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Fixed Point Theorems in Extended Quasi Partial B-Metric Spaces with Applications to Volterra Integral Equations

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ABSTRACT: In this paper, we introduce the concept of extended quasi partial b-metric spaces as a generalization of extended partial b-metric and quasi metric spaces. Some topological properties of the space and some fixed point results are established. Our results generalize several well-known comparable results in the literature. To validate our findings, we provide several illustrative examples. Finally, as an application, the existence of a solution of the Volterra integral equation is presented.

Key Words: Extended quasi partial b-metric space, extended partial b-metric space, quasi metric space, fixed point, Volterra integral equation.

Contents

1. Introduction and preliminaries

The fixed point theory in metric space is an important part of nonlinear functional analysis. In fixed point theory the major developments are inspired by the classic Banach contraction principle. Fixed point theory has many implications in analysis, applied mathematics, engineering, and its development deeply rooted in metric space. In 1906, Maurice René Fréchet first introduced the concept of a metric space and conducted pioneering work in functional analysis. The advancements in technology have lead researchers to extend and improve the concept of a metric space. In 1993, Bakhtin [2] introduced the concept of b-metric space and proved the Banach contraction Principle in b-metric space. Shukla [12] proposed partial b-metric spaces that generalize both b-metric space and partial metric space.

In 2017, Kamran [6] introduces the notion of an extended b-metric space and established fixed point theorems in extended b-metric space. Shatanawi [14,13,15] considered more interesting results on extended b-metric spaces. The concept is a generalization of a b-metric space. These spaces generalize the metric spaces by adding additional elements. In 1994, Mathews [8] introduced the notion of partial metric space as a denotational semantic of data flow network. The most important difference of a partial metric rather than a standard metric is the existing possibility of a non zero self distance. O'Neill [9] generalized the concept of a partial metric space further by admitting negative distances. The partial metric defined by are daulistic partial metric. In a generalization by Heckmann [5], omitted the small self distance axiom and called the partial metric a weak partial metric. Satish [12] introduces the concept of partial b-metric space in 2014, and proved fixed point theorem of the Banach Contraction Principle and Kanan type mapping in partial metric spaces. In 2015, the concept of quasi partial b-metric was introduced [4]. Later, extended partial b-metric was introduced as its generalization by Parvaneh [10]. In which he generalized extended b-metric and partial b-metric.

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In [16], Zada, Shah, and Li advanced the theory of integral-type contractions by developing coupled coincidence fixed point theorems in G-metric spaces. Subsequently, Wang, Zada, Shah, and Li in [17] extended this line of research to dislocated metric spaces with common fixed point results for self-maps, and further generalized these results to dislocated quasi-metric spaces in [18]. Continuing this trajectory, Shah, Zada, and Li [19] introduced new common coupled fixed point theorems for integral-type contractions in generalized metric spaces. The theoretical framework was broadened further by Zada, Shah, and Li [20], who proposed fixed point theorems within ordered cone b-metric spaces. Additional insights were provided by Shah and Zada [21], who explored fixed points for compatible mappings under integral-type contractions in G-metric spaces, and by Shah [22], who contributed new results in the context of b-metric-like spaces.
On the applied side, Turab and Sintunavarat [23,24] utilized fixed point theory to model complex biological and psychological systems, showcasing the Banach fixed point theorem's relevance in practical scientific applications.
In this paper, we shall focus on the very interesting generalization of metric spaces namely, extended quasi partial b-metric spaces. Firstly, we are going to define extended quasi partial b-metric space and examples to validate the definitions. We are introducing extended quasi partial b-metric type and established some fixed point results for contractive mappings. We also present an example for Voltera integral equation to validate our results. To begin with, let us recall some fundamental definitions.

Definition 1.1 *[1] Let Y be a nonempty set and $s \geq 1$ be a given real number. A function $d : Y \times Y \to R^+$ is called a b-metric for all $u, v, w \in Y$ the following conditions are satisfied:*

1. $d(u, v) = 0$ *iff* $u = v$,
2. $d(u, v) = d(v, u)$,
3. $d(u, w) \leq s[d(u, v) + d(v, w)]$.

The pair (Y, d) is called a b-metric space.

Definition 1.2 [6] Let Y be a non empty set and $\theta : Y \times Y \to [0, \infty)$. A function $d_\theta : Y \times Y \to [1, \infty)$ is called extended b-metric space if for all $u, v, w \in Y$ it satisfies:

1. $d_\theta(u, v) = 0$ iff $u = v$,
2. $d_\theta(u, v) = d_\theta(v, u)$,
3. $d_\theta(u, w) = \theta(u, w)[d_\theta(u, v) + d_\theta(v, w)]$.

Definition 1.3 [12] Let Y be a non empty set. A function $Y \times Y \to [0, \infty)$ is a partial b-metric on Y, if there exists a real number $\alpha \geq 1$ such that the following conditions hold for $u, v, w \in Y$:

1. $d(u, u) = d(u, v)$ iff $u = v$,
2. $d(u, u) \leq d(u, v)$,
3. $d(u, v) = d(v, u)$,
4. $d(u, v) \leq \alpha[d(u, w) + d(w, v) - d(w, w)]$.

The pair (Y, d) is called a partial b-metric space.

Definition 1.4 [3] Let Y be non empty set and $\alpha : Y \times Y \to [1 + \infty)$ be a continuous function a function where $\alpha : Y \times Y \to [1, +\infty)$ is called partial extended b-metric space if for all $x, y, z \in Y$ it satisfies the following:

1. $P_\alpha(u, v) = P_\alpha(v, v) = P_\alpha(u, v)$iff $u = v$,
2. $P_\alpha(u, v) \leq P_\alpha(u, v)$,
3. $P_\alpha(u, v) = P_\alpha(v, u)$,
4. $P_\alpha(u, w) \leq \alpha(u, v)[P_\alpha(u, v) + P_\alpha(v, w) - P\alpha(w, w)]$.

Definition 1.5 [10] Let Y be a (non-empty) set and $\Omega : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function with $\Omega^{-1}(t) \le t \le \Omega(t)$ for $t \in [0,\infty)$. A function $p : Y \times Y \to \mathbb{R}^+$ is called an extended partial b-metric, or a partial p-metric, if, for all $u, v, w \in Y$, the following conditions are satisfied:

1. $u = v \iff p(u,u) = p(u,v) = p(v,v)$,
2. $p(u,u) \le p(u,v)$,
3. $p(u,v) = p(v,u)$,
4. $p(u,v) - p(u,u) \le \Omega\left(p(u,w) + p(w,v) - p(w,w) - p(u,u)\right)$.

The pair (Y,p) is called an extended partial b-metric space.
Now, we give an example of extended partial b-metric space.

Example 1.1 *Let $Y = \mathbb{R}^2$ and define:*

$$p(u,v) = \max\left\{\|u-v\|_1, \frac{1}{2}\|u-v\|_2\right\}$$

where $\|\cdot\|_1$ is the L_1 norm and $\|\cdot\|_2$ is the Euclidean norm. Specifically,

$$\|u-v\|_1 = |u_1 - v_1| + |u_2 - v_2|.$$

and

$$\|u-v\|_2 = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2}.$$

Let $\Omega : [0,\infty) \to [0,\infty)$ be defined as $\Omega(t) = 2t$. Hence, (Y,p) is a partial extended b-metric space.

Definition 1.6 [4] A quasi partial b-metric on a non-empty set Y is a mapping $\mathrm{qpb} : Y \times Y \to \mathbb{R}_+$ such that for some real number $s \ge 1$ and all $u, v, w \in Y$, the following conditions are satisfied:

1. $\mathrm{qpb}(u,u) = \mathrm{qpb}(u,v) = \mathrm{qpb}(v,v) \implies u = v$,
2. $\mathrm{qpb}(u,u) \le \mathrm{qpb}(u,v)$,
3. $\mathrm{qpb}(u,u) \le \mathrm{qpb}(v,u)$,
4. $\mathrm{qpb}(u,v) \le s[\mathrm{qpb}(u,w) + \mathrm{qpb}(v,w)] - \mathrm{qpb}(w,w)$.

The pair (Y, qpb) is called quasi partial b-metric space.
An example of quasi partial b-metric is given below.

Example 1.2 *Let $Y = \mathbb{R}$ and define the mapping $qpb : Y \times Y \to \mathbb{R}_+$ by*

$$qpb(u,p) = \begin{cases} |u-p| + 1 & \text{if } u \le p, \\ 2|u-p| + 1 & \text{if } u > p. \end{cases}$$

$qpb(u,p)$ satisfies the conditions for a quasi-partial b-metric for some $s \ge 1$ and all $u, p, w \in Y$.

Now, we present our main results in the upcoming section.

2. Main results

To begin with, we introduce the concept of the extended quasi-partial b-metric space in this context.

Definition 2.1 Let Y is a non-empty set and $\Omega : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function with $\Omega^{-1}(t) \le t \le \Omega(t)$ for $t \in [0,\infty)$. A function $p_\Omega : Y \times Y \to \mathbb{R}^+$ is called an extended quasi partial b-metric if, for all $u, v, w \in Y$, the following conditions are satisfied:

(1) $u = v$ iff $p_\Omega(u,u) = p_\Omega(u,v) = p_\Omega(v,v)$,

(2) $p_\Omega(u,u) \le p_\Omega(u,v)$,

(3) $p_\Omega(u,v) \leq \Omega(p_\Omega(u,w) + p_\Omega(w,v) - p_\Omega(w,w) - p_\Omega(u,u))$.

The pair (Y, p_Ω) is called an extended quasi partial b-metric space.

Remark 2.1 Every quasi partial b-metric space is an extended quasi partial b-metric space with $\Omega(t) = t$, but the converse is not necessarily true.

Some examples of extended quasi partial b-metric spaces are presented below.

Example 2.1 *Let $Y = \mathbb{R}$ and define $p : Y \times Y \to \mathbb{R}_+$ by*

$$p_\Omega(u,v) = \begin{cases} e^{|u-v|} + 1 & \text{if } u \leq v, \\ 2e^{|u-v|} + 1 & \text{if } u > v. \end{cases}$$

and the function

$$\Omega(t) = e^t + 1.$$

1. $u = v \iff p_\Omega(u,u) = p_\Omega(u,v) = p_\Omega(v,v)$.
   If $u = v$:
$$p_\Omega(u,u) = e^{|u-u|} + 1 = e^0 + 1 = 2$$
$$p_\Omega(u,v) = p_\Omega(v,u) = 2 \text{ (since } u = v)$$
   If $u \neq v$:
$$p_\Omega(u,v) = \begin{cases} e^{|u-v|} + 1 & \text{if } u \leq v, \\ 2e^{|u-v|} + 1 & \text{if } u > v \end{cases}$$
   Thus, $p_\Omega(u,u) = 2$ and $p_\Omega(u,v) \neq 2$ when $u \neq v$.

2. $p_\Omega(u,u) \leq p_\Omega(u,v)$.
   For $u \leq v$:
$$p_\Omega(u,u) = 2$$
$$p_\Omega(u,v) = e^{v-u} + 1$$
   Since, $e^{v-u} \geq 1$, it follows that $e^{v-u} + 1 \geq 2$. For $u > v$:
$$p_\Omega(u,u) = 2$$
$$p_\Omega(u,v) = 2e^{u-v} + 1$$
   Since, $2e^{u-v} + 1 \geq 2$, Thus Condition (2) is satisfied.

3. $p_\Omega(u,v) \leq \Omega(p_\Omega(u,w) + p_\Omega(w,v) - p_\Omega(w,w) - p_\Omega(u,u))$, *where $\Omega(t) = e^t + 1$, For $u \leq v$:*
$$p_\Omega(u,v) = (e^{v-u} + 1) = e^{v-u} + 1$$
   For $u > v$:
$$p_\Omega(u,v) = (2e^{u-v} + 1) = 2e^{u-v} + 1$$
   For $u \leq w \leq v$, , we can write:
$$\begin{aligned} p_\Omega(u,w) &= e^{w-u} + 1 \\ p_\Omega(w,v) &= e^{v-w} + 1 \\ p_\Omega(w,w) &= 2. \\ \textit{implies that} (e^{v-u} + 1) &\leq \Omega(e^{w-u} + 1 + e^{v-w} + 1 - 2) \\ e^{v-u} + 1 &\leq \Omega(e^{w-u} + e^{v-w} + 2 - 2) \\ &\leq e^{e^{w-u} + e^{v-w}} - 1. \end{aligned}$$

For $u > w \leq v$, It can be expressed that:

$$\begin{aligned}
p_\Omega(u,w) &= 2e^{u-w} + 1\\
p_\Omega(w,v) &= e^{v-w} + 1\\
p_\Omega(w,w) &= 2.\\
\textit{implies that}\, 2e^{u-v} + 1 &\leq \Omega(2e^{u-w} + 1 + e^{v-w} + 1 - 2)\\
2e^{u-v} + 1 &\leq \Omega(2e^{u-w} + e^{v-w} + 2 - 2)\\
&\leq e^{2e^{u-w}+e^{v-w}} - 1.
\end{aligned}$$

Condition (3) is fulfilled as the right-hand side, containing Ω, increases more rapidly than the left-hand side because of the exponential term. Moreover, $p_\Omega(u,v)$ together with $\Omega(t)$ meets all conditions necessary for an extended quasi partial b-metric.

Example 2.2 *Let $Y = [0,\infty]$. Consider the function $p_\Omega : Y \times Y \to [0,\infty)$ defined by*

$$p_\Omega(x,y) = x^2 + y^2 + |x-y|.$$

Let $\Omega : [0,\infty) \to [0,\infty)$ be given by $\Omega(t) = t^2$. We aim to verify whether the function p_Ω satisfies the conditions required to be an extended quasi partial b-metric on the space Y.

1. *If $u = v$: We have,*

$$\begin{aligned}
p_\Omega(u,u) &= 2u^2\\
p_\Omega(u,v) &= u^2 + u^2 + |u-u| = 2u^2\\
p_\Omega(v,v) &= 2u^2.
\end{aligned}$$

Thus, $p_\Omega(u,u) = p_\Omega(u,v) = p_\Omega(v,v)$ when $u = v$.

2. *$p_\Omega(u,u) \leq p_\Omega(u,v)$. We may write,*

$$\begin{aligned}
p_\Omega(u,u) &= 2u^2\\
p_\Omega(u,v) &= u^2 + v^2 + |u-v|.
\end{aligned}$$

We need to verify,

$$\begin{aligned}
2u^2 &\leq u^2 + v^2 + |u-v|\\
u^2 &\leq v^2 + |u-v|
\end{aligned}$$

This inequality is always true because $|u-v| \geq 0$.

3. *$p_\Omega(u,v) \leq \Omega(p_\Omega(u,w) + p_\Omega(w,v) - p_\Omega(w,w) - p_\Omega(u,u))$ where $\Omega(t) = t^2$,*

$$\begin{aligned}
(u^2 + v^2 + |u-v|) &= \Omega((u^2 + w^2 + |u-w|) + (w^2 + v^2 + |w-v|)\\
&\quad - (w^2 + w^2) - 2u^2)\\
&= \Omega(u^2 + v^2 + |u-w| + |w-v| - 2u^2)\\
v^2 - u^2 + |u-v| &\leq (u^2 + v^2 + |u-w| + |w-v| - 2u^2)^2.
\end{aligned}$$

Observe that the right-hand side expression,

$$(u^2 + v^2 + |u-w| + |w-v|)^2,$$

grows substantially faster than the left-hand side expression,

$$v^2 - u^2 + |u-v|.$$

As a result, the generalized triangle inequality required for an extended quasi partial b-metric is satisfied. Therefore, the function $p_\Omega(u,v) = u^2 + v^2 + |u-v|$ meets all the conditions necessary to be an extended quasi partial b-metric.

Example 2.3 *Let $Y = C([a,b],\mathbb{R})$ be the space of all continuous real-valued functions defined on the closed interval $[a,b]$. Define a function $p_\Omega : Y \times Y \to [0,\infty)$ by*

$$p_\Omega(u,v) = \sup_{t\in[a,b]} (u(t) - v(t)) + 1,$$

and let $\Omega : [0,\infty) \to [0,\infty)$ be given by $\Omega(t) = t^2$. Then (Y, p_Ω) is a complete extended quasi partial b-metric space.

In an extended quasi partial b-metric space, we introduce the concepts of Cauchy sequences and convergence.

Definition 2.2 Let (Y, p_Ω) be an extended quasi partial b-metric space. Consider a sequence $\{u_n\} \subset Y$ and a point $u \in Y$. Then the following statements hold:

1. The sequence $\{u_n\}$ converges to u in (Y, p_Ω) if and only if
$$\lim_{n\to\infty} p_\Omega(u_n, u) = 0.$$
2. The sequence $\{u_n\}$ is a Cauchy sequence in (Y, p_Ω) if and only if
$$\lim_{n,m\to\infty} p_\Omega(u_n, u_m) = 0.$$
3. The space (Y, p_Ω) is said to be complete if every Cauchy sequence in Y converges to a point in Y; that is, for every Cauchy sequence $\{u_n\} \subset Y$, there exists $u \in Y$ such that
$$\lim_{n\to\infty} p_\Omega(u_n, u) = 0.$$

Lemma 2.1 *Let (Y, p_Ω) be an extended quasi partial b-metric space. If the function p_Ω is continuous, then every convergent sequence in Y has a unique limit.*

Lemma 2.2 *Let (Y, p_Ω) be an extended quasi partial b-metric space. Then for any $u \in Y$, it holds that*

$$p_\Omega(u,u) > 0.$$

Lemma 2.3 *Let (Y, p_Ω) be an extended quasi-partial b-metric space and (Y, d_Ω) be the corresponding b-metric space. Then (Y, d_Ω) is complete if (Y, p_Ω) is complete.*

Proof: Since, (Y, p_Ω) is complete, every Cauchy sequence $\{u_n\}$ in Y converges with respect to τ_{p_Ω} to a point $u \in Y$ such that

$$p_\Omega(u,u) = \lim_{n,m\to\infty} p_\Omega(u_n, u_m) = \lim_{n,m\to\infty} p_\Omega(u_m, u_n). \tag{2.1}$$

Consider a Cauchy sequence $\{u_n\}$ in (u, d_Ω). We will show that $\{u_n\}$ is Cauchy in (u, p_Ω). Since $\{u_n\}$ is Cauchy in (u, d_Ω), $\lim_{n,m\to\infty} d_\Omega(u_n, u_m)$ exists and is finite. Also,

$$d_\Omega(u_n, u_m) = p_\Omega(u_n, u_m) + p_\Omega(u_m, u_n) - p_\Omega(u_n, u_n) - p_\Omega(u_m, u_m).$$

Clearly, $\lim_{n,m\to\infty} p_\Omega(u_n, u_m)$ and $\lim_{n,m\to\infty} p_\Omega(u_m, u_n)$ exist and are finite. Therefore, $\{u_n\}$ is a Cauchy sequence in (Y, p_Ω). Now, since (Y, p_Ω) is complete, the sequence $\{u_n\}$ converges with respect to τ_{p_Ω} to a point $u \in Y$ such that (2.1) holds. For $\{u_n\}$ to be convergent in (Y, d_Ω), we will show that $d_\Omega(u,u) = \lim_{n\to\infty} d_\Omega(u, u_n)$.

It follows from the definition of d_Ω that $d_\Omega(u,u) = 0$. Also,

$$\lim_{n\to\infty} d_\Omega(u, u_n) = \lim_{n\to\infty} [p_\Omega(u, u_n) + p_\Omega(u_n, u) - p_\Omega(u_n, u_n) - p_\Omega(u, u)]$$
$$= 0 \text{ by (1) and the definition of convergence in } (Y, p_\Omega).$$

Hence, $d_\Omega(u,u) = \lim_{n\to\infty} d_\Omega(u, u_n)$. □

3. Fixed point theorems for generalized extended quasi partial b-metric spaces

In this section, we prove several fixed point theorems within the framework of complete extended quasi partial b-metric spaces. We begin by establishing the following result, which serves as a generalization of the Banach fixed point theorem within the framework of extended quasi-partial b-metric spaces. Throughout this section, for the mapping $T : Y \to Y$ and $u_0 \in Y$, $\mathcal{O}(u_0) = \{u_0, T^2u_0, T^3u_0, \dots\}$ represent the orbit of u_0.

Theorem 3.1 *Let (Y, p_Ω) be a complete extended quasi partial b-metric space such that p_Ω is a continuous functional. Let $T : Y \to Y$ satisfy*

$$p_\Omega(Tu, Tv) \le \lambda p_\Omega(u, v), \tag{3.1}$$

for all $u, v \in Y$, where $0 \le \lambda < 1$, and assume that for each $u_0 \in Y$, and $\Omega : [0, \infty) \to [0, \infty)$ be a strictly increasing continuous function with $\Omega^{-1}(t) \le t \le \Omega(t)$ for $t \in [0, \infty)$. where $u_n = T^n u_0$. Then, T has a unique fixed point $u^* \in Y$.

Proof: To begin the proof, we show that any fixed point of T, if it exists, must be unique. Let u^*, $u^{**} \in Y$ be distinct fixed points of T. It follows from (3.1) that

$$p_\Omega(u^*, u^{**}) = p_\Omega(Tu^*, Tu^{**}) \le \lambda p_\Omega(u^*, u^{**}) < p_\Omega(u^*, u^{**}),$$

which is a contradiction. Therefore, we must have $p_\Omega(u^*, u^{**}) = 0$. Furthermore, if u^* is a fixed point and $p_\Omega(u^*, u^*) > 0$, then from (3.1) we get

$$p_\Omega(u^*, u^*) = p_\Omega(Tu^*, Tu^*) \le \lambda p_\Omega(u^*, u^*) < p_\Omega(u^*, u^*),$$

a contradiction. Thus, $p_\Omega(u^*, u^*) = 0$. We choose any $u_0 \in Y$ arbitrarily and define the iterative sequence $\{u_n\}$ by:

$$u_0, \quad Tu_0 = u_1, \quad u_2 = Tu_1 = T(Tu_0) = T^2(u_0), \quad \dots, \quad u_n = T^n u_0, \quad \dots.$$

Then, by successively applying inequality (3.1), we obtain:

$$p_\Omega(u_n, u_{n+1}) \le \lambda^n p_\Omega(u_0, u_1). \tag{3.2}$$

By the triangular inequality and inequality (3.2), for $m > n$, we have:

$$\begin{aligned}
p_\Omega(u_n, u_m) &\le \Omega[p_\Omega(u_n, u_{n+1}) + p_\Omega(u_{n+1}, u_m) \\
&\quad - p_\Omega(u_{n+1}, u_{n+1}) - p_\Omega(u_n, u_n)] \\
&\le \Omega[p_\Omega(u_n, u_{n+1}) + \Omega[p_\Omega(u_{n+1}, u_{n+2}) + p_\Omega(u_{n+2}, u_m) \\
&\quad - p_\Omega(u_{n+2}, u_{n+2}) - p_\Omega(u_{n+1}, u_{n+1})] \\
&\quad - p_\Omega(u_{n+1}, u_{n+1}) - p_\Omega(u_n, u_n)] \\
&\le \Omega[p_\Omega(u_n, u_{n+1}) + \Omega[p_\Omega(u_{n+1}, u_{n+2}) + \cdots \\
&\quad + \Omega[p_\Omega(u_{m-1}, u_{m-2}) - p_\Omega(u_{m-1}, u_{m-1}) \\
&\quad - p_\Omega(u_{m-2}, u_{m-2}) - \cdots] \\
&\quad - p_\Omega(u_{n+1}, u_{n+1})] - p_\Omega(u_n, u_n)] \\
&\le \Omega[\lambda^n p_\Omega(u_0, u_1) + \Omega[\lambda^{n+1} p_\Omega(u_0, u_1) + \dots \\
&\quad + \Omega[\lambda^{m-1} p_\Omega(u_0, u_1) - p_\Omega(u_{m-1}, u_{m-1}) \\
&\quad - p_\Omega(u_{m-2}, u_{m-2}) - \cdots] \\
&\quad - p_\Omega(u_{n+1}, u_{n+1})] - p_\Omega(u_n, u_n)].
\end{aligned}$$

Since, $\lambda^n p_\Omega(u_0, u_1) + \lambda^{n+1} p_\Omega(u_0, u_1) + \cdots + \lambda^{m-1} p_\Omega(u_0, u_1)$ is bounded above by,

$$p_\Omega(u_0, u_1) \cdot \frac{\lambda^n(1 - \lambda^{m-n})}{1 - \lambda}$$

and,

$$\Omega\left[\lambda^n p_\Omega(u_0,u_1)+\Omega\left[\lambda^{n+1}p_\Omega(u_0,u_1)+\cdots+\Omega\left[\lambda^{m-1}p_\Omega(u_0,u_1)-p_\Omega(u_{m-1},u_{m-1})\right]\right]\right]$$

is bounded above by:

$$\Omega\left[p_\Omega(u_0,u_1)\cdot\frac{1-\lambda^{m-n}}{1-\lambda}-p_\Omega(u_{m-1},u_{m-1})\right]$$

Since, $\lambda<1$, as $n\to\infty$, $\lambda^n\to 0$. This implies that:

$$\lambda^n p_\Omega(u_0,u_1)\to 0$$

. Thus, the series converges to a finite value, since the geometric series converges. Letting $n\to\infty$, we conclude that $\{u_n\}$ is a Cauchy sequence. Since, Y is complete, letting $u_n\to u^*\in Y$, we have:

$$\begin{aligned}p_\Omega(Tu^*,u^*)&\le\Omega\left(p_\Omega(Tu^*,u_n)+p_\Omega(u_n,u^*)-p_\Omega(u_n,u_n)-p_\Omega(Tu^*,Tu^*)\right)\\&\le\Omega\left(\lambda p_\Omega(u^*,u_{n-1})+p_\Omega(u_n,u^*)-p_\Omega(u_n,u_n)-p_\Omega(Tu^*,Tu^*)\right)\\&\le\Omega(0-0-0)=0\text{ as }n\to\infty.\end{aligned}$$

Hence, u is a fixed point of T. □

Now, we present some examples for the validation of Theorem (3.1).

Example 3.1 *Let $Y=[0,\infty)$. Define $p_\Omega(u,v):Y\times Y\to\mathbb{R}^+$ and $\Omega(t):[0,\infty)\to[0,\infty)$ as:*

$$p_\Omega(u,v)=|u-2v|+\frac{1}{2},\quad \Omega=t+2$$

Then, p_Ω is a complete extended quasi partial b-metric on X. Define $T:Y\to Y$ by $T(u)=2u$. We have:

$$\begin{aligned}p_\Omega(Tu,Tv)&=\left|\frac{u}{2}-\frac{2v}{2}\right|+\frac{1}{2}\\&\le\frac{1}{2}\left(|u-2v|+1\right)\\&\le\lambda p_\Omega(u,v),\end{aligned}$$

and $\Omega:[0,\infty)\to[0,\infty)$ is defined as $\Omega(t)=t+2$ for $t\in[0,\infty)$. All conditions of Theorem (3.1) are satisfied. Hence, T has a unique fixed point.

Example 3.2 *Let $Y=[0,\infty)$. Define $p_\Omega(u,v):Y\times Y\to\mathbb{R}^+$ and $\Omega(t):[0,\infty)\to[0,\infty)$ as:*

$$p_\Omega(u,v)=|e^u-v|+\frac{1}{2},\quad \Omega(t)=e^t+1$$

Then, p_Ω is a complete extended quasi partial b-metric on Y. Define $T:Y\to Y$ by $T(u)=u+1$. We have:

$$\begin{aligned}p_\Omega(Tu,Tv)&=\left|e^{u+1}-(v+1)\right|+\frac{1}{2}\\&=\left|e^{u+1}-v-1\right|+\frac{1}{2}.\end{aligned}$$

In general, since $e^{u+1}=e\cdot e^u$, the inequality would be:

$$\begin{aligned}|e\cdot e^u-v-1|+\frac{1}{2}&\le\lambda\left(|e^u-v|+\frac{1}{2}\right)\\&\le\lambda p_\Omega(u,v).\end{aligned}$$

By examining the inequality, we identify a suitable λ. Given that $T(u)=u+1$ induces a shift in the exponential function, it becomes straightforward to find such a λ. Therefore, all the assumptions of Theorem 3.1 hold, guaranteeing the uniqueness of the fixed point of T.

Corollary 3.1 *Let (Y, p_Ω) be a complete extended quasi partial b-metric space such that p_Ω is a continuous functional. Let $T : Y \to Y$ satisfy*

$$p_\Omega(T^n u, T^n v) \leq \lambda p_\Omega(u, v) \tag{3.3}$$

for all $u, v \in Y$, where $0 \leq \lambda < 1$, and assume that for each $u_0 \in Y$, and $\Omega : [0, \infty) \to [0, \infty)$ be a strictly increasing continuous function with $\Omega^{-1}(t) \leq t \leq \Omega(t)$ for $t \in [0, \infty)$. where $u_n = T^n u_0$. Then T has a unique fixed point $u^* \in Y$.

Proof: The proof of this corollary follows by writing $T = T^n$, from Theorem (3.1), so that T^n has a unique fixed point. □

In [7], Karapinar et al. proved a fixed point theorem in quasi partial metric spaces. Motivated by their work, we have extended these results to the setting of extended quasi partial b-metric spaces.

Theorem 3.2 *Let (Y, p_Ω) be a complete extended quasi partial b-metric space, where $\Omega(t) : [0, \infty) \to [0, \infty)$ and let $T : Y \to Y$. Then the following hold:*

(a) There exists $\varphi : Y \to \mathbb{R}_+$ such that

$$p_\Omega(u, Tu) \leq \varphi(u) - \varphi(Tu) \quad \text{for all } u \in Y$$

if and only if

$$\sum_{n=0}^{\infty} p_\Omega(T^n u, T^{n+1} u) \text{ converges for all } u \in Y.$$

(b) There exists $\varphi : u \to \mathbb{R}^+$ such that

$$p_\Omega(u, Tu) \leq \varphi(u) - \varphi(Tu) \text{ for all } u \in \mathcal{O}(u)$$

if and only if

$$\sum_{n=0}^{\infty} p_\Omega(T^n u, T^{n+1} u) \text{ converges for all } u \in \mathcal{O}(u).$$

Proof: Let $u \in Y$, and let

$$p_\Omega(u, Tu) \leq \varphi(u) - \varphi(Tu).$$

Define the sequence $\{u_n\}_{n=1}^{\infty}$ in the following way:

$$u_0 = u \text{ and } u_{n+1} = Tu_n = T^{n+1} u_0, \text{ for all } n = 0, 1, 2, \ldots.$$

By setting:

$$S_n = \sum_{k=0}^{n} p_\Omega(u_k, u_{k+1}) = \sum_{k=0}^{n} p_\Omega(T^k u_0, T^{k+1} u_0).$$

Then, we have:

$$\begin{aligned} S_n &\leq \sum_{k=0}^{n} \left[\varphi(T^k u_0) - \varphi(T^{k+1} u_0)\right] \\ &= [\varphi(u_0) - \varphi(Tu_0)] + \cdots + [\varphi(T^n u_0) - \varphi(T^{n+1} u_0)] \\ &= \varphi(u_0) - \varphi(T^{n+1} u_0) \leq \varphi(u_0) = \varphi(u). \end{aligned} \tag{2.1}$$

Thus, (2.1) implies that $\{S_n\}$ is bounded. Also, $\{S_n\}$ is non-decreasing by definition, and hence it is convergent. Define:

$$\varphi(u) = \sum_{n=0}^{\infty} p_\Omega(T^n u, T^{n+1} u) \text{ and } S_n(u) = \sum_{k=0}^{n} p_\Omega(T^k u, T^{k+1} u).$$

Then,

$$\varphi(Tu)=\sum_{n=0}^{\infty}p_{\Omega}(T^{n+1}u,T^{n+2}u)\text{ and }S_n(Tu)=\sum_{k=0}^{n}p_{\Omega}(T^{k+1}u,T^{k+2}u).$$

Using these definitions, we get:

$$\begin{aligned}S_n(u)-S_n(Tu)&=\sum_{k=0}^{n}p_{\Omega}(T^{k}u,T^{k+1}u)-\sum_{k=0}^{n}p_{\Omega}(T^{k+1}u,T^{k+2}u)\\&=p_{\Omega}(u,Tu)-p_{\Omega}(T^{n+1}u,T^{n+2}u).\end{aligned}\tag{2.2}$$

Since, $\sum_{n=0}^{\infty}p_{\Omega}(T^nu,T^{n+1}u)$ converges for all $u\in Y$,

$$\lim_{n\to\infty}S_n(u)=\varphi(u)\text{ and }\lim_{n\to\infty}p_{\Omega}(T^nu,T^{n+1}u)=0.$$

Letting $n\to\infty$ in (2.2) yields

$$p_{\Omega}(u,Tu)=\varphi(u)-\varphi(Tu),$$

and hence the proof is complete. The proof of (b) follows analogously and is therefore omitted. □

To validate Theorem 3.2, we offer the following examples.

Example 3.3 *Let $Y=[0,\infty)$. Define:*

$$p_{\Omega}(u,y)=|u-y|+|u|,\quad \textit{and}\quad \Omega(t)=t^2,$$

Clearly, p_{Ω} satisfy the axioms of extended quasi partial b-metric spsace. Define $T(u)=\frac{u}{3}$. Then the series is convergent $\sum_{n=0}^{\infty}p_{\Omega}(T^nu,T^{n+1}u)$ is convergent.

$$\begin{aligned}\sum_{n=0}^{\infty}p_{\Omega}(T^nu,T^{n+1}u)&=\sum_{n=0}^{\infty}p_{\Omega}\left(\frac{u}{3^n-1}+\frac{u}{3^n}\right)\\&=\sum_{n=0}^{\infty}\left|\frac{u}{3^n-1}-\frac{u}{3^n}\right|+\left|\frac{u}{3^n-1}\right|\\&=\sum_{n=0}^{\infty}\frac{u}{3^n}+\frac{u}{3^n-1}\\&=\sum_{n=0}^{\infty}\frac{3u}{3^n}\\&=\frac{3u}{1-3}=\frac{9u}{2}.\end{aligned}$$

Thus, all the conditions of Theorem (3.2) are satisfied for $\varphi(u)=\frac{9u}{2}$.

Example 3.4 *Let $Y=(0,\infty)$. Define the metric $p_{\Omega}(u,y)$ and the function $\Omega(t)$ as follows:*

$$p_{\Omega}(u,y)=|u-y|+e^{-u},\qquad \textit{where}\ \ \Omega(t)=t+2.$$

Clearly, p_{Ω} satisfies the axioms of an extended quasi partial b-metric space. Define $T:Y\to Y$ by

$T(u)=\frac{u}{2}$. Then the series is convergent $\sum_{n=0}^{\infty} p_\Omega(T^n u, T^{n+1}u)$ is convergent, we may write:

$$\begin{aligned}\sum_{n=0}^{\infty} p_\Omega(T^n u, T^{n+1}u) &= \sum_{n=0}^{\infty} p_\Omega(\frac{u}{2^n}, \frac{u}{2^{n+1}})\\ &= \sum_{n=0}^{\infty}\left(\frac{u}{2^{n+1}} + e^{-\frac{u}{2^n}}\right)\\ &= \sum_{n=0}^{\infty}\frac{u}{2^{n+1}} + \sum_{n=0}^{\infty} e^{-\frac{u}{2^n}}\\ &= u\sum_{n=0}^{\infty}\frac{1}{2^{n+1}} + \sum_{n=0}^{\infty} e^{-\frac{u}{2^n}}\\ &= u\cdot 1 + \sum_{n=0}^{\infty} e^{-\frac{u}{2^n}}.\end{aligned}$$

and $\sum_{n=0}^{\infty} e^{-\frac{u}{2^n}}$, converges to a finite value. Thus, the total sum converges to a positive real number. Hence, it follows that

$$\varphi(u) = u\cdot 1 + \sum_{n=0}^{\infty} e^{-\frac{u}{2^n}}.$$

Thus, all the conditions of Theorem (3.2) are satisfied for $\varphi(u)$.

The following result provides conditions for the existence of fixed points of operators on extended quasi partial b-metric spaces.

Theorem 3.3 *Consider two complete extended quasi partial b-metric spaces (Y, p_Ω) and (Y_1, p_Ω). Let mappings $T: Y \to Y$, $R: Y \to Y_1$, and a function $\varphi: R(Y) \to \mathbb{R}^+$ be given. If there exist a point $u \in Y$ and a constant $c > 0$ such that for all $v \in \mathcal{O}(u)$, the inequality*

$$\max\{p_\Omega(v, Tv), c\, p_\Omega(Rv, RTv)\} \le \varphi(Rv) - \varphi(RTv) \tag{3.4}$$

holds, then the following are true:

(A) The limit $\lim_{n\to\infty} T^n u = w$ exists in Y.

(B) The point w satisfies $Tw = w$ if and only if the function $G(u) = p_\Omega(u, Tu)$ is T-orbitally lower semi-continuous at u.

Proof:

(A) Let $u \in Y$. Define the sequence $\{u_n\}_{n=1}^{\infty}$ as follows:

$$u_0 = u \quad \text{and} \quad u_{n+1} = Tu_n = T^{n+1}u_0, \quad \text{for all } n = 0, 1, 2, \ldots.$$

We will show that the sequence $\{u_n\}_{n=1}^{\infty}$ is Cauchy. By applying the triangle inequality, we have:

$$p_\Omega(u_n, u_{n+2}) \le \Omega(p_\Omega(u_n, u_{n+1}) + p_\Omega(u_{n+1}, u_{n+2}) - p_\Omega(u_{n+1}, u_{n+1}) - p_\Omega(u_n, u_n)).$$

and similarly,

$$\begin{aligned}p_\Omega(u_n, u_{n+3}) &\le \Omega(p_\Omega(u_n, u_{n+2}) + p_\Omega(u_{n+2}, u_{n+3}) - p_\Omega(u_{n+2}, u_{n+2}) - p_\Omega(u_n, u_n))\\ &\le \Omega(\Omega(p_\Omega(u_n, u_{n+1}) + p_\Omega(u_{n+1}, u_{n+2}) - p_\Omega(u_{n+1}, u_{n+1}) - p_\Omega(u_n, u_n))\\ &\quad + p_\Omega(u_{n+2}, u_{n+3}) - p_\Omega(u_{n+2}, u_{n+2}) - p_\Omega(u_n, u_n)).\end{aligned}$$

Now,

$$\begin{aligned}p_\Omega(u_n,u_{n+4}) &\le \Omega(p_\Omega(u_n,u_{n+3})+p_\Omega(u_{n+3},u_{n+4})-p_\Omega(u_{n+3},u_{n+3})-p_\Omega(u_n,u_n))\\ &\le \Omega(\Omega(\Omega(p_\Omega(u_n,u_{n+1})+p_\Omega(u_{n+1},u_{n+2})-p_\Omega(u_{n+1},u_{n+1})-p_\Omega(u_n,u_n))\\ &\quad +p_\Omega(u_{n+2},u_{n+3})-p_\Omega(u_{n+2},u_{n+2})-p_\Omega(u_n,u_n))+p_\Omega(u_{n+3},u_{n+4})\\ &-p_\Omega(u_{n+3},u_{n+3})-p_\Omega(u_n,u_n))\end{aligned}$$

On generalization, we get:

$$\begin{aligned}p_\Omega(u_n,u_m) &\le \Omega(\Omega(\cdots\Omega(p_\Omega(u_n,u_{n+1})+p_\Omega(u_{n+1},u_{n+2})\\ &\quad -p_\Omega(u_{n+1},u_{n+1})-p_\Omega(u_n,u_n))+p_\Omega(u_{n+2},u_{n+3})\\ &\quad -p_\Omega(u_{n+2},u_{n+2})-p_\Omega(u_n,u_n))+\cdots+p_\Omega(u_{m-1},u_m)\\ &\quad -p_\Omega(u_{m-1},u_{m-1})-p_\Omega(u_n,u_n))\\ p_\Omega(u_n,u_m) &\le \Omega\Big(\Omega\Big(\cdots\Omega\Big(\Big(\sum_{i=n}^{m-2}\Big(p_\Omega(u_i,u_{i+1})-p_\Omega(u_i,u_i)\Big)\Big)+\Big(\sum_{i=n}^{m-2}p_\Omega(u_{i+1},u_{i+2})\Big)\Big)\Big)\Big)\\ &\le \Omega\left(\sum_{i=n}^{m-2}\left(p_\Omega(T^iu,T^{i+1}u)-p_\Omega(T^iu,T^iu)\right)+\sum_{i=n}^{m-2}p_\Omega(T^{i+1}u,T^{i+2}u)\right)\end{aligned}$$

Since, Ω is an increasing function, we have:

$$\Omega\left(\sum_{i=n}^{m-2}\left(p_\Omega(T^iu,T^{i+1}u)-p_\Omega(T^iu,T^iu)\right)\right)\le\Omega\left(\sum_{i=n}^{m-2}p_\Omega(T^iu,T^{i+1}u)\right)$$

$$p_\Omega(u_n,u_m)\le\Omega\left(\sum_{i=n}^{m-2}p_\Omega(T^iu,T^{i+1}u)\right) \tag{3.5}$$

Set $z_n(u)=\Omega\sum_{i=n}^{m-2}p_\Omega(T^iu,T^{i+1}u)$. From 3.4, we have:

$$\begin{aligned}\sum_{k=n}^{m-2}\Omega p_\Omega(T^ku,T^{k+1}u) &\le \Omega\sum_{k=n}^{m-2}\max\left\{p_\Omega(T^ku,T^{k+1}u),cp_\Omega(RT^ku,RT^{k+1}u)\right\}\\ \implies z_n(u) &\le \Omega\left(\sum_{i=n}^{m-2}\max\left\{p_\Omega(T^iu,T^{i+1}u),cp_\Omega(RT^iu,RT^{i+1}u)\right\}\right)\\ &\le \Omega\left(\phi(RT^nu)-\phi(RT^{m-1}u)\right)\\ &\le \Omega\left(\phi(RT^nu)-\phi(RT^{n+1}u)\right)+\left(\phi(RT^{n+1}u)-\phi(RT^{n+2}u)\right)\\ &\quad+\cdots+\left(\phi(RT^{m-2}u)-\phi(RT^{m-1}u)\right)\\ &\le \Omega(\phi(RT^nu)-\phi(RT^{m-1}u))\\ &\le \Omega(\phi(RT^nu)).\end{aligned}$$

Thus, $\Omega\sum_{i=n}^{m-2}p_\Omega(T^iu,T^{i+1}u)$ is convergent.

$$\text{it implies } \sum_{n=0}^{\infty}s^{m-n}\{p_\Omega(T^nu,T^{n+1}u)\} \text{ is convergent.}$$

Taking the limit as $n,m\to\infty$ in the equation (3.5) above, we get:

$$\lim_{m,n\to\infty}p_\Omega(u_n,u_m)=\lim_{m,n\to\infty}(z_{m-1}(u)-z_{n-1}(u))=0. \tag{3.6}$$

Using similar arguments, we can write:

$$\lim_{m,n\to\infty}p_\Omega(u_m,u_n)=0. \tag{3.7}$$

Thus, the sequence $\{u_n\}$ is Cauchy in (Y, p_Ω). Moreover,

$$\lim_{n\to\infty} p_\Omega(T^n u, w) = 0,$$

which implies

$$\lim_{n\to\infty} T^n u = w.$$

(B) Assume that $Tw = w$ and that u_n is a sequence in $O(u)$ with $u_n \to w$. By Lemma (2.3),

$$\lim_{n\to\infty} d_\Omega(w, u_n) = 0 \iff d_\Omega(w, w) = \lim_{n\to\infty} d_\Omega(w, u_n) = \lim_{n,m\to\infty} d_\Omega(u_n, u_m). \tag{3.8}$$

Then, $G(w) = d_\Omega(w, Tw) = d_\Omega(w, w) \leq \lim_{n\to\infty} \inf d_\Omega(u_n, Tu_n) = \lim_{n\to\infty} \inf G(u_n)$. Thus, G is T-orbitally lower semi-continuous at u. Conversely, suppose that $u_n = T^n u \to w$ and that G is T-orbitally lower semi-continuous at u. Then

$$\begin{aligned} 0 &\leq d_\Omega(w, Tw) = G(w) \leq \lim_{n\to\infty} \inf G(u_n) = \lim_{n\to\infty} \inf d_\Omega(T^n u, T^{n+1} u) \\ &= \lim_{n\to\infty} \inf d_\Omega(u_n, u_{n+1}) = d_\Omega(w, w) = 0. \end{aligned} \tag{3.9}$$

Thus, we have $Tw = w$.

□

Example 3.5 *Let $Y = Y_1 = [0, 2]$. Define*

$$p_\Omega(u, v) = |u - v| + 2v.$$

Then p_Ω is a quasi-partial b-metric with constant $s = 1$. Consider the mappings

$$T : Y \to Y, \quad T(u) = \frac{u}{2}, \quad R : Y \to Y_1, \quad R(u) = 2u,$$

and the function

$$\phi : R(Y) \to \mathbb{R}^+, \quad \phi(u) = 2u.$$

For $c = 2$ and $v \in [0, 2]$, we analyze

$$\max\{p_\Omega(v, Tv), c \cdot p_\Omega(Rv, RTv)\}.$$

Note that

$$\begin{aligned} p_\Omega(v, Tv) &= \left|v - \frac{v}{2}\right| + 2 \cdot \frac{v}{2} = \frac{3v}{2}, \\ p_\Omega(Rv, RTv) &= |2v - v| + 2 \cdot v = 3v, \\ c \cdot p_\Omega(Rv, RTv) &= 2 \cdot 3v = 6v, \\ \max\left\{\frac{3v}{2}, 6v\right\} &= 6v, \\ \phi(Rv) - \phi(RTv) &= 4v - v = 3v. \end{aligned}$$

Since,

$$\max\{p_\Omega(v, Tv), c \cdot p_\Omega(Rv, RTv)\} = 6v > 3v = \phi(Rv) - \phi(RTv),$$

the inequality condition from the theorem is not satisfied. Nevertheless, we proceed to verify claims (A) and (B) of the above theorem.

(A)

$$\lim_{n\to\infty} T^n u = \lim_{n\to\infty} \frac{u}{2^n} = 0 = w.$$

Thus, the limit $\lim_{n\to\infty} T^n u = w$ exists.

(B) By part (A), we have $w = 0$. Therefore,

$$T(w) = T(0) = 0 = w,$$

which holds trivially. Hence, if $G(u) = p_\Omega(u, Tu)$ is T-orbitally lower semi-continuous at u, then $Tw = w$.

Conversely, suppose $Tw = w$. We aim to prove that the function G is T-orbitally lower semi-continuous at u, i.e.,

$$G(w) \leq \liminf G(u_n) \quad \forall \{u_n\} \subseteq O(u),\ u_n \to w.$$

Let $\{u_n\} \subseteq O(u)$ be a sequence converging to w. Then,

$$G(w) = p_\Omega(w, Tw) = p_\Omega(w, w) = w.$$

$$\begin{aligned} G(w) &= p_\Omega(w, Tw) = p_\Omega(w, w) = w \\ &= \frac{3w}{2} \quad (\textit{since } w = 0) = \liminf_{n\to\infty} \frac{3u_n}{2} \\ &= \liminf_{n\to\infty} p_\Omega\left(u_n, \frac{u_n}{2}\right) \\ &= \liminf_{n\to\infty} p_\Omega(u_n, Tu_n) \\ &= \liminf_{n\to\infty} G(u_n). \end{aligned}$$

Hence,

$$G(w) = \liminf_{n\to\infty} G(u_n).$$

Corollary 3.2 *Let (Y, p_Ω) be a complete extended quasi partial b-metric space. Let $T : Y \to Y$ and $\phi : Y \to \mathbb{R}^+$. Suppose there exists $u \in Y$ such that*

$$p_\Omega(v, Tv) \leq \phi(v) - \phi(Tv) \quad \textit{for all } v \in O(u). \tag{3.10}$$

Then the following hold:

(A) $\lim_{n\to\infty} T^n u = w$ exists.

(B) $Tw = w$ if and only if $G(u) = p_\Omega(u, Tu)$ is T-orbitally lower semi-continuous at u.

Proof: Take $Y = Y_1$, $R = I$ (the identity mapping), and $c = 1$ in Theorem 3.3. □

Corollary 3.3 *Let (Y, p_Ω) be a complete be a complete extended quasi partial b-metric space, and let $0 < k < 1$. Suppose that $T : Y \to Y$ and that there exists $u \in Y$ such that*

$$p_\Omega(Tv, T^2v) \leq k p_\Omega(v, Tv) \textit{ for all } v \in O(u). \tag{3.11}$$

Then the following hold:

(A) $\lim_{n\to\infty} T^n u = w$ exists.

(B) $Tw = w$ if and only if $G(u) = p_\Omega(u, Tu)$ is T-orbitally lower semi-continuous at u.

Proof: Set $\phi(v) = \frac{1}{1-k} p_\Omega(v, T)$ for $v \in O(u)$. Let $v = T^n u$ in (16). Then

$$p_\Omega(T^{n+1}u, T^{n+2}u) \le k p_\Omega(T^n u, T^{n+1}u)$$

and

$$p_\Omega(T^n u, T^{n+1}u) - k p_\Omega(T^n u, T^{n+1}u) \le p_\Omega(T^n u, T^{n+1}u) - p_\Omega(T^{n+1}u, T^{n+2}u).$$

Thus,

$$p_\Omega(T^n u, T^{n+1}u) \le \frac{1}{1-k}[p_\Omega(T^n u, T^{n+1}u) - p_\Omega(T^{n+1}u, T^{n+2}u)]$$

or

$$p_\Omega(v, Tv) \le [\phi(v) - \phi(Tv)].$$

The result follows immediately from Corollary 3.2. □

In the following section, we apply the established fixed point results to study the existence and uniqueness of solutions for a class of Volterra integral equations.

4. Application to Volterra Integral Equations

In this section, we present an existence theorem for the Volterra integral equation.

Theorem 4.1 *Let $Y = C([a, b], \mathbb{R})$ denote the space of all continuous real-valued functions defined on the interval $[a, b]$. Note that Y is a complete extended quasi partial b-metric space when equipped with the function*

$$p_\Omega(u, v) = \sup_{t \in [a,b]} \big(u(t) - v(t)\big) + 1,$$

where $\Omega : [0, \infty) \to [0, \infty)$ is defined by $\Omega(t) = t^2$. Consider the following Volterra integral equation:

$$u(t) = \int_a^t F(t, s, u(s))\, ds + f(t), \quad t, s \in [a, b], \tag{4.1}$$

where $f : [a, b] \to \mathbb{R}$ and $F : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Define the operator $T : Y \to Y$ by

$$(Tu)(t) = \int_a^t F(t, s, u(s))\, ds + f(t), \quad t \in [a, b].$$

Assume further that the following condition holds:

$$F(t, s, u(s)) - F(t, s, v(s)) \le \frac{3}{4}\big(u(s) - v(s) + 1\big) \quad \textit{for all } t, s \in [a, b], \textit{ and } u, v \in Y.$$

Then the Volterra integral equation (4.1) has a solution.

Proof: *We need to show that the operator T satisfies the conditions of Theorem 3.1. For any $u \in Y$, we have*

$$p_\Omega(Tu, T^2u) = \sup_{t \in [a,b]} \big((Tu)(t) - (T^2u)(t)\big) + 1.$$

Using the definition of T, it follows that

$$\begin{aligned}
(Tu)(t) - (T^2u)(t) &= \int_a^t F(t, s, u(s))\, ds + f(t) - \int_a^t F(t, s, (Tu)(s))\, ds - f(t) \\
&= \int_a^t \big(F(t, s, u(s)) - F(t, s, (Tu)(s))\big)\, ds \\
&\le \int_a^t \frac{3}{4}\big(u(s) - (Tu)(s) + 1\big)\, ds \\
&\le \frac{3}{4}(b - a) \cdot p_\Omega(u, Tu).
\end{aligned}$$

Thus,

$$p_\Omega(Tu, T^2u) \leq \frac{3}{4}(b-a)\, p_\Omega(u, Tu) + 1.$$

By choosing the interval length $(b-a)$ suitably or considering the constant, all conditions of Theorem 3.1 are satisfied. Hence, the operator T has a fixed point, which is a solution to the Volterra integral equation (4.1). □

5. Conclusion

We established fixed point theorems for several types of contractive mappings in the context of extended quasi partial b-metric spaces. An application to a Volterra integral equation was provided to illustrate the practical relevance of the results. These findings extend classical fixed point theory by confirming that contractive mappings admit fixed points within this generalized framework.

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