Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.76753

Seidel Energy of Some Large Graphs

D. P. Pushpa, S. V. Roopa*, M. A. Sriraj and P. Siva Kota Reddy

ABSTRACT: The Seidel energy of a simple graph G is the sum of the absolute values of the eigenvalues of the Seidel matrix of G. In this paper, we construct some large graphs using graph operations like lexicographic product, corona and join operations on regular graphs and study their spectra and energy. As a consequence of this, we obtain Seidel energy of particular graphs like $C_n[K_m]$, $K_n[C_m]$, $C_n[C_m]$, $K_n[K_m]$, $C_n[Cay(Z_m; U_m)]$, $K_n[Cay(Z_m; U_m)]$, $C_n \circ Cay(Z_m; U_m)$, $K_n \circ Cay(Z_m; U_m)$, $C_n \circ N_m$, $C_n \circ K_m$, $C_n \circ C_m$, $K_n \circ K_m$, $W_{1,m}$, $W_{m,n}$, W^n_{m+1}. By applying the above operations, we construct new classes of non co-spectral Seidel equienergetic graphs.

Key Words: Seidel matrix of graphs, Seidel energy of graphs, Circulant and block circulant matrix, Lexicographic product, Equienergetic graphs.

Contents

1. Introduction

In spectral graph theory, energy of a graph, color energy, partition energy and many more energy like quantities are studied extensively for more than 2 decades, see [1,4,16,28,29]. In [9], the Seidel matrix of a graph G is defined as follows. Let G be a simple graph on n vertices and m edges with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The Seidel matrix of a graph G is an $n \times n$ real symmetric matrix $S(G) = [s_{ij}]$, where $s_{ij} = -1$ if the vertices v_i and v_j are adjacent, $s_{ij} = 1$ if the vertices v_i and v_j are non adjacent and $s_{ij} = 0$ if $i = j$.

The collection of the eigenvalues of the Seidel matrix of a graph is called the Seidel spectrum of G. The Seidel energy $SE(G)$ of a graph G is defined as:

$$SE(G) = \sum_{i=1}^{n} |\sigma_i|. \tag{1.1}$$

An interesting fact in literature is that, if σ_i is an eigenvalue of G then $-\sigma_i$ is the corresponding eigenvalue of $\overline{G}$. Hence G and $\overline{G}$ are non-cospectral and Seidel equienergetic graphs. Constructing new classes of non-cospectral Seidel equienergetic graphs using graph operations like join on the existing Seidel equienergetic graphs is explored in [21]. More details about Seidel energy of a graph can be found in [9,19,20].

For graph terminologies and definitions like regular graph, circulant, block circulant, factor circulant matrices, etc., one can refer to [7]. Numerous matrices can be related to a graph, and their spectrums provide certain helpful information about the graph, see [2,3,10,14,15,17,18,22-27,29-36].

* Corresponding author

Submitted May 12, 2025. Published July 11, 2025
2010 *Mathematics Subject Classification:* 05C76, 05C70, 05C50

Driven by this, we obtain Seidel energy of some large graphs built using graph operations like lexicographic product, corona and join operations on regular graphs. We also attempt to construct new classes of Seidel equienergetic graphs.

Some results which are used for computation of spectrum of the Seidel matrices are stated below:

Theorem 1.1 *[7] All matrices in $BCCB_{m,n}$ are simultaneously diagonalizable by the unitary matrix $F_m \otimes F_n$. Hence they commute. If the eigenvalues of the circulant blocks are given by $\wedge_{k+1}, k = 0, 1, 2, \ldots, m-1$, the diagonal matrix of the eigenvalues of the $BCCB_{m,n}$ matrix is given by*

$$\sum_{k=0}^{m-1} \Omega_m^k \otimes \wedge_{k+1},$$

where $\wedge_{k+1} = diag(\lambda_1^{(k+1)}, \lambda_2^{(k+1)}, \ldots, \lambda_n^{(k+1)}), \Omega_m^k = diag(1^k, \omega^k, \ldots, \omega^{(m-1)k})$ and $\omega = exp(\frac{2\pi i}{m})$.

Theorem 1.2 *[5] Let C be an A-factor block circulant. Then*

$$C = V_A P(D_A) V_A^{-1},$$

where V_A is block Vandermonde matrix and $P(z)$ is the representor of C. Moreover, the set of A-factor circulants coincides with the set of matrices of the form

$$V_A diag[M_1, M_2, \ldots, M_m] V_A^{-1},$$

that is, $P(D_A) = diag[M_1, M_2, \ldots, M_m]$ for a matrix polynomial

$$P(z) = C_1 + C_2 z + \ldots + C_m z^{m-1} \text{if and only if } [C_1 C_2 \ldots C_m] V_A = [M_1 M_2 \ldots M_m].$$

The following result is a consequence of the above Theorem 1.2.

Corollary 1.1 *[5] The factor circulant C can also be expressed as*

$$C = \Re F_{mn}^* P(K\Omega) F_{mn} \Re^{-1},$$

where F_{mn} is a block Fourier matrix, $\Omega = diag[I, \omega I, \omega^2 I, \ldots, \omega^{m-1} I]$ $(\omega = exp(\frac{2\pi i}{m})), K$ is the principal m^{th} root of the non -singular matrix A and $\Re = diag[IKK^2 \ldots K^{m-1}]$. In particular if C is a block circulant then it can be represented as

$$C = F_{mn}^* P(\Omega) F_{mn}.$$

2. Seidel Energy of Lexicographic Product of Two Regular Graphs

Let G and H be two graphs with n and m vertices respectively. Then lexicographic product of two graphs G and H is formed by taking n copies of H and joining any two vertices (u, v) and (x, y) if, and only if, either u is adjacent to x in G or $u = x$ and v is adjacent to y. It is represented by $G[H]$ [11].

In this section, we consider the graphs $G[H]$, $\overline{G}[H]$ where G and H are regular graphs and obtain Seidel energy of these graphs in terms of the Seidel energy of the component graph H. Also, we come up with Seidel energy of some particular graphs like $C_n[K_m]$, $K_n[C_m]$, $C_n[C_m]$, $K_n[K_m]$, $C_n[Cay(Z_m; U_m)]$ and $K_n[Cay(Z_m; U_m)]$.

We prove the following result which is necessary while discussing the spectra of the main graphs considered in this manuscript.

Theorem 2.1 *Let H be a r-regular graph with m vertices, then the characteristic polynomial of $S(H)+uJ_{m\times m}$ is*

$$\phi_{(S(H)+uJ_{m\times m})}(\sigma)=\frac{\phi_{S(H)}(\sigma)}{\sigma-(m-2r-1)}\left[\sigma-(m-2r-1+um)\right],$$

where u is a constant.

Proof: Let $\sigma_0,\ \sigma_1,\ \cdots,\ \sigma_{m-1}$ be the eigenvalues of $S(H)$. Since H is r-regular, the Seidel matrix H will have the row sum $\sigma_0=m-2r-1$ as an eigenvalue. It can be observe that the corresponding eigenvector is $J_{m\times 1}$. Also the row sum in any row of $J_{m\times m}$ is m. This implies that m is an eigenvalue of $J_{m\times m}$ with the corresponding eigenvector $J_{m\times 1}$ and its remaining eigenvalues are zeros.
Consider

$$\begin{aligned}(S(H)+uJ_{m\times m})J_{m\times 1}&=(S(H)+uJ_{m\times m})J_{m\times 1}\\&=S(H)J_{m\times 1}+(uJ_{m\times m})J_{m\times 1}\\&=(m-2r-1)J_{m\times 1}+umJ_{m\times 1}\\&=(m-2r-1+um)J_{m\times 1}\end{aligned}$$

which implies that $m-2r-1+um$ is an eigenvalue of $S(H)+uJ_{m\times m}$.
Let $X_i=(x_{(i1)},\ x_{(i2)},\ \cdots,\ x_{(im)})^T$ be the eigenvector of $S(H)$ corresponding to the eigenvalue σ_i for $i=1,\ 2,\ \cdots,\ m-1$. This implies that

$$x_{(i1)}+x_{(i2)}+\cdots+x_{(im)}=0. \tag{2.1}$$

It can be observed from equation (2.1) that X_i is also is an eigenvector of $J_{m\times m}$ corresponding to the zero eigenvalues.
Therefore for $i=1,\ 2,\ \cdots,\ m-1$.

$$(S(H)+uJ_{m\times m})X_i=\lambda_iX_i.$$

Hence

$$\phi_{(S(H)+uJ_{m\times m})}(\sigma)=\frac{\phi_{S(H)}(\sigma)}{\sigma-(m-2r-1)}\left[\sigma-(m-2r-1+um)\right].$$

□

Theorem 2.2 *Let G be a r_1-regular circulant graph with n vertices and H be r_2-regular graph with m vertices. Then*

$$(i)SE(G[H])=nSE(H)-n|m-2r_2-1|+|nm-2(r_2+r_1m)-1|+\sum_{t=1}^{n-1}|2r_2+1+2m\alpha_t|$$

$$(ii)SE(\overline{G}[H])=nSE(H)-n|m-2r_2-1|+|m(2r_1-n+2)-2r_2-1|+\sum_{t=1}^{n-1}|2m(1+\alpha_t)-2r_2-1|,$$

where α_t are adjacency eigenvalues of G.

Proof: (i) The Seidel matrix of $G[H]$ is a block circulant of order nm denoted by

$$S(G[H])=bcirc(H_1,H_2,H_3,\ldots,H_n)_{n\times n}$$

The block circulant structure of $S(G[H])$ is due to the circulant nature of the graph G. Also, since G is r_1- regular, $S(G[H])$ has only three different types of blocks namely $S(H)$ along the principal diagonal, r_1 numbers of $-J_{m\times m}$ matrices and $(n-r_1-1)$ numbers of $J_{m\times m}$ matrices. Without loss of generality we denote $H_1 = S(H),\ H_2 = -J_{m\times m}$ and $H_3 = J_{m\times m}$. From Corollary 1.1, the diagonal form of $S(G[H])$ is

$$diag(S(H) + (n - 2r_1 - 1)J_{m\times m},\ S(H) - (1 + 2\alpha_1)J_{m\times m},\ \cdots,\ S(H) - (1 + 2\alpha_{n-1})J_{m\times m})$$

where α_t are adjacency eigenvalues of G.
Since H is r_2 -regular, let the Seidel eigenvalues of H be $\sigma_0 = m - 2r_2 - 1,\ \sigma_1,\ \cdots,\ \sigma_{n-1}$.
Then from Theorem 2.1, the characteristic polynomial of $S(H) + u_t J_{m\times m}$ is

$$\frac{\phi_{S(H)}(\sigma)}{\sigma - (m - 2r_2 - 1)}\left[\sigma - (m - 2r_2 - 1 + um)\right].$$

The characteristic polynomial of $S(G[H])$ is the product of the characteristic polynomial of $S(H) + u_t J_{m\times m}$, for $t = 0,\ 1,\ \ldots,\ n-1$.
Hence the characteristic polynomial of $S(G[H])$ is

$$\left(\frac{\phi_{S(H)}(\sigma)}{\sigma - (m - 2r_2 - 1)}\right)^n \left[\sigma - (m - 2r_2 - 1 + m(n - 2r_1 - 1))\right]\prod_{t=1}^{n-1}\left[\sigma - (m - 2r_2 - 1 + m(-1 - 2\alpha_t))\right]. \tag{2.2}$$

Hence,

$$SE(G[H]) = nSE(H) - n|m - 2r_2 - 1| + |nm - 2(r_2 + r_1 m) - 1| + \sum_{t=1}^{n-1}|2r_2 + 1 + 2m\alpha_t|.$$

(ii) As $\overline{G}$ is $(n - r_1 - 1)$ regular, The characteristic polynomial of $S(\overline{G}[H])$ is obtained by replacing r_1 by $(n - r_1 - 1)$ and α_t by $(-1 - \alpha_t)$ in equation (2.2) which are adjacency eigenvalues of $\overline{G}$

$$\left(\frac{\phi_{S(H)}(\sigma)}{\sigma - (m - 2r_2 - 1)}\right)^n \left[\sigma - (m - 2r_2 - 1 + m(2r_1 - n + 1))\right]\prod_{t=1}^{n-1}\left[\sigma - (m - 2r_2 - 1 + m(1 + 2\alpha_t))\right].$$

Hence,

$$SE(\overline{G}[H]) = nSE(H) - n|m - 2r_2 - 1| + |m(2r_1 - n + 2) - 2r_2 - 1| + \sum_{t=1}^{n-1}|2m(1 + \alpha_t) - 2r_2 - 1|.$$

□

Observation
As $\overline{G[H]} = \overline{G}[\overline{H}],\ \overline{\overline{G}[H]} = G[\overline{H}]$. It follows that

1. $G[H]$ and $\overline{G}[\overline{H}]$ are Seidel equienergetic.
2. $\overline{G}[H]$ and $G[\overline{H}]$ are Seidel equienergetic.

Following are some of the particular cases which are deduced from Theorem 2.2.
Case-1. If $G = C_n,\ H = K_m,\ r_1 = 2,\ r_2 = m - 1$ and $\alpha_t = 2\cos\left(\frac{2\pi t}{n}\right)$, then

$$SE(C_n[K_m]) = nSE(K_m) - n(m - 1) + |nm - 6m + 1| + \sum_{t=1}^{n-1}\left|1 - 2m\left(1 + 2\cos\left(\frac{2\pi t}{n}\right)\right)\right|.$$

Case-2. If $G = K_n,\ H = C_m,\ r_1 = n - 1,\ r_2 = 2$ and $\alpha_t = -1$, then

$$SE(K_n[C_m]) = nSE(C_m) - n|m - 5| + |2m - nm - 5| + (n - 1)|2m - 5|.$$

Case-3. If $G = C_n,\ H = C_m,\ r_1 = 2,\ r_2 = 2$ and $\alpha_t = 2\cos\left(\frac{2\pi t}{n}\right)$, then

$$SE(C_n[C_m]) = nSE(C_m) - n|m - 5| + |nm - 4m - 5| + \sum_{t=1}^{n-1}\left|-4m\cos\left(\frac{2\pi t}{n}\right) - 5\right|.$$

Case-4. If $G = K_n,\ H = K_m,\ r_1 = n - 1,\ r_2 = m - 1$ and $\alpha_t = -1$, then

$$SE(K_n[K_m]) = nSE(K_m) + 2(n - 1).$$

Case-5. If $G = C_n, H = Cay(Z_m; U_m)$ which represents the unitary Cayley graph on m vertices with degree $\phi(m)$, $r_1 = 2$, $r_2 = \phi(m)$ and $\alpha_t = 2\cos\left(\frac{2\pi t}{n}\right)$, then

$$\begin{aligned} SE(C_n[Cay(Z_m; U_m)]) &= nSE(Cay(Z_m; U_m)) + |nm - 2(\phi(m) + 2m) - 1| \\ &\quad - n|m - 2\phi(m) - 1| + \sum_{t=1}^{n-1} \left| -2\phi(m) - 1 - 4m\cos\left(\frac{2\pi t}{n}\right) \right|. \end{aligned}$$

Case-6. If $G = K_n, H = Cay(Z_m; U_m)$ which represents the unitary Cayley graph on m vertices with degree $\phi(m)$, $r_1 = n - 1$, $r_2 = \phi(m)$ and $\alpha_t = -1$, then

$$\begin{aligned} SE(K_n[Cay(Z_m; U_m)]) &= nSE(Cay(Z_m; U_m)) + |nm - 2(\phi(m) + m(n-1)) - 1| \\ &\quad - n|m - 2\phi(m) - 1| + (n-1)\left|-2\phi(m) - 1 + 2m\right|. \end{aligned}$$

3. Seidel Energy of Wheel, Multi-Wheel and Windmill Graphs

In this section, we take r-regular graph H and construct two semi-regular graphs $H_1 = H\nabla K_1$ and $G = (\bigcup_n H)\nabla K_1$. For $H = C_m$, H_1 will be a wheel and G will be a multi-wheel graph. Also for $H = K_m$, G will be windmill graph. We obtain Seidel spectra and hence energy of H_1 and G, and in particular for wheel, multi-wheel and windmill graphs.

Definition 3.1 *[12] An n-wheel graph $W_{m,\,n}$ is a graph obtained from n copies of cycles C_m and one copy of vertex v, such that all vertices of every copy of C_m are adjacent to v.*

Definition 3.2 *[13] The (n, m)-windmill graph W_m^n is the graph obtained by taking n-copies of complete graph K_m with a vertex in common.*

Theorem 3.1 *Let H be a r-regular graph with m vertices and $H_1 = H\nabla K_1$. Then*

$$SE(H_1) = SE(H) - |m - 2r - 1| + \sqrt{(2r - m + 1)^2 + 4m}.$$

Proof: Let $H_1 = H\nabla K_1$, the matrix $S(H_1)$ is of the form,

$$\begin{pmatrix} S(H) & -J_{m\times 1} \\ -J^T_{1\times m} & 0 \end{pmatrix}$$

From ([6], p. 62, Lemma 2.2).

$$|\sigma I - S(H_1)| = |\sigma I - S(H)| \left|\sigma - J^T_{1\times m}(\sigma I - S(H))^{-1} J_{m\times 1}\right|$$

$$|\sigma I - S(H_1)| = |\sigma I - S(H)| \left|\sigma - J^T_{1\times m}\frac{adj\,(\sigma I - S(H))}{|\sigma I - S(H)|}J_{m\times 1}\right|. \tag{3.1}$$

Consider

$$\begin{aligned} J^T_{1\times m} adj(\sigma I - S(H)) J_{m\times 1} &= sum\left\{\sigma I - S(H)\right\} \\ &= |\sigma I - S(H) + J_{m\times m}| - |\sigma I - S(H)|. \end{aligned}$$

$$\frac{J^T_{1\times m} adj(\sigma I - S(H)) J_{m\times 1}}{|\sigma I - S(H)|} = \frac{|\sigma I - (S(H) - J_{m\times m})|}{|\sigma I - S(H)|} - 1. \tag{3.2}$$

Substituting (3.2) in (3.1), we get

$$|\sigma I - S(H_1)| = |\sigma I - S(H)| \left|\sigma - \left(\frac{|\sigma I - (S(H) - J_{m\times m})|}{|\sigma I - S(H)|} - 1\right)\right|$$

$$\phi_{S(H_1)}(\sigma) = \phi_{S(H)}(\sigma)\left[\sigma + 1 - \frac{\phi_{(S(H)-J_{m\times m})}(\sigma)}{\phi_{S(H)}(\sigma)}\right]. \tag{3.3}$$

Since H is r-regular, the eigenvalues of $S(H)$ will be of the form $\sigma_0 = m-2r-1,\ \sigma_1,\ \ldots,\ \sigma_{m-1}$. Setting $u=-1$ in Theorem 2.1 and substituting in equation (3.3), we get

$$\phi_{S(H_1)}(\sigma) = \phi_{S(H)}(\sigma)\left[\sigma + 1 - \left(\frac{1}{\phi_{S(H)}(\sigma)}\right)\left(\frac{\phi_{S(H)}(\sigma)(\sigma-(m-2r-1-m))}{\sigma-(m-2r-1)}\right)\right].$$

$$\phi_{S(H_1)}(\sigma) = \frac{\phi_{S(H)}(\sigma)}{\sigma - m + 2r + 1}\left[\sigma^2 + \sigma(2r-m+1) - m\right].$$

Hence the spectra of $S(H_1)$ is

$$\begin{cases} \sigma_i \ \ for\ \ i = 1, 2, \ldots, m-1 & once \\ \frac{-(2r-m+1)\pm\sqrt{(2r-m+1)^2+4m}}{2} & once, \end{cases}$$

further

$$-(2r-m+1) + \sqrt{(2r-m+1)^2+4m} > 0$$

and

$$-(2r-m+1) - \sqrt{(2r-m+1)^2+4m} < 0.$$

Hence,

$$SE(H_1) = SE(H) - |m-2r-1| + \sqrt{(2r-m+1)^2+4m}.$$

□

Corollary 3.1 *Let $W_{1,m}$ represent the wheel graph. Then*

$$SE(W_{1,m}) = SE(C_m) - |m-5| + \sqrt{(5-m)^2+4m},$$

where C_m is the cycle with m vertices.

Proof: If we choose $H = C_m$ in Theorem 3.1, then $H_1 = W_{1,m}$

$$\therefore \quad SE(W_{1,m}) = SE(C_m) - |m-5| + \sqrt{(5-m)^2+4m}.$$

□

Theorem 3.2 *Let $G' = \bigcup_n H$, where H is a r-regular graph with m-vertices. Then*

$$SE(G') = nSE(H) - n|m-2r-1| + |nm-2r-1| + (n-1)(2r+1).$$

Proof: The Seidel matrix of G' is a block circulant matrix with circulant blocks of order nm which is denoted as follows:

$$S(G') = bcirc(H_1, H_2, \cdots, H_n)$$

where $H_1 = S(H), H_2 = H_3 = ... = H_n = J_{m\times m}$.

Let $\sigma_0,\ \sigma_1,\ \cdots,\ \sigma_{m-1}$ be the Seidel eigenvalues of H. Since H is r-regular, $\sigma_0 = m-2r-1$. We know that $\wedge_1 = diag(m-2r-1,\ \sigma_1,\ \cdots,\ \sigma_{m-1})$ and $\wedge_2 = diag(m,\ 0,\ \cdots,\ 0)$ are the matrices of eigenvalues of $S(H)$ and H_2 respectively. Then from Theorem 1.1, the diagonal form $S(G')$ is

$$diag(\wedge_1 + (n-1)\wedge_2,\ \wedge_1 - \wedge_2,\ \cdots,\ \wedge_1 - \wedge_2)_{mn\times mn}.$$

Here

$$\wedge_1 + (n-1)\wedge_2 = diag(nm - 2r - 1,\ \sigma_1,\ \cdots,\ \sigma_{m-1})$$

and

$$\wedge_1 - \wedge_2 = diag(-2r - 1,\ \sigma_1,\ \cdots,\ \sigma_{m-1}).$$

Hence, the eigenvalues of $S(G')$ are

$$\begin{cases} \sigma_i \ \ for \ \ i = 1, 2, \ldots, m-1 & n \ \ times \\ nm - 2r - 1 & once \\ -2r - 1 & (n-1) \ \ times. \end{cases}$$

Hence,

$$SE(G') = nSE(H) - n|m - 2r - 1| + |nm - 2r - 1| + (n-1)(2r+1).$$

□

Theorem 3.3 *Let $G = G'\nabla K_1$ where $G' = \bigcup_n H$ and H is a r-regular graph with m-vertices. Then*

$$SE(G) = nSE(H) - n|m - 2r - 1| + (n-1)(2r+1) + \sqrt{(2r - mn + 1)^2 + 4mn}.$$

Proof: Since G' is r-regular graph with mn-vertices. From Theorem 3.1, the characteristic polynomial of $S(G)$ is

$$\phi_{S(G)}(\sigma) = \frac{\phi_{S(G')}(\sigma)}{\sigma - mn + 2r + 1}\left[\sigma^2 + \sigma(2r - mn + 1) - mn\right].$$

From Theorem 3.2,

$$\phi_{S(G)}(\sigma) = \left(\frac{\phi_{S(H)}(\sigma)}{\sigma - m + 2r + 1}\right)^n (\sigma + 2r + 1)^{(n-1)}\left[\sigma^2 + \sigma(2r - mn + 1) - mn\right].$$

Hence, the spectrum is

$$\begin{cases} \sigma_i \ \ for \ \ i = 1, 2, \ldots, m-1 & n \ times \\ -2r - 1 & (n-1) \ times \\ \frac{-(2r - mn + 1) \pm \sqrt{(2r - mn + 1)^2 + 4mn}}{2} & once, \end{cases}$$

where σ_i are Seidel eigenvalues of H.
Further

$$-(2r - mn + 1) + \sqrt{(2r - mn + 1)^2 + 4mn} > 0$$

and

$$-(2r - mn + 1) - \sqrt{(2r - mn + 1)^2 + 4mn} < 0.$$

Hence,

$$SE(G) = nSE(H) - n|m - 2r - 1| + (n-1)(2r+1) + \sqrt{(2r - mn + 1)^2 + 4mn}.$$

□

Corollary 3.2 *Let $W_{m,n}$ represent the multi-wheel graph. Then*

$$SE(W_{m,n}) = nSE(C_m) - n|m-5| + 5(n-1) + \sqrt{(5-mn)^2 + 4mn},$$

where C_m is the cycle with m vertices.

Proof: If we choose $H = C_m$ in Theorem 3.3, then $G = W_{m,n}$

$$\therefore\ SE(W_{m,n}) = nSE(C_m) - n|m-5| + 5(n-1) + \sqrt{(5-mn)^2 + 4mn}.$$

□

Corollary 3.3 *Let W^n_{m+1} represent the windmill graph. Then*

$$SE(W^n_{m+1}) = nSE(K_m) + m(n-2) + 1 + \sqrt{(2m-mn-1)^2 + 4mn},$$

where K_m is the complete graph with m vertices.

Proof: If we choose $H = K_m$ in Theorem 3.3, then $G = W^n_{m+1}$

$$\therefore\ SE(W^n_{m+1}) = nSE(K_m) + m(n-2) + 1 + \sqrt{(2m-mn-1)^2 + 4mn}.$$

□

4. Seidel Energy of Corona of Two Regular Graphs

In this section, we consider two regular graphs G and H, where G is a circulant and construct a new graph called corona of G with H denoted by $G \circ H$. We obtain Seidel spectra of this graph in terms of spectra of its component graph H and obtain Seidel energy of particular graphs like $C_n \circ N_m$, $C_n \circ K_m$, $C_n \circ C_m$, $K_n \circ K_m$, $C_n \circ Cay(Z_m; U_m)$ and $K_n \circ Cay(Z_m; U_m)$.

Definition 4.1 *[8] Let G_1 and G_2 be two graphs on n and m vertices respectively. The corona $G_1 \circ G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and n copies of G_2, and then joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2.*

We first obtain the following result which is necessary to find spectra of large graphs like $(G \circ H)$.

Theorem 4.1 *Let H be a r-regular graph with m vertices and $H_1 = H \nabla K_1$ and u is a constant. Then the characteristic polynomial of $S(H_1) + uJ_{(m+1)\times(m+1)}$ is*

$$\frac{\phi_{S(H)}(\sigma)}{(\sigma - m + 2r + 1)}[\sigma^2 + \sigma(2r - u(m+1) - m + 1) + u(3m - 2r - 1) - m].$$

Proof: The matrix of $S(H_1) + uJ_{(m+1)\times(m+1)}$ will be of the form

$$\begin{pmatrix} S(H) + uJ_{m\times m} & (u-1)J_{m\times 1} \\ (u-1)J^T_{1\times m} & u \end{pmatrix}.$$

From ([6], p. 62, Lemma 2.2).

$$\begin{aligned} &|\sigma I - (S(H_1) + uJ_{(m+1)\times(m+1)})| \\ &\quad = |\sigma I - (S(H) + uJ_{m\times m})| \left|(\sigma I - u) - (u-1)^2 J^T_{1\times m}\left(\sigma I - (S(H) + uJ_{m\times m})\right)^{-1} J_{m\times 1}\right| \end{aligned} \tag{4.1}$$

with similar discussion as in Theorem 3.1, we get

$$J^T_{1\times m}(\sigma I-(S(H)+uJ_{m\times m}))^{-1}J_{m\times 1}=\frac{\phi_{[S(H)+(u-1)J_{m\times m}]}(\sigma)}{\phi_{[S(H)+uJ_{m\times m}]}(\sigma)}-1.$$

using Theorem 2.1, this can be further simplified as

$$\frac{m}{\sigma-m+2r+1-mu}$$

substituting this in (4.1) and simplifying, we get $\phi_{(S(H_1)+uJ_{(m+1)\times(m+1)})}(\sigma)$ as follows:

$$\frac{\phi_{S(H)}(\sigma)}{(\sigma-m+2r+1)}[\sigma^2+\sigma(2r-u(m+1)-m+1)+u(3m-2r-1)-m].$$

□

Theorem 4.2 *Let G be a r_1-regular circulant graph and H be r_2-regular graph with n and m vertices respectively. Then*

$$SE(G\circ H)=nSE(H)-n|m-2r_2-1|+|\beta+\rho|+|\beta-\rho|+\sum_{t=1}^{n-1}[|\chi_t+\xi_t|+|\chi_t-\xi_t|],$$

where
$\beta=\frac{-(2r_1+2r_2+2-nm-n)}{2}$, $\rho=\frac{\sqrt{(2r_1+2r_2+2-nm-n)^2-4[n(3m-2r_2-1)+2r_1(2r_2+1-mn)+2r_2+1-4m]}}{2}$,

$\chi_t=-(r_2+1+\alpha_t)$, $\xi_t=\sqrt{(r_2+1+\alpha_t)^2-[2r_2-4m+1+2\alpha_t(2r_2+1)]}$ *and α_t are adjacency eigenvalues of G .*

Proof: Since G is circulant, the Seidel matrix $G\circ H$ is a block circulant matrix of order $n(m+1)$ which is denoted by

$$S(G\circ H)=bcirc(S(H_1),\ H_2,\ \ldots,\ H_n)$$

where $S(H_1)=S(H\nabla K_1)$ and remaining $H_i's$ are either J matrices or matrices with first entry -1 and all other entries as one's. These matrices are of order $(m+1)$.
Let us take H_2 as the matrix in which first entry is -1 and all other entries are one's and $H_3=J_{(m+1)\times(m+1)}$. Since G is r_1- regular, there will be r_1 numbers of H_2 matrices and $(n-r_1-1)$ numbers of $J_{(m+1)\times(m+1)}$ matrices. Thus the diagonal form of $S(G\circ H)$ is

$$diag(A_0,\ A_1,\ \ldots,\ A_{n-1})$$

where
$A_0=S(H_1)+r_1H_2+(n-r_1-1)J_{(m+1)\times(m+1)}$,
$A_t=S(H_1)+\alpha_tH_2+(-1-\alpha_t)J_{(m+1)\times(m+1)}$ for $t=1,\ 2,\ \ldots,\ n-1$
and $\alpha_t=\sum_{k=1}^{n}a_k\ e^{\frac{2\pi it(k-1)}{n}}$ for $t=0,\ 1,\ \ldots,\ n-1$ are adjacency eigenvalues of G.
Consider

$$\begin{aligned}A_0&=S(H_1)+r_1H_2+(n-r_1-1)J_{(m+1)\times(m+1)}\\&=S(H_1)+r_1(J_{(m+1)\times(m+1)}-2H')+(n-r_1-1)J_{(m+1)\times(m+1)}\\&=S(H_1)+(n-1)J_{(m+1)\times(m+1)}-2r_1H',\end{aligned}$$

where $H_2=J_{(m+1)\times(m+1)}-2H'$ and H' is the matrix in which first entry is one and all other entries are zero's. It can be observed that

$$\phi_{(A_0)}(\sigma)=\phi_{(S(H_1)+uJ_{(m+1)\times(m+1)})}(\sigma)+2r_1\phi_{(S(H)+uJ_{m\times m})}(\sigma). \tag{4.2}$$

Choosing $u = n-1$ and $r = r_2$ in Theorem 2.1 and 4.1 and substituting in equation 4.2, we get the characteristic polynomial of A_0 as

$$\frac{\phi_{S(H)}(\sigma)}{(\sigma - m + 2r + 1)}[\sigma^2 + \sigma(2r_1 + 2r_2 + 2 - nm - n) + n(3m - 2r_2 - 1) + 2r_1(2r_2 + 1 - mn) + 2r_2 + 1 - 4m].$$

Thus, the spectra of A_0 is

$$\begin{cases} \sigma_i \ \ for \ \ i = 1, 2, \ldots, m-1 \quad\quad once \\ \\ \beta \pm \rho = \frac{-(2r_1+2r_2+2-nm-n) \pm \sqrt{(2r_1+2r_2+2-nm-n)^2 - 4[n(3m-2r_2-1)+2r_1(2r_2+1-mn)+2r_2+1-4m]}}{2} \quad\quad once, \end{cases}$$

where σ_i are Seidel eigenvalues of H.
Let $A_t = S(H_1) + \alpha_t H_2 + (-1-\alpha_t)J_{(m+1)\times(m+1)} = S(H_1) - J_{(m+1)\times(m+1)} - 2\alpha_t H'$ for $t = 1,\ 2,\ \ldots,\ n-1$. The characteristic polynomial of A_t as

$$\phi_{(A_t)}(\sigma) = \phi_{(S(H_1)-J_{(m+1)\times(m+1)})}(\sigma) + 2\alpha_t \phi_{(S(H)-J_{m\times m})}(\sigma). \tag{4.3}$$

Choosing $u = -1$ and $r = r_2$ in Theorem 2.1 and 4.1 and substituting in equation 4.3, we get $\phi_{(A_t)}(\sigma)$ as

$$\frac{\phi_{S(H)}(\sigma)}{(\sigma - m + 2r + 1)}[\sigma^2 + \sigma(2r_2 + 2 + 2\alpha_t) + 2r_2 - 4m + 1 + 2\alpha_t(2r_2 + 1)].$$

Thus, the spectra of A_t is

$$\begin{cases} \sigma_i \ \ for \ \ i = 1, 2, \ldots, m-1 \quad\quad (n-1) times \\ \\ \chi_t \pm \xi_t = -(r_2 + 1 + \alpha_t) \pm \sqrt{(r_2 + 1 + \alpha_t)^2 - [2r_2 - 4m + 1 + 2\alpha_t(2r_2 + 1)]} \quad 1 \le t \le n-1 \quad once, \end{cases}$$

where σ_i are Seidel eigenvalues of H and α_t are adjacency eigenvalues of G.
Hence, the spectra of $S(G \circ H)$ is:

$$\begin{cases} \sigma_i \ \ for \ \ i = 1, 2, \ldots, m-1 \quad\quad n \ \ times \\ \\ \beta \pm \rho = \frac{-(2r_1+2r_2+2-nm-n) \pm \sqrt{(2r_1+2r_2+2-nm-n)^2 - 4[n(3m-2r_2-1)+2r_1(2r_2+1-mn)+2r_2+1-4m]}}{2} \quad\quad once \\ \\ \chi_t \pm \xi_t = -(r_2 + 1 + \alpha_t) \pm \sqrt{(r_2 + 1 + \alpha_t)^2 - [2r_2 - 4m + 1 + 2\alpha_t(2r_2 + 1)]} \quad 1 \le t \le n-1 \quad once. \end{cases}$$

Hence,

$$SE(G \circ H) = nSE(H) - n|m - 2r_2 - 1| + |\beta + \rho| + |\beta - \rho| + \sum_{t=1}^{n-1} [|\chi_t + \xi_t| + |\chi_t - \xi_t|],$$

where
$\beta = \frac{-(2r_1+2r_2+2-nm-n)}{2}$, $\rho = \frac{\sqrt{(2r_1+2r_2+2-nm-n)^2 - 4[n(3m-2r_2-1)+2r_1(2r_2+1-mn)+2r_2+1-4m]}}{2}$,

$\chi_t = -(r_2 + 1 + \alpha_t)$, $\xi_t = \sqrt{(r_2 + 1 + \alpha_t)^2 - [2r_2 - 4m + 1 + 2\alpha_t(2r_2 + 1)]}$

and α_t are adjacency eigenvalues of G. □

Some particular cases obtained from Theorem 4.2 are:
Case-1. If $G = C_n,\ H = N_m,\ r_1 = 2,\ r_2 = 0$ and $\alpha_t = 2\cos\left(\frac{2\pi t}{n}\right)$, then

$$\begin{aligned} SE(C_n \circ N_m) &= nSE(N_m) - n(m-1) \\ &+ \left| \frac{-(6-mn-n) + \sqrt{(6-mn-n)^2 - 4(5-4m-mn-n)}}{2} \right| \\ &+ \left| \frac{-(6-mn-n) - \sqrt{(6-mn-n)^2 - 4(5-4m-mn-n)}}{2} \right| \\ &+ \sum_{t=1}^{n-1} \left| -\left(1+2\cos\left(\frac{2\pi t}{n}\right)\right) + \sqrt{\left(1+2\cos\left(\frac{2\pi t}{n}\right)\right)^2 - 1 + 4m - 4\cos\left(\frac{2\pi t}{n}\right)} \right| \\ &+ \sum_{t=1}^{n-1} \left| -\left(1+2\cos\left(\frac{2\pi t}{n}\right)\right) - \sqrt{\left(1+2\cos\left(\frac{2\pi t}{n}\right)\right)^2 - 1 + 4m - 4\cos\left(\frac{2\pi t}{n}\right)} \right|. \end{aligned}$$

Case-2. If $G = C_n,\quad H = K_m,\quad r_1 = 2,\quad r_2 = m-1$ and $\alpha_t = 2\cos\left(\frac{2\pi t}{n}\right)$, then

$$\begin{aligned} SE(C_n \circ K_m) &= nSE(K_m) - n(m-1) \\ &+ \left| \frac{-(4+2m-mn-n) + \sqrt{(4+2m-mn-n)^2 - 4(n-3mn+6m-5)}}{2} \right| \\ &+ \left| \frac{-(4+2m-mn-n) - \sqrt{(4+2m-mn-n)^2 - 4(n-3mn+6m-5)}}{2} \right| \\ &+ \sum_{t=1}^{n-1} \left| -\left(m+2\cos\left(\frac{2\pi t}{n}\right)\right) + \sqrt{\left(m+2\cos\left(\frac{2\pi t}{n}\right)\right)^2 - 4(2m-1)\cos\left(\frac{2\pi t}{n}\right) + 2m + 1} \right| \\ &+ \sum_{t=1}^{n-1} \left| -\left(m+2\cos\left(\frac{2\pi t}{n}\right)\right) - \sqrt{\left(m+2\cos\left(\frac{2\pi t}{n}\right)\right)^2 - 4(2m-1)\cos\left(\frac{2\pi t}{n}\right) + 2m + 1} \right|. \end{aligned}$$

Case-3. If $G = C_n,\quad H = C_m,\quad r_1 = 2,\quad r_2 = 2$ and $\alpha_t = 2\cos\left(\frac{2\pi t}{n}\right)$, then

$$\begin{aligned} SE(C_n \circ C_m) &= nSE(C_m) - n|m-5| \\ &+ \left| \frac{-(10-mn-n) + \sqrt{(10-mn-n)^2 - 4(n(3m-5) + 4(5-mn) + 5 - 4m)}}{2} \right| \\ &+ \left| \frac{-(10-mn-n) - \sqrt{(10-mn-n)^2 - 4(n(3m-5) + 4(5-mn) + 5 - 4m)}}{2} \right| \\ &+ \sum_{t=1}^{n-1} \left| -\left(3+2\cos\left(\frac{2\pi t}{n}\right)\right) + \sqrt{\left(3+2\cos\left(\frac{2\pi t}{n}\right)\right)^2 - \left(5 - 4m + 20\cos\left(\frac{2\pi t}{n}\right)\right)} \right| \\ &+ \sum_{t=1}^{n-1} \left| -\left(3+2\cos\left(\frac{2\pi t}{n}\right)\right) - \sqrt{\left(3+2\cos\left(\frac{2\pi t}{n}\right)\right)^2 - \left(5 - 4m + 20\cos\left(\frac{2\pi t}{n}\right)\right)} \right|. \end{aligned}$$

Case-4. If $G=K_n,\quad H=K_m,\quad r_1=n-1,\quad r_2=m-1$ and $\alpha_t=-1$, where $t=1,2,\ldots,n-1$ then

$$\begin{aligned}SE(K_n\circ K_m) &= nSE(K_m)-n(m-1)\\ &+\left|\frac{-(2m-2-mn+n)+\sqrt{(2m-2-mn+n)^2-4(7mn-6m+1-n-2mn^2)}}{2}\right|\\ &+\left|\frac{-(2m-2-mn+n)-\sqrt{(2m-2-mn+n)^2-4(7mn-6m+1-n-2mn^2)}}{2}\right|\\ &+(n-1)\left[\left|-(m-1)+\sqrt{(m-1)^2-(1-6m)}\right|+\left|-(m-1)-\sqrt{(m-1)^2-(1-6m)}\right|\right].\end{aligned}$$

Case-5. If $G=C_n, H=Cay(Z_m;U_m)$ which represets the unitary cayley graph on m vertices with degree $\phi(m),\quad r_1=2,\quad r_2=\phi(m)$ and $\alpha_t=2\cos\left(\frac{2\pi t}{n}\right)$, then $SE(C_n\circ Cay(Z_m;U_m))=$

$$\begin{aligned}&nSE(Cay(Z_m;U_m))-n|m-2\phi(m)-1|\\ &+\left|\frac{-(6+2\phi(m)-n(m+1))+\sqrt{(6+2\phi(m)-n(m+1))^2-4(2\phi(m)(5-n)+5-n(m+1)-4m)}}{2}\right|\\ &+\left|\frac{-(6+2\phi(m)-n(m+1))-\sqrt{(6+2\phi(m)-n(m+1))^2-4(2\phi(m)(5-n)+5-n(m+1)-4m)}}{2}\right|\\ &+\sum_{t=1}^{n-1}\left|-\left(\phi(m)+1+2\cos\left(\frac{2\pi t}{n}\right)\right)+R\right|+\sum_{t=1}^{n-1}\left|-\left(\phi(m)+1+2\cos\left(\frac{2\pi t}{n}\right)\right)-R\right|,\end{aligned}$$

where $R=\sqrt{\left(\phi(m)+1+2\cos\left(\frac{2\pi t}{n}\right)\right)^2-2\phi(m)+4m-1-4(2\phi(m)+1)\cos\left(\frac{2\pi t}{n}\right)}$.

Case-6. If $G=K_n,\quad H=Cay(Z_m;U_m)$ which represents the unitary Cayley graph on m vertices with degree $\phi(m),\quad r_1=n-1,\quad r_2=\phi(m)$ and $\alpha_t=-1$, then

$$\begin{aligned}&SE(K_n\circ Cay(Z_m;U_m))=nSE(Cay(Z_m;U_m))-n|m-2\phi(m)-1|\\ &+\left|\frac{-(n(1-m)+2\phi(m))+\sqrt{(n(1-m)+2\phi(m))^2-4(n(5m+1)+2\phi(m)(n-1)-1-2m(n^2+2))}}{2}\right|\\ &+\left|\frac{-(n(1-m)+2\phi(m))-\sqrt{(n(1-m)+2\phi(m))^2-4(n(5m+1)+2\phi(m)(n-1)-1-2m(n^2+2))}}{2}\right|\\ &+(n-1)\left[\left|-\phi(m)+\sqrt{(\phi(m))^2+2\phi(m)+4m+1}\right|+\left|-\phi(m)-\sqrt{(\phi(m))^2+2\phi(m)+4m+1}\right|\right].\end{aligned}$$

5. Construction of Seidel Equienergetic Graphs

In [21], the authors have constructed non-cospectral Seidel equienergetic graphs which are having same number of vertices and are same regular. They have also constructed a new class of graphs which are Seidel equienergetic using join operations. Motivated by this, we also construct new class of Seidel equienergetic graphs using the graph operations as: Consider the graphs H_a and H_b with same number of vertices and are 3-regular as shown in Figure 1.

The characteristic polynomials of H_a and H_b are: $\phi(H_a:\sigma)=(\sigma-5)(\sigma-3)^3(\sigma-1)^3(\sigma+1)^2(\sigma+5)^3$ and $\phi(H_b:\sigma)=(\sigma-5)^2(\sigma-3)^2(\sigma-1)(\sigma+1)^4(\sigma+3)(\sigma+5)^2$. Then $SE(H_a)=SE(H_b)=34$. Hence they are non co-spectral Seidel equienergetic.

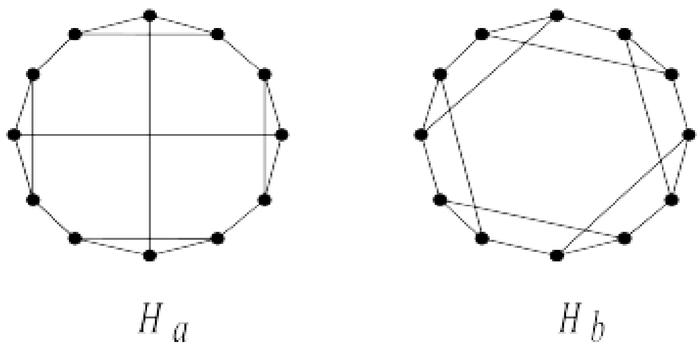

Figure 1: Seidel equienergetic graphs H_a and H_b

Let G be any r_1-regular circulant graph. Then by Theorem 2.2

$$\begin{aligned} SE(G[H_a]) &= nSE(H_a) - 5n + |12n - 24r_1 - 7| + \sum_{t=1}^{n-1} |-7 - 24\alpha_t| \\ &= nSE(H_b) - 5n + |12n - 24r_1 - 7| + \sum_{t=1}^{n-1} |-7 - 24\alpha_t| \\ &= SE(G[H_b]). \end{aligned}$$

Thus, $G[H_a]$ and $G[H_b]$ are non co-spectral Seidel equienergetic.
Similarly, from Theorem 4.2, we have:

$$\begin{aligned} SE(G \circ H_a) &= nSE(H_a) - 5n \\ &+ \left| \frac{-(2r_1 + 8 - 13n) + \sqrt{(2r_1 + 8 - 13n)^2 - 4(29n + 2r_1(7 - 12n) - 41)}}{2} \right| \\ &+ \left| \frac{-(2r_1 + 8 - 13n) - \sqrt{(2r_1 + 8 - 13n)^2 - 4(29n + 2r_1(7 - 12n) - 41)}}{2} \right| \\ &+ \sum_{t=1}^{n-1} \left| -(4 + \alpha_t) + \sqrt{(4 + \alpha_t)^2 + 41 - 14\alpha_t} \right| \\ &+ \sum_{t=1}^{n-1} \left| -(4 + \alpha_t) - \sqrt{(4 + \alpha_t)^2 + 41 - 14\alpha_t} \right| \\ &= SE(G \circ H_b). \quad (\because \; SE(H_a) = SE(H_b)) \end{aligned}$$

The above construction lead to the following Theorem:

Theorem 5.1 *Let G be a r_1-regular circulant graph with n vertices. If H_1 and H_2 are r_2-regular graphs with m vertices which are non co-spectral Seidel equienergetic graphs then*
(i) $G[H_1]$ and $G[H_2]$ are Seidel equienergetic.
(ii) $G \circ H_1$ and $G \circ H_2$ are Seidel equienergetic.

Note: We know that if H_1 and H_2 are Seidel equienergetic then their complements also Seidel equienergetic. Using this in the above theorem, we have the following:

1. The pairs $\overline{G}[H_1]$ and $\overline{G}[H_2]$, $G[\overline{H_1}]$ and $G[\overline{H_2}]$, $\overline{G[H_1]}$ and $\overline{G[H_2]}$ are Seidel equienergetic
2. The pairs $\overline{G} \circ [H_1]$ and $\overline{G} \circ [H_2]$, $G \circ [\overline{H_1}]$ and $G \circ [\overline{H_2}]$, $\overline{G \circ [H_1]}$ and $\overline{G \circ [H_2]}$ are Seidel equienergetic.

Acknowledgments

The authors thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

References

1. Adiga, C., Sampathkumar, E., Sriraj, M. A., and Shrikanth, A. S., *Color energy of a graph*, Proc. Jangjeon Math. Soc., 16(3), 335–351, (2013).
2. Adiga, C., Sampathkumar, E. and Sriraj, M. A., *Color energy of a unitary Cayley graph*, Discuss. Math., Graph Theory, 34(4), 707-721, (2014).
3. AlFran, H. A., Rajendra, R., Siva Kota Reddy, P., Kemparaju, R. and Altoum, Sami H., *Spectral Analysis of Arithmetic Function Signed Graphs*, Glob. Stoch. Anal., 11(3), 50-59, (2024).
4. Brouwer, A. E. and Haemers, W. H., *Spectra of Graphs-Monograph*, Springer, (2011).
5. Ruiz Claeyssen, J. C. and dos Santos Leal, L. A., *Diagonalization and spectral decomposition of factor block circulant matrices*, Linear Algebra Appl., 99, 41-61, (1988).
6. Cvetković, D. M., Doob, M. and Sachs, H., *Spectra of Graphs*, Academic Press, (1979).
7. Davis, P. J., *Circulant matrices*, 2^{nd} Edition, AMS Chelsea Publishing, New York, (1994).
8. Frucht, R. and Harary, F., *On the corona of two graphs*, Aequationes Math., 4, 322-325, (1970).
9. Haemers, W. H., *Seidel switching and graph energy*, MATCH Commun. Math. Comput. Chem., 68(3), 653-659, (2012).
10. Hemavathi, P. S., Mangala Gowramma, H., Kirankumar, M., Pavithra, M. and Siva Kota Reddy, P., *On Minimum Stress Energy of Graphs*, J. Appl. Math. Inform., 43(2), 543-557, (2025).
11. Imrich, W. and Klavžar, S., *Product Graphs: Structure and Recognition*, Wiley-Interscience Series in Discrete Mathematics and Optimization, New York, (2000).
12. Liu, J.-B., Munir, M., Yousaf, A., Naseem, A., and Ayub, K., *Distance and Adjacency Energies of Multi-Level Wheel Networks*, Mathematics, 7(1), Article No. 43, 9 Pages, (2019).
13. Gallian, J. A., *A dynamic survey of graph labeling*, Electron. J. Comb., DS6, Version 27, 1-712, (2023).
14. Kirankumar, M., Ruby Salestina, M., Harshavardhana, C. N., Kemparaju, R. and Siva Kota Reddy, P., *On Stress Product Eigenvalues and Energy of Graphs*, *Glob. Stoch. Anal.*, 12(1), 111-123, (2025).
15. Kirankumar, M., Harshavardhana, C. N., Ruby Salestina, M., Pavithra, M. and Siva Kota Reddy, P., *On Sombor Stress Energy of Graphs*, J. Appl. Math. Inform., 43(2), 475-490, (2025).
16. Li, X., Shi, Y., and Gutman, I., *Graph Energy*, Springer, Berlin, (2012).
17. Lokesha, V., Shanthakumari, Y. and Siva Kota Reddy, P., *Skew-Zagreb Energy of Directed Graphs*, Proc. Jangjeon Math. Soc., 23(4), 557-568, (2020).
18. Nalina, C., Siva Kota Reddy, P., Kirankumar, M. and Pavithra, M., *On Stress Sum Eigenvalues and Stress Sum Energy of Graphs*, Bol. Soc. Parana. Mat. (3), 43, Article Id: 75954, 15 Pages, (2025).
19. Oboudi, M. R., *Energy and Seidel energy of graphs*, MATCH Commun. Math. Comput. Chem., 75(2), 291-303, (2016).
20. Ramane, H. S. and Walikar, H. B., *Construction of equienergetic graphs*, MATCH Commun. Math. Comput. Chem., 57(1), 203-210, (2007).
21. Ramane, H. S., Gundloor, M. M. and Hosamani, S. M., *Seidel equienergetic graphs*, Bull. Math. Sci. Appl., 16, 62–69, (2016).
22. Prakasha, K. N., Siva kota Reddy, P. and Cangul, I. N., *Partition Laplacian Energy of a Graph*, Adv. Stud. Contemp. Math., Kyungshang, 27(4), 477-494, (2017).
23. Prakasha, K. N., Siva Kota Reddy, P. and Cangul, I. N., *Minimum Covering Randic energy of a graph*, Kyungpook Math. J., 57(4), 701-709, (2017).
24. Prakasha, K. N., Siva Kota Reddy, P. and Cangul, I. N., *Sum-Connectivity Energy of Graphs*, Adv. Math. Sci. Appl., 28(1), 85-98, (2019).
25. Prakasha, K. N., Siva kota Reddy, P., Cangul, I. N. and Purushotham, S., *Atom-Bond-Connectivity Energy of Graphs*, TWMS J. App. Eng. Math., 14(4), 1689-1704, (2024).
26. Rajendra, R., Siva Kota Reddy, P. and Kemparaju, R., *Eigenvalues and Energy of Arithmetic Function Graph of a Finite Group*, Proc. Jangjeon Math. Soc., 27(1), 29-34, (2024).
27. Rakshith, B. R., Das, K. C. and Sriraj, M. A., *On (distance) signless Laplacian spectra of graphs*, J. Appl. Math. Comput., 67(1-2), 23-40, (2021).

28. Sampathkumar, E. and Sriraj, M. A., *Vertex labeled/colored graphs, matrices and signed graphs*, J. Comb. Inf. Syst. Sci., 38(1-4), 113-120, (2013).

29. Sampathkumar, E., Roopa S. V., Vidya, K. A. and Sriraj, M. A., *Partition Energy of a graph*, Proc. Jangjeon Math. Soc., 18(4), 473-493, (2015).

30. Sampathkumar, E., Roopa, S. V., Vidya, K. A. and Sriraj, M. A., *Partition energy of complete product of circulant graphs and some new class of graphs*, Adv. Stud. Contemp. Math. (Kyungshang), 28(2), 269–283, (2018).

31. Sampathkumar, E., Roopa, S. V., Vidya, K. A. and Sriraj, M. A., *Partition Laplacian energy of a graph*, Palest. J. Math., 8(1), 272–284, (2019).

32. Sampathkumar, E., Roopa, S. V., Vidya, K. A. and Sriraj, M. A., *Partition energy of some trees and their generalized complements*, TWMS J. App. Eng. Math., 10(2), 521–531, (2020).

33. Shwetha, B. C., Sriraj, M. A., Roopa, S. V. and Veena, C. R., *Partition energy of m-splitting graph and their generalized complements*, Proyecciones, 43(5), 1113–1139, (2024).

34. Somashekar, P., AlFran, H. A., Siva Kota Reddy, P., Kirankumar, M. and Pavithra, M., *On Cangul Stress Energy of Graphs*, Bol. Soc. Parana. Mat. (3), 43, Article Id: 76033, 12 Pages, (2025).

35. Sriraj, M. A., Shwetha, B. C., Veena, C. R. and Roopa, S. V., Partition energy of some lexicographic product of two graphs, Proc. Jangjeon Math. Soc., 25(4), 435-454, (2022).

36. Sureshkumar, S., Mangala Gowramma, H., Kirankumar, M., Pavithra, M. and Siva Kota Reddy, P., *On Maximum Stress Energy of Graphs*, Glob. Stoch. Anal., 12(2), 56-69, (2025).

D. P. Pushpa,
Research Scholar,
Department of Mathematics,
The National Institute of Engineering,
(Autonomous under Visvesvaraya Technological University, Belagavi-590 018, India.)
Mysuru-570 008, India.
E-mail address: `npushpadp@gmail.com`

and

S. V. Roopa (Corresponding author),
Department of Mathematics
The National Institute of Engineering
(Autonomous under Visvesvaraya Technological University, Belagavi-590 018, India.)
Mysuru-570 008, India.
E-mail address: `svrroopa@gmail.com`

and

M. A. Sriraj,
Department of Mathematics
Vidyavardhaka College of Engineering,
(Autonomous under Visvesvaraya Technological University, Belagavi-590 018, India.),
Mysuru-570 002, India.
E-mail address: `masriraj@gmail.com; sriraj@vvce.ac.in`

and

P. Siva Kota Reddy,
Department of Mathematics,
JSS Science and Technology University,
Mysuru-570 006, India.

and

Universidad Bernardo O'Higgins,
Facultad de Ingeniería, Ciencia y Tecnología,
Departamento de Formación y Desarrollo Científico en Ingeniería,
Av. Viel 1497, Santiago, Chile
E-mail address: pskreddy@jssstuniv.in