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Generalized Kannan Type Fixed Point Theorems in Equivalent Distance Spaces

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ABSTRACT: This paper develops new fixed point theorems for self-maps on metric spaces endowed with $\mathcal{E}_{A,B}$-distances. We consider families of operators $\{\mathcal{T}^t\}$ that satisfy a generalized Kannan-type contraction condition and establish corresponding existence and uniqueness fixed point results under suitable assumptions on the parameters involved. The proposed framework generalizes classical fixed point theory while retaining its versatility and wide range of applications. The $\mathcal{E}_{A,B}$-distance structure proves particularly effective for analyzing nonlinear operators in functional analysis. An illustrative example demonstrates the verifiability and practical utility of the proposed theoretical conditions.

Key Words: Fixed point theory, nonlinear operators, generalized metric spaces, common fixed point, equivalent distance.

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1. Introduction

The Kannan fixed-point theorem [1], introduced in 1968, provides an important alternative to the Banach contraction principle. While Banach contractions require the operator to reduce distances uniformly, Kannan's condition involves a combination of distances from the point to its image. A fundamental result by Subrahmanyam [2] establishes that Kannan contractions provide a complete characterization of metric completeness, as a metric space $(\mathcal{X}, d)$ is complete if and only if every Kannan contraction on $\mathcal{X}$ admits a unique fixed point.

Recent developments in fixed point theory have significantly extended the applicability of fixed point theory across various generalized metric spaces. The foundational work begins with Malkawi et al. [3] in MR-metric spaces, complemented by their subsequent contributions to M*-metric spaces [4]. Further generalizations include (Ψ, L)-M-weak contractions in Mb-metric spaces [5] and Ω-distance results via simulation functions [6]. Notable extensions encompass the $(\alpha(s), F)$-contractions for four mappings by Nazam et al. [7] and the (α, p)-convex contractions with asymptotic regularity by Khan et al. [8]. Additional refinement appear in Suzuki-type ω-distance theorems [9]. The theory's interdisciplinary impact is evidenced by fractional derivative modifications [10] and applications to Lie symmetry analysis [11]. Collectively, these works demonstrate the enduring versatility and evolution of Kannan's original framework. In [12] extends fixed point theory for generalized weak contractions in b-metric spaces, demonstrating new coincidence and fixed point results under relaxed metric conditions. Recent advances have been made in the study of fixed point theorems in neutrosophic metric spaces.

Building upon the foundational work of Bataihah [13] on fixed point theorems in generalized distance spaces, recent research has extended these results to more abstract settings. Hazaymeh [14] established new fixed point theorems in complete b-metric spaces with T-distance, while Bataihah et al. [15,16] further developed the theory in neutrosophic metric spaces. These advances have significantly expanded the applicability of fixed point theory to nonlinear problems.

These developments have opened new directions in the field.

This paper extends Kannan's theory to time-dependent metric spaces with $\mathcal{E}_{A,B}$-distances.

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Definition 1.1 *[17] A function $A : [0,\infty) \to [1,\infty)$ belongs to Λ if for any sequence (t_n) in $[0,\infty)$:*

$$\lim_{n\to\infty} A(t_n) = 1 \iff \lim_{n\to\infty} t_n = 0.$$

An example of a function belonging to Λ is $A(t) = 1 + t$.

Definition 1.2 *[17] Given a metric space $(\mathcal{X}, d)$, a function $\mathcal{E} : [0,\infty) \times \mathcal{X} \times \mathcal{X} \to [0,\infty)$ is an $E_{A,B}$-distance if there exist $A, B \in \Lambda$ such that:*

$$A(t)d(\zeta,\xi) \le \mathcal{E}(t,\zeta,\xi) \le B(t)d(\zeta,\xi).$$

Clearly, a metric induces an $\mathcal{E}_{A,B}$-distance. In particular, if $(\mathcal{X}, d)$ is a metric space, then the function

$$\mathcal{E}(t,\zeta,\xi) = (1+t)d(\zeta,\xi)$$

defines an $\mathcal{E}_{A,B}$-distance.

Bataihah [19] established novel fixed point theorems for Geraghty-type contractions in the framework of equivalent distances.

Lemma 1.1 *[17] Let $(\mathcal{X}, d)$ be a metric space, $A, B \in \Lambda$ and $\mathcal{E}$ be $\mathcal{E}_{A,B}$-distance over $(\mathcal{X}, d)$. Then, for all $\zeta, \xi, w \in \mathcal{X}$ and all $t \ge 0$, we have*

1. *$\mathcal{E}(t,\zeta,\xi) = 0 \iff \zeta = \xi$,*
2. *$\mathcal{E}(t,\xi,\zeta) = \mathcal{E}(t,\zeta,\xi)$, whenever $A(t) = B(t)$,*
3. *$\mathcal{E}(t,\zeta,\xi) \le \frac{B(t)}{A(t)}[\mathcal{E}(t,\zeta,w) + \mathcal{E}(t,w,\xi)]$.*

Remark 1.1 [17] The $\mathcal{E}_{A,B}$-distance framework generalizes several well-known metric extensions. In particular, if $(\mathcal{X}, d)$ is a metric space, then $\mathcal{E}_{A,B}$-distance induced a quasi extended b -metric [18], where the triangle inequality takes the form

$$d_\gamma(\zeta,\xi) \le \gamma(\zeta,\xi)\left[d_\gamma(\zeta,w) + d_\gamma(w,\xi)\right],$$

where $\gamma(\zeta,\xi) = B(d(\zeta,\xi))$.

However, If $A(t) = B(t)$ then d_γ is an extended b -metric on $\mathcal{X}$.

This work leverages the structure of $\mathcal{E}_{A,B}$-distances, where scaling functions ensure the distance dominates the base metric. Time-dependent operators with carefully chosen contraction properties form the basis of our analysis. The key idea involves constructing iterative sequences and controlling their behavior through the interplay between the scaling functions and contraction conditions. By exploiting the inherent properties of these generalized distances, we establish convergence and uniqueness results without relying on traditional continuity assumptions. This approach extends classical fixed-point theory to accommodate parameter-dependent operators and non-uniform metric spaces.

2. Main Results

Let $(\mathcal{X}, d)$ be a metric space, and let $A, B \in \Lambda$. Suppose $\mathcal{E}$ is an $\mathcal{E}_{A,B}$-distance defined on $(\mathcal{X}, d)$. Throughout the following, unless otherwise specified, we assume that for every fixed $\zeta, \xi \in \mathcal{X}$ and each $t \ge 0$, the mappings

$$\mathcal{E}(t,\zeta,\cdot),\ \mathcal{E}(t,\cdot,\xi) : \mathcal{X} \to \mathcal{X}$$

are continuous.

Theorem 2.1 *Let $(\mathcal{X}, d)$ be a complete metric space with $\mathcal{E}_{A,B}$-distance $\mathcal{E}$. Suppose $\{\mathcal{T}^t\}_{t\ge0}$ satisfies*

1. *For all $\zeta, y \in \mathcal{X}$ and $t \ge 0$:*

$$\mathcal{E}(t,\mathcal{T}^t\zeta,\mathcal{T}^t y) \le k(t)\left[\mathcal{E}(t,\zeta,\mathcal{T}^t\zeta) + \mathcal{E}(t,y,\mathcal{T}^t y)\right],$$

where $A(t) - B(t)k(t) < 1$ and $\sup_t k(t) < \frac{1}{2}$.

2. For any sequence (t_n),

$$\mathcal{E}(t_n, \mathcal{T}^{t_n}\zeta, \mathcal{T}^{t_{n+1}}\zeta) \le C(t_n, t_{n+1})\mathcal{E}(t_n, \zeta, \mathcal{T}^{t_n}\zeta),$$

where $C(t_n, t_{n+1}) \to 0$ *as* $n \to \infty$.

Then for any $\zeta_0 \in \mathcal{X}$ *and sequence* (t_n) *with*

$$\sup_n \frac{B(t_n)k(t_n)[1 + C(t_n, t_{n+1})]}{A(t)(1 - k(t_n))} < 1,$$

the iterates $\zeta_{n+1} = \mathcal{T}^{t_n}\zeta_n$ *converge to the unique common fixed point* ζ^* *of* $\{\mathcal{T}^{t_n} : \ n = 0, 1, 2, \cdots\}$.

Proof: Define the iterative sequence

$$\zeta_1 = \mathcal{T}^{t_0}\zeta_0, \ \zeta_2 = \mathcal{T}^{t_1}\zeta_1, \ \cdots, \ \zeta_{n+1} = \mathcal{T}^{t_n}\zeta_n$$

From the Kannan condition, we get

$$\begin{aligned}
\mathcal{E}(t_n, \zeta_{n+1}, \zeta_{n+2}) &\le \frac{B(t)}{A(t)}[\mathcal{E}(t_n, \mathcal{T}^{t_n}\zeta_n, \mathcal{T}^{t_n}\zeta_{n+1}) + \mathcal{E}(t_n, \mathcal{T}^{t_n}\zeta_{n+1}, \mathcal{T}^{t_{n+1}}\zeta_{n+1})] \\
&\le \frac{B(t)}{A(t)}k(t_n)\left[\mathcal{E}(t_n, \zeta_n, \zeta_{n+1}) + \mathcal{E}(t_n, \zeta_{n+1}, \mathcal{T}^{t_n}\zeta_{n+1})\right] \\
&\quad + \frac{B(t)}{A(t)}C(t_n, t_{n+1})\mathcal{E}(t_n, \zeta_{n+1}, \mathcal{T}^{t_n}\zeta_{n+1}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathcal{E}(t_n, \zeta_{n+1}, \mathcal{T}^{t_n}\zeta_{n+1}) &= \mathcal{E}(t_n, \mathcal{T}^{t_n}\zeta_n, \mathcal{T}^{t_n}\zeta_{n+1}) \\
&\le k(t_n)[\mathcal{E}(t_n, \zeta_n, \zeta_{n+1}) + \mathcal{E}(t_n, \zeta_{n+1}, \mathcal{T}^{t_n}\zeta_{n+1})]
\end{aligned}$$

Rearranging

$$\mathcal{E}(t_n, \zeta_{n+1}, \mathcal{T}^{t_n}\zeta_{n+1}) \le \frac{k(t_n)}{1 - k(t_n)}\mathcal{E}(t_n, \zeta_n, \zeta_{n+1})$$

Hence,

$$\mathcal{E}(t_n, \zeta_{n+1}, \zeta_{n+2}) \le \frac{B(t_n)k(t_n)[1 + C(t_n, t_{n+1})]}{A(t)(1 - k(t_n))}\mathcal{E}(t_n, \zeta_n, \zeta_{n+1}).$$

Let $\lambda(t_n) = \dfrac{B(t_n)k(t_n)[1 + C(t_n, t_{n+1})]}{A(t)(1 - k(t_n))}$. Then,

$$\mathcal{E}(t_n, \zeta_{n+1}, \zeta_{n+2}) \le \lambda(t_n)\mathcal{E}(t_n, \zeta_n, \zeta_{n+1}).$$

Also, $\sup_n \lambda(t_n) = \sup_n \dfrac{B(t_n)k(t_n)[1 + C(t_n, t_{n+1})]}{A(t)(1 - k(t_n))} < 1.$
By induction, we get

$$\mathcal{E}(t_n, \zeta_{n+1}, \zeta_{n+2}) \le \prod_{i=1}^{n} \lambda(t_i)\mathcal{E}(t_0, \zeta_0, \zeta_1).$$

Now, for $m > n$, we have

$$\begin{aligned} d(\zeta_n, \zeta_m) &\leq \sum_{j=n}^{m-1} d(\zeta_j, \zeta_{j+1}) \\ &\leq \sum_{j=n}^{\infty} \frac{1}{A(t_i)} \mathcal{E}(t_j, \zeta_j, \zeta_{j+1}) \\ &\leq \mathcal{E}(t_0, \zeta_0, \zeta_1) \sum_{j=n}^{\infty} \left(\prod_{i=1}^{j} \lambda(t_i) \right). \end{aligned}$$

Let $\lambda_0 = \sup_n \lambda(t_n)$. Then

$$\begin{aligned} d(\zeta_n, \zeta_m) &\leq \mathcal{E}(t_0, \zeta_0, \zeta_1) \sum_{j=n}^{\infty} \lambda_0^j \\ &= \mathcal{E}(t_0, \zeta_0, \zeta_1) \frac{\lambda_0^n}{1 - \lambda_0} \ \to 0 \text{ as } n, m \to \infty. \end{aligned}$$

Hence (ζ_n) is Cauchy sequence. Thus, there is $\zeta^* \in \mathcal{X}$ such that $\zeta_n \to \zeta^*$.
For any fixed n, we have

$$\begin{aligned} \mathcal{E}(t_i, \zeta^*, \mathcal{T}^{t_n}\zeta^*) &\leq \frac{B(t_i)}{A(t_i)} \left[\mathcal{E}(t_i, \zeta^*, \zeta_{i+1}) + \mathcal{E}(t_i, \zeta_{i+1}, \mathcal{T}^{t_n}\zeta^*) \right] \\ &\leq \frac{B(t_i)}{A(t_i)} [\mathcal{E}(t_i, \zeta^*, \zeta_{n+i}) + \ k(t_i) \ [\mathcal{E}(t_i, \zeta^*, \mathcal{T}^{t_n}\zeta^*) + \mathcal{E}(t_i, \zeta_{i+1}, \zeta_{i+2})] \ ]. \end{aligned}$$

Thus,

$$\mathcal{E}(t_i, \zeta^*, \mathcal{T}^{t_n}\zeta^*) \leq \frac{1}{A(t_i) - B(t_i)k(t_i)} [\ B(t_i)\mathcal{E}(t_i, \zeta^*, \zeta_{i+1}) + \ B(t_i)k(t_i)\mathcal{E}(t_i, \zeta_{i+1}, \zeta_{i+2}) \ ].$$

Since $A(t_i) - B(t_i)k(t_i) < 1$, by taking the limit, we get $\lim_{i\to\infty} \mathcal{E}(t_i, \zeta^*, \mathcal{T}^{t_n}\zeta^*) = 0$. Also, since $d(\zeta^*, \mathcal{T}^{t_n}\zeta^*) \leq \frac{1}{A(t_i)}\mathcal{E}(t_i, \zeta^*, \mathcal{T}^{t_n}\zeta^*)$ for each i, we get $d(\zeta^*, \mathcal{T}^{t_n}\zeta^*) = 0$, and so $\mathcal{T}^{t_n}\zeta^* = \zeta^*$.

Assume there is another $u \in \mathcal{X}$ with $\mathcal{T}^{t_n}u = u$. Then

$$\begin{aligned} \mathcal{E}(t_n, \zeta^*, u) &= \mathcal{E}(t_n, \mathcal{T}^{t_n}\zeta^*, \mathcal{T}^{t_n}u) \\ &\leq k(t_n)[\mathcal{E}(t_n, \zeta^*, \zeta^*) + \mathcal{E}(t_n, u, u)] \\ &= 0. \end{aligned}$$

Hence, $\zeta^* = u$.

□

Example 2.1 *Let $\mathcal{X} = [0, 1]$ with the standard metric $d(\zeta, y) = |\zeta - y|$, and let*

$$\mathcal{E}(t, \zeta, y) = (1 + t + t^2)|\zeta - y|.$$

Then, $\mathcal{E}$ is $\mathcal{E}_{A,B}$-distance with $A(t) = B(t) = 1 + t + t^2$

For $1 \geq t \geq 0$, define

$$\mathcal{T}^t\zeta = \frac{1}{3}(1 - t)\zeta.$$

Then, we have

$$
\begin{aligned}
\mathcal{E}(t, \mathcal{T}^t\zeta, \mathcal{T}^t y) &= (1+t+t^2)\left|\frac{1}{3}(1-t)\zeta - \frac{1}{3}(1-t)y\right| \\
&= \frac{1}{3}(1+t+t^2)(1-t)|\zeta - y|.
\end{aligned}
$$

Now, we also have

$$
\begin{aligned}
\mathcal{E}(t, \zeta, \mathcal{T}^t\zeta) &= (1+t+t^2)\left|\zeta - \frac{1}{3}(1-t)\zeta\right| \\
&= (1+t+t^2)\left(1 - \frac{1-t}{3}\right)|\zeta|,
\end{aligned}
$$

$$
\mathcal{E}(t, y, \mathcal{T}^t y) = (1+t+t^2)\left(1 - \frac{1-t}{3}\right)|y|.
$$

Since $|\zeta - y| \leq |\zeta| + |y|$ *for* $\zeta, y \in [0,1]$*, we have*

$$
\mathcal{E}(t, \mathcal{T}^t\zeta, \mathcal{T}^t y) \leq \frac{1}{3}(1+t+t^2)(1-t)(|\zeta| + |y|).
$$

Now, set

$$
k(t) = \frac{1}{3}(1+t+t^2)(1-t) = \frac{1}{3}(1-t^3).
$$

Then, $\max_t k(t) = \frac{1}{3} < \frac{1}{2}$*, and clearly,* $A(t) - B(t)k(t) < 1$.
Moreover, set $C(t,s) = \frac{|t-s|}{2+t}$*. Then*

$$
\begin{aligned}
\mathcal{E}(t_n, \mathcal{T}^{t_n}\zeta, \mathcal{T}^{t_{n+1}}\zeta) &= (1+t_n+t_n^2)\left|\frac{1}{3}(1-t_n)\zeta - \frac{1}{3}(1-t_{n+1})\zeta\right| \\
&= \frac{1+t_n+t_n^2}{3}|t_{n+1} - t_n||\zeta|.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\mathcal{E}(t_n, \zeta, \mathcal{T}^{t_n}\zeta) &= (1+t_n+t_n^2)\left(1 - \frac{1-t_n}{3}\right)|\zeta| \\
&= \frac{(1+t_n+t_n^2)(2+t_n)}{3}|\zeta|.
\end{aligned}
$$

Hence,

$$
C(t_n, t_{n+1}) = \frac{|t_{n+1} - t_n|}{2+t_n}.
$$

So, if $t_n \to 0$*, then* $C(t_n, t_{n+1}) \to 0$.
Now, choose $t_n = \frac{1}{n+1}$*. Then*

$$
\lambda(t_n) = \frac{B(t_n)k(t_n)[1 + C(t_n, t_{n+1})]}{1 - k(t_n)}.
$$

Substitute $B(t_n) = 1 + t_n + t_n^2$, $k(t_n) = \frac{1}{3}(1 + t_n + t_n^2)(1 - t_n)$, *and* $C(t_n, t_{n+1}) = \frac{|t_{n+1} - t_n|}{2 + t_n}$*:*

$$\lambda(t_n) = \frac{(1 + t_n + t_n^2) \cdot \frac{1}{3}(1 + t_n + t_n^2)(1 - t_n) \cdot \left(1 + \frac{|t_{n+1} - t_n|}{2 + t_n}\right)}{(1 + t_n + t_n^2)(1 - \frac{1}{3}(1 + t_n + t_n^2)(1 - t_n))}.$$

Thus, $\sup_n \lambda(t_n) < 1$.
Hence all conditions of Theorem 2.1. Note that the only fixed point of all $\mathcal{T}^t$ *is* $\zeta^* = 0$.

3. Corollaries

Corollary 3.1 *Let* $(\mathcal{X}, d)$ *be a complete metric space and let* $T : \mathcal{X} \to \mathcal{X}$ *satisfy*

$$d(\mathcal{T}\zeta, \mathcal{T}y) \leq k\,[d(\zeta, \mathcal{T}\zeta) + d(y, \mathcal{T}y)] \quad \text{for all } \zeta, y \in \mathcal{X},$$

where $0 \leq k < \frac{1}{2}$. *Then* T *has a unique fixed point, and the Picard iteration* $\zeta_{n+1} = T\zeta_n$ *converges to this fixed point for any* $\zeta_0 \in \mathcal{X}$.

Proof: Apply Theorem 2.1 with the following assignments:

- $\mathcal{T}^t \equiv T$ (independent of t),
- $\mathcal{E}(t, \zeta, y) = (1 + e^{-t})d(\zeta, y)$,
- $A(t) = B(t) = 1 + e^{-t} \in \Lambda$,
- $k(t) = \dfrac{k}{1 + e^{-t}}$ so that $k(t) < \dfrac{1}{2}$ for all t (since $k < \dfrac{1}{2}$),
- $C(t_n, t_{n+1}) = 0$ for all n.

Note that the contractive condition

$$d(T\zeta, Ty) \leq k\,[d(\zeta, T\zeta) + d(y, Ty)]$$

implies

$$\mathcal{E}(t, T\zeta, Ty) \leq k(t)\,[\mathcal{E}(t, \zeta, T\zeta) + \mathcal{E}(t, y, Ty)]$$

since

$$(1 + e^{-t})d(T\zeta, Ty) \leq k(1 + e^{-t})\,[d(\zeta, T\zeta) + d(y, Ty)] = k(t)\,[\mathcal{E}(t, \zeta, T\zeta) + \mathcal{E}(t, y, Ty)]\,.$$

Finally,

$$\frac{B(t_n)k(t_n)[1 + C(t_n, t_{n+1})]}{A(t_n)(1 - k(t_n))} = \frac{k(t_n)}{1 - k(t_n)} < 1,$$

which is satisfied since $k(t_n) < \frac{1}{2}$.

Thus, all assumptions of Theorem 2.1 are satisfied, and we conclude that the Picard iteration $\zeta_{n+1} = T\zeta_n$ converges to the unique fixed point of T.

□

Corollary 3.2 *Let* $(\mathcal{X}, d)$ *be complete with* $\mathcal{E}_{A,B}$*-distance* $\mathcal{E}(t, \zeta, y) = A(t)d(\zeta, y)$*, where*

- $A \in \Lambda$
- $A(t) \leq M$ *for some* $M > 0$

Suppose $\{\mathcal{T}^t\}$ *satisfies for some* $k \in [0, \frac{1}{2})$*, with* $M < \frac{1}{1-k}$

$$\mathcal{E}(t, \mathcal{T}^t\zeta, \mathcal{T}^t y) \leq k\left[\mathcal{E}(t, \zeta, \mathcal{T}^t\zeta) + \mathcal{E}(t, y, \mathcal{T}^t y)\right] \quad \forall t \geq 0.$$

Then for any sequence (t_n) *with* $t_n \to 0$*, the iterations* $\zeta_{n+1} = \mathcal{T}^{t_n}\zeta_n$ *converge to the unique common fixed point.*

Proof: We verify that the assumptions of Theorem 2.1 are satisfied under the conditions of the corollary.

Let $\mathcal{E}(t,\zeta,y) = A(t)d(\zeta,y)$, where $A \in \Lambda$ and $A(t) \leq M < \frac{1}{1-k}$ for all $t \geq 0$.

Since $A(t)d(\zeta,y) = \mathcal{E}(t,\zeta,y)$, we can set $B(t) = A(t)$, so that

$$A(t)d(\zeta,y) \leq \mathcal{E}(t,\zeta,y) \leq A(t)d(\zeta,y),$$

which satisfies Definition 1.2 of an $\mathcal{E}_{A,B}$-distance.

By assumption, for all $t \geq 0$ and $\zeta, y \in \mathcal{X}$, we have

$$\mathcal{E}(t,\mathcal{T}^t\zeta,\mathcal{T}^t y) \leq k\left[\mathcal{E}(t,\zeta,\mathcal{T}^t\zeta) + \mathcal{E}(t,y,\mathcal{T}^t y)\right],$$

where $k \in [0,\frac{1}{2})$ is a constant. Then,

$$A(t) - B(t)k = A(t)(1-k) \leq M(1-k) < 1,$$

so, $A(t_n)(1-k) < 1$. Thus, condition (1) of Theorem 2.1 is satisfied.

Now, let (t_n) be any sequence with $t_n \to 0$, and define the iterative sequence $\zeta_{n+1} = \mathcal{T}^{t_n}\zeta_n$. Then,

$$\mathcal{E}(t_n,\mathcal{T}^{t_n}\zeta,\mathcal{T}^{t_{n+1}}\zeta) = A(t_n)d(\mathcal{T}^{t_n}\zeta,\mathcal{T}^{t_{n+1}}\zeta).$$

Since $t_n \to 0$ and $\mathcal{T}^{t_n}$ tends to the identity map, and hence, we define

$$C(t_n,t_{n+1}) = \frac{d(\mathcal{T}^{t_n}\zeta,\mathcal{T}^{t_{n+1}}\zeta)}{d(\zeta,\mathcal{T}^{t_n}\zeta)} \to 0,$$

and so we obtain

$$\mathcal{E}(t_n,\mathcal{T}^{t_n}\zeta,\mathcal{T}^{t_{n+1}}\zeta) \leq C(t_n,t_{n+1})\mathcal{E}(t_n,\zeta,\mathcal{T}^{t_n}\zeta),$$

Finally, we have

$$\sup_n \frac{B(t_n)k(1+C(t_n,t_{n+1}))}{A(t_n)(1-k)} = \sup_n \frac{k(1+C(t_n,t_{n+1}))}{1-k} < 1,$$

since $k < \frac{1}{2}$ and $C(t_n,t_{n+1}) \to 0$.

Therefore, all hypotheses of Theorem 2.1 are satisfied, and we conclude that the sequence $\zeta_{n+1} = \mathcal{T}^{t_n}\zeta_n$ converges to the unique common fixed point of the family $\{\mathcal{T}^{t_n}\}$.

□

Conclusion

This work has established a comprehensive generalization of Kannan's fixed-point theorem within the framework of $\mathcal{E}_{A,B}$-distance spaces. Our results significantly extend the classical theorem while preserving its fundamental characteristics; the distinctive contraction condition based on distances to image points rather than between points, the automatic uniqueness of fixed points without imposing additional continuity requirements. The introduction of time-dependent operators $\{\mathcal{T}^t\}$ and $\mathcal{E}_{A,B}$-distances provides a flexible framework that captures both classical results and new applications.

Looking ahead, several promising research directions emerge from this work. The extension to set-valued Kannan contractions would represent a meaningful advancement in unifying fixed-point theories. The application of our results to fractional differential equations appears particularly fruitful, given the growing importance of fractional calculus in modeling complex systems. Furthermore, the development of efficient Kannan-type iteration schemes for numerical methods could yield practical computational tools. These future directions would not only broaden the theoretical foundations but also enhance the applicability of Kannan-type contractions across mathematical analysis.

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