[x,y]_{\mathbb{T}},\mathfrak{X},\psi\}$ with integral $m_r^{**}(m^*)$. If $\check{\mathcal{Q}} = \Big\{\big(\vartheta_z,[t_{z-1},t_z)_{\mathbb{T}}\big)\Big\}_{z=1}^{i}$ be any Δ-tagged partition of $[x,y]_{\mathbb{T}}$ and if $r\geq N$, then

$$\left|m^*\left(\sum_{z=1}^{i}\Big[\psi(t_z)-\psi(t_{z-1})\Big]\cdot k_r(\vartheta_z)\right) - m^*\left(\sum_{z=1}^{i}\Big[\psi(t_z)-\psi(t_{z-1})\Big]\cdot k(\vartheta_z)\right)\right|$$
$$\leq \sum_{z=1}^{i}\Big[\psi(t_z)-\psi(t_{z-1})\Big]\Big|m^*\big(k_r(\vartheta_z)\big)-m^*\big(k(\vartheta_z)\big)\Big| \leq \varepsilon(y-x).$$

We now choose $p\geq N$ such that $\Big|\overline{\mathrm{RS}}\int_x^y m^*\big(k_p(t)\big)\Delta\psi(t) - m_r^{**}(m^*)\Big| < \varepsilon$, and we let $\delta_{p,\varepsilon}>0$ be such that $\Big|\overline{\mathrm{RS}}\int_x^y m^*\big(k_p(t)\big)\Delta\psi(t) - m^*\Big((\overline{RS})(k_p;\check{\mathcal{Q}}_{\delta_{p,\varepsilon}};\psi)\Big)\Big| < \varepsilon$ whenever length of the Δ-subintervals in the Δ-partition is less than $\delta_{p,\varepsilon}$. Then we have,

$$\Big|m^*\Big((\overline{RS})(k;\check{\mathcal{Q}}_{\delta_{p,\varepsilon}};\psi)\Big) - m_r^{**}(m^*)\Big| \leq \Big|m^*\Big((\overline{RS})(k;\check{\mathcal{Q}}_{\delta_{p,\varepsilon}};\psi)\Big) - m^*\Big((\overline{RS})(k_p;\check{\mathcal{Q}}_{\delta_{p,\varepsilon}};\psi)\Big)\Big| +$$
$$\Big|m^*\Big((\overline{RS})(k_p;\check{\mathcal{Q}}_{\delta_{p,\varepsilon}};\psi)\Big) - \overline{\mathrm{RS}}\int_x^y m^*\big(k_p(t)\big)\Delta\psi(t)\Big| + \Big|\overline{\mathrm{RS}}\int_x^y m^*\big(k_p(t)\big)\Delta\psi(t) - m_r^{**}(m^*)\Big| \leq \varepsilon(y-x+2).$$

But since $\varepsilon>0$ is arbitrary, it follows that $k\in(\mathrm{RSD})_\Delta\{[x,y]_{\mathbb{T}},\mathfrak{X},\psi\}$ and $\overline{\mathrm{RS}}\int_x^y m^*\big(k(t)\big)\Delta\psi(t) = m_r^{**}(m^*)$, and for all $r\geq N$ it follows that $\overline{\mathrm{RS}}\int_x^y m^*\big(k(t)\big)\Delta\psi(t) = \lim_{r\to\infty}\overline{\mathrm{RS}}\int_x^y m^*\big(k_r(t)\big)\Delta\psi(t)$. □

Theorem 4.1 is called the uniform convergence theorem for sequence of functions, here the sequence of functions under consideration are RSD Δ-equiintegrable.

Remark 4.1 If $\{k_r\}$ be a sequence of functions that are RSD Δ-equiintegrable on $[x,y]_{\mathbb{T}}$ that converges uniformly to a Riemann-Stieltjes-Dunford Δ-integrable function k on $[x,y]_{\mathbb{T}}$. Then the function k is unique.

4.2. RSP-equiintegrable convergence theorem

Definition 4.3 RSP Δ-*equiintegrable: A collection* $(\mathrm{RSP})_\Delta$ *of functions is called Riemann-Stieltjes-Pettis* Δ*-equiintegrable on* $[x,y]_{\mathbb{T}}$ *if for every* $k \in (\mathrm{RSP})_\Delta$, $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ *with respect to a monotone increasing function* $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$*, there exists an element* $m \in \mathfrak{X}$ *such that for any* $\varepsilon > 0$ *and every functional* $m^* \in \mathfrak{X}^*$ *there exists* $\delta > 0$ *hence for any* Δ*-tagged partition* $\breve{\mathcal{Q}}_\delta$ *we have,*

$$\left| m^*\left(\sum_{z=1}^{i} \Big[\psi(t_z) - \psi(t_{z-1})\Big] \cdot k(\vartheta_z) \right) - m(m^*) \right| < \varepsilon,$$

for each $k \in (\mathrm{RSP})_\Delta$.

Definition 4.4 RSP ∇-*equiintegrable: A collection* $(\mathrm{RSP})_\nabla$ *of functions is called Riemann-Stieltjes-Pettis* ∇*-equiintegrable on* $[x,y]_{\mathbb{T}}$ *if for every* $k \in (\mathrm{RSP})_\nabla$, $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ *with respect to a monotone increasing function* $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$*, there exists an element* $m \in \mathfrak{X}$ *such that for any* $\varepsilon > 0$ *and every functional* $m^* \in \mathfrak{X}^*$ *there exists* $\delta > 0$ *hence for any* ∇*-tagged partition* $\breve{\mathcal{R}}_\delta$ *we have,*

$$\left| m^*\left(\sum_{z=1}^{i} \Big[\psi(t_z) - \psi(t_{z-1})\Big] \cdot k(\xi_z) \right) - m(m^*) \right| < \varepsilon,$$

for each $k \in (\mathrm{RSP})_\nabla$.

Using the concept of RSP Δ-equiintegrable we present the convergence theorem for Riemann-Stieltjes-Pettis Δ-integrals. The case of the ∇-integral can be obtained in a similar manner using the above ∇-integral definition (Definition 3.9 and Definition 4.4) hence not explicitly stated here.

Theorem 4.2 *Let* $\{k_r\}$ *be a sequence of functions that are* RSP Δ*-equiintegrable on* $[x,y]_{\mathbb{T}}$ *that converges uniformly to* k *on* $[x,y]_{\mathbb{T}}$ *then,*k *is also Riemann-Stieltjes-Pettis* Δ*-integrable on* $[x,y]_{\mathbb{T}}$ *and*

$$\overline{\mathrm{RS}} \int_x^y m^*\big(k(t)\big)\Delta\psi(t) = \lim_{r\to\infty} \overline{\mathrm{RS}} \int_x^y m^*\big(k_r(t)\big)\Delta\psi(t).$$

Remark 4.2 If $\{k_r\}$ be a sequence of functions that are RSP Δ-equiintegrable on $[x,y]_{\mathbb{T}}$ that converges uniformly to a Riemann-Stieltjes-Pettis Δ-integrable function k on $[x,y]_{\mathbb{T}}$. Then the function k is unique.

4.3. RSG-equiintegrable convergence theorem

Definition 4.5 RSG Δ-*equiintegrable: A collection* $(\mathrm{RSG})_\Delta$ *of functions is called Riemann-Stieltjes-Gelfand* Δ*-equiintegrable on* $[x,y]_{\mathbb{T}}$ *if for every* $k \in (\mathrm{RSG})_\Delta$, $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ *with respect to a monotone increasing function* $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$*, there exists an element* $m^* \in \mathfrak{X}^*$ *such that for any* $\varepsilon > 0$ *and every* $m \in \mathfrak{X}$ *there exists* $\delta > 0$ *hence for any* Δ*-tagged partition* $\breve{\mathcal{Q}}_\delta$ *we have,*

$$\left| m\left(\sum_{z=1}^{i} k(\vartheta_z).\Big[\psi(t_z) - \psi(t_{z-1})\Big] \right) - m^*(m) \right| < \varepsilon,$$

for each $k \in (\mathrm{RSG})_\Delta$.

Definition 4.6 RSG ∇-*equiintegrable: A collection* $(\mathrm{RSG})_\nabla$ *of functions is called Riemann-Stieltjes-Gelfand* ∇*-equiintegrable on* $[x,y]_{\mathbb{T}}$ *if for every* $k \in (\mathrm{RSG})_\nabla$, $k : [x,y]_{\mathbb{T}} \to \mathfrak{X}$ *with respect to a monotone increasing function* $\psi : [x,y]_{\mathbb{T}} \to \mathbb{R}$*, there exists an element* $m^* \in \mathfrak{X}^*$ *such that for any* $\varepsilon > 0$ *and every* $m \in \mathfrak{X}$ *there exists* $\delta > 0$ *hence for any* ∇*-tagged partition* $\breve{\mathcal{R}}_\delta$ *we have,*

$$\left| m\left(\sum_{z=1}^{i} k(\xi_z).\Big[\psi(t_z) - \psi(t_{z-1})\Big] \right) - m^*(m) \right| < \varepsilon,$$

for each $k \in (\mathrm{RSG})_\nabla$.

Using the concept of RSG Δ-equiintegrable we present the convergence theorem for Riemann-Stieltjes-Gelfand Δ-integrals. The case of the ∇-integral can be obtained in a similar manner using the above ∇-integral definition (Definition 3.14 and Definition 4.6) hence not explicitly stated here.

Theorem 4.3 *Let* $\{k_r\}$ *be a sequence of functions that are* RSG Δ*-equiintegrable on* $[x,y]_{\mathbb{T}}$ *that converges uniformly to* k *on* $[x,y]_{\mathbb{T}}$ *then,* k *is also Riemann-Stieltjes-Gelfand* Δ*-integrable on* $[x,y]_{\mathbb{T}}$ *and*

$$\overline{\mathrm{RS}}\int_x^y m\big(k(t)\big)\Delta\psi(t) = \lim_{r\to\infty}\overline{\mathrm{RS}}\int_x^y m\big(k_r(t)\big)\Delta\psi(t).$$

Remark 4.3 If $\{k_r\}$ be a sequence of functions that are RSG Δ-equiintegrable on $[x,y]_{\mathbb{T}}$ that converges uniformly to a Riemann-Stieltjes-Gelfand Δ-integrable function k on $[x,y]_{\mathbb{T}}$. Then the function k is unique.

5. Conclusion

This paper explores the theory of Riemann-Stieltjes and Riemann-Stieltjes-type integrals for Banach-valued functions on time scales and discuss a few fascinating results.

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