Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.76726

On Six Consecutive Pairs of Lucas and Fibonacci-Type p Entities connected to Chebyshev Polynomials

R. Rangarajan, C. Goutham, K. Bharatha* and P. Siva Kota Reddy

ABSTRACT: Let p be a chosen positive integer parameter. Let $N_p = (p+1)^2 + 1$ and $M_p = 2(p+1)^2 + 1$. We denote the p-entities by $\{(\lambda_{6n+k}, \theta_{6n+k}) : k = 0, 1, 2, 3, 4, 5; n = 0, 1, 2, 3, \cdots\}$. When $p = 1$ they are equal to (L_{6n+k}, F_{6n+k}) where L_n and F_n are the well known Lucas and Fibonacci numbers. Also, if $x = \lambda_{6n+k}$ and $y = \theta_{6n+k}$, they satisfy generalised Pell's equation $x^2 - N_p y^2 = \pm(p+1)^2$. In the present paper, it will be shown that λ_{6n+k} and θ_{6n+k} are expressible in terms of $T_n(M_p)$ and $U_{n-1}(M_p)$ where, $T_n(x)$ and $U_{n-1}(x)$ are well known Chebyshev polynomials of the first and the second kind. Some interesting combinatorial identities are also derived for each $p-$ entities.

Key Words: Fibonacci sequence, Lucas sequence, Generalized Combinatorial Entities, Chebyshev Polynomials.

Contents

1. Introduction

Combinatorial algebra of solving Pell's equation [3,10], Combinatorial number theory of Lucas and Fibonacci numbers, Pell and Pell-Lucas numbers and their generalizations [1,2,5,8,9,11,12,15,16,18,19,22] and concrete Mathematics of combinatorics have [7] became very important for their well described theory, endless scope for computation and many useful applications not only to pure Mathematics but also to applied Mathematics. The main motive of the research works will be to create new combinatorial entities and work with their beautiful combinatorial identities. The identities can also be approached via geometrical configurations and their countings. For instance one can refer [4,14].

In [6], the D'Ocagne's identity for the generalized Fibonacci and Lucas sequences is established in terms of log convex identity of generalized Fibonacci and Lucas sequence by using mathematical induction. In [13], the authors constructed the sequences of Fibonacci and Lucas in any quadratic field $\mathbb{Q}(\sqrt{d})$ with $d > 0$ square free, noting that the general properties remain valid as those given by the classical sequences of Fibonacci and Lucas for the case $d = 5$, under the respective variants. Further, for both sequences, the authors [13] obtained the generating function, Golden ratio, Binet's formula and some identities that they keep.

Chebyshev polynomials are fine jewels of Mathematics to decorate in many places of number theory, approximation theory and related areas. The present paper mainly focuses on generalized p entities

* Corresponding author
Submitted May 10, 2025. Published July 11, 2025
2010 *Mathematics Subject Classification:* 11B39, 11B83, 05A19.

of Lucas and Fibonacci numbers and their interconnections to Chebyshev polynomials. Some of the interesting combinatorial identities are derived for those p entities. In [20], the required p entities are obtained by the generalising the following identities of (L_{3n}, F_{3n}):

$$L_{3n} : 2, 4, 18, 76, \ldots, \ F_{3n} : \ 0, 2, 8, 34, \ldots$$

are even Lucas and Fibonacci numbers [9,12,19,22] given by the following binet forms:

$$\begin{aligned} L_{3n} &= \left(\frac{L_3 + F_3\sqrt{5}}{2}\right)^n + \left(\frac{L_3 - F_3\sqrt{5}}{2}\right)^n = \left(2+\sqrt{5}\right)^n + \left(2-\sqrt{5}\right)^n \\ \sqrt{5}F_{3n} &= \left(\frac{L_3 + F_3\sqrt{5}}{2}\right)^n + \left(\frac{L_3 - F_3\sqrt{5}}{2}\right)^n = \left(2+\sqrt{5}\right)^n + \left(2-\sqrt{5}\right)^n, \ n = 0, 1, 2 \cdots . \end{aligned}$$

It is very interesting to note that

$$\begin{aligned} &2L_{3n+1} = L_{3n+3} - L_{3n}; \ 2F_{3n+1} = F_{3n+3} - F_{3n}, \\ &2L_{3n+2} = L_{3n+3} + L_{3n}; \ 2F_{3n+2} = F_{3n+3} + F_{3n}, \ n = 0, 1, 2 \cdots . \end{aligned}$$

Definition: [20] Let p be chosen positive integer parameter. Put $N_p = (p+1)^2 + 1; \alpha_p = (p+1) + \sqrt{N_p}$ and $\beta_p = (p+1) + \sqrt{N_p}$.
Define

(i) $\lambda_{3n} = \dfrac{p+1}{2}\left(\alpha_p^n + \beta_p^n\right); \ \theta_{3n} = \dfrac{p+1}{2\sqrt{N_p}}\left(\alpha_p^n - \beta_p^n\right)$

(ii) $(p+1)\lambda_{3n+1} = \lambda_{3n+3} - p\lambda_{3n}; \quad (p+1)\theta_{3n+1} = \theta_{3n+3} - p\theta_{3n}$

(iii) $(p+1)\lambda_{3n+2} = p\ \lambda_{3n+3} + \lambda_{3n}; \quad (p+1)\theta_{3n+2} = p\ \theta_{3n+3} + \theta_{3n}$, where $n = 0, 1, 2, 3 \cdots$.

n	λ_n	θ_n
0	$p+1$	0
1	1	1
2	p^2+p+1	p
3	$(p+1)^2$	$p+1$
4	p^2+3p+3	$p+2$
5	$2p^3+4p^2+4p+1$	$2p^2+2p+1$
6	$2(p+1)^2+(p+1)$	$2(p+1)^2$

Table 1. Initial Values

If $x_n = L_{3n}, L_{3n+1}, L_{3n+2}, F_{3n}, F_{3n+1}, F_{3n+2}$ then it satisfies simple three term relations $x_{n+1} = 2(p+1)x_n + x_{n-1}, n = 0, 1, 2 \cdots$ with appropriate two initial points (See [20]).

If $p = 1, \lambda_{3n+k} = L_{3n+k}$ and $\theta_{3n+k} = F_{3n+k}, k = 0, 1, 2; n = 0, 1, 2, \cdots$. Also, if $x = L_{3n+k}$ and $y = F_{3n+k}$ it is shown that $x^2 - N_p y^2 = \pm(p+1)^2$ (See [20]). They are regarded as three pairs of Lucas and Fibonacci type $p-$entities.

The Chebyshev polynomials of first and second kind [14] are given by their binet forms as:

$$\begin{aligned} T_n(x) &= \frac{1}{2}\left[\left(x + \sqrt{x^2-1}\right)^n + \left(x - \sqrt{x^2-1}\right)^n\right]; \\ U_n(x) &= \frac{1}{2\sqrt{x^2-1}}\left[\left(x + \sqrt{x^2-1}\right)^{n+1} - \left(x - \sqrt{x^2-1}\right)^{n+1}\right], \end{aligned}$$

where $n = 0, 1, 2, \ldots$

First, we would like to connect (L_{6n}, F_{6n}) to $(T_n(9), U_{n-1}(9))$ as given below:

$$\begin{aligned}
L_{6n} &= \left(\frac{L_6 + F_6\sqrt{5}}{2}\right)^n + \left(\frac{L_6 - F_6\sqrt{5}}{2}\right)^n \\
&= \left(9 + 4\sqrt{5}\right)^n + \left(9 - 4\sqrt{5}\right)^n \\
&= 2\ T_n(9). \\
\sqrt{5}F_{6n} &= \left(\frac{L_6 + F_6\sqrt{5}}{2}\right)^n + \left(\frac{L_6 - F_6\sqrt{5}}{2}\right)^n \\
&= \left(9 + 4\sqrt{5}\right)^n + \left(9 - 4\sqrt{5}\right)^n \\
&= 8\sqrt{5}\ U_{n-1}(9).
\end{aligned}$$

In the present paper, 9 is generalised as $M_p = 2(p+1)^2 + 1$. We introduce a pair of sequences, namely, (t_n, u_n) given by $t_n = T_n(M_p),\ u_n = U_{n-1}(M_p)$. They fit so well that $(\lambda_{6n+k}, \theta_{6n+k})$ have neat expressions in terms of t_n and u_n (please see identities (2.1)). The coming sections are devoted to derive some interesting combinatorial identities of $\{(t_n, u_n) : n = 0, 1, 2, 3\cdots\}$ as well as $\{(\lambda_{6n+k}, \theta_{6n+k}) : k = 0, 1, 2, 3, 4, 5; n = 0, 1, 2, 3, \cdots\}$.

2. Recurrence Relations

In this section, combinatorial identities of $t_n, u_n, \lambda_{6n+k}, \theta_{6n+k}$ in the form of interrelations are stated as main theorems. Ideas of proof are given before each theorem.

2.1. Identities for 3-term recurrence relations

The standard recurrence relations satisfied by $(T_n(x), U_n(x))$ directly yield the following identities satisfied by $t_n = T_n(M_p)$ and $u_n = U_{n-1}(M_p)$, where $M_p = 2(p+1)^2 + 1$:

Theorem 2.1.

$$\begin{aligned}
&(a)\ t_{n+1} = 2\ M_p\ t_n - t_{n-1}, t_0 = 1, t_1 = M_p, n = 1, 2, 3\cdots; \\
&\quad\ u_{n+1} = 2\ M_p\ u_n - u_{n-1}, u_0 = 0, u_1 = 1. \\
&(b)\ t_n = M_p\ u_n - u_{n-1}, (M_p^2 - 1)u_n = M_p t_n - t_{n-1}. \\
&(c)\ t_{n+1} = M_p\ t_n + (M_p^2 - 1)u_n; u_{n+1} = t_n + M_p u_n.
\end{aligned}$$

2.2. Connecting Identities I

By definition,

$$\begin{aligned}
\lambda_{6n} &= \frac{p+1}{2}\left[\alpha_p^{2n} + \beta_p^{2n}\right]; \\
\theta_{6n} &= \frac{p+1}{2\sqrt{N_p}}\left[\alpha_p^{2n} - \beta_p^{2n}\right].
\end{aligned}$$

$$\begin{aligned}
\alpha_p &= (p+1) + \sqrt{N_p}; \\
\alpha_p^2 &= \big((p+1)^2 + N_p\big) + 2(p+1)\sqrt{N_p}, \text{ where, } N_p = (p+1)^2 + 1, \\
\text{Hence, } \alpha_P^2 &= M_P + \sqrt{M_p^2 - 1}. \\
\text{Similarly, } \beta_P^2 &= M_P - \sqrt{M_p^2 - 1}.
\end{aligned}$$

As a result,

$$\lambda_{6n} = (p+1)t_n; \quad \theta_{6n} = 2(p+1)^2 u_n.$$

One may rewrite,

$$\begin{aligned}
2(p+1)\lambda_{6n} &= \theta_0 t_{n+1} + \theta_6 t_n; \\
\theta_{6n} &= \theta_0 u_{n+1} + \theta_6 u_n \\
2(p+1)\lambda_{6n+6} &= \theta_6 t_{n+1} + \theta_0 t_n; \\
\theta_{6n+6} &= \theta_6 u_{n+1} + \theta_0 u_n.
\end{aligned}$$

Which implies,

$$\begin{aligned}
4(p+1)^2\lambda_{6n+3} &= 2(p+1)[\lambda_{6n+6} - \lambda_{6n}] \\
&= (\theta_6 - \theta_0)t_{n+1} - (\theta_6 - \theta_0)t_n \\
&= 2(p+1)[\theta_3 t_{n+1} - \theta_3 t_n] \\
\text{So, } 2(p+1)\lambda_{6n+3} &= \theta_3 t_{n+1} - \theta_3 t_n \\
\text{Similarly, } \theta_{6n+3} &= \theta_3 u_{n+1} - \theta_3 u_n \\
\text{Also, } 2(p+1)^2\lambda_{6n+1} &= 2(p+1)[\lambda_{6n+3} - p\lambda_{6n}] \\
&= [\lambda_{6n+6} - \lambda_{6n}] - (2p^2 + 2p)\lambda_{6n} \\
&= \lambda_{6n+6} - (2p^2 + 2p + 1)\lambda_{6n} \\
&= (p+1)\theta_1 t_{n+1} - (p+1)\theta_5 t_n \\
\text{So, } 2(p+1)\lambda_{6n+1} &= \theta_1 t_{n+1} - \theta_5 t_n. \\
\text{Similarly, } \theta_{6n+1} &= \theta_1 u_{n+1} - \theta_5 u_n.
\end{aligned}$$

The computation for the cases $6n+2, 6n+4$ and $6n+5$ are quite similar. Hence one can combine all of them and write the following compact identities:

Theorem 2.2.

$$\begin{aligned}
&(a)\ 2(p+1)\lambda_{6n+k} = \theta_k t_{n+1} + (-1)^k \theta_{6-k} t_n. \\
&(b)\ \theta_{6n+k} = \theta_k u_{n+1} + (-1)^k \theta_{6-k} u_n.
\end{aligned}$$

2.3. Connecting Identities II

One may also write,

$$\begin{aligned}
\lambda_{6n} &= (p+1)t_n \\
&= (p+1)[M_p u_n - u_{n-1}] \\
&= (p+1)[u_{n+1} - M_P u_n] \\
&= (p+1)u_{n+1} - [2(p+1)^3 + (p+1)]u_n \\
&= \lambda_0 u_{n+1} - \lambda_6 u_n.
\end{aligned}$$

$$\begin{aligned}
\text{and}\quad \theta_{6n} &= 2(p+1)^2 u_n \\
&= (M_p - 1)u_n. \\
\text{This further implies,}\quad (M_p+1)\theta_{6n} &= (M_p^2 - 1)u_n \\
&= (M_p)t_n - t_{n-1} \\
&= t_{n+1} - M_p t_n \\
2N_p\theta_{6n} &= t_{n+1} - M_p t_n \\
\text{So,}\quad 2(p+1)N_p\theta_{6n} &= \lambda_0 t_{n+1} - \lambda_6 t_n.
\end{aligned}$$

By a similar derivation, one obtain the following compact identities:

Theorem 2.3.

$$\begin{aligned}
&(a)\ \lambda_{6n+k} = \lambda_k\ u_{n+1} + (-1)^{k+1}\ \lambda_{6-k}\ u_n. \\
&(b)\ 2(p+1)\ N_p\ \theta_{6n+k} = \lambda_k\ t_{n+1} + (-1)^{k+1}\ t_n.
\end{aligned}$$

2.4. Connecting Identities III

One may write in another way as follows:

$$\begin{aligned}
\lambda_{6n+6} &= (p+1)\ t_{n+1} \\
&= (p+1)\ [M_p\ t_n + (M_p^2 - 1)\ u_n]. \\
&\text{Since, } \lambda_6 = (p+1)\ M_p,\ \theta_6 = 2(p+1)^2 \text{ and } N_p = (p+1)^2 + 1, \\
\lambda_{6n+6} &= \lambda_6 t_n + 2(p+1)N_p\theta_6 u_n, \\
\theta_{6n+6} &= 2(p+1)^2 u_{n+1} \\
&= 2(p+1)^2[t_n + M_p u_n] \\
&= \theta_6 t_n + 2(p+1)\lambda_6 u_n \\
\text{Also, } \lambda_{6n} &= (p+1)t_n \\
&= \lambda_0 t_n + 2(p+1)N_p\theta_0 u_n; \\
\theta_{6n} &= 2(p+1)^2 u_n \\
&= \theta_0 t_n + 2(p+1)\lambda_0 u_n.
\end{aligned}$$

The computations for the cases $6n+k, k = 1, 2, 3, 4, 5$ are quite similar, hence one may write the following compact identities:

Theorem 2.4.

$$\begin{aligned}
&(a)\ \lambda_{6n+k} = \lambda_k t_n + 2(p+1)N_p\theta_k u_n. \\
&(b)\ \theta_{6n+k} = \theta_k t_n + 2(p+1)\lambda_k u_n, \qquad k = 0, 1, 2, 3, 4, 5; n = 0, 1, 2, 3\ldots
\end{aligned}$$

2.5. Computational Identities

As a consequence of the above identities, one can derive the following computational identities:

Theorem 2.5.

(a) $\lambda_{6(n+1)+k} = 2M_P\lambda_{6n+k} - \lambda_{6(n-1)+k}$

$$\lambda_{6+k} = \begin{cases} 2(p+1)\lambda_{3+k} + \lambda_k & \textit{if} \;\; k = 0, 1, 2. \\ (4(p+1)^2+1)\lambda_k + 2(p+1)\lambda_{k-3} & \textit{if} \;\; k = 3, 4, 5. \end{cases}$$

$\lambda_k : \lambda_0 = p+1, \lambda_1 = 1, \lambda_2 = p^2+p+1, \lambda_3 = (p+1)^2, \lambda_4 = p^2+3p+3, \lambda_5 = 2p^3+4p^2+4p+1.$

$n = 1, 2, 3 \cdots$

(b) $\theta_{6(n+1)+k} = 2M_P\theta_{6n+k} - \theta_{6(n-1)+k}$

$$\theta_{6+k} = \begin{cases} 2(p+1)\theta_{3+k} + \theta_k & \textit{if} \;\; k = 0, 1, 2. \\ (4(p+1)^2+1)\theta_k + 2(p+1)\theta_{k-3} & \textit{if} \;\; k = 3, 4, 5. \end{cases}$$

$\theta_k : \theta_0 = 0, \theta_1 = 1, \theta_2 = p, \theta_3 = (p+1), \theta_4 = p+2, \theta_5 = 2p^2+2p+1.$

These identities will be used to derive some more interesting identities in the next two sections.

3. Matrix Identities, Cassini-type Identities and Generating Functions

Identities 2.1 (c) may be written as

$$\begin{bmatrix} t_n & u_n \\ (M_p^2-1)u_n & t_n \end{bmatrix} = \begin{bmatrix} M_p & 1 \\ (M_p^2-1) & M_p \end{bmatrix} \begin{bmatrix} t_{n-1} & u_{n-1} \\ (M_p^2-1)u_{n-1} & t_{n-1} \end{bmatrix}$$

and identities 2.1 (a) may be written as

$$\begin{bmatrix} t_{n-1} & t_n \\ t_n & t_{n+1} \end{bmatrix} = \begin{bmatrix} t_{n-2} & t_{n-1} \\ t_{n-1} & t_n \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2M_p \end{bmatrix}$$

and

$$\begin{bmatrix} u_{n-1} & u_n \\ u_n & u_{n+1} \end{bmatrix} = \begin{bmatrix} u_{n-2} & u_{n-1} \\ u_{n-1} & u_n \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2M_p \end{bmatrix}.$$

By iterating the above identities, one obtain the following matrix identities:

Theorem 3.1.

$$(a)\ \begin{bmatrix} t_n & u_n \\ (M_p^2-1)u_n & t_n \end{bmatrix} = \begin{bmatrix} M_p & 1 \\ (M_p^2-1) & M_p \end{bmatrix}^n$$

$$(b)\ \begin{bmatrix} t_{n-1} & t_n \\ t_n & t_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & M_p \\ M_p & 2M_p^2-1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2M_p \end{bmatrix}^{n-1}$$

$$(c)\ \begin{bmatrix} u_{n-1} & u_n \\ u_n & u_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2M_p \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2M_p \end{bmatrix}^{n-1}.$$

$n = 1, 2, 3, \ldots$
Identities (2.4) may be rewritten as (with $M_p^2 - 1 = 4(p+1)^2 N_p$)

$$\begin{bmatrix} 2(p+1)\ \lambda_{6n+k} & \theta_{6n+k} \\ 2(p+1)\ N_p\ \theta_{6n+k} & \lambda_{6n+k} \end{bmatrix} = \begin{bmatrix} 2(p+1)\ \lambda_k & \theta_k \\ 2(p+1)\ N_p\ \theta_k & \lambda_k \end{bmatrix} \begin{bmatrix} t_n & u_n \\ (M_p^2-1)u_n & t_n \end{bmatrix}.$$

Using 3.1(a), we deduce the following identity:

Corollary 3.2.

$$\begin{bmatrix} 2(p+1)\ \lambda_{6n+k} & \theta_{6n+k} \\ 2(p+1)\ N_p\ \theta_{6n+k} & \lambda_{6n+k} \end{bmatrix} = \begin{bmatrix} 2(p+1)\ \lambda_k & \theta_k \\ 2(p+1)\ N_p\ \theta_k & \lambda_k \end{bmatrix} \begin{bmatrix} M_p & 1 \\ (M_p^2-1) & M_p \end{bmatrix}^n,\ k = 0, 1, ..., 5;\ n = 1, 2, 3, \ldots$$

By computing determinants on both sides of the identities 3.1 $(a), (b), (c)$ and (3.2) one obtain the following Cassini-type identities:

Corollary 3.3.

(a) $t_n^2 - (M_p^2 - 1)\ u_n^2 = 1$

(b) $t_{n-1}\ t_{n+1} - t_n^2 = M_p^2 - 1$

(c) $u_{n-1}\ u_{n+1} - u_n^2 = -1$

(d) $\lambda_{6n+k}^2 - N_P\ \theta_{6n+k}^2 = \lambda_k^2 - N_p\ \theta_k^2 = (-1)^k\ (p+1)^2,$
$k = 0, 1, 2, 3, 4, 5; n = 0, 1, 2, 3, \ldots$

The identities in 2.1(a) and 2.5$(a), (b)$ directly help us to obtain the following generating functions (The derivations are exactly same as those for obtaining generating functions of $T_n(x), U_n(x), F_n$ and L_n):

Theorem 3.4.

$$(a)\ \sum_{n=0}^{\infty} t_n\ x^n = \frac{1 - M_p\ x}{1 - 2\ M_p\ x + x^2}$$

$$(b)\ \sum_{n=0}^{\infty} u_n\ x^n = \frac{x}{1 - 2\ M_p\ x + x^2}$$

$$(c)\ \sum_{n=0}^{\infty} \lambda_{6n+k}\ x^n = \frac{\lambda_k - (\lambda_{6+k} - 2\ M_p\ \lambda_k)\ x}{1 - 2\ M_p\ x + x^2}$$

$$(d)\ \sum_{n=0}^{\infty} \theta_{6n+k}\ x^n = \frac{\theta_k - (\theta_{6+k} - 2\ M_p\ \theta_k)\ x}{1 - 2\ M_p\ x + x^2}, \quad k = 0, 1, 2, 3, 4, 5.$$

4. Summation and Convolution Identities

A simple consequences of identities in $2.1(a), (b)$ are:

$$2\,(M_p - 1)\, t_k = \begin{cases} 2(t_1 - t_0) & \text{if } \ k = 0 \\ (t_{k+1} - t_k) - (t_k - t_{k-1}) & \text{if } \ k = 1, 2, \cdots, n. \end{cases}$$

$$2\,(M_p - 1)\, u_k = \begin{cases} 0 & \text{if } \ k = 0 \\ (u_{k+1} - u_k) - (u_k - u_{k-1}) & \text{if } \ k = 1, 2, \cdots, n. \end{cases}$$

As a result one can directly derive the following identities:

Theorem 4.1.

$$(a)\ \ (M_p - 1) \sum_{k=0}^{n} t_k = [t_{n+1} - t_n] + [t_1 - t_0]$$

$$(b)\ 2\,(M_p - 1) \sum_{k=0}^{n} u_k = [u_{n+1} - u_n] - [u_1 - u_0], n = 0, 1, 2, \cdots.$$

Applying the identities $2.4(a), (b)$ and the identities $4.1(a), (b)$, one can directly deduce the following identities:

Corollary 4.2.

$$(a)\quad 2\,(M_{p-1}) \sum_{r=0}^{n} \lambda_{6r+k} = \lambda_k[(t_{n+1} - t_n) + (t_1 - t_0)] + 2(p+1)N_p\theta_k[(u_{n+1} - u_n) - (u_1 - u_0)];$$

$$(b)\quad 2\,(M_p - 1) \sum_{r=0}^{n} \theta_{6r+k} = \theta_k[(t_{n+1} - t_n) + (t_1 - t_0)] + 2(p+1)\lambda_k[(u_{n+1} - u_n) - (u_1 - u_0)].$$

Put $\gamma = M_p + \sqrt{M_p^2 - 1}$ and $\delta = M_p - \sqrt{M_p^2 - 1}$. Then, $t_n = \frac{1}{2}\,[\gamma^n + \delta^n]$; $u_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$.

Applying the above binet forms and summing suitable geometric series into powers of $\frac{\gamma}{\delta}$ and $\frac{\delta}{\gamma}$, one can deduce the following convolution identities:

Theorem 4.3.

$$(a)\quad \sum_{k=0}^{n} t_k t_{n-k} = \frac{1}{2}[(n+1)t_n + u_{n+1}];$$

$$(b)\quad \sum_{k=0}^{n} u_k u_{n-k} = \frac{1}{2(M_p^2 - 1)}[(n+1)t_n - u_{n+1}];$$

$$(c)\quad \sum_{k=0}^{n} t_k u_{n-k} = \sum_{k=0}^{n} u_k t_{n-k} = \frac{n+1}{2} u_n.$$

Applying the identities 2.4$(a),(b)$ and identities 4.3$(a),(b)$ and (c), with suitable simplifications one can finally derive the following convolution identities:

Corollary 4.4.

$$(a)\quad \sum_{r=0}^{n} \lambda_{6r+k}\lambda_{6(n-r)+k} = \left(\frac{\lambda_k^2 + N_p\theta_k^2}{2}\right)(n+1)t_n + 2N_p(p+1)(n+1)\lambda_k\theta_k u_n + \left(\frac{\lambda_k^2 - N_p\theta_k^2}{2}\right)(u_{n+1})$$

$$(b)\quad \sum_{r=0}^{n} \theta_{6r+k}\theta_{6(n-r)+k} = \left(\frac{\lambda_k^2 + N_p\theta_k^2}{2N_P}\right)(n+1)t_n + 2(p+1)(n+1)\lambda_k\theta_k u_n + \left(\frac{\lambda_k^2 - N_p\theta_k^2}{2N_p}\right)(u_{n+1})$$

$$(c)\quad \sum_{r=0}^{n} \lambda_{6r+k}\theta_{6(n-r)+k} = \sum_{r=0}^{n} \theta_{6r+k}\lambda_{6(n-r)+k} = \lambda_k\theta_k(n+1)t_n + (p+1)(\lambda_k^2 + N_p\theta_k^2)(n+1)u_n.$$

Acknowledgments

The authors would like to thank the referees for their invaluable comments and suggestions which led to the improvement of the manuscript.

References

1. Abd-Elhameed, W. M. and Zeyada, N. A., *New identities involving generalized Fibonacci and generalized Lucas numbers*, Indian J. Pure Appl. Math., 49(3), 527–537, (2018).
2. Adegoke, K., Frontczak, R. and Goy, T., *Some notes on a Fibonacci-Lucas identity*, Proc. Jangjeon Math. Soc., 27(4), 859–869, (2024).
3. Barbeau, E. J., *Pell's equation*, Problem Books in Mathematics, Springer, New York, (2003).
4. Benjamin, A. T. and Quinn, J. J., *Fibonacci and Lucas identities through colored tilings*, Util. Math., 56, 137–142, (1999).
5. Bhatnagar, G., *Analogues of a Fibonacci-Lucas identity*, Fibonacci Q., 54(2), 166–171, (2016).
6. Dhanya, P., Nagaraja, K. M. and Siva Kota Reddy, P., *A Note on D'Ocagne's Identity on Generalized Fibonacci and Lucas numbers*, Palest. J. Math., 10(2), 751–755, (2021).
7. Graham, R. L., Knuth, D. E. and Patashnik, O., *Concrete mathematics: a foundation for computer science*, Second Edition, Addison-Wesley Publishing Group, Amsterdam, (1994).
8. Honnegowda C. K., *Studies on Sum Aspects of Combinatorial Number Theory*, Ph.D. Thesis, University of Mysore, Mysore, (2018).
9. Honsberger, R., *Mathematical gems III*, Dolciani Math. Expo., Volume 9, The Mathematical Association of America, Washington, DC, (1985).
10. Jacobson, M. and Williams, H., *Solving the Pell Equation*, CMS Books in Mathematics. Springer, Berlin, (2010).
11. Karadeniz Gözeri, G, *On Pell, Pell-Lucas, and balancing numbers*, J. Inequal. Appl., Volume 2016, Paper No. 3, 16 Pages, (2018).
12. Koshy, T., *Fibonacci and Lucas numbers with applications*, Volume I, Pure and Applied Mathematics, A Wiley-Interscience Series of Texts, Monographs and Tracts, New York, Wiley, (2001).
13. Lam–Estrada, P., Maldonado–Ramírez, M. R., López–Bonilla, J. L., Jarquín–Zárate, F., Rajendra, R. and Siva Kota Reddy, P., *The Sequences of Fibonacci and Lucas for Real Quadratic Number Fields*, *Bol. Soc. Parana. Mat. (3)*, 43, Article Id: 68136, 15 Pages, (2025).
14. Mason, J. C. and Handscomb, D. C., *Chebyshev Polynomials*, Chapman and Hall/CRC, New York, (2002).
15. Niven, I., Zuckerman, H. S. and Montgomery, H. L., *An Introduction to the Theory of Numbers*, 5th Edition, John Wiley & Sons, Inc., (1991).
16. Rangarajan, R., Mukund, R. and Honnegowda, C. K. and Mayura, R., *On combinatorial identities of Pell-Chebyshev twin number pairs*, Palest. J. Math., 11(2), 308–317, (2022).
17. Rangaswamy, *Studies on Brahmagupta polynomials and continued fractions with some interconnections*, Ph.D. Thesis, University of Mysore, Mysore, (2007).
18. Ribenboim, P., *My numbers, my friends: Popular lectures on number theory*, Springer, New York, (2000).
19. Robbins, N., *Beginning number theory*, Dubuque, IA: Wm. C. Brown Publishers, (1993).

20. R. Rangarajan, R., Goutham, C. and Siva Kota Reddy, P., *On Three Pairs of Lucas and Fibonacci-type Combinatorial* $\boldsymbol{p}$*-entities*, *Bol. Soc. Parana. Mat. (3)*, accepted for publication.

21. Suryanarayan, E. R., *The Brahmagupta Polynomials*, The Fibonacci Quarterly, 34, 30–39, (1996).

22. Vajda S., *Fibonacci and Lucas Numbers and Golden Section: Theory and Applications* , Ellis-Horwood, New York, (1989).

R. Rangarajan
Department of Studies in Mathematics
University of Mysore, Mysuru-570 006, India.
E-mail address: rajra63@gmail.com

and

C. Goutham
Department of Studies in Mathematics
University of Mysore, Mysuru-570 006, India.
E-mail address: gautamc600@gmail.com

and

K. Bharatha
Post Graduate Department of Mathematics
St. Philomena's College
Mysuru-570 015, India.
E-mail address: bharatha.k12@gmail.com

and

P. Siva Kota Reddy
Department of Mathematics
JSS Science and Technology University
Mysuru-570 006, India.

and

Universidad Bernardo O'Higgins
Facultad de Ingeniería, Ciencia y Tecnología
Departamento de Formación y Desarrollo Científico en Ingeniería
Av. Viel 1497, Santiago, Chile
E-mail address: pskreddy@jssstuniv.in