Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–18.
ISSN-0037-8712
doi:10.5269/bspm.76720

Generalized Homogeneous q-Shift Operator Applied for Generalized Cigler's Polynomials

Husam L. Saad* and Samaher A. Abdul-Ghani

ABSTRACT: In this paper, we establish the generalized homogeneous q-shift operator $E(q^{\alpha} z\theta_{xy})$ and the generalised Cigler's polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$. Then, we apply this operator to derive some q-identities such as: the generating function and its extension, Rogers formula and its extension, Mehler's formula and its extension, Srivastava-Agarwal type bilinear generating functions to the polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$. In addition, we supply some special values for the identities of $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$ in order to establish the same identities for the polynomials $D_n^{(\alpha-n)}(x,y,b)$ and $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$.

Key Words: the homogeneous q-shift operator, Cigler's polynomials, generating function, Rogers formula, Mehler's formula, Srivastava-Agarwal type bilinear generating function.

Contents

1. Introduction

In this paper, we will follow common notations and definitions for the q-series that used in [11]. We assume that $0 < q < 1$.

For $\delta \in \mathbb{C}$, the q-shifted factorial is defined by [9,10,11]

$$(\delta;q)_0 = 1, \quad (\delta;q)_n = \prod_{k=0}^{n-1}(1-\delta q^k), \quad (\delta;q)_\infty = \prod_{k=0}^{\infty}(1-\delta q^k),$$

and the multiple q-shifted factorials by:

$$(\delta_1,\delta_2,\ldots,\delta_r;q)_m = (\delta_1;q)_m(\delta_2;q)_m\cdots(\delta_r;q)_m,$$

where $m \in \mathbb{Z}$ or ∞.

The basic hypergeometric series ${}_r\phi_s$ is presented as follows [11,15]:

$${}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,x\right) = \sum_{n=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} x^n,$$

---

* Corresponding author.
2010 *Mathematics Subject Classification:* 05A30, 33D45.
Submitted May 09, 2025. Published August 31, 2025

where $q \neq 0$ when $r > s+1$. Note that

$$ {}_{r+1}\phi_r\left(\begin{array}{c} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_r \end{array};q,x\right) = \sum_{n=0}^{\infty} \frac{(a_1,\ldots,a_{r+1};q)_n}{(q,b_1,\ldots,b_r;q)_n} x^n, \quad |x|<1. $$

The q-binomial coefficient is defined as [11]:

$$ \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}} \quad for \quad 0 \leqslant k \leqslant n, $$

where $n,k \in \mathbb{N}$.

The Cauchy identity is given by [11,17]:

$$ \sum_{m=0}^{\infty} \frac{(\delta;q)_m}{(q;q)_m}\, x^m = \frac{(\delta x;q)_\infty}{(x;q)_\infty}, \quad |x|<1. \tag{1.1} $$

For $\delta = 0$, Cauchy identity becomes Euler's identity [11]:

$$ \sum_{m=0}^{\infty} \frac{x^m}{(q;q)_m} = \frac{1}{(x;q)_\infty}, \quad |x|<1 \tag{1.2} $$

which has the following inverse relation:

$$ \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} x^m}{(q;q)_m} = (x;q)_\infty. \tag{1.3} $$

The q-Chu-Vandermonde's sum is given by [11]:

$$ {}_2\phi_1(q^{-n},a;c;q,q) = \frac{(c/a;q)_n}{(c;q)_n} a^n. \tag{1.4} $$

We will use the following identities in this paper [4,13]:

$$ (aq^{-n};q)_n = (q/a;q)_n(-a)^n q^{-n-\binom{n}{2}}. \tag{1.5} $$

$$ \frac{(q^{-n};q)_k}{(q;q)_k} = \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}-nk}. \tag{1.6} $$

The Cauchy polynomials are defined as follows [14,16,18,21]:

$$ P_n(x,y) = (x-y)(x-qy)\cdots(x-q^{n-1}y) = (y/x;q)_n\ x^n, $$

which have the following generating function [2,3,8,12]:

$$ \sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \quad |xt|<1. \tag{1.7} $$

The Mehler's formula for Cauchy polynomials $P_n(x,y)$ is [20]:

$$ \sum_{n=0}^{\infty} P_n(x,y)P_n(u,v)\frac{t^n}{(q;q)_n} = \frac{(xvt;q)_\infty}{(xut;q)_\infty}\, {}_1\phi_1\left(\begin{array}{c} v/u \\ xvt \end{array};q,yut\right), \quad |xut|<1. \tag{1.8} $$

In 2010, Saad and Sukhi [19] introduced the dual homogeneous q-difference operator θ_{xy} as follows:

$$ \theta_{xy}\left\{f(x,y)\right\} = \frac{f(q^{-1}x,y) - f(x,qy)}{q^{-1}x - y}. $$

Furthermore, the homogeneous q-shift operator $\mathbb{L}(z\theta_{xy})$ was introduced as follows:

$$ \mathbb{L}(z\theta_{xy}) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}(z\theta)_{xy}^k}{(q;q)_k}. $$

Proposition 1.1 [19]. *We have*

$$\theta_{xy}^k\left\{P_n(y,x)\right\}=(-1)^k\frac{(q;q)_n}{(q;q)_{n-k}}\ P_{n-k}(y,x).$$
$$\theta_{xy}^k\left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\}=(-t)^k\frac{(xt;q)_\infty}{(yt;q)_\infty}.$$

Abdlhusein [1] proposed the trivariate q-polynomials $F_n(x,y,z;q)$ in 2016:

$$F_n(x,y,z;q)=(-1)^nq^{-\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^kq^{\binom{k}{2}}z^kP_{n-k}(y,x).$$

The q-identities for the trivariate q-polynomials were derived through the utilization of the homogeneous q-shift operator $\mathbb{L}(z\theta_{xy})$. Also, Abdlhusein [1] defined the homogeneous q-shift operator $\mathbb{L}(a,b;\theta_{xy})$ as follows:

$$L(a,b;\theta_{xy})=\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(b\theta_{xy}). \tag{1.9}$$

In 2016, Cao and Niu [6] studied Cigler's polynomials

$$D_n^{(\alpha-n)}(x,b)=\sum_{k=0}^{n}\begin{bmatrix}\alpha\\k\end{bmatrix}q^{k^2-nk}b^k\frac{(q;q)_n}{(q;q)_{n-k}}x^{n-k}.$$

Wang and Cao [25] in 2018 introduced an extension of Cigler's polynomials as follows:

$$D_n^{(\alpha-n)}(x,y,b)=(-1)^nq^{-\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix}\alpha\\k\end{bmatrix}(-1)^kq^{\binom{k}{2}}b^k\frac{(q;q)_n}{(q;q)_{n-k}}P_{n-k}(y,x). \tag{1.10}$$

The following are some results for Cigler's polynomials $D_n^{(\alpha-n)}(x,y,b)$:

Theorem 1.1 *The generating function for $D_n^{(\alpha-n)}(x,y,b)$ is* [5,25]

$$\sum_{n=0}^{\infty}(-1)^nq^{\binom{n}{2}}D_n^{(\alpha-n)}(x,y,b)\frac{t^n}{(q;q)_n}=\frac{(xt,bt;q)_\infty}{(yt,btq^\alpha;q)_\infty},\quad \max\{|yt|,|btq^\alpha|\}<1. \tag{1.11}$$

The extended generating function for $D_n^{(\alpha-n)}(x,y,b)$ is [5,25]

$$\sum_{n=0}^{\infty}(-1)^{n+k}q^{\binom{n+k}{2}}D_n^{(\alpha-n-k)}(x,y,b)\frac{t^{n+k}}{(q;q)_n}$$
$$=\frac{(xt;q)_\infty(bt;q)_\alpha}{(yt;q)_\infty}\ {}_3\phi_2\left(\begin{matrix}q^{-n},yt,btq^\alpha\\ xt,bt\end{matrix};q,q\right),\quad |yt|<1. \tag{1.12}$$

The Rogers formula for $D_n^{(\alpha-n)}(x,y,b)$ is [5]

$$\sum_{n=0}^{\infty}(-1)^{n+m}q^{\binom{n+m}{2}}D_{n+m}^{(\alpha-n-m)}(x,y,b)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}$$
$$=\frac{(x\ell,b\ell;q)_\infty}{(t/\ell,y\ell,q^\alpha b\ell;q)_\infty}\ {}_4\phi_3\left(\begin{matrix}y\ell,q^\alpha b\ell,0,0\\ q\ell/t,x\ell,b\ell\end{matrix};q,q\right),\quad \max\{|t/\ell|,|y\ell|,|q^\alpha b\ell|\}<1. \tag{1.13}$$

Srivastava-Agarwal type generating function for $D_n^{(\alpha-n)}(x,y,b)$ is [25]

$$\sum_{n=0}^{\infty}(-1)^nq^{\binom{n}{2}}D_n^{(\alpha-n)}(x,y,b)(\lambda;q)_n\frac{t^n}{(q;q)_n}$$

$$= \frac{(\lambda, xt;q)_\infty (bt;q)_\alpha}{(yt;q)_\infty} {}_3\phi_2 \left(\begin{matrix} yt, btq^\alpha, 0 \\ xt, bt \end{matrix} ;q,q \right), \quad |yt|<1. \tag{1.14}$$

Srivastava-Agarwal type bilinear generating function for $D_n^{(\alpha-n)}(x,y,b)$ *is* [5]

$$\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}} \psi_n^{(\alpha)}(u|q) D_n^{(\beta-n)}(x,y,b)\frac{t^n}{(q;q)_n}$$
$$= \frac{(q/u, xutq, butq;q)_\infty}{(\alpha q, yutq, butq^{1+\beta};q)_\infty} {}_3\phi_3 \left(\begin{matrix} 1/\alpha u, 1/xut, 1/but \\ q/u, 1/yut, q^{-\beta}/but \end{matrix} ;q, \alpha x q^{1-\beta}/y \right), \tag{1.15}$$

where $\max\{|\alpha q|, |yutq|, |butq^{1+\beta}|\}<1$.

Proposition 1.2 [5].

$$\sum_{n=0}^{\infty} \psi_n^{(c)}(u|q)(\lambda;q)_n \frac{(\lambda q t)^n}{(q;q)_n} = \frac{(q/u, utq;q)_\infty}{(cq, \lambda utq;q)_\infty} {}_2\phi_2 \left(\begin{matrix} 1/cu, 1/ut \\ q/u, 1/\lambda ut \end{matrix} ;q, cq/\lambda \right). \tag{1.16}$$

In 2019, Srivastava et al. [22] introduced the generalized Cauchy polynomials $P_n(x,y,a)$ as follows:

$$P_n(x,y,a) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a;q)_k (-1)^k q^{\binom{k}{2}} x^{n-k} y^k. \tag{1.17}$$

Also, they constructed the homogeneous q-shift operator

$$\tilde{L}(a,b;\theta_{xy}) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (a;q)_k}{(q;q)_k} (-b\theta_{xy}). \tag{1.18}$$

In 2020, Srivastava and Arjika [23] established the generalized Al-Salam-Carlitz q-polynomials $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ in the following form:

$$\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_1,a_2,\ldots,a_{s+1};q)_k}{(b_1,b_2,\ldots,b_s;q)_k} q^{\binom{k+1}{2}-nk}\ x^k y^{n-k}.$$

In 2021, the family of generalized q-hypergeometric polynomials were defined by Srivastava and Arjika [24] as follows:

$$\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q) = (-1)^n q^{-\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_1,\cdots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k P_{n-k}(y,x). \tag{1.19}$$

They [24] defined the homogeneous q-shift operator as follows:

$${}_r\Phi_s \left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix} ;q,-z\theta_{xy} \right) = \sum_{k=0}^{\infty} \frac{(a_1,\cdots,a_r;q)_k}{(q,b_1,\ldots,b_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (-z\theta_{xy})^k. \tag{1.20}$$

Also, they [24] gave the following results:

Theorem 1.2 [24].
The generating function for $\Psi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,z|q)$ *is:*

$$\sum_{n=0}^{\infty} \Psi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,z|q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} = \frac{(xt;q)_\infty}{(yt;q)_\infty} {}_r\phi_s \left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix} ;q,zt \right), \tag{1.21}$$

The extended generating function for $\Psi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,z|q)$ *is:*

$$\sum_{n=0}^{\infty}\Psi_{n+k}^{(\boldsymbol{a},\boldsymbol{b})}(x,y,z|q)\frac{(-1)^{n+k}q^{\binom{n+k}{2}}t^n}{(q;q)_n}$$
$$=\frac{t^{-k}(xt;q)_\infty}{(yt;q)_\infty}\sum_{i=0}^{k}\frac{(q^{-k},yt;q)_iq^i}{(q,xt;q)_i}\,{}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,ztq^i\right),\quad |yt|<1. \tag{1.22}$$

The Rogers formula for $\Psi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,z|q)$ *is:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Psi_{n+m}^{(\boldsymbol{a},\boldsymbol{b})}(x,y,z|q)(-1)^{n+m}q^{\binom{n+m}{2}}\frac{t^n}{(q;q)_n}\frac{w^m}{(q;q)_m}$$
$$=\frac{(xw;q)_\infty}{(t/w,yw;q)_\infty}\sum_{k=0}^{\infty}\frac{(yw;q)_kq^k}{(q,xw,qw/t;q)_k}\,{}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,zwq^k\right), \tag{1.23}$$

where $\max\{|t/w|,|yw|\}<1$.

The Srivastava-Agarwal type bilinear generating functions for $\Psi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,z|q)$ *is:*

$$\sum_{n=0}^{\infty}\Psi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,z|q)\psi_n^{(\alpha)}(u|q)(-1)^nq^{\binom{n+1}{2}}\frac{t^n}{(q;q)_n}=\frac{(q/u,xutq;q)_\infty}{(\alpha q,yutq;q)_\infty}$$
$$\times\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}(1/\alpha u,1/xut;q)_n}{(q,q/u,1/yut;q)_k}(\alpha xq/y)^k\,{}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,uztq^{1-k}\right), \tag{1.24}$$

provided that $\max\{|\alpha q|,|uyt|\}<1$.

In 2021, Cao et al. [7] constructed the following generalized trivariate q-Hahn polynomials as follows:

$$\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{q^{\binom{k}{2}}(a,b,c;q)_k}{(d,e;q)_k}P_{n-k}(y,x)z^k.$$

The structure of this paper is as follows: In section 2, we will introduce the generalised homogeneous q-shift operator $E(q^\alpha z\theta_{xy})$ and then find some of its identities. In section 3, the generating function and its extension for the polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$ are presented. The operator approach to Rogers formula and its extension for the polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$ will be applied in section 4. For the polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$, we establish Mehler's formula and its extension in section 5. In section 6, for the polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$, we develop a Srivastava-Agarwal type bilinear generating function.

2. Generalized homogeneous q-shift operator and some of its operator identities

In this section, we will begin by introducing the generalised homogeneous q-shift operator $E(q^\alpha z\theta_{xy})$, the generalised Cigler's polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$ and then we find some identities for the operator $E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})$.

Definition 2.1 We define the generalized homogeneous q-shift operator as follows:

$$E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})$$
$$=\sum_{k=0}^{\infty}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_k}{(\rho_1,\cdots,\rho_s;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}(-z\theta_{xy})^k. \tag{2.1}$$

- Setting $\alpha\to\infty$, $r=1$, $s=0$ and $z=-b$ in equation (2.1), we get the operator $L(a,b;\theta_{xy})$ defined by Abdlhusein [1] (equation (1.9)).

- When $\alpha \to \infty$, $r = s = 1$, and $z = b$ in equation (2.1), we get the operator $\tilde{L}(a, b; \theta_{xy})$ defined by Srivastava et al. [22] (equation (1.18)).
- Letting $\alpha \to \infty$ in equation (2.1), we get the operator ${}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, -z\theta_{xy}\right)$ introduced by Srivastava and Arjika [24] (equation (1.20)).

Definition 2.2 We define the generalised Cigler's polynomials as follows:

$$\begin{aligned} &\mathbb{D}_n^{(\alpha-n)}(\sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s, x, y, z) \\ &= (-1)^n q^{-\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} \alpha \\ k \end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r; q)_k}{(\rho_1, \ldots, \rho_s; q)_k} (q; q)_k \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k P_{n-k}(y, x). \end{aligned} \tag{2.2}$$

- Setting $r = s = 0$, $z = b$ in equation (2.2), we get an extension of Cigler's polynomials $D_n^{(\alpha-n)}(x, y, b)$ [25] (equation (1.10)).
- Setting $\alpha \to \infty$ in equation (2.2), we get the generalized q-hypergeometric polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x, y, z|q)$ [24] (equation (1.19)).

Several previously mentioned polynomials can be generated by providing specific values to the generalised Cigler's polynomials; for additional information, see [1,6,22,23].

The following theorem is straightforward to prove:

Theorem 2.1 *Let the operator $E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z\theta_{xy})$ be defined as in equation (2.1). Then*

$$\begin{aligned} &E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z\theta_{xy}) \left\{(-1)^n q^{-\binom{n}{2}} P_n(y, x)\right\} \\ &\quad = \mathbb{D}_n^{(\alpha-n)}(\sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s, x, y, z). \end{aligned} \tag{2.3}$$

$$\begin{aligned} &E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z\theta_{xy}) \left\{\frac{(xt; q)_\infty}{(yt; q)_\infty}\right\} \\ &\quad = \frac{(xt; q)_\infty}{(yt; q)_\infty} \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ k \end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r; q)_k}{(\rho_1, \cdots, \rho_s; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (zt)^k, \quad |yt| < 1. \end{aligned} \tag{2.4}$$

Lemma 2.1 *Let the operator $E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z\theta_{xy})$ be defined as in equation (2.1). Then*

$$\begin{aligned} &E(q^{-\alpha}, \sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s; q, q^\alpha z\theta_{xy}) \left\{\frac{P_k(y, x)}{(xt; q)_k} \frac{(xt; q)_\infty}{(yt; q)_\infty}\right\} \\ &= \frac{(xt; q)_\infty}{(yt; q)_\infty} t^{-k} \sum_{j=0}^{k} \frac{(q^{-k}, yt; q)_j}{(q, xt; q)_j} q^j \sum_{\varrho=0}^{\infty} \begin{bmatrix} \alpha \\ \varrho \end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r; q)_\varrho}{(\rho_1, \cdots, \rho_s; q)_\varrho} \left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r} (ztq^j)^\varrho, \end{aligned} \tag{2.5}$$

provided that $|yt| < 1$.

Proof: By using q-Chu-Vandermonde sum (1.4), we find that

$$\begin{aligned} \frac{P_k(y, x)}{(xt; q)_k} \frac{(xt; q)_\infty}{(yt; q)_\infty} &= \frac{(xt; q)_\infty}{(yt; q)_\infty} \frac{(x/y; q)_k y^k}{(xt; q)_k} \\ &= \frac{(xt; q)_\infty}{(yt; q)_\infty} t^{-k} {}_2\phi_1(q^{-k}, yt; xt; q, q) \\ &= t^{-k} \sum_{j=0}^{k} \frac{(q^{-k}; q)_j}{(q; q)_j} \frac{(q^j xt; q)_\infty}{(q^j yt; q)_\infty} q^j. \end{aligned} \tag{2.6}$$

By using (2.6) and (2.4), we obtain

$$
\begin{aligned}
&E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{\frac{P_k(y,x)}{(xt;q)_k}\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\}\\
&=t^{-k}\sum_{j=0}^{k}\frac{(q^{-k};q)_j}{(q;q)_j}q^jE(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{\frac{(q^jxt;q)_\infty}{(q^jyt;q)_\infty}\right\}\\
&=t^{-k}\sum_{j=0}^{k}\frac{(q^{-k};q)_j}{(q;q)_j}q^j\frac{(xtq^j;q)_\infty}{(ytq^j;q)_\infty}\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\ \varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^{\varrho}q^{\binom{\varrho}{2}}\right]^{1+s-r}(ztq^j)^{\varrho}.
\end{aligned}
$$

□

3. The generating function for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$

In this section, we use the operator representation (2.3) to drive an extension for the generating function for the generalised Cigler's polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$. Special values are supplied for the parameters of the extension of the generating function for the polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$ to get the generating function for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$, the generating function and its extension for the polynomials $D_n^{(\alpha-n)}(x,y,b)$ and $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$.

Theorem 3.1 (Extension to the generating function for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$). *For $\alpha\in\mathbb{R}$, then*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^{n+k}q^{\binom{n+k}{2}}\mathbb{D}_{n+k}^{(\alpha-n-k)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)\frac{t^n}{(q;q)_n}\\
&=\frac{(xt;q)_\infty}{(yt;q)_\infty}t^{-k}\sum_{j=0}^{k}\frac{(q^{-k},yt;q)_j}{(q,xt;q)_j}\,q^j\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\ \varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^{\varrho}q^{\binom{\varrho}{2}}\right]^{1+s-r}(ztq^j)^{\varrho},
\end{aligned}\tag{3.1}
$$

where $|yt|<1$.

Proof: By using (2.3)), we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^{n+k}q^{\binom{n+k}{2}}\mathbb{D}_{n+k}^{(\alpha-n-k)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)\frac{t^n}{(q;q)_n}\\
&=\sum_{n=0}^{\infty}(-1)^{n+k}q^{\binom{n+k}{2}}\\
&\quad\times E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{(-1)^{n+k}q^{-\binom{n+k}{2}}P_{n+k}(y,x)\right\}\frac{t^n}{(q;q)_n}\\
&=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{\sum_{n=0}^{\infty}P_{n+k}(y,x)\frac{t^n}{(q;q)_n}\right\}\\
&=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{P_k(y,x)\sum_{n=0}^{\infty}P_n(y,q^kx)\frac{t^n}{(q;q)_n}\right\}\\
&=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{P_k(y,x)\frac{(q^kxt;q)_\infty}{(yt;q)_\infty}\right\}\quad\text{(by using (1.7))}\\
&=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{\frac{P_k(y,x)}{(xt;q)_k}\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\}
\end{aligned}
$$

$$= \frac{(xt;q)_\infty}{(yt;q)_\infty} t^{-k} \sum_{j=0}^{k} \frac{(q^{-k}, yt;q)_j}{(q, xt;q)_j} q^j \sum_{\varrho=0}^{\infty} \begin{bmatrix}\alpha\\ \varrho\end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r;q)_\varrho}{(\rho_1, \cdots, \rho_s;q)_\varrho} \left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r} (ztq^j)^\varrho.$$

(by using (2.5))

□

- Setting $k=0$ in equation (3.1), we obtain the generating function for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s, x, y, z)$.

Corollary 3.1 (Generating function for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s, x, y, z)$). *For $\alpha \in \mathbb{R}$, we have*

$$\sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \mathbb{D}_n^{(\alpha-n)}(\sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s, x, y, z) \frac{t^n}{(q;q)_n} = \frac{(xt;q)_\infty}{(yt;q)_\infty}$$
$$\times \sum_{k=0}^{\infty} \begin{bmatrix}\alpha\\ k\end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r;q)_k}{(\rho_1, \cdots, \rho_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (zt)^k, \quad |yt|<1. \tag{3.2}$$

  - ∗ Setting $r=s=0$, $z=b$ in equation (3.2), we recover the generating function for the polynomials $D_n^{(\alpha-n)}(x,y,b)$ (equation (1.11)).
  - ∗ Setting $\alpha \to \infty$ in equation (3.2), we regain the generating function for the polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$ (equation (1.21)).

- For $r=s=0$, $z=b$ in equation (3.1), we recover the extension of the generating function for $D_n^{(\alpha-n)}(x,y,b)$ (equation (1.12)).

- Letting $\alpha \to \infty$ in equation (3.1), we regain the extension of the generating function for $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$ (equation (1.22)).

4. Rogers formula for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s, x, y, z)$

In this section, we will present an operator approach to the extension of the Rogers formula for the generalized Cigler's polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s, x, y, z)$. The Rogers formula for the polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s, x, y, z)$, the Rogers formula and its extension for the polynomials $D_n^{(\alpha-n)}(x,y,b)$ and $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$ are obtained by incorporating special values for the parameters in the extension of the Rogers formula for the polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s, x, y, z)$.

Theorem 4.1 (Extension of Rogers formula for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s, x, y, z)$). *For $\alpha \in \mathbb{R}$, then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} (-1)^{n+m+k} q^{\binom{n+m+k}{2}} \mathbb{D}_{n+m+k}^{(\alpha-n-m-k)}(\sigma_1, \cdots, \sigma_r, \rho_1, \cdots, \rho_s, x, y, z) \frac{t^n}{(q;q)_n} \frac{\ell^m}{(q;q)_m} \frac{\tau^k}{(q;q)_k}$$
$$= \frac{(x\tau;q)_\infty}{(y\tau, t/\tau, \ell/\tau;q)_\infty} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(y\tau,q)_{i+j}(-1)^{i+j} q^{-\binom{i+j}{2}}}{(x\tau, q^{-(i+j)}\ell/\tau;q)_{i+j}(q^{-i}t/\tau;q)_i} \frac{(t/\tau)^i}{(q;q)_i} \frac{(\ell/\tau)^j}{(q;q)_j}$$
$$\times \sum_{\varrho=0}^{\infty} \begin{bmatrix}\alpha\\ \varrho\end{bmatrix} \frac{(\sigma_1, \cdots, \sigma_r;q)_\varrho}{(\rho_1, \cdots, \rho_s;q)_\varrho} \left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r} (z\tau q^{i+j})^\varrho, \tag{4.1}$$

where $\max\{|y\tau|, |t/\tau|, |\ell/\tau|\} < 1$.

Proof:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+m+k}q^{\binom{n+m+k}{2}}\mathbb{D}_{n+m+k}^{(\alpha-n-m-k)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)\frac{t^n}{(q;q)_{n+m}}\frac{\ell^m}{(q;q)_m}\frac{\tau^k}{(q;q)_k}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+m+k}q^{\binom{n+m+k}{2}}\\
&\quad\times E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{(-1)^{n+m+k}q^{-\binom{n+m+k}{2}}P_{n+m+k}(y,x)\right\}\\
&\quad\times\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}\frac{\tau^k}{(q;q)_k}\\
&=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}P_{n+m}(y,x)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}\right.\\
&\qquad\qquad\left.\times\sum_{k=0}^{\infty}P_k(y,q^{n+m}x)\frac{\tau^k}{(q;q)_k}\right\}\\
&=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}P_{n+m}(y,x)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}\frac{(q^{n+m}x\tau,q)_\infty}{(y\tau;q)_\infty}\right\}\\
&\qquad\qquad\text{(by using (1.7))}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{\frac{P_{n+m}(y,x)}{(x\tau;q)_{n+m}}\frac{(x\tau;q)_\infty}{(y\tau;q)_\infty}\right\}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}\tau^{-(n+m)}\frac{(x\tau;q)_\infty}{(y\tau;q)_\infty}\sum_{k=0}^{n+m}\frac{(q^{-(n+m)},y\tau,q)_kq^k}{(q,x\tau;q)_k}\\
&\quad\times\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\ \varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^{\varrho}q^{\binom{\varrho}{2}}\right]^{1+s-r}(z\tau q^k)^{\varrho}\quad\text{(by using (2.5))}\\
&=\frac{(x\tau;q)_\infty}{(y\tau;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(t/\tau)^n}{(q;q)_n}\frac{(\ell/\tau)^m}{(q;q)_m}\sum_{k=0}^{n+m}\begin{bmatrix}n+m\\ k\end{bmatrix}(-1)^kq^{\binom{k}{2}-(n+m)k+k}\frac{(y\tau,q)_k}{(x\tau;q)_k}\\
&\quad\times\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\ \varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^{\varrho}q^{\binom{\varrho}{2}}\right]^{1+s-r}(z\tau q^k)^{\varrho}\\
&=\frac{(x\tau;q)_\infty}{(y\tau;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(t/\tau)^n(\ell/\tau)^m}{(q;q)_n(q;q)_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\begin{bmatrix}n\\ i\end{bmatrix}\begin{bmatrix}m\\ j\end{bmatrix}q^{(n-i)j}(-1)^{(i+j)}q^{\binom{i+j}{2}-(n+m)(i+j)+i+j}\\
&\quad\times\frac{(y\tau,q)_{i+j}}{(x\tau;q)_{i+j}}\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\ \varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^{\varrho}q^{\binom{\varrho}{2}}\right]^{1+s-r}(z\tau q^{i+j})^{\varrho}\quad\text{(by using (1.6))}\\
&=\frac{(x\tau;q)_\infty}{(y\tau;q)_\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=i}^{\infty}\sum_{m=j}^{\infty}\frac{(t/\tau)^n(\ell/\tau)^m}{(q;q)_{n-i}(q;q)_{m-j}(q;q)_i(q;q)_j}q^{(n-i)j}(-1)^{(i+j)}q^{\binom{i+j}{2}-(n+m)(i+j)+i+j}\\
&\quad\times\frac{(y\tau,q)_{i+j}}{(x\tau;q)_{i+j}}\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\ \varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^{\varrho}q^{\binom{\varrho}{2}}\right]^{1+s-r}(z\tau q^{i+j})^{\varrho}\\
&=\frac{(x\tau;q)_\infty}{(y\tau;q)_\infty}\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{(t/\tau)^{n+i}(-1)^iq^{-\binom{i}{2}-ni}}{(q;q)_i(q;q)_n}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\ell/\tau)^{m+j}(-1)^jq^{-\binom{j}{2}-m(i+j)-ij}}{(q;q)_m(q;q)_j}\\
&\quad\times\frac{(y\tau,q)_{i+j}}{(x\tau;q)_{i+j}}\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\ \varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^{\varrho}q^{\binom{\varrho}{2}}\right]^{1+s-r}(z\tau q^{i+j})^{\varrho}
\end{aligned}$$

$$= \frac{(x\tau;q)_\infty}{(y\tau;q)_\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(t/\tau)^i(\ell/\tau)^j(-1)^{i+j}q^{-\binom{i}{2}-\binom{j}{2}-ij}}{(q;q)_i(q;q)_j}\frac{1}{(q^{-i}t/\tau;q)_\infty(q^{-(i+j)}\ell/\tau;q)_\infty}$$
$$\times \frac{(y\tau,q)_{i+j}}{(x\tau;q)_{i+j}}\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\ \varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r}(z\tau q^{i+j})^\varrho$$
$$= \frac{(x\tau;q)_\infty}{(y\tau,t/\tau,\ell/\tau;q)_\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(t/\tau)^i(\ell/\tau)^j(-1)^{i+j}q^{-\binom{i}{2}-\binom{j}{2}-ij}}{(q;q)_i(q;q)_j}\frac{1}{(q^{-i}t/\tau;q)_i(q^{-(i+j)}\ell/\tau;q)_{i+j}}$$
$$\times \frac{(y\tau,q)_{i+j}}{(x\tau;q)_{i+j}}\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\ \varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r}(z\tau q^{i+j})^\varrho.$$

□

- Letting $\ell=0$ and $\tau=\ell$ in equation (4.1), we get Rogers formula for the polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$.

  Corollary 4.1 (Rogers formula for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$). *For $\alpha\in\mathbb{R}$, we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}q^{\binom{n+m}{2}}\mathbb{D}_{n+m}^{(\alpha-n-m)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m} = \frac{(x\ell;q)_\infty}{(t/\ell,y\ell;q)_\infty}$$
$$\times\sum_{k=0}^{\infty}\frac{(y\ell;q)_k}{(q,x\ell,q\ell/t;q)_k}q^k\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\ \varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r}(z\ell q^k)^\varrho, \tag{4.2}$$

  where $\max\{|t/\ell|,|y\ell|\}<1$.

  * Rogers formula for the polynomials $D_n^{(\alpha-n)}(x,y,b)$ (equation (1.13)) can be preserved by setting $r=s=0$, $z=b$ in equation (4.2).
  * Setting $\alpha\to\infty$ in equation (4.2), we regain the Rogers formula for the polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$ (equation (1.23)).

- Setting $r=s=0$, $z=b$ in equation (4.1), we get an extension to the Rogers formula for the polynomials $D_n^{(\alpha-n)}(x,y,b)$.

  Corollary 4.2 (Extension of Rogers Formula for $D_n^{(\alpha-n)}(x,y,b)$). *For $\alpha\in\mathbb{R}$, we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+m+k}q^{\binom{n+m+k}{2}}D_{n+m+k}^{(\alpha-n-m-k)}(x,y,b)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}\frac{\tau^k}{(q;q)_k}$$
$$= \frac{(x\tau,b\tau;q)_\infty}{(t/\tau,\ell/\tau,b\tau q^\alpha,y\tau;q)_\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(y\tau,b\tau q^\alpha;q)_{i+j}(-1)^{i+j}q^{-\binom{i+j}{2}}}{(b\tau,x\tau,q^{-(i+j)}\ell/\tau;q)_{i+j}(q^{-i}t/\tau;q)_i}\frac{(t/\tau)^i}{(q;q)_i}\frac{(\ell/\tau)^j}{(q;q)_j},$$

  where $\max\{|t/\tau|,|\ell/\tau|,|y\tau|,|b\tau q^\alpha|\}<1$.

- Setting $\alpha\to\infty$ in equation (4.1), we get an extension to the Rogers formula for the polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$.

  Corollary 4.3 (Extension of Rogers Formula for $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$. *For $\alpha\in\Re$, we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+m+k}q^{\binom{n+m+k}{2}}\Psi_{n+m+k}^{(\boldsymbol{a},\boldsymbol{b})}(x,y,z|q)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}\frac{\tau^k}{(q;q)_k}$$

$$= \frac{(x\tau;q)_\infty}{(y\tau,t/\tau,\ell/\tau;q)_\infty}\sum_{i=0}^\infty\sum_{j=0}^\infty \frac{(-1)^{i+j}q^{-\binom{i+j}{2}}(y\tau,q)_{i+j}}{(x\tau,q^{-(i+j)}\ell/\tau;q)_{i+j}(q^{-i}t/\tau;q)_i}\frac{(t/\tau)^i}{(q;q)_i}\frac{(\ell/\tau)^j}{(q;q)_j}$$
$$\times \quad {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,z\tau q^{i+j}\right),\quad \max\{|y\tau|,|t/\tau|,|\ell/\tau|\}<1.$$

5. Mehler's formula for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$

In this section, we construct an extension of the Mehler's formula for the generalized Cigler's polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$ using the operator representation (2.3). Mehler's formula for the polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$, Miller's formula and its extension for polynomials $D_n^{(\alpha-n)}(x,y,b)$ and $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$ are derived by providing special values for the variables in the extension of Mehler's formula for the polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$.

Theorem 5.1 (Extension of Mehler's formula for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$). *For $\alpha\in\mathbb{R}$ and $\beta\in\mathbb{R}$, we have*

$$\sum_{n=0}^\infty q^{n^2-n}\mathbb{D}_{n+m}^{(\alpha-n-m)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)\mathbb{D}_n^{(\beta-n)}(c_1,\cdots,c_r,d_1,\cdots,d_s,u,v,w)\frac{t^n}{(q;q)_n}$$
$$= \frac{(q^{-m}xvt;q)_\infty}{(q^{-m}yvt;q)_\infty}\sum_{k=0}^\infty \begin{bmatrix}\beta\\ k\end{bmatrix}\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}(w/v)^k(-1)^mq^{\binom{m+1}{2}}(vt)^{-m}$$
$$\times\sum_{n=0}^\infty\frac{(-1)^nq^{\binom{n}{2}}(u/v)^n}{(q;q)_n}\sum_{j=0}^{n+m+k}\frac{(q^{-(n+m+k)},q^{-m}yvt;q)_jq^j}{(q,q^{-m}xvt;q)_j}$$
$$\times\sum_{\varrho=0}^\infty\begin{bmatrix}\alpha\\ \varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r}(zvtq^{j-m})^\varrho,\quad |yvt|<1. \tag{5.1}$$

Proof:

$$\sum_{n=0}^\infty q^{(n^2-n)}\mathbb{D}_{n+m}^{(\alpha-n-m)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)\mathbb{D}_n^{(\beta-n)}(c_1,\cdots,c_r,d_1,\cdots,d_s,u,v,w)\frac{t^n}{(q;q)_n}$$
$$=\sum_{n=0}^\infty q^{(n^2-n)}E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{(-1)^{n+m}q^{-\binom{n+m}{2}}P_{n+m}(y,x)\right\}$$
$$\times\mathbb{D}_n^{(\beta-n)}(c_1,\cdots,c_r,d_1,\cdots,d_s,u,v,w)\frac{t^n}{(q;q)_n}$$
$$=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\sum_{n=0}^\infty(-1)^{n+m}q^{\binom{n}{2}-\binom{m}{2}-nm}P_{n+m}(y,x)\right.$$
$$\left.\times\mathbb{D}_n^{(\beta-n)}(c_1,\cdots,c_r,d_1,\cdots,d_s,u,v,w)\frac{t^n}{(q;q)_n}\right\}$$
$$=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\sum_{n=0}^\infty(-1)^mq^{-\binom{m}{2}-nm}P_{n+m}(y,x)\frac{t^n}{(q;q)_n}\right.$$
$$\left.\times\sum_{k=0}^n\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}\beta\\ k\end{bmatrix}\frac{(c_1,\ldots,c_r;q)_k}{(d_1,\ldots,d_s;q)_k}P_{n-k}(v,u)(q;q)_kw^k\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\right\}\quad\text{(by using (2.2))}$$
$$=E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\sum_{k=0}^\infty\sum_{n=k}^\infty\begin{bmatrix}\beta\\ k\end{bmatrix}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\right.$$

$$\times (-1)^m q^{-\binom{m}{2}-nm} P_{n+m}(y,x)P_{n-k}(v,u)\frac{w^k t^n}{(q;q)_{n-k}}\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}\Bigg\}$$

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Bigg\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}$$

$$\times (-1)^m q^{-\binom{m}{2}-(n+k)m} P_{n+m+k}(y,x)P_n(v,u)\frac{w^k t^{n+k}}{(q;q)_n}\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}\Bigg\}$$

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Bigg\{\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}P_{m+k}(y,x)(wt)^k\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}$$

$$\times (-1)^m q^{-\binom{m}{2}-mk}\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}\sum_{n=0}^{\infty}P_n(v,u)P_n(y,q^{m+k}x)\frac{(q^{-m}t)^n}{(q;q)_n}\Bigg\}$$

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Bigg\{\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}P_{m+k}(y,x)(wt)^k\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}$$

$$\times (-1)^m q^{-\binom{m}{2}-mk}\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}\frac{(q^{m+k-m}xvt;q)_\infty}{(q^{-m}yvt;q)_\infty}\,{}_1\phi_1\left(\begin{matrix}q^{m+k}x/y\\q^k xvt\end{matrix};q,q^{-m}uyt\right)\Bigg\}$$

(by using (1.8))

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Bigg\{\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}P_{m+k}(y,x)(wt)^k\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}$$

$$\times (-1)^m q^{-\binom{m}{2}-mk}\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}\frac{(q^k xvt;q)_\infty}{(q^{-m}yvt;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{P_n(y,q^{m+k}x)(q^{-m}ut)^n}{(q,q^k xvt;q)_n}\Bigg\}$$

$$= \sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}(wt)^k\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}(-1)^m q^{-\binom{m}{2}-mk}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}-mn}\frac{(ut)^n}{(q;q)_n}$$

$$\times E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Bigg\{\frac{P_{n+m+k}(y,x)}{(q^{-m}xvt;q)_{n+m+k}}\frac{(q^{-m}xvt;q)_\infty}{(q^{-m}yvt;q)_\infty}\Bigg\}$$

$$= \sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}(wt)^k\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}(-1)^m q^{-\binom{m}{2}-mk}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}-nm}$$

$$\times \frac{(ut)^n}{(q;q)_n}\frac{(q^{-m}xvt;q)_\infty}{(q^{-m}yvt;q)_\infty}(q^{-m}vt)^{-(n+m+k)}\sum_{j=0}^{n+m+k}\frac{(q^{-(n+m+k)},q^{-m}yvt;q)_j q^j}{(q,q^{-m}xvt;q)_j}$$

$$\times \sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\\varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\,(zvtq^{j-m})^\varrho\left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r}$$

$$= \frac{(q^{-m}xvt;q)_\infty}{(q^{-m}yvt;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}(w/v)^k\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}(-1)^m q^{\binom{m+1}{2}}(vt)^{-m}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}$$

$$\times \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(u/v)^n}{(q;q)_n}\sum_{j=0}^{n+m+k}\frac{(q^{-(n+m+k)},q^{-m}yvt;q)_j q^j}{(q,q^{-m}xvt;q)_j}$$

$$\times \sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\\varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\,(zvtq^{j-m})^\varrho\left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r}.$$

$\square$

- Setting $m=0$ in equation (5.1), we get recover Mehler's formula for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$.

Corollary 5.1 (Mehler's formula for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$). *For $\alpha\in\mathbb{R}$ and $\beta\in\mathbb{R}$, then*

$$\sum_{n=0}^{\infty} q^{(n^2-n)}\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)\mathbb{D}_n^{(\beta-n)}(c_1,\cdots,c_r,d_1,\cdots,d_s,u,v,w)\frac{t^n}{(q;q)_n}$$
$$=\frac{(xvt;q)_\infty}{(yvt;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}\frac{(c_1,...,c_r;q)_k}{(d_1,...,d_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(w/v)^k\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{(u/v)^n}{(q;q)_n}$$
$$\times\sum_{j=0}^{n+k}\frac{(q^{-(n+k)},yvt;q)_j}{(q,xvt;q)_j}\;q^j\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\\varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r}(zvtq^j)^\varrho, \tag{5.2}$$

where $|yvt|<1$.

* Setting $r=s=0$, $z=b$ in equation (5.2), we get Mehler's formula for the polynomials $D_n^{(\alpha-n)}(x,y,b)$.

  Corollary 5.2 (Mehler's formula for $D_n^{(\alpha-n)}(x,y,b)$). *For $\alpha\in\mathbb{R}$ and $\beta\in\mathbb{R}$, we have*

$$\sum_{n=0}^{\infty}q^{(n^2-n)}D_n^{(\alpha-n)}(x,y,b)D_n^{(\beta-n)}(u,v,w)\frac{t^n}{(q;q)_n}$$
$$=\frac{(xvt,bvt;q)_\infty}{(yvt,bvtq^\alpha;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}(w/v)^k\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(u/v)^n}{(q;q)_n}$$
$$\times\ {}_3\phi_2\left(\begin{matrix}q^{-(n+k)},yvt,bvtq^\alpha\\ xvt,bvt\end{matrix};q,q\right),\quad \max\{|yvt|,|bvtq^\alpha|\}<1.$$

* Setting $\alpha\to\infty$, $\beta\to\infty$ in equation (5.2), we get Mehler's formula for the polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$.

  Corollary 5.3 (Mehler's formula for $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$). *We have*

$$\sum_{n=0}^{\infty}q^{(n^2-n)}\Psi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,z|q)\Psi_n^{(\boldsymbol{c},\boldsymbol{d})}(u,v,w|q)\frac{t^n}{(q;q)_n}$$
$$=\frac{(xvt;q)_\infty}{(yvt;q)_\infty}\sum_{k=0}^{\infty}\frac{(c_1,...,c_r;q)_k}{(q,d_1,...,d_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(w/v)^k\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{(u/v)^n}{(q;q)_n}$$
$$\times\sum_{j=0}^{n+k}\frac{(q^{-(n+k)},yvt;q)_j}{(q,xvt;q)_j}\;q^j\ {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,zvtq^j\right),\quad |yvt|<1.$$

- Setting $r=s=0$, $z=b$ in equation (5.1), we get an extension of Mehler's formula for $D_n^{(\alpha-n)}(x,y,b)$.

Corollary 5.4 (Extension of Mehler's formula for $D_n^{(\alpha-n)}(x,y,b)$). *For $\alpha\in\mathbb{R}$ and $\beta\in\mathbb{R}$, we have*

$$\sum_{n=0}^{\infty}q^{(n^2-n)}D_{n+m}^{(\alpha-n-m)}(x,y,b)D_n^{(\beta-n)}(u,v,w)\frac{t^n}{(q;q)_n}=\frac{(q^{-m}xvt,bvt;q)_\infty}{(q^{-m}yvt,bvtq^\alpha;q)_\infty}$$
$$\times\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}(w/v)^k\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(u/v)^n}{(q;q)_n}(-1)^m q^{\binom{m+1}{2}}(vt)^{-m}$$
$$\times\sum_{j=0}^{n+m+k}\frac{(q^{-(n+m+k)},q^{-m}yvt;q)_j q^j}{(q,q^{-m}xvt;q)_j}\frac{(bvtq^\alpha;q)_{j-m}}{(bvt;q)_{j-m}},\quad \max\{|yvt|,|bvtq^\alpha|\}<1.$$

- Setting $\alpha \to \infty$, $\beta \to \infty$ in equation (5.1), we get Mehler's formula for the polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$.

Corollary 5.5 (Extension of Mehler's formula for $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$). *We have*

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{(n^2-n)} \Psi_{n+m}^{(\boldsymbol{a},\boldsymbol{b})}(x,y,z|q)\Psi_n^{(\boldsymbol{c},\boldsymbol{d})}(u,v,w|q)\frac{t^n}{(q;q)_n} \\
&\quad = \frac{(q^{-m}xvt;q)_\infty}{(q^{-m}yvt;q)_\infty}\sum_{k=0}^{\infty}\frac{(c_1,...,c_r;q)_k}{(q,d_1,...,d_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(w/v)^k \sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{(u/v)^n}{(q;q)_n} \\
&\quad\times (-1)^m q^{\binom{m+1}{2}}(vt)^{-m}\sum_{j=0}^{n+m+k}\frac{(q^{-(n+m+k)},q^{-m}yvt;q)_j}{(q,q^{-m}xvt;q)_j}\, q^j \\
&\quad\times\ {}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s\end{matrix};q,zvtq^{j-m}\right), \quad |yvt|<1.
\end{aligned}$$

6. The Srivastava-Agarwal type bilinear generating functions for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$

In this section, we create the Srivastava-Agarwal type bilinear generating function and Srivastava-Agarwal type generating function for the generalized Cigler's polynomials $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$. In addition, we use parameter values to reconstruct the Srivastava-Agarwal type generating function for the polynomials $D_n^{(\alpha-n)}(x,y,b)$ and $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$.

Theorem 6.1 (Srivastava-Agarwal type bilinear generating function for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$). *For $\alpha\in\mathbb{R}$, we have*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\psi_n^{(c)}(u|q)\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)\frac{(-1)^n q^{\binom{n+1}{2}}t^n}{(q;q)_n} \\
&= \frac{(q/u,xutq;q)_\infty}{(cq,yutq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(1/cu,xutq^{1-n};q)_n}{(q,q/u,yutq^{1-n};q)_n}(cq)^n \\
&\times\sum_{k=0}^{\infty}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r)_k}{(\rho_1,\cdots,\rho_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(zutq^{1-n})^k, \quad \max\{|cq|,|qyut|\}<1.
\end{aligned} \tag{6.1}$$

Proof:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\psi_n^{(c)}(u|q)\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)\frac{(-1)^n q^{\binom{n+1}{2}}t^n}{(q;q)_n} \\
&= \sum_{n=0}^{\infty}\psi_n^{(c)}(u|q)E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{(-1)^n q^{-\binom{n}{2}}P_n(y,x)\right\}\frac{(-1)^n q^{\binom{n+1}{2}}t^n}{(q;q)_n} \\
&= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\sum_{n=0}^{\infty}\psi_n^{(c)}(u|q)P_n(y,x)\frac{(qt)^n}{(q;q)_n}\right\} \\
&= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\sum_{n=0}^{\infty}\psi_n^{(c)}(u|q)(x/y;q)_n\frac{(qyt)^n}{(q;q)_n}\right\} \\
&= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^\alpha z\theta_{xy})\left\{\frac{(q/u,xutq;q)_\infty}{(cq,yutq;q)_\infty}\,{}_2\phi_2\left(\begin{matrix}1/cu,1/xut\\ q/u,1/yut\end{matrix};q,cqx/y\right)\right\}
\end{aligned}$$

(by using (1.16))

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\Big\{\frac{(q/u,xutq;q)_\infty}{(cq,yutq;q)_\infty} \times\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(1/cu,xutq^{1-n};q)_n(cq)^n}{(q,q/u,yutq^{1-n};q)_n}\Big\}$$

$$= \frac{(q/u;q)_\infty}{(cq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(1/cu;q)_n(cq)^n}{(q,q/u;q)_n} \times E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{\frac{(xutq^{1-n};q)_\infty}{(yutq^{1-n};q)_\infty}\right\}$$

$$= \frac{(q/u;q)_\infty}{(cq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(1/cu;q)_n(cq)^n}{(q,q/u;q)_n}\frac{(xutq^{1-n};q)_\infty}{(yutq^{1-n};q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r)_k}{(\rho_1,\cdots,\rho_s;q)_k} \times\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(zutq^{1-n})^k \quad \text{(by using (2.4))}$$

$$= \frac{(q/u,xutq;q)_\infty}{(cq,yutq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(1/cu,xutq^{1-n};q)_n(cq)^n}{(q,q/u,yutq^{1-n};q)_n}\sum_{k=0}^{\infty}\begin{bmatrix}\alpha\\k\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r)_k}{(\rho_1,\cdots,\rho_s;q)_k} \times\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(zutq^{1-n})^k.$$

□

- Setting $r=s=0$, $z=b$ in equation (6.1), we get Srivastava-Agarwal type bilinear generating functions for $D_n^{(\alpha-n)}(x,y,b)$ (equation (1.15)).
- Setting $\alpha\to\infty$ in equation (6.1), we regain Srivastava-Agarwal type bilinear generating functions for $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$ (equation (1.24)).

Theorem 6.2 *For $\alpha\in\mathbb{R}$, we have*

$$\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}P_n(v,u)\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)\frac{t^n}{(q;q)_n} = \frac{(xvt;q)_\infty}{(yvt;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(u/v)^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(q^{-n},yvt;q)_k q^k}{(q,xvt;q)_k} \times\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\\varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^\varrho q^{\binom{\varrho}{2}}\right]^{1+s-r}(ztq^k)^\varrho,\quad |yvt|<1. \tag{6.2}$$

Proof:

$$\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}P_n(v,u)\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)\frac{t^n}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}P_n(v,u)E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{(-1)^n q^{-\binom{n}{2}}P_n(y,x)\right\}\frac{t^n}{(q;q)_n}$$

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{\sum_{n=0}^{\infty}P_n(v,u)P_n(y,x)\frac{t^n}{(q;q)_n}\right\}$$

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{\frac{(vxt;q)_\infty}{(vyt;q)_\infty}\,{}_1\phi_1\left(\begin{matrix}x/y\\vxt\end{matrix};q,uyt\right)\right\}$$

(by using (1.8))

$$= E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{\frac{(vxt;q)_\infty}{(vyt;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(ut)^n}{(q;q)_n}\frac{P_n(y,x)}{(vxt;q)_n}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(ut)^n}{(q;q)_n}E(q^{-\alpha},\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s;q,q^{\alpha}z\theta_{xy})\left\{\frac{P_n(y,x)}{(vxt;q)_n}\frac{(vxt;q)_\infty}{(vyt;q)_\infty}\right\}$$
$$=\frac{(xvt;q)_\infty}{(yvt;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(u/v)^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(q^{-n},yvt;q)_k}{(q,xvt;q)_k}q^k\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\ \varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}$$
$$\times\left[(-1)^{\varrho}q^{\binom{\varrho}{2}}\right]^{1+s-r}(ztq^k)^{\varrho}.$$

□

- Setting $u/v=\lambda$ and $vt\to t$ in equation (6.2), we get the following corollary:

Corollary 6.1 (Srivastava-Agarwal type generating function for $\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)$). *For $\alpha\in\mathbb{R}$, we have*

$$\sum_{n=0}^{\infty}(-1)^nq^{\binom{n}{2}}\mathbb{D}_n^{(\alpha-n)}(\sigma_1,\cdots,\sigma_r,\rho_1,\cdots,\rho_s,x,y,z)(\lambda;q)_n\frac{t^n}{(q;q)_n}$$
$$=\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}\lambda^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(q^{-n},yt;q)_k}{(q,xt;q)_k}q^k$$
$$\times\sum_{\varrho=0}^{\infty}\begin{bmatrix}\alpha\\ \varrho\end{bmatrix}\frac{(\sigma_1,\cdots,\sigma_r;q)_\varrho}{(\rho_1,\cdots,\rho_s;q)_\varrho}\left[(-1)^{\varrho}q^{\binom{\varrho}{2}}\right]^{1+s-r}(ztq^k)^{\varrho},\quad |yt|<1. \tag{6.3}$$

- Substituting $r=s=0$, $z=b$ into equation (6.3), we Srivastava-Agarwal type generating function for $D_n^{(\alpha-n)}(x,y,b)$ (equation (1.14)).
- Setting $\alpha\to\infty$ in equation (6.3), we regain Srivastava-Agarwal type generating functions for $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$.

Corollary 6.2 (Srivastava-Agarwal type generating function for $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$). *We have*

$$\sum_{n=0}^{\infty}(-1)^nq^{\binom{n}{2}}\Psi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,z|q)(\lambda;q)_n\frac{t^n}{(q;q)_n}$$
$$=\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}\lambda^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(q^{-n},yt;q)_k}{(q,xt;q)_k}q^k\ {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,ztq^k\right),\quad |yt|<1.$$

References

1. Abdlhusein, M. A., *Two operator representations for the trivariate q-polynomials and Hahn polynomials*, Ramanujan J. 40, 491-509, (2016).

2. Abdul-Ghani, S. A. and Saad, H. L., *Applications of q-difference equation and homogeneous q-shift operator ${}_r\Phi_s(D_{xy})$ in q-polynomials*, Partial Differ. Equ. Appl. Math. 8, (2023) https://doi.org/10.1016/j.padiff.2023.100536.

3. Abdul-Ghani, S. A. and Saad, H. L., *The generalized homogeneous q-shift operator ${}_r\Phi_s(D_{xy})$ for q-identities and q-integrals*, Iraqi J. Science 64(11), 5815-5829, (2023).

4. Abdul-Ghani, S. A. and Saad, H. L., *Applications of the generalized homogeneous q-shift operator in q-polynomials*, TWMS J. App. and Eng. Math. 14(2), 473-494, (2024).

5. Arjika, S., *Certain generating functions for Cigler's polynomials*, Montes Taurus J. Pure Appl. Math. 3(3), 284-296, (2021).

6. Cao, J. and Niu, D.-W., *A note on q-difference equations for Cigler's polynomials*, J. Differ. Equ. Appl. 22, 1880—1892, (2016).

7. Cao, J., Zhou H. and Arjika. S., *Generalized homogeneous q-difference equations for q-polynomials and their applications to generating functions and fractional q-integrals*, Adv. Difference Equ. 2021, 1-18, (2021).

8. Chen, W. Y. C., Fu, A. M. and Zhang, B. Y., *The homogeneous q-difference operator*, Adv. Appl. Math. 31, 659-668, (2003).

9. Chen, W. Y. C., Paule, P. and Saad, H. L., *Converging to Gosper's algorithm*, Adv. Appl. Math. 41, 351–364, (2008).

10. Chen, W. Y. C. and Saad, H. L., *On the Gosper–Petkovšek representation of rational functions*, J. Symbolic Computation 40, 955–963, (2005).

11. Gasper, G. and Rahman, M., *Basic Hypergeometric Series*, 2^{nd} ed., Cambridge University Press, Cambridge, MA, (2004).

12. Jaber, R. H. and Saad, H. L., *Applications of the finite Operator* ${}_3\mathscr{E}_2\left(\begin{array}{c} q^{-N}, a, b \\ c, d \end{array}; q, -f\theta\right)$ *for the polynomials* $B_n(a, b, c, d, f, x, y \mid q)$, *Iraqi J. Science* **64**(2), 783–797, (2023).

13. Reshem, F. A. and Saad, H. L., *Generalized q-difference equation of the generalized q-operator* ${}_r\Phi_s(\theta)$ *and its application in q-integrals*, Partial Differ. Equ. Appl. Math. 7 (2023) https://doi.org/10.1016/j.padiff.2023.100517.

14. Saad, H. L. and Abdlhusein, M. A., *New application of the Cauchy operator on the homogeneous Rogers-Szegö polynomials*, The Ramanujan J. 56(1), 347-367, (2021).

15. Saad, H. L. and Hassan, H. J., *Applications of the operator* ${}_r\Phi_s$ *in q-polynomials*, TWMS J. App. and Eng. Math. 13(2) 696-709, (2023).

16. Saad, H. L. and Jaber, R. H., *Application of the operator* $\phi\left(\begin{array}{c} a, b, c \\ d, e \end{array}; q, fD_q\right)$ *for the polynomials* $Y_n(a, b, c; d, e; x, y|q)$, TWMS J. App. and Eng. Math. 12(2), 691-702, (2022).

17. Saad, H. L. and Khalaf, S. M., *The generalized q-operator* ${}_r\Phi_s$ *and its applications in q-identities*, TWMS J. App. and Eng. Math. 13(3), 1137-1149, (2023).

18. Saad, H. L. and Reshem, F. A., *The operator* $\mathbb{S}(a, b; \theta_x)$ *for the polynomials* $Z_n(x, y, a, b|q)$, Iraqi J. Science 63(10), 4397-4409, (2022).

19. Saad, H. L. and Sukhi, A. A., *Another homogeneous q-difference operator*, Appl. Math. Comput. 215, 4332—4339, (2010).

20. Saad, H. L. and Sukhi, A. A., *The q-exponential operator*, Appl. Math. Sci. 7, 6369-6380, (2013).

21. Srivastava, H. M. and Abdlhusein, M. A., *New forms of the Cauchy operator and some of their applications*, Russ. J. Math. Phys. 23(1), 124-134, (2016).

22. Srivastava, H. M., Arjika, S. and Kelil, A. S., *Some homogeneous q-difference operators and the associated generalized Hahn polynomials*, Appl. Set-Valued Anal. Optim. 1, 187–201, (2019).

23. Srivastava, H. M. and Arjika, S., *Generating functions for some families of the generalized Al-Salam-Carlitz q-polynomials*, Adv. Difference Equ. (2020) 498.https://doi.org/10.1186/s13662-020-02963-9.

24. Srivastava, H. M. and Arjika, S., *A General family of q-hypergeometric polynomials and associated generating functions*, Mathematics 9(11), (2021) 1161. https://doi.org/10.3390/math9111161.

25. Wang, X. F. and Cao, J., *q-difference equations for the generalized Cigler's polynomials*, J. Difference Equ. Appl. 24, 479-502, (2018).

Husam L. Saad,
Department of Mathematics,
College of Science,
University of Basrah,
Iraq.
E-mail address: `hus6274@hotmail.com`

and

Samaher A. Abdul-Ghani,

Department of Mathematics,
College of Science,
University of Basrah,
Iraq.
E-mail address: samaheradnanmath@gmail.com