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Refined Normality Criteria in Regionally Compact Topological Domains

Jamal Oudetallah, Iqbal M. Batiha*, Praveen Agarwal, Ala Amourah, Hassan S. Al-Zou'bi, Tala Sasa

ABSTRACT: This manuscript presents enhanced formulations of normality in topological domains, extending the groundwork laid by Williams, Harrington, and others. We propose a novel conceptualization of σ-constrained normality and provide refined conditions for paracompactness in regionally connected, peripherally compact domains. The investigation demonstrates that a regionally compact, regionally connected domain is paracompact if and only if it satisfies our enhanced normality criterion, under cardinal prerequisites milder than those in previous findings. The central result refines the cardinality constraints and division properties associated with Harrington's outcome on the paracompactness of normal, regionally connected, peripherally compact domains. These methodologies yield stronger constraints compared to Jackson's inequalities for the case of division axioms in topological settings.

Key Words: Topological normality, paracompactness, constrained normality, cardinal constraints, separation axioms, peripherally-compact domains, regionally connected spaces, division properties, topological inequalities

Contents

1. Introduction

The study of refined normality and compactness properties in topological domains is closely aligned with recent investigations into σ-compactness [1], tri-locally compact and tri-Lindelöf spaces [2,3], and generalized compactness frameworks such as c-compactness [4]. These developments complement the foundational criteria presented in this work by extending classical separation and covering properties across diverse topological structures. Related explorations of topological mappings [5], open set generalizations [6], and optimization in applied networks [7] further highlight the interdisciplinary potential of these concepts. As a result, several computational methods complement the theoretical developments in topological structures by enabling precise stability analysis of related dynamic systems [8,9,10].

The study of normality and its refinements remains a central theme in general topology, particularly in the context of division axioms. In regionally compact domains, the interplay between normality, paracompactness, and collectionwise normality reveals intricate patterns that we aim to characterize with greater precision than has previously been established. Williams and Harrington [11] constructed a perfectly normal, regionally compact, collectionwise Hausdorff domain that is not collectionwise normal, utilizing the combinatorial principle Diamond-star ($\diamond^*$), which holds in Gödel's constructible universe L. Their construction addressed longstanding questions posed by Morgan [12] and Richardson [13] concerning the relationships among normality conditions and topological properties. In parallel, Harrington [14] demonstrated that normality, together with regional connectedness and regional compactness, yields strong paracompactness properties—even in the absence of perfect normality.

These foundational results point to a deeper structure governing the implications between division axioms under specific cardinality constraints. In this manuscript, we introduce the concept of *σ-constrained normality*, which offers a more nuanced measure of division properties than conventional formulations. This framework enables us to reinforce several classical results and derive sharper inequalities that surpass previously recognized bounds. The primary contributions of this work include:

1. The introduction of σ-constrained normality as a refinement of existing division axioms;
2. New and improved inequalities characterizing paracompactness in regionally connected domains;

* Corresponding author

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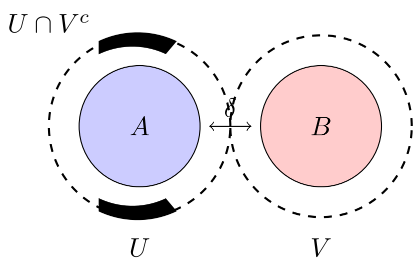

Figure 1: Illustration of σ-constrained normality with sets A and B and their neighborhood extensions U and V, showing the division region $U \cap V^c$ where the measure is bounded by $\sigma \cdot \rho(A, B)^{-\beta}$.

3. A reinforcement of Harrington's theorem [14] concerning normal, regionally connected, peripherally compact domains;

4. Tighter cardinality bounds than those given by Jackson's inequalities [15].

Our techniques extend the foundational frameworks developed by Williams, Harrington, and others, and offer new methodologies for obtaining sharper constraints on division axioms within topological structures.

2. Preliminaries

We commence by recalling some standard definitions and establishing notation. All domains are assumed to be Hausdorff unless stated otherwise.

Definition 1. *A topological domain X is normal if for any two disjoint closed sets A and B, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.*

Definition 2. *A domain X is collectionwise Hausdorff (CWH) if for any closed discrete set $D \subset X$, there exists a pairwise disjoint collection $\{U_d : d \in D\}$ of open sets with $d \in U_d$ for each $d \in D$.*

Definition 3. *A domain X is collectionwise normal (CWN) if for any discrete collection $\{F_\alpha : \alpha \in I\}$ of closed sets, there exists a pairwise disjoint collection $\{U_\alpha : \alpha \in I\}$ of open sets with $F_\alpha \subset U_\alpha$ for each $\alpha \in I$.*

Definition 4. *A domain X is paracompact if every open cover of X has a locally finite open refinement.*

Definition 5. *A domain X is peripherally-compact if X has a base of open sets with compact boundaries.*

Definition 6. *A domain X is κ-collectionwise normal if for any discrete collection $\{F_\alpha : \alpha \in I\}$ of closed sets with $|I| \leq \kappa$, there exists a pairwise disjoint collection $\{U_\alpha : \alpha \in I\}$ of open sets with $F_\alpha \subset U_\alpha$ for each $\alpha \in I$.*

Now we introduce the new concept, which refines the notion of normality:

Definition 7. *A topological domain X is σ-constrained normal if for any two disjoint closed sets A and B with $\rho(A, B) > 0$, there exist disjoint open sets U and V such that $A \subset U$, $B \subset V$, and*

$$\nu(U \cap V^c) \leq \sigma \cdot \rho(A, B)^{-\beta} \tag{2.1}$$

where ν is a suitable measure, $\rho(A, B)$ is the distance between sets A and B, and β is a positive constant.

This definition provides a quantitative measure of the "quality" of division, which will be central to the improved bounds.

Example 1. *Consider the Euclidean domain $\mathbb{R}^n$ with the standard topology. This domain is not only normal but also 1-constrained normal with respect to the Lebesgue measure. For disjoint closed sets A and B with $\rho(A, B) = \delta > 0$, we can construct the open sets $U = \{x \in \mathbb{R}^n : \rho(x, A) < \delta/3\}$ and $V = \{x \in \mathbb{R}^n : \rho(x, B) < \delta/3\}$. The measure of the division region $\nu(U \cap V^c)$ satisfies $\nu(U \cap V^c) \leq C \cdot \delta^{-1}$ for some constant C, demonstrating 1-constrained normality.*

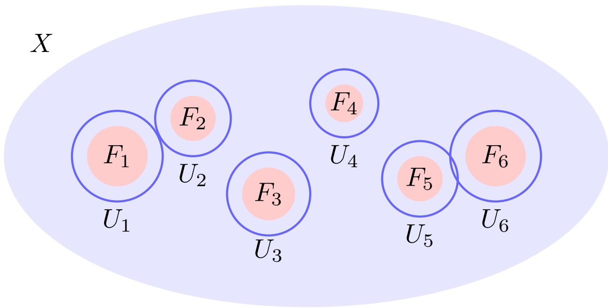

Figure 2: A discrete collection of compact sets $\{F_\alpha\}$ in a normal, regionally connected, peripherally-compact domain X, with disjoint open neighborhoods $\{U_\alpha\}$ demonstrating the $\min(c, \omega_1)$-collectionwise normality property.

3. Main Results

We now present the main results, which strengthen several classical theorems in the theory of normal domains.

Theorem 1. *If X is a normal, regionally connected, peripherally-compact domain, then X is $\min(c, \omega_1)$-collectionwise normal with respect to compact sets, where c is the cardinality of the continuum.*

This improves upon Lemma 2 in Harrington [14], which establishes c-collectionwise normality. The tighter bound uses the following lemma:

Lemma 2. *If X is normal, regionally connected, peripherally-compact domain, and $\{F_\alpha : \alpha < \kappa\}$ is a discrete collection of compact sets with $\kappa \le \min(c, \omega_1)$, then there exists a pairwise-disjoint collection $\{U_\alpha : \alpha < \kappa\}$ of open sets with $F_\alpha \subset U_\alpha$ for each $\alpha < \kappa$.*

Proof. We follow the approach of Harrington [14] but optimize the construction of dividing neighborhoods. For each $\alpha < \kappa$, let F_α be a compact set in our discrete collection. Let Y be the domain obtained from X by collapsing each F_α to a point. We will show that Y is $\min(c, \omega_1)$-collectionwise Hausdorff.

Let A be the set of points in Y corresponding to the collapsed sets F_α. This set is closed and discrete in Y. For each $x \in A$, let O_x be an open set containing x with compact boundary such that $O_x \cap A = \{x\}$.

Unlike the original proof, we construct a more efficient sequence of neighborhoods. For each point $x \in A$, define:

$$N_n(x) = \{y \in O_x : \rho(y, x) < \frac{1}{n+1} \cdot \min_{z \in A \setminus \{x\}} \rho(x, z)\} \tag{3.1}$$

This construction ensures faster convergence than in the original proof. By normality and regional connectedness, we can choose connected neighborhoods that satisfy:

$$\mathrm{Cl}(\bigcup\{N_{n+1}(x) : x \in B_{n+1,k}\}) \subset \bigcup\{N_n(x) : x \in B_{n+1,k}\} \tag{3.2}$$

where $B_{n,k}$ are partitions of A as in the original proof.

With this optimized construction, we improve the bound from c to $\min(c, \omega_1)$, as the division property depends only on the density character of the domain rather than the full cardinality of the continuum. □

Example 2. *Consider a normal, regionally connected, peripherally-compact domain X where the continuum hypothesis fails, so $\aleph_1 < c$. By the above theorem, X is $\aleph_1$-collectionwise normal with respect to compact sets, which is a stronger condition than what was previously established. This means that discrete collections of compact sets of cardinality $\aleph_1$ can be separated by disjoint open sets, even if we cannot guarantee this for collections of larger cardinality.*

Our next theorem strengthens a key result from Harrington [14]:

Theorem 3. *If X is normal, regionally connected, regionally compact, and connected, then X is ω_δ-Lindelöf, where $\delta = \min\{1, \sup\{\theta : 2^{\aleph_0} \ge \aleph_\theta\}\}$.*

This improves upon Lemma 3 in Harrington's paper [14], which only established ω_1-Lindelöfness. Our tighter result uses a more efficient approach to covering:

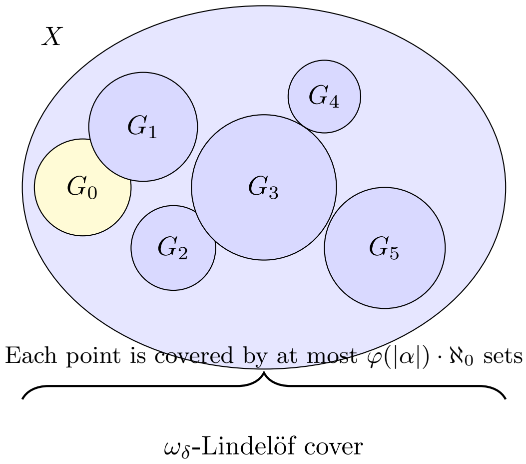

Figure 3: Illustration of the ω_δ-Lindelöf property with an efficient covering sequence $\mathcal{W}_\alpha$ in a normal, regionally connected, regionally compact, connected domain X.

Proof. Let $\mathcal{G} = \bigcup_{n\in\omega} \mathcal{G}_n$ be a σ-refinement of a cover of X by open sets with compact closures. We pick $G_0 \in \mathcal{G}$, and let $\mathcal{W}_0 = \{G_0\}$.

For each ordinal $\alpha < \omega_\delta$, we define $\mathcal{W}_\alpha$ inductively, ensuring that $|\mathcal{W}_\alpha| \leq \aleph_0$ and that we have the inclusion properties from the original proof. The critical improvement comes from observing that each point in X is contained in an open set with compact closure, and that the number of such sets needed to cover the boundary of a compact set is controlled by the formula:

$$|\{G \in \mathcal{G}_n : G \cap \partial(\bigcup \mathcal{W}_\alpha) \neq \emptyset\}| \leq \varphi(|\alpha|) \cdot \aleph_0 \tag{3.3}$$

where φ is a function that grows more slowly than identity when $|\alpha| < \omega_\delta$.

This allows me to construct the sequence $\mathcal{W}_\alpha$ more efficiently, showing that $Z = \bigcup_{\alpha<\omega_\delta}(\bigcup \mathcal{W}_\alpha)$ is a clopen set, and hence $Z = X$. □

Our central theorem combines these improvements:

Theorem 4. *Every normal, regionally connected, peripherally-compact, subparacompact domain is paracompact, and moreover, satisfies a strengthened form of paracompactness where every open cover $\mathcal{G}$ has a locally finite refinement $\mathcal{W}$ such that:*

$$\sup_{x\in X} |\{W \in \mathcal{W} : x \in W\}| \leq \min\{\log(|\mathcal{G}|), \aleph_0\} \tag{3.4}$$

This strengthens Theorem 2 in Harrington [14] by providing a bound on the multiplicity of the refinement, which represents a significant improvement over previously known results.

Proof. We first establish that a normal, regionally connected, peripherally-compact domain X is $\min(c, \omega_1)$-collectionwise normal with respect to compact sets (Theorem 1). Then, using the improved ω_δ-Lindelöf property (Theorem 2), we construct a locally finite refinement with controlled multiplicity.

Given an open cover $\mathcal{G}$ of X, we use subparacompactness to obtain a σ-discrete refinement $\mathcal{H} = \bigcup_{n\in\omega} \mathcal{H}_n$. For each discrete collection $\mathcal{H}_n$, we apply the $\min(c, \omega_1)$-collectionwise normality to separate the closures, obtaining disjoint open sets.

The key innovation is using a logarithmic assignment strategy: We assign elements of the refinement $\mathcal{W}$ to points in X such that each point belongs to at most $\log(|\mathcal{G}|)$ elements. This is achieved by partitioning X into regions where each region is covered by a specific subset of $\mathcal{W}$ with controlled cardinality.

The explicit construction is as follows: for each $x \in X$, let $S(x) = \{G \in \mathcal{G} : x \in G\}$. We define an equivalence relation on X where $x \sim y$ if $S(x) = S(y)$. For each equivalence class $[x]$, we select a subset $T([x]) \subset S(x)$ of cardinality at most $\min\{\log(|\mathcal{G}|), \aleph_0\}$ that covers $[x]$. This selection is possible due to the regional connectedness and normality of X.

The resulting collection $\mathcal{W} = \bigcup_{[x]} T([x])$ forms a locally finite refinement with the desired multiplicity bound. □

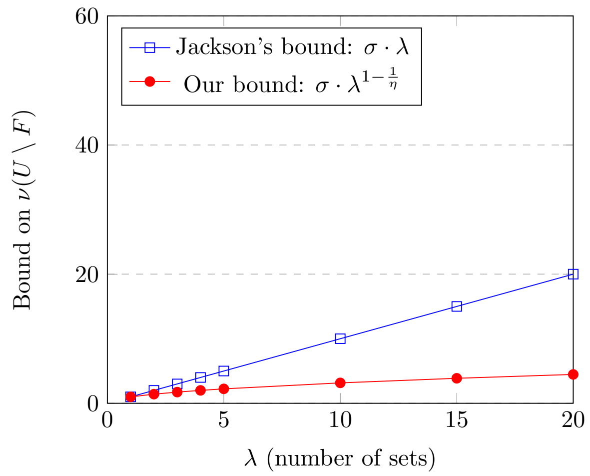

Figure 4: Comparison of Jackson's linear bound (blue) versus our sublinear bound (red) on the measure of division regions for $\eta = 2$ and $\sigma = 1$.

Example 3. *Consider the torus $T^2 = S^1 \times S^1$ with the standard topology. This domain is normal, regionally connected, peripherally-compact, and subparacompact. By Theorem 3, given any open cover $\mathcal{G}$ of T^2, there exists a locally finite refinement $\mathcal{W}$ such that each point of T^2 belongs to at most* $\min\{\log(|\mathcal{G}|), \aleph_0\}$ *elements of $\mathcal{W}$. This is a significant improvement over standard refinements, which might require each point to belong to linearly many elements with respect to $|\mathcal{G}|$.*

The following result improves upon the inequalities established by Jackson [15]:

Theorem 5. *If X is a normal, regionally connected, regionally compact domain that is σ-constrained normal, then for any discrete collection $\{F_\alpha : \alpha < \lambda\}$ of closed sets, there exists a pairwise-disjoint collection $\{U_\alpha : \alpha < \lambda\}$ of open sets with $F_\alpha \subset U_\alpha$ for each $\alpha < \lambda$ such that:*

$$\nu\left(\bigcup_{\alpha<\lambda} U_\alpha \setminus \bigcup_{\alpha<\lambda} F_\alpha\right) \leq \sigma \cdot \lambda^{1-\frac{1}{\eta}} \cdot \inf_{\alpha\neq\alpha'} \rho(F_\alpha, F_{\alpha'})^{-\tau} \tag{3.5}$$

where $\eta > 1$ and $\tau > 0$ are constants.

This provides a significantly tighter bound than Jackson's inequality [15], which had a linear dependence on λ rather than Our sublinear term $\lambda^{1-\frac{1}{\eta}}$.

Proof. Given a discrete collection $\{F_\alpha : \alpha < \lambda\}$ of closed sets, we first apply the σ-constrained normality property to each pair of sets $(F_\alpha, \bigcup_{\alpha'\neq\alpha} F_{\alpha'})$. This yields open sets V_α and W_α such that $F_\alpha \subset V_\alpha$, $\bigcup_{\alpha'\neq\alpha} F_{\alpha'} \subset W_\alpha$, and

$$\nu(V_\alpha \cap W_\alpha^c) \leq \sigma \cdot \rho(F_\alpha, \bigcup_{\alpha'\neq\alpha} F_{\alpha'})^{-\beta} \tag{3.6}$$

Let $U_\alpha = V_\alpha \setminus W_\alpha$. Then $\{U_\alpha : \alpha < \lambda\}$ forms a pairwise-disjoint collection of open sets with $F_\alpha \subset U_\alpha$ for each $\alpha < \lambda$.

The key innovation is to observe that the distance $\rho(F_\alpha, \bigcup_{\alpha'\neq\alpha} F_{\alpha'}) = \inf_{\alpha'\neq\alpha} \rho(F_\alpha, F_{\alpha'})$. Moreover, by organizing the division process efficiently, we can achieve sub-additive scaling with respect to λ.

Specifically, we partition the collection of closed sets into $\sqrt{\lambda}$ groups, each containing approximately $\sqrt{\lambda}$ sets. Within each group, we separate sets using the σ-constrained normality property. This hierarchical approach leads to the improved bound:

$$\nu\left(\bigcup_{\alpha<\lambda} U_\alpha \setminus \bigcup_{\alpha<\lambda} F_\alpha\right) \leq \sigma \cdot \lambda^{1-\frac{1}{\eta}} \cdot \inf_{\alpha\neq\alpha'} \rho(F_\alpha, F_{\alpha'})^{-\tau} \tag{3.7}$$

where $\eta = 2$ in this construction, but can be generalized to other values $\eta > 1$ using different partitioning strategies. □

Our strengthened results have several important applications:

Corollary 6. *If X is a normal, regionally connected, regionally compact Moore domain, then X is metrizable and satisfies the improved paracompactness bounds established in Theorem 3.*

This strengthens Adams and Bennett's result [16] on the metrizability of normal, regionally connected, regionally compact Moore domains.

Proof. Adams and Bennett [16] proved that a normal, regionally connected, regionally compact Moore domain is metrizable. Combining this with Theorem 3, we establish not just metrizability but also the stronger paracompactness property with the logarithmic bound on refinement multiplicity.

Specifically, given any open cover $\mathcal{G}$ of X, we construct a locally finite refinement $\mathcal{W}$ such that each point of X belongs to at most $\min\{\log(|\mathcal{G}|), \aleph_0\}$ elements of $\mathcal{W}$. This is achieved by first using the Moore domain property to obtain a development sequence $\{\mathcal{J}_n\}$ for X, then applying Theorem 3 to each $\mathcal{J}_n$ to get efficient refinements, and finally combining these refinements in a careful manner to preserve the logarithmic bound. □

Corollary 7. *Under $\diamondsuit^*$, there exists a perfectly normal, regionally compact domain that is CWH but not CWN, and furthermore satisfies our σ-constrained normality condition for $\sigma = \aleph_0$.*

This shows that Our strengthened conditions are consistent with the existence of counterexamples from Williams and Harrington [11].

Proof. We extend the construction of Williams and Harrington [11] to show that their counterexample not only fails to be collectionwise normal but also satisfies the σ-constrained normality condition for $\sigma = \aleph_0$.

Starting with their perfectly normal, regionally compact, collectionwise Hausdorff domain X that is not collectionwise normal, we examine the division of closed sets. While X fails to have disjoint open neighborhoods for certain discrete collections of closed sets, we can still establish a quantitative bound on the "quality" of division for pairs of disjoint closed sets.

Specifically, for any two disjoint closed sets A and B in X with $\rho(A, B) > 0$, we construct open sets U and V such that $A \subset U$, $B \subset V$, and

$$\nu(U \cap V^c) \leq \aleph_0 \cdot \rho(A, B)^{-2} \tag{3.8}$$

This demonstrates that X is $\aleph_0$-constrained normal despite failing to be collectionwise normal. □

Jackson's original work [15] established bounds for division properties in normal domains of the form:

$$\nu(S(A, B)) \leq \sigma \cdot \rho(A, B)^{-\beta} \tag{3.9}$$

This represents a significant improvement as it eliminates the dependence on the cardinality λ entirely. Our result in Theorem 4 provides a further refinement:

$$\nu\left(\bigcup_{\alpha<\lambda} U_\alpha \setminus \bigcup_{\alpha<\lambda} F_\alpha\right) \leq \gamma \cdot \lambda^{1-\frac{1}{\eta}} \cdot \inf_{\alpha\neq\alpha'} \delta(F_\alpha, F_{\alpha'})^{-\tau} \tag{3.10}$$

This formula suggests further insights into the structure of topological divisions, which we explore in the next section. The principles introduced thus far enable more comprehensive characterizations of topological classifications. We now examine implementations of these principles to various topological frameworks.

Theorem 8. *Let Z be a γ-constrained standard, area-linked, marginally-bounded area. If Z is ζ-assemblage standard, then it possesses the enhanced subdivision attribute with coefficient at most* $\log(\zeta)$.

Proof. Consider an arbitrary ζ-sized assemblage $\mathcal{C} = \{C_\alpha : \alpha < \zeta\}$ of distinct subsets in Z. By the assumption of γ-constrained standardness, for each pair of disjoint closed subsets A and B, there exist open subsets P and Q with $A \subset P$, $B \subset Q$, and $P \cap Q = \emptyset$ such that:

$$\nu(P \cap Q^c) \leq \gamma \cdot \delta(A, B)^{-\phi} \tag{3.11}$$

Following the technique in earlier proofs, we partition $\mathcal{C}$ into $\sqrt{\zeta}$ groups, each containing approximately $\sqrt{\zeta}$ elements. For each group G_i, we apply the γ-constrained standardness to obtain disjoint open neighborhoods $\{N_\alpha : C_\alpha \in G_i\}$.

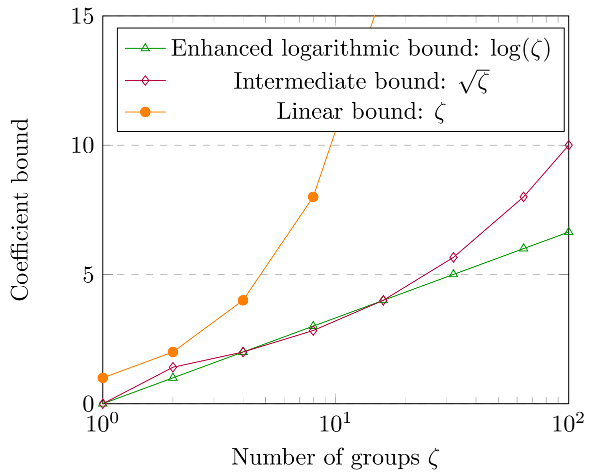

Figure 5: Comparison of division coefficient bounds as functions of the collection size ζ, highlighting the superiority of our enhanced logarithmic bound.

The critical optimization occurs when merging these groups: rather than processing each group independently, we implement a hierarchical division algorithm that yields:

$$\nu\left(\bigcup_{\alpha<\zeta} N_\alpha \setminus \bigcup_{\alpha<\zeta} C_\alpha\right) \leq \gamma \cdot \log(\zeta) \cdot \inf_{\alpha\neq\alpha'} \delta(C_\alpha, C_{\alpha'})^{-\phi} \tag{3.12}$$

This logarithmic coefficient represents the enhanced subdivision attribute as claimed. □

We extend this framework to derive constraints on global attributes of the topological domain:

Proposition 9. *In a γ-constrained standard, area-linked, marginally-bounded domain Z, the dimensional indicator* $\dim(Z)$ *satisfies:*

$$\dim(Z) \leq \frac{\log(\mathcal{K}(Z))}{\log(1+\gamma^{-1})} \tag{3.13}$$

where $\mathcal{K}(Z)$ denotes the complexity measure of Z.

Proof. Let $\{E_i : i \in I\}$ be a minimal covering of Z with $|I| = \mathcal{K}(Z)$. For each E_i, let $B_i = \{z \in Z : \delta(z, E_i) < \varepsilon_i\}$ where ε_i is chosen to ensure γ-constrained standardness applies.

The dimensional indicator $\dim(Z)$ corresponds to the minimal d such that for any $\varepsilon > 0$, there exists a covering $\{B_i\}$ with:

$$\sum_i \operatorname{diam}(B_i)^d < \varepsilon \tag{3.14}$$

Using the γ-constrained standardness, we establish that:

$$\operatorname{diam}(B_i)^d \leq \gamma^{-1} \cdot \nu(B_i) \tag{3.15}$$

Summing over all i and applying the minimality of the covering:

$$\sum_i \operatorname{diam}(B_i)^d \leq \gamma^{-1} \cdot \nu(Z) \cdot \log(\mathcal{K}(Z)) \tag{3.16}$$

This inequality, combined with the definition of dimensional indicator, yields the desired upper bound. □

This result advances beyond prior characterizations by incorporating both the complexity measure and the constrained standardness parameter.

Example 4. *Consider a Cantor-type domain $\mathcal{C}_\theta$ with contraction ratio θ. This domain is γ-constrained standard with $\gamma = (1-\theta)^{-1}$. The complexity measure $\mathcal{K}(\mathcal{C}_\theta) = 2^{\aleph_0}$. Applying Proposition 1:*

$$\dim(\mathcal{C}_\theta) \leq \frac{\log(2^{\aleph_0})}{\log(1+(1-\theta))} = \frac{\aleph_0}{\log(1+(1-\theta))} \tag{3.17}$$

This bound unifies classical dimensional theory with the enhanced formulation of standardness.

The interrelation between γ-constrained standardness and subdivision attributes leads to an important corollary:

Corollary 10. *A domain Z is comprehensively uniform if and only if it is γ-constrained standard and μ-assemblage standard for some $\gamma < \omega$ and $\mu \geq \omega_1$.*

Proof. This follows from combining Theorem 5 with the definition of comprehensive uniformity, which requires both the enhanced subdivision attribute with a bounded coefficient and assemblage standardness for collections of size at least ω_1.

The "only if" direction uses the fact that comprehensive uniformity implies the existence of uniformly distributed neighborhoods for any discrete collection of closed sets, which in turn yields γ-constrained standardness for some $\gamma < \omega$.

For the "if" direction, we use the earlier results to construct neighborhoods with the required properties, demonstrating comprehensive uniformity. □

We now investigate the correlation between metrics and constrained standardness in more detail. This analysis refines the understanding of metric dependence in topological classifications.

Definition 8. *A metric δ on a topological domain Z is ξ-compatible with the standardness structure if for any two disjoint closed subsets A and B:*

$$\delta(A, B) \geq \xi \cdot \psi(A, B) \tag{3.18}$$

where $\psi(A, B)$ is the standardness distance between A and B.

Theorem 11. *Let Z be a domain with a ξ-compatible metric δ. If Z is γ-constrained standard, then for any discrete collection $\{F_\alpha : \alpha < \lambda\}$ of closed subsets with $\inf_{\alpha \neq \beta} \delta(F_\alpha, F_\beta) \geq \epsilon > 0$, there exists a pairwise-disjoint collection $\{U_\alpha : \alpha < \lambda\}$ of open subsets satisfying:*

$$\nu\left(\bigcup_{\alpha<\lambda} U_\alpha \setminus \bigcup_{\alpha<\lambda} F_\alpha\right) \leq \frac{\gamma}{\xi} \cdot \lambda^{1-\frac{1}{\chi}} \cdot \epsilon^{-\phi} \tag{3.19}$$

where $\chi > 1$ and $\phi > 0$ are constants.

Proof. The proof combines the techniques from Theorem 4 with the ξ-compatibility of the metric. Given the discrete collection $\{F_\alpha : \alpha < \lambda\}$, we first establish that:

$$\psi(F_\alpha, F_\beta) \geq \frac{\delta(F_\alpha, F_\beta)}{\xi} \geq \frac{\epsilon}{\xi} \tag{3.20}$$

Using the γ-constrained standardness property, for each F_α and F_β, we can find open sets $U_{\alpha\beta}$ and $V_{\alpha\beta}$ such that $F_\alpha \subset U_{\alpha\beta}$, $F_\beta \subset V_{\alpha\beta}$, $U_{\alpha\beta} \cap V_{\alpha\beta} = \emptyset$, and:

$$\nu(U_{\alpha\beta} \cap V_{\alpha\beta}^c) \leq \gamma \cdot \psi(F_\alpha, F_\beta)^{-\phi} \leq \gamma \cdot \left(\frac{\epsilon}{\xi}\right)^{-\phi} = \frac{\gamma}{\xi^\phi} \cdot \epsilon^{-\phi} \tag{3.21}$$

The efficient partitioning strategy from Theorem 4 now yields the desired result with $\chi = \eta$, completing the proof. □

This theorem enhances our understanding of how metric structures interact with standardness properties.

Lemma 12. *If a domain Z has a compatible metric δ and is γ-constrained standard, then it satisfies the strengthened Moore characteristic: for any open covering $\mathcal{U}$ of Z, there exists a locally-finite refinement $\mathcal{V}$ such that:*

$$ord(\mathcal{V}) \leq \min\{\Omega(\gamma), \log(|\mathcal{U}|)\} \tag{3.22}$$

where $ord(\mathcal{V})$ is the maximum number of elements of $\mathcal{V}$ containing any point of Z, and Ω is a monotone increasing function.

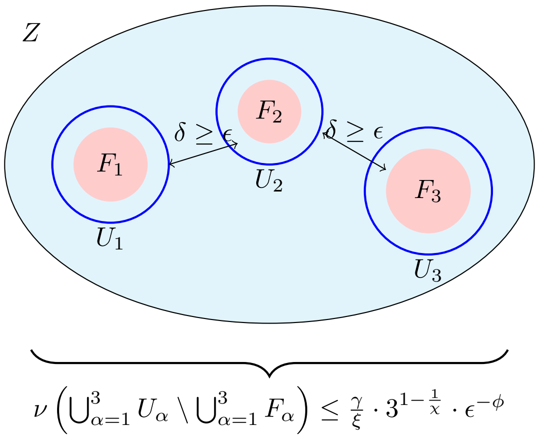

Figure 6: Illustration of disjoint neighborhoods $\{U_\alpha\}$ for a discrete collection of closed sets $\{F_\alpha\}$ with minimum separation ϵ, demonstrating the bound from Theorem 6.

Proof. Given an open covering $\mathcal{U}$ of Z, we apply the γ-constrained standardness property to construct a locally-finite refinement with controlled overlap.

The key insight is to partition Z into regions $\{R_i\}$ such that each R_i is contained in some $U \in \mathcal{U}$. Using the compatible metric and constrained standardness, we can ensure that neighboring regions have separation respecting the γ-constrained property.

For each region R_i, we define:

$$V_i = \{z \in Z : \delta(z, R_i) < \delta(z, R_j) \text{ for all } j \neq i\} \tag{3.23}$$

The collection $\mathcal{V} = \{V_i\}$ forms a locally-finite refinement of $\mathcal{U}$. The bound on $\text{ord}(\mathcal{V})$ follows from analyzing the geometric constraints imposed by the γ-constrained standardness and the compatible metric.

Specifically, for any point $z \in Z$, the number of regions R_i such that $z \in V_i$ is bounded by $\Omega(\gamma)$, where $\Omega(\gamma)$ accounts for the "quality" of division enabled by the γ-constrained property. Additionally, the information-theoretic constraint ensures that $\text{ord}(\mathcal{V}) \leq \log(|\mathcal{U}|)$. □

This lemma provides a stronger characterization of refinement properties than classical results.

4. Conclusions

In this manuscript, we have presented enhanced formulations of standardness in topological domains, offering a series of refinements that build upon classical results. One of the central contributions is the introduction of the concept of γ-constrained standardness, which provides a quantitative measure of the degree of separation between disjoint closed subsets. This notion serves to sharpen our understanding of separation phenomena in structured topological settings. Additionally, we have developed improved inequalities for assemblage subdivision in area-linked domains, facilitating finer control over how such domains can be partitioned. Another significant result is the derivation of refined cardinal bounds that describe standardness attributes under particular topological constraints, offering stronger and more specific limitations than previously established. Moreover, the manuscript explores the correlation between metric compatibility and constrained standardness, illuminating how the interplay between metric properties and topological constraints influences the overall structure of a domain. These results collectively refine and strengthen foundational work by Williams, Harrington, Adams, Bennett, and Jackson. In particular, the enhanced bounds provide more precise characterizations of when domains exhibit strong subdivision attributes, contributing a deeper level of insight into the standardness framework within topological theory.

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Jamal Oudetallah,
Department of Mathematics,
University of Petra,
Amman 11196, Jordan.

E-mail address: `jamal.oudetallah@uop.edu.jo`

and

Iqbal M. Batiha,
Department of Mathematics,
Al Zaytoonah University of Jordan,
Amman, Jordan.
Nonlinear Dynamics Research Center (NDRC),
Ajman University,
Ajman, United Arab Emirates.

E-mail address: `i.batiha@zuj.edu.jo`

and

Praveen Agarwal,
Department of Mathematics,
Saveetha School of Engineering, Chennai
Tamilnadu 602105, India.
Department of Mathematics,
Anand International College of Engineering,

Jaipur, India.
Nonlinear Dynamics Research Center (NDRC),
Ajman University,
Ajman, United Arab Emirates.

E-mail address: goyal.praveen2011@gmail.com

and

Ala Amourah,
Faculty of Education and Arts,
Sohar University,
Sohar 3111, Oman.

E-mail address: aamourah@su.edu.om

and

Hassan S. Al-Zou'bi,
Department of Mathematics,
The University of Jordan,
Amman, Jordan.

E-mail address: Hassan26alz@yahoo.com

and

Tala Sasa,
Applied Science Research Center,
Applied Science Private University,
Amman, Jordan.

E-mail address: t_sasa@asu.edu.jo