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P-decomposition matrix of derived Chevalley group

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ABSTRACT: Character theory has diverse applications in the applied sciences. In particular, it plays a crucial role in understanding symmetries and features of quantum systems in quantum mechanics, constructing error-correcting codes in coding theory, and analyzing and predicting molecular structures in computational chemistry. In addition, the characters of finite groups are utilized in algebraic geometry, graph theory, cryptography, number theory, network analysis, and cohomology theory. This article studies the derived Chevalley group $G_2'(2)$, which is isomorphic to the group of all 3×3 invertible matrices preserving a non-singular Hermitian form over the Galois field $\mathbb{F}_9$ of order 9. The computation of p-decomposition matrices relative to all prime divisors of the group order enables the investigation of the decomposition matrices of its subgroups and supergroups. Absolutely irreducible p-characters provide valuable insights into the structure of the group.

Key Words: Decomposition matrices, Chevalley group, irreducible modular characters, principal indecomposable characters.

Contents

1. Introduction

Assume that G is a finite group, $\mathrm{GL}(r, \mathbb{C})$ is the general linear group of degree r over r, the field of complex numbers of characteristic zero and p is a prime number. Representations of G over $\mathbb{C}$ are regarded as ordinary representations while over the algebraically closed field of characteristic p are termed as p-modular representation. Representation of G is equivalent to the representation ρ of G over a finite extension of the rationals and in addition, over a maximal subring S of the latter field not containing p. Furthermore, S has a unique maximal ideal T such that $E = S/T$ is a field of characteristic p. Thus, if $\rho(G) \subseteq \mathrm{GL}(r, \mathbb{C})$, for some $r \in \mathbb{N}$ (set of natural numbers), then the representation $\overline{\rho} : G \to \mathrm{GL}(r, E)$ is normally called the reduction of ρ modular$p$. $\overline{\rho}$ may not need to necessarily be irreducible even if ρ is irreducible. The multiplicities of irreducible of G over $\overline{E}$, the Algebraic closure of E as a composition factor of $\overline{\rho}$ are known as the decomposition number , they are numerically invariant that remarkably connect the characteristic zero and characteristic p representations of G. This work is in continuous to our investigation of the decomposition matrices of finite groups. Now we compute the absolutely irreducible p-characters of the derived Chevalley group $G_2'(2)$, isomorphic to the group of all 3×3 matrices preserving a non- singular hermitian from over the field $\mathbb{F}_9$, relative to the all prime characteristic dividing the group order . The results are given in the form of the decomposition matrices. For such matrices, the hard problem is getting them, that is constructing an enormous number of projective characters. There are methods of generating these: induction, restriction, tensor product and symmetrization but no apparent algorithm on how to choose the optimal among them. The task thus entails intensive searching in a large

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set for optimal projective characters. This usually is done with the computational help of a computer; in our case, this was accomplished by purely character-theoretic arguments.

2. Methodology

First, we use the central characters to split the ordinary characters into p-blocks (for general information about p-blocks, see [1,3,5,7,9,10]). Let B be a p-block of a group G, and let $\psi_1, \ldots, \psi_s$ and $\phi_1, \ldots, \phi_r$ respectively denote the ordinary and p-modular irreducible characters of G in the block B. The restriction of each ψ_i to p-regular classes of G, denoted by $\overline{\psi_i}$, is a p-modular character of G, and

$$\overline{\psi_i} = \sum_{j=1}^{r} d_{ij}\phi_j \quad (1 \leq i \leq s)$$

where d_{ij} are non-negative integers called the decomposition numbers of B for the prime p. The $(s \times r)$ matrix $D_P^B(G) = (d_{ij})$ is called the decomposition matrix of G for the prime p. Furthermore, a principal character $\sum_{j=1}^{r} c_i\psi_i$ corresponds to the column of integers $c = c_i$ and may be indecomposable or a sum of principal indecomposable characters in B. For more details, see [2,4,6,8,11,12,13].

One way to construct $D_P^B(G)$ is to find the principal indecomposable characters in B. This method is described by James and Kerber [6] as follows: let

$$R_P^B(G) = \left(c^{(0)}, c^{(1)}, c^{(2)}, \ldots, c^{(n_1)}, \ldots, c_{r-1}^{(0)}, c_{r-1}^{(1)}, \ldots, c_{r-1}^{(n_{r-1})}, c_r\right)$$

be the matrix of principal characters that has been determined. With an appropriate arrangement of the ordinary irreducible characters, for $0 \leq k_i \leq n_i$ and $i = 1, \ldots, r-1$, every matrix

$$R_P^B(G) = \left(c_1^{(k_1)}, \ldots, c_{r-1}^{(k_{r-1})}, c_r\right)$$

is upper triangular, with zeroes above the main diagonal and ones on the main diagonal.

The matrix $D_P^B(G)$ has the same structure as $R_P^B(G)$, where the entries of $R_P^B(G)$ are always greater than or equal to the corresponding entries in $D_P^B(G)$. Moreover, for all $d_r = c_r$ and for all $i = 1, \ldots, r-1$ and $k_i = 0, 1, \ldots, n_i$, we have:

$$d_i = c_i^{(k_i)} \to \sum_{j=i+1}^{r} a_j^{(k_i)} d_j$$

where $a_j^{(k_i)}$ are non-negative integers and d_i $(i = 1, \ldots, r)$ are the columns of $D_P^B(G)$.

If G is a group and p a prime, then the symbols $\mathrm{irr}(G)$, $\mathrm{irmc}(G)$, and $\mathrm{indec}_p(G)$ denote respectively the full set of ordinary irreducible, p-modular irreducible, and principal p-indecomposable characters of G. If there is no confusion about p, we omit it. Furthermore, $\uparrow^G$ $(\downarrow_G)$ denotes the induction from (restriction to) a subgroup of G, and the symbol ϕ denotes the least character corresponding to that column in the matrix. From now onward, assume that $G = G_2'(2)$ and $H = L_2(7)$, a maximal subgroup of G of index 36.

3. Decomposition number for $p = 2$

Here $\mathrm{irmc}(G) = 5$. Certainly, the character $\overline{(32)}$, $32 \in \mathrm{irr}(G)$ resides in $\mathrm{irmc}(G)$ and in $\mathrm{indec}(G)$ as well, since they fit in 2-blocks of defect zero. All the remaining elements of $\mathrm{irr}(G)$ lie in the principal 2-block of G. The integral relations on 2-regular classes are as follows:

$$7_a = 7_b = \overline{7_b} = 1 + 6 \tag{2.1}$$

$$21_a = 21_b = \overline{21_b} = 1 + 6 + 14 \tag{2.2}$$

$$27 = 1 + 2.6 + 14 \tag{2.3}$$

$$28 = \overline{28} = 2.1 + 2.6 + 14 \tag{2.4}$$

It is evident from these relations that we need only work with the characters 1, 6, and 14 in order to complete the remaining 3 elements of $\mathrm{irmc}(G)$ lying in the principal block. If the character 6 splits, there must exist two new elements, namely $\phi(3)$ and $\overline{\phi(3)}$, each of degree 3, in irmc(G) since $6\downarrow_H = \phi(3)+\overline{\phi(3)}$, where $\phi(3), \overline{\phi(3)} \in$ irmc(H). Hence, these elements of irmc(G) together with the trivial character give the required number of elements of irmc(G) in the principal 2-block.

However, $14\downarrow_H = \phi(3)+\phi(3)+\phi(8)$ implies that the character of degree 14 of G cannot be obtained from the elements of irmc(G) computed so far. Thus, the character 6 belongs to irmc(G). From the restriction $14\downarrow_H$ and the above information, it follows that $14 \in$ irr(G) yields the remaining element of irmc(G), connected by $\phi(14)$.

Furthermore, $\phi(6)$ appears in 14 at most once, and $\phi(1)$ is not a composition factor of 14. Thus, we only need to prove that the irreducible modular character of degree 6, $\phi(6)$, is not a constituent of 14. Consider the following tensor products:

$$\phi(6)\otimes\phi(6) = 2\phi(1)+\phi(6)+2\phi(14) \tag{2.5}$$

$$14\otimes\phi(6) = 2\phi(1)+3\phi(6)+\phi(32)+\overline{\phi(32)} \tag{2.6}$$

Relation (2.5) shows that $\phi(14)$ appears in $\phi(6)\otimes\phi(6)$ exactly twice. Moreover, all the constituents in relation (2.6) are in irmc(G), and none of them is $\phi(14)$. From all the above information, we conclude that $\phi(14)$ is a constituent of $\phi(6)\otimes\phi(6)$ but not of $14\otimes\phi(6)$. Thus, the character $\phi(6)$ is not contained in 14. Therefore, the required decomposition matrix is as follows:

	d_1	d_2	d_3	d_4	d_5
1	1				
6		1			
7_a	1	1			
7_b	1	1			
$\overline{7_b}$	1	1			
14			1		
21_a	1	1	1		
21_b	1	1	1		
$\overline{21_b}$	1	1	1		
27	1	2	1		
28	1	2	1		
$\overline{28}$	2	2	1		
32				1	
$\overline{32}$					1

4. Decomposition number for $p = 3$

The element of degree $27 \in$ irr(G), being of defect zero, is in indec(G) and also in $B_e(G)$ for $p = 3$. The 3-regular bases of G imply that the size of the required 3-decomposition matrix of G is 14×9. Clearly, all but the character of degree 27 are associated with the principal 3-block of G. Moreover, this block contains 8 elements of the set irmc(G). The difficulty of finding irmc(G) for groups makes it interesting to compute the basic sets explicitly.

We begin with a basic set of nine 3-projective characters of G, as shown in Table **??**, which is the first approximation towards the required 3-decomposition matrix. The origins of these characters are as follows: the columns c_1, c_2, c_3, and c_5 respectively come from inducing elements $(1+8)$, 6, $(7+8)$, and $3 \in$ Indec(H) to G. Additional columns c_4, c_5, c_6, c_7, and c_8 are constructed by taking tensor products of elements from indec(G) with their own 3-modular (or modular irreducible) characters:

$$7_d\otimes 27,\quad \overline{7_b}\otimes 27,\quad 7_b\otimes 27,\quad \phi_2\otimes 27,\quad \phi_3\otimes 27$$

Construction of $\phi_2, \phi_3 \in$ irmc(G):
Note that $6\downarrow_H = \phi(3)+\overline{\phi(3)}$, where $\phi(3), \overline{\phi(3)} \in$ irmc(H). Now, we show that $6 \in$ irr(G) splits into two elements of irmc(G), say ϕ_2, ϕ_3, each of degree 3. We calculate these characters explicitly on 3-regular classes of G, where $6 = \phi_2+\phi_3$.

The existence of ϕ_2 and ϕ_3 of degree 3 follows from subgroup H of G, clearly:

$$\phi_2\downarrow_H = \phi(3),\quad \phi_3\downarrow_H = \overline{\phi(3)}$$

The values of $\phi(3)$ and $\overline{\phi(3)} \in \mathrm{irmc}(H)$ and the 3-modular character $\phi(3) \otimes \overline{\phi(3)}$ on 3-regular classes are:

Classes	$1A$	$2A$	$4A$	$7A$	B^{**}
$\phi(3)$	3	-1	1	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$
$\overline{\phi(3)}$	3	-1	1	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$
$\phi(3) \otimes \overline{\phi(3)}$	9	1	1	2	2

Moreover, the following relations exist among the 3-modular characters of G:

$$14 = 1 + 6 + 7_a \tag{3.1}$$

$$21_a = 7_a + 7_b + \overline{7_b} \tag{3.2}$$

$$14 + 21_b = 28_a + 7_b \tag{3.3}$$

$$14 + \overline{21_b} = \overline{28_a} + \overline{7_b} \tag{3.4}$$

From these relations and character values, the values of ϕ_2 and ϕ_3 on 3-regular classes of G are given explicitly.

Table 1: Values of ϕ_2 and ϕ_3

Classes	$1A$	$2A$	$4A$	B^*	$4C$	$7A$	B^{**}	$8A$	B^{***}
ϕ_2	3	-1	$2i-1$	$-2i-1$	1	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$	i	$-i$
ϕ_3	3	-1	$-2i-1$	$2i-1$	1	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$	$-i$	i

Hence, the required decomposition matrix, constructed from these arguments and computations, is denoted by: where d_i are derived explicitly based on the previous character relations and constraints.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	
1	1								
6		1	1						
7_a				1					
7_b	1				1				1
$\overline{7_b}$	1					1			1
14	1	1	1	1					
21_a	2			1	1	1			2
21_b	1				1	1	1		1
$\overline{21_b}$	1				1	1		1	1
28	1	1	1	1		1	1		
$\overline{28}$	1	1	1	1	1			1	
32	2	1		1	1	1	1		1
$\overline{32}$	2		1	1	1	1		1	1

The proactive character $\overline{3} \uparrow_H^G$ of G, in addition to c_5, contains c_6 once. The comparison of the multiplicity of the character 28 in c_5 and c_6 with c_5 shows that c_7 must be a constituent of c_6 exactly once. Consequently, $d_6 = c_5 - d_7$ is indecomposable. Since c_5 and c_6 are conjugates, and d_7 and d_8 are also conjugate under the outer automorphism of G, it follows that $d_5 = c_5 - d_8$ is also an indecomposable projective character of G.

Clearly, $7_d \otimes 27$ is a self-dual projective character by ([13], 1.4). We conclude that $7_d \otimes 27 \in \mathrm{indec}(G)$. Although $33 = |G|_3$ divides the degree of the following characters:

$$21_a + 28_a + 32_a, \quad 21_a + 28_a + \overline{32_a}, \quad 21_a + \overline{28_a} + 32_a, \quad 21_a + \overline{28_a} + \overline{32_a}$$

none vanish on the 3-singular classes of G. Thus, we obtain $d_4 = c_4$. A standard subsume argument shows that c_2 and c_3 are projective indecomposable characters of G, hence $d_2 = c_2$ and $d_3 = c_3$.

Since $14 = 1 + 6 + 7_a$, the integral relation on 3-regular classes of G implies the trivial 3-modular character of G occurs as a composition factor of the character of degree 14. Hence, d_5 is not a constituent of c_1. Therefore,

$$d_1 = c_1 - \alpha d_7 - \beta d_8, \quad \text{where} \quad \alpha, \beta \in \{0, 1\}.$$

Since d_7 and d_8 are conjugates, we have

$$d_1 = c_1 \quad \text{or} \quad d_1 = c_1 - c_7 - d_8.$$

Now,

$$6 \otimes 21_b = 7_a + 28_a + 32 + \overline{32} + \text{(other blocks)}. \tag{3.5}$$

Since $6 = \phi(3) + \overline{\phi(3)}$, it follows that

$$(6 \otimes 21_b) = (\phi(3) \otimes 21_b) + (\overline{\phi(3)} \otimes 21_b) \tag{4.1}$$
$$= 21_b + \overline{21_b} + 28_a + \overline{28_a} + 2.7_b + 2.1 + 6 + \overline{6}. \tag{3.6}$$

Comparing (3.5) and (3.6), we obtain:

$$21_b + \overline{21_b} + 28_a + \overline{28_a} + 2.7_b + 2.1 + 6 + \overline{6} = 7_a + 32 + \overline{32}.$$

If the trivial module were contained in $21_b + \overline{21_b} + 28_b$ exactly once, it would appear as a composition factor of $32 + \overline{32}$ with multiplicity 5, which is impossible. Hence, the trivial module is neither contained in $21_b, 28$ nor in their conjugate characters. Thus, d_r and d_s are constituents of c_1 and must be discarded from c_1, yielding:

$$d_1 = c_1 - c_7 - c_8.$$

Hence, the required decomposition matrix is:

	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
1	1							
6		1	1					
7_a				1				
7_b	1				1			
$\overline{7_b}$	1					1		
14	1	1	1	1				
21_a	2			1	1	1		
21_b					1		1	
$\overline{21_b}$						1		1
28		1	1	1			1	
$\overline{28}$		1	1	1				1
32	1	1		1	1		1	
$\overline{32}$	1		1	1		1		1

5. Decomposition number for $p = 7$

The rank of the required decomposition is 12, as there are twelve 7-regular classes in G. All characters except those of degrees 1, 6, 27, 32, and $\overline{32}$ are in irmc(G) and also in indec(G). On 7-regular classes, we have:

$$32 = \overline{32}, \quad 1 + 32 = 6 + 27.$$

Thus, the Brauer tree for the principal 7-block of G, as the trivial module is not a composition factor of $G \downarrow_H$, is given by:

6. Conclusion

In this paper, we presented a novel method for constructing the irreducible modular characters and principal indecomposable characters of the group $G_2'(2)$. By employing decomposition matrices, we systematically analyzed the modular representation theory of this group with respect to all its prime divisors.

This approach not only simplifies the process of character construction but also provides deeper insight into the structure and behavior of the group's modular representations. Our results contribute to the broader understanding of the representation theory of finite groups of Lie type, particularly in modular settings, and may serve as a foundation for future studies involving similar groups or more complex decomposition techniques.

Future work may extend this approach to other finite groups of Lie type, exploring broader classes of decomposition matrices. Additionally, algorithmic implementation of these constructions could enhance computational tools in modular representation theory.

Certainly, these results can be very useful and helpful in investigating properties and applications in various fields such as finite geometry, coding theory, homology groups, cryptography, and group actions.

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