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Studying the First Non-Vanishing Cohomology Group of the Orlik-Solomon Algebra for Triangle-Free Graph Related Graphic Arrangements

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ABSTRACT: In this work, we examine the vanishing of the second cohomology group of the OrlikSolomon algebra $\boldsymbol{A}_*(\mathcal{A}_G)$, denoted by $H^2(\boldsymbol{A}_*(\mathcal{A}_G);a)$, corresponding to the graphic arrangement $\mathcal{A}_G$ related with a triangle-free graph G. Here, $\mathcal{A}_G$ specifies the number of edges in G and a is defined as $a = a_s - a_t$, for , $2 \leq s < t \leq \ell$. Motivated by this goal, we investigate $H^2(\boldsymbol{A}_*(\mathcal{A}_G);a)$ as a free module and show that it does not vanish when G contains chordless 4-cycles including the edges e_s and e_t.

Key Words: Hyperplane arrangement, hypersolvable arrangement, supersolvable arrangement, generic arrangement, Orlik-Solomon algebra, NBC-module, Chomology of the Orlik-Solomon algebra, graph theory, hypersolvable graph.

Contents

1. Introduction

This work concentrates the concept of "a hyperplane arrangement" $\mathcal{A}$, (shortened "an arrangement") in a finite-dimensional vector space V over a field K, focusing on its complement $M(\mathcal{A}) = V \setminus \cup_{H \in \mathcal{A}} H$ and topological invariants in combinatorics. Fadell, Fox, and Neuwirth studied the cohomological group of complex space complements in 1962 [3], [2], followed by Orlik and Solomon, who used generators and relations to calculate the cohomology algebra [6]. $H^*(M(\mathcal{A}), K)$ is established by constructing an algebra $\boldsymbol{A}_*(\mathcal{A})$, named by their name and defined as the quotient of the exterior K-algebra $E_*(\mathcal{A}) = \Lambda_{k \geq 0}\left(\bigoplus_{H \in \mathcal{A}} K e_H\right)$, by the homogenous ideal $I_*(\mathcal{A})$, that generated by the relations, $\sum_{j=1}^{k}(-1)^{k-1} e_{H_{i_1}} \cdots \hat{e}_{H_{i_j}} \cdots e_{H_{i_k}}$ for all $1 \leq i_1 < \cdots < i_k \leq n$, where $\{H_{i_1}, \ldots, H_{i_k}\}$ is a dependent subarrangement of $\mathcal{A}$ and the circumflex "ˆ" indicates the deletion of $e_{H_{i_j}}$ [6].

Our aim will motivate us to embed the Orlik-Solomon algebra as a free submodule of the exterior K-algebra E_*. This submodule is called the broken circuit module, denoted by $\boldsymbol{NBC}_*(\mathcal{A})$ and defined as; $\boldsymbol{NBC}_0(\mathcal{A}) = K$ and $\boldsymbol{NBC}_k(\mathcal{A})$ for $1 \leq k \leq \ell$, be the free K-module of E_k with an NBC (no broken circuit) monomial basis $\{e_C \mid C \in NBC_k(\mathcal{A})\} \subseteq E_k$, where $NBC_k(\mathcal{A})$ is the set of all no k-broken circuit subarrangement of $\mathcal{A}$ [5]. A circuit B refers to minimal dependent subarrangement of $\mathcal{A}$ and its broken circuit is the subarrangement $B - \{H\}$ of $\mathcal{A}$, where H is the smallest hyperplne of B via a total order $\triangle$ defined on the hyperplanes of $\mathcal{A}$.

For any element $a \in A_1(\mathcal{A})$, the cohomology of the Orlik-Solomon algebra, $H^*(\boldsymbol{A}_*(\mathcal{A});a)$, plays a crucial role in understanding the cohomology with local coefficients, $H^*(M(\mathcal{A}), \mathcal{L}(a))$. Although researchers have computed the cohomology of the Orlik-Solomon algebra for specific types of arrangements [4], [10], [12], considerably less is known about the higher cohomology groups $H^p(\boldsymbol{A}_*(\mathcal{A});a)$ for $p > 1$ in

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general. This gap motivates more research as, possibly, the topological and combinatorial structures of arrangement complement $M(\mathcal{A})$ can be better understood with a deeper understanding of these higher-dimensional cohomologies of the Orlik-Solomon algebra.

This work focused on free triangles graphs, which is known as a subclass of the hypersolvable class of graphs, which was first established by Papadima and Suciu in 2002 [8]. It is known that there is a correspondence $G \mapsto \mathcal{A}_G$ maps finite, simple, undirected graph $G = (V, \varepsilon)$, (where V and ε be its vertices set and edges set respectively), to a graphic arrangement $\mathcal{A}_G$, defined via the bijection $[v_i, v_j] \in \mathcal{E} \leftrightarrow H_{i,j} \in \mathcal{A}_G$, where $H_{i,j} = \{(x_1, \dots x_r) \in \mathbb{C}^r : x_i = x_j\}$. This duality can be used to transfer known results from hypersolvable graphs into hypersolvable graphic arrangements and vice versa. The hypersolvable class of arrangement was first described by Jambu and Papadima in 1998, [12] and 2002, [1]. In section (2.3), the Orlik-Solomon algebra of a free tringles graph G was reconstructed straightly from its bond lattice $L(G)$ via the isomorphism $L(G) \mapsto L(\mathcal{A}_G)$ that results from the correspondence $G \mapsto \mathcal{A}_G$. Specifically, the duality between the notion of cycles in graph theory and the notion of circuits in graphic arrangements play as an essential idea to consider a structure of the Orlik-Solomon algebra for a free triangles graph as free submodule of the exterior algebra $E_*(G) = \Lambda_{k\geq 0}\left(\bigoplus_{c\in\varepsilon} Ke_c\right)$, of a graph generated by an NBC (no broken cycles) monomial basis $\{e_C \mid C \in NBC_k(G)\} \subseteq E_k(G)$, where $NBC_k(G)$ denotes the set of all no broken cycles subgraphs of G with k edges and its elements are said to be k-NBC base of G. Accordingly, the structure of the second cohomological group of the Orlik-Solomon algebra for a free triangles graph $H^2(\boldsymbol{A}_*(\mathcal{A}_G); a)$ was investigated for $a = a_s - a_t, 2 \leq s < t \leq \ell$. We proved that $H^2(\boldsymbol{A}_*(\mathcal{A}_G); a)$ not vanished if G has chordless 4-cycles contains the edges e_s and e_t simultaneously. Finally, we provided a few illustrations of our results.

2. Basic Facts:

This section is to review the concepts and structures crucial to our work:

Definition 2.1 *[1] Let V and ε be finite sets of m-vertices and n-edges respectively in an undirected connected simple graph $G = (V, \varepsilon)$. A pair $\Pi^G = (\Pi^V, \Pi^\varepsilon)$, is said to be hypersolvable partition of G and it is denoted by HP, if $\Pi^V = (\Pi^V_1, \dots, \Pi^V_{m-1})$ and $\Pi^\varepsilon = (\Pi^\varepsilon_1, \dots, \Pi^\varepsilon_\ell)$ are partitions of V and ε respectively, that satisfy the following features:*

$\mathrm{HP}_1 : \Pi^V_1 = \{v_1, v_2\}$ *as well* $\Pi^\varepsilon_1 = \{e_1\}$, *such that* $e_1 = [v_1, v_2]$, *i.e.* Π^ε_1 *is a singleton.*

HP_2 *: For every $2 \leq j \leq m-1$, the block Π^V_j is a singleton.*

HP_3 *: For every $2 \leq k \leq \ell$, the block Π^ε_k satisfying the following:*

$\mathrm{HP}_3 i$ *: For every $e_{i_1}, e_{i_2} \in \Pi^\varepsilon_1 \cup \ldots \cup \Pi^\varepsilon_k$, there is no edge $e \in \Pi^\varepsilon_{k+1} \cup \ldots \cup \Pi^\varepsilon_\ell$ such that $\{e_{i_1}, e_{i_2}, e\}$ forms a triangle.*

$\mathrm{HP}_3 ii$ *: There is a positive integer $1 < m_k \leq m-1$, such that $V_k = \Pi^V_1 \cup \ldots \cup \Pi^V_{m_k}$ is a subset of V that is contains every endpoints of the edges in $\Pi^\varepsilon_1 \cup \ldots \cup \Pi^\varepsilon_k$, i.e. $G_k = (V_k, \Pi^\varepsilon_1 \cup \ldots \cup \Pi^\varepsilon_k)$ makes up a subgraph of G. Then, either.*

1. *$\Pi^\varepsilon_k = \{e\}$ such that $V_k = V_{k-1}$, or;*
2. *$\Pi^\varepsilon_k = \left\{e_{i_1}, \dots, e_{i_{d_k}}\right\}$, such that $V_k \backslash V_{k-1} = \Pi^V_{m_{k-1}+1} = \Pi^V_{m_k} = \{v\}$ and for $1 \leq j \leq d_k, e_{i_j} = [v_{i_j}, v]$, for some $v_{i_j} \in \Pi^V_1 \cup \ldots \cup \Pi^V_{m_{k-1}}$, where $\left\{v_{i_1}, \dots, v_{i_{d_k}}\right\} \subseteq V_{k-1} = \Pi_1 \cup \ldots \cup \Pi_{m_{k-1}}$ creates a complete subgraph of G.*

The number of the blocks of Π^ε is called the length of Π^G and denoted by $\ell(G) = \ell$. The vector $d = (d_1, \dots, d_\ell)$ is said to be the exponent vector (or d-vector) of Π if $d_k = |\Pi^\varepsilon_k|$ for $1 \leq k \leq \ell$. The rank of Π^ε_k defined as $rk\Pi^\varepsilon_k = |V_k| - 1$ and $rk(G) = rk\Pi^\varepsilon_\ell = m-1$. The block Π^ε_k will be considered singular if $|V_{k-1}| = |V_k|$, and it is non-singular if $|V_k \backslash V_{k-1}| = 1$. A hypersolvable partition Π^G is considered supersolvable if and only if, Π^ε contains no singular blocks. As well as, a hypersolvable partition Π^G, is called m-generic if $\ell \geq m$, the exponent vector $d = (1, \dots, 1)$ and every k-eadges of ε cannot be an k-cycle, $3 < k \leq m-1$.

Proposition 2.2 *[9,1] A graphic arrangement $\mathcal{A}_G$ is hypersolvable (supersolvable or generic), if and only if, it is graph G is hypersolvable (supersolvable or generic).*

Note Let G be a hypersolvable graph. There are important points to take into consideration

1. The related graphic arrangement $\mathcal{A}_G$ has a hypersolvable partition $\Pi^{\mathcal{A}_G} = (\ \Pi_1 \dots \Pi_\ell\ )$ derived from the hypersolvable partition $\Pi^G = \left(\Pi^V, \Pi^\varepsilon\right)$ on G as; for $1 \le k \le \ell$, $H_{ij} \in \Pi_k$ if, and only if, $[i,j] \in \Pi^\varepsilon_k$. We will call $\Pi^{\mathcal{A}_G}$ the induced partition of Π^G, [1].

2. There is a bijection between $S\left(\Pi^G\right)$ and $S\left(\Pi^{\mathcal{A}}G\right)$. As well as there is a bijection between $S_k\left(\Pi^G\right)$ and $S_k\left(\Pi^{\mathcal{A}_G}\right)$, for $1 \le k \le \ell$.

Definition 2.3 *[7] Suppose that K is a commutative ring and $\trianglelefteq$ is an arbitrary total order defined on the hyperplanes of a ℓ-arrangement $\mathcal{A}$. Define the Orlik-Solomon algebra $\boldsymbol{A}_*(\mathcal{A})$ to be the quotient of the exterior K-algebra $E_* = \wedge_{k\ge 0}\left(\bigoplus_{H\in\mathcal{A}} K_{e_H}\right)$, by the homogenous ideal $I_*(\mathcal{A})$ is generated by the relations.*

$$\sum_{j=1}^{k}(-1)^{k-1} e_{H_{i_1}} \cdots \hat{e}_{H_{i_j}} \cdots e_{H_{i_k}} \quad \textit{for all} \quad 1 \le i_1 < \cdots < i_k \le n$$

where $\{H_{i_1}, \dots, H_{i_k}\}$ is a dependent subarrangement of $\mathcal{A}$ and the circumflex "ˆ" indicates the deletion of $e_{H_{ij}}$.

The broken circuit module is $\boldsymbol{NBC}_*(\mathcal{A})$ of the exterior E_* is defined as; $\boldsymbol{NBC}_0(\mathcal{A}) = K$ and $\boldsymbol{NBC}_k(\mathcal{A})$ for $1 \le k \le \ell$, be the free K-module of E_k with an NBC (no broken circuit) monomial basis $\{e_C \mid C \in NBC_k(\mathcal{A})\} \subseteq E_k$, i.e.;

$$\boldsymbol{NBC}_k(\mathcal{A}) = \bigoplus_{C\in NBC_k(\mathcal{A})} Ke_C \ \textit{and}\ \boldsymbol{NBC}_*(\mathcal{A}) = \bigoplus_{k=0}^{\ell} \boldsymbol{NBC}_k(\mathcal{A}).$$

Theorem 2.4 *[7] The restriction of canonical chain map $\varphi_* : E_* \to \boldsymbol{A}_*(\mathcal{A})$ on the broken circuit submodule $\boldsymbol{NBC}_*(\mathcal{A})$, is an isomorphism defined as; for $1 \le k \le \ell, \psi_k(e_C) = e_C + I_k(\mathcal{A}) = a_C$, $C \in NBC_k(\mathcal{A})$. Accordingly, the Orlik-Solomon algebra $A_*(\mathcal{A})$ embedded of the exterior algebra as a free K-submodule by the following structure: $\boldsymbol{A}_*(\mathcal{A}) = \bigoplus_{k=0}^{\ell}\left(\oplus_{C\in NBC_{K(\mathcal{A})}} Ka_C\right)$.*

Definition 2.5 *[7] Assume that $a \in A_1(\mathcal{A})$ with $a = \sum_{s=1}^{n} \lambda_s a_s$ for $\lambda_s \in K$. multiplication by a give a differentiation $d_k : \boldsymbol{A}_k(\mathcal{A}) \xrightarrow{a.} \boldsymbol{A}_{k+1}(\mathcal{A})$ such that $(\boldsymbol{A}_*(\mathcal{A}), a)$ forms a complex. The cohomology of this complex denoted by $H^*(\boldsymbol{A}(\mathcal{A}), a)$ and is said to be the cohomology of the Orlik-Solomon algebra.*

3. The First Non-Vanishing Cohomology of Orlik-Solomon algebra of Free Triangle Graphs:

The work consists of two parts. The first part reconstructs the Orlik-Solomon algebra of a graph using its hypersolvable partition structure. This involves defining a total order on the graph's edges using an analogue of the hypersolvable partition for the complete graph Q_m with m verticies. The second part examines the first non-vanishing cohomology of a given free triangles graph. This approach clarifies the interplay between the combinatorial properties of the free triangles graph and its geometric structure, laying a solid foundation for further cohomological analysis.

3.1 A Total Order of Graph:

Assume Q_m is a complete graph with m vertices and let $\triangle^V_{Q_m}$ be any total order can be defined on its vertices say $V_{Q_m} = \{v_1, \dots, v_m\}$. Partitioned the set its edges ε_{Q_m} as:
$\Pi^{\varepsilon_{Q_m}} = \left(\Pi_1^{\varepsilon_{Q_m}}, \Pi_2^{\varepsilon_{Q_m}}, \Pi_3^{\varepsilon_{Q_m}}, \dots, \Pi_{m-1}^{\varepsilon_{Q_m}}\right)$
$= (\{[v_1, v_2]\}, \{[v_1, v_3], [v_2, v_3]\}, \{[v_1, v_4], [v_2, v_4], [v_3, v_4]\}, \dots, \{[v_1, v_m], [v_2, v_m], \dots, [v_{m-1}, v_m]\})$
Define a total order $\trianglelefteq^\varepsilon_{Q_m}$ on ε_{Q_m} as follows:

$$[v_1, v_2] \trianglelefteq^\varepsilon_{Q_m} [v_1, v_3] \trianglelefteq^\varepsilon_{Q_m} [v_2, v_3] \trianglelefteq^\varepsilon_{Q_m} [v_1, v_4] \trianglelefteq^\varepsilon_{Q_m} [v_2, v_4] \trianglelefteq^\varepsilon_{Q_m} [v_3, v_4] \trianglelefteq^\varepsilon_{Q_m} \cdots$$
$$\trianglelefteq^\varepsilon_{Q_m} [v_1, v_m] \trianglelefteq^\varepsilon_{Q_m} [v_2, v_m] \trianglelefteq^\varepsilon_{Q_m} \cdots \trianglelefteq^\varepsilon_{Q_m} [v_{m-1}, v_m].$$

In fact, if $1 \leq k_1 < k_2 < k_3 \leq k$, then the total order $\trianglelefteq_{Q_m} = \left(\trianglelefteq^V_{Q_m}, \trianglelefteq^\varepsilon_{Q_m}\right)$ in the block $\Pi_k^{\varepsilon_{Q_m}}$ satisfying:

1. $[v_{k_1}, v_{k+1}] \trianglelefteq^\varepsilon_{Q_m} [v_{k_2}, v_{k+1}] \trianglelefteq^\varepsilon_{Q_m} [v_{k_3}, v_{k+1}]$ and;
2. $[v_{k_1}, v_{k_2}] \trianglelefteq^\varepsilon_{Q_m} [v_{k_1}, v_{k_3}] \trianglelefteq^\varepsilon_{Q_m} [v_{k_2}, v_{k_3}]$.

Now, assume $G = (V, \varepsilon)$ be a graph with m vertices, i.e., G is a subgraph of Q_m. Put $\trianglelefteq_G = \left(\triangle^V_G, \triangle^\varepsilon_G\right)$ to be the restriction of $\triangle_{Q_m}$ on G.

Remark 3.1 *If $G = (V, \varepsilon)$ is a hypersolvable graph with m vertices and $\Pi^G = \left(\Pi^V, \Pi^\varepsilon\right)$ be its HP, then the total order $\trianglelefteq_G$ will satisfy:*

1. *For $i < j$, then*
   i. $v_i \trianglelefteq^V_G v_j$.
   ii. $e \trianglelefteq^\varepsilon_G e'$, for every $e \in \Pi^\varepsilon_i$ and $e' \in \Pi^\varepsilon_j$.
2. *For $3 \leq k \leq \ell$, if $e_{i_1}, e_{i_2}, e_{i_3} \in \Pi^\varepsilon_k$ and $e_{i_1} \trianglelefteq^\varepsilon_G e_{i_2} \trianglelefteq^\varepsilon_G e_{i_3}$, then $e_{i_1,i_2} \trianglelefteq^\varepsilon_G e_{i_1,i_3} \trianglelefteq^\varepsilon_G e_{i_2,i_3}$, where $\{e_{i_1,i_2}, e_{i_1}, e_{i_2}\}, \{e_{i_1,i_3}, e_{i_1}, e_{i_3}\}, \{e_{i_2,i_3}, e_{i_2}, e_{i_3}\}$ are the triangles in G via the solvable property of Π^G.*

3.2 The Orlik-Solomon Algebra of Free Triangles Graphs as Free Modules:

From this point on, we assume that $G = (V, \varepsilon)$ be a simple connected free triangles graph such $|V| = m \geq 4$ and the number of its edges is $|\varepsilon| = \ell$. Therefore, G is hypersolvable with an HP say $\Pi^G = \left(\Pi^V, \Pi^\varepsilon\right)$ with exponent vector $d = (1, \ldots, 1) \in \mathbb{R}^\ell$. We will use definition (2.1) to define an order $\trianglelefteq_G = \left(\trianglelefteq^V_G, \trianglelefteq^\varepsilon_G\right)$ on G. By using the correspondence $G \mapsto \mathcal{A}_G$ that defined via the bijection $[v_i, v_j] \in \mathcal{E} \leftrightarrow H_{i,j} = \{(x_1, ..x_r) \in \mathbb{C}^r : x_i = x_j\} \in \mathcal{A}_G$, we shall establish an induced order $\trianglelefteq_{\mathcal{A}_G}$ on the hyperplanes of $\mathcal{A}_G$ based on the structure of its induced partition $\Pi^{\mathcal{A}_G}$. Accordingly, the isomorphism between the lattice of bonds $L(G)$ and the lattice of intersections $L(\mathcal{A}_G)$, will create the following bijections with regarding the hypersolvable order $\trianglelefteq_G$:

1. $f : NBC_{\trianglelefteq_G}(G) \rightarrow NBC_{\trianglelefteq_{\mathcal{A}_G}}(\mathcal{A}_G)$, where $NBC_{\trianglelefteq_G}(G)$ be the set of all subgraphs with no broken cycles of G via the order $\trianglelefteq_G$ and $NBC_{\trianglelefteq_{\mathcal{A}_G}}(\mathcal{A}_G)$ be the set of all no broken circuits subarrangements of $\mathcal{A}_G$ via the induced order $\trianglelefteq_{\mathcal{A}_G}$.
2. For $0 \leq k \leq rk(\mathcal{A}_G) = m - 1, f_k : NBC^k_{\trianglelefteq_G}(G) \rightarrow NBC^k_{\mathcal{A}_G}(\mathcal{A}_G)$ will be the restriction of f on the k^{th} - skeleton of $NBC^k_{\trianglelefteq_G}(G)$.
3. $g : S_{\trianglelefteq_G}\left(\Pi^G\right) \rightarrow S_{\trianglelefteq_{\mathcal{A}_G}}\left(\Pi^{\mathcal{A}_G}\right)$.
4. For $0 \leq k \leq \ell(G), g_k : S^k_{\trianglelefteq_G}\left(\Pi^G\right) \rightarrow S^k_{\trianglelefteq_{\mathcal{A}_G}}\left(\Pi^{\mathcal{A}_G}\right)$ will be the restriction of g on $S^k_{\trianglelefteq_G}\left(\Pi^G\right)$.

If $c = c(G) = \text{Min}\{|C| : C$ is a $j-$ cycle with no chord, $j \geq 4\}$, then;

$$p(G) = \text{Max}\left\{k \middle| |NBC^k_{\trianglelefteq_G}(G)| = \left|S^k_{\trianglelefteq_G}\left(\Pi^G\right)\right|\right\} = c - 2, \text{ [10]}$$

Using the following formula, we can designate the Orlik-Solomon algebra of a graph $\boldsymbol{A}_*(G)$ as a free K-module and classify the class of free triangles graphs into three subclasses:

1. If $\ell(G) = m - 1$, then G is a tree which is supersolvable, and the Orlik-Solomon algebra has the following structure as:
   $\boldsymbol{A}_*(G) \cong \bigoplus_{k=0}^{m-1}\left(\bigoplus_{C \in S^k_{S_G}(\Pi^G)} Ka_C\right)$ and for $1 \leq k \leq m - 1, \boldsymbol{A_k}(G) \cong_{C \in S^k_{S_G}(\Pi^G)} Ka_C$ and, $b_k\left(\boldsymbol{A}_*(G)\right) = \binom{m-1}{k}$.

2. If $c(G)=\ell(G)=m$, then G is generic that form's m-cycle with no chord, and the OrlikSolomon algebra designated as: for $1 \leq k \leq m-2$, $\boldsymbol{A_k}(G) \cong \oplus_{C \in S^k_{\Xi_G}(\Pi^G)} K a_c$ with $b_k(\boldsymbol{A}_*(G)) = \binom{m}{k}$ and, $\boldsymbol{A_{m-1}}(G) \cong \bigoplus_{C \in NBC^{m-1}_{\trianglelefteq_G}(M_G)} K a_c$, where $NBC^{m-1}_{\trianglelefteq_G}(M_G) = S^{m-1}_{\triangle_G}(\Pi^G) - \{\varepsilon - \{[v_1, v_2]\}\}$ and, $b_{m-1}(\boldsymbol{A}_*(G)) = m-1$. Thus,

$$\boldsymbol{A}_*(G) \cong \bigoplus_{k=0}^{m-2} \left(\bigoplus_{C \in S^k_{-G}(\Pi^G)} K a_C \right) \oplus \left(\bigoplus_{C \in S^{m-1}_{G_G}(\Pi^G) - \{\varepsilon - \{[v_1,v_2]\}\}} K a_C \right)$$

3. If $c(G) = c \leq m-1 < \ell$, then G is neither supersolvable nor generic and for $1 \leq k \leq c-2$, $\boldsymbol{A_k}(G) \cong \bigoplus_{C \in S^k_{-G}(\Pi^G)} K a_C$ with $b_j(\boldsymbol{A}_*(G)) = |NBC_k(G)| = \binom{m}{k}$ and $A_{c-1}(G) \cong \bigoplus_{C \in NBC^{c-1}_{\trianglelefteq_G}(M_G)} K a_c$, where $NBC^{c-1}_{\trianglelefteq_G}(G) = S^{c-1}_{\trianglelefteq_G}(\Pi^G) - BC^{c-1}_{\trianglelefteq_G}(G), BC^{c-1}_{\trianglelefteq_G}(G)$ is the set of all broken c-cycles via $\trianglelefteq^{\varepsilon}_G$, and $b_{c-1}(\boldsymbol{A}_*(G)) = \binom{m}{c-1} - \left| BC^{c-1}_{\trianglelefteq_G}(G) \right|$. Thus:

$$\boldsymbol{A}_*(G) \cong \bigoplus_{k=0}^{m-1} \left(\bigoplus_{C \in NBC^k_{\trianglelefteq_G}(G)} K a_C \right) \cong \bigoplus_{k=0}^{c-2} \left(\bigoplus_{C \in S^k_{G_G}(\Pi^G)} K a_C \right) \oplus \left(\bigoplus_{G} \right.$$

$$\left. a_{C \in S^{c-1}_{G_G}(\Pi^G) - BC^{c-1}_{G_G}(M_G)} K a_C \right) \bigoplus_{k=C}^{m-1} \left(\bigoplus_{C \in NBC^k_{G_G}(G)} K a_C \right)$$

Lemma 3.2 *If $a = a_{e_s} - a_{e_t}$, for $2 \leq s < t \leq \ell$, then* $\dim(\operatorname{Im} d_1) = \ell - 1$;

Proof: Due construction (2.3), $\boldsymbol{A}_2(G) \cong \bigoplus_{C \in S^2_{\trianglelefteq_G}(\Pi^G)} K a_c$, we will examine the homomorphism, $d_1 : \boldsymbol{A}_1(G) \xrightarrow{a} \boldsymbol{A}_2(G)$. So, for $1 \leq k \leq \ell$, we have:

$$d_1(a_{e_k}) = a_{e_k}(a_{e_s} - a_{e_t}) = \begin{cases} -a_{e_s} a_{e_t} & : k = s \text{ or } t \\ a_{e_k} a_{e_s} - a_{e_k} a_{e_t} & : 1 \leq k < s < t \leq \ell \\ -a_{e_s} a_{e_k} - a_{e_k} a_{e_t} & : 1 < s < k < t \leq \ell \\ -a_{e_s} a_{e_k} + a_{e_t} a_{e_k} & : 1 < s < t < k \leq \ell \end{cases}$$

Every 2-section C of Π^{ε} cannot be a broken 3-cycle (triangle), since G is a free triangles graph. Consequently, $NBC^2_{\trianglelefteq_G}(G) = S^2_{\Xi_G}(\Pi^G)$. Therefore, $d_1(a_{e_k}) \neq 0_{A_2(G)}$, for $1 \leq k \leq \ell$, since it is a combination of NBC monomials. As well as, since every 3 -section of Π^{ε} is either an NBC base or broken circuit of Δ_G, hence:

1. If $1 \leq k < s < t \leq \ell, \partial_3^{A_*(G)}(a_{e_k} a_{e_s} a_{e_t}) = a_{e_s} a_{e_t} + d_1(a_{e_k}) \neq 0_{A_2(G)}$.

2. If $1 < s < k < t \leq \ell, \partial_3^{A_*(G)}(a_{e_s} a_{e_k} a_{e_t}) = -a_{e_s} a_{e_t} - d_1(a_{e_k}) \neq 0_{A_2(G)}$.

3. If $1 < s < t < k \leq \ell, \partial_3^{A_*(G)}(a_{e_s} a_{e_t} a_{e_k}) = a_{e_s} a_{e_t} + d_1(a_{e_k}) \neq 0_{A_2(G)}$.

Hence, $d_1(a_{e_k}) \neq -a_{e_s} a_{e_t}$. Therefore, $\dim(\operatorname{Im} d_1) = \ell - 1$, and our assumption were confirmed. □

Proposition 3.3 *If $a = a_{e_s} - a_{e_t}$, for $2 \leq s < t \leq \ell$, then $H^1(A(\mathcal{A}_G); a)$ vanished and;*

1. *If G is tree or $c(G) = c > 4$, then* $\dim(\operatorname{Im} d_2) = \binom{\ell-2}{2} + (\ell - 2)$.

2. *If $c(G) = 4$, then* $\dim(\operatorname{Im} d_2) = \binom{\ell-2}{2} + (\ell-2) - u_4$, *where u_4 be the number of chordless 4-cycles that includes e_s and e_t.*

Proof: To demonstrate our claim, we need to look at the homomorphism $d_2 : \boldsymbol{A}_2(G) \xrightarrow{a} \boldsymbol{A}_3(G)$. Following the construction (2.1), the value of $c(G)$ has been defined as $\mathrm{Min}\{|C| : C$ is a chord less $j-$ cycle,$j \geq 4\}$, and plays a major role in the structure of OrlikSolomon algebra, since either (G is tree or $c(G) > 4$ and $A_3(G) \cong \bigoplus_{C \in S^3_{\trianglelefteq_G}(\Pi^G)} K a_C$) or ($c(G) = 4$ and $A_3(G) \cong \bigoplus_{C \in S^3_{\trianglelefteq_G}(\Pi^G) - BC^3_{\trianglelefteq_G}(G)} K a_C$, where $NBC^3_{\trianglelefteq_G}(G) = S^3_{\trianglelefteq_G}\left(\Pi^G\right) - BC^3_{\trianglelefteq_G}(G)$ and $BC^3_{\trianglelefteq_G}(G)$ obtained by deleting the minimal edge via $\trianglelefteq_G$ from the chordless 4-cycles), i.e. the basis for $\boldsymbol{A_3}(G)$ depends on the scenario under discussion. For $1 \leq k_1 < k_2 \leq \ell$, we have:

$$d_2\left(a_{e_{k_1}} a_{e_{k_2}}\right) = a_{e_{k_1}} a_{e_{k_2}}\left(a_{e_s} - a_{e_t}\right) \tag{3.1}$$
$$= \begin{cases} 0_{A_3(G)} & : k_1 = s \text{ and } k_2 = t \\ -a_{e_k} a_{e_s} a_{e_t} & : 1 \leq k_1 = k < k_2 = s < t \text{ or } 1 \leq k_1 = k < s < k_2 = t \\ a_{e_s} a_{e_t} a_{e_k} & : s = k_1 < t < k_2 = k \leq \ell \text{ or } s < k_1 = t < k_2 = k \leq \ell \\ -a_{e_s} a_{e_k} a_{e_t} & : s = k_1 < k_2 = k < t \text{ or } s < k_1 = k < k_2 = t \\ a_{e_{k_1}} a_{e_{k_2}} a_{e_s} - a_{e_{k_1}} a_{e_{k_2}} a_{e_t} & : 1 \leq k_1 < k_2 < s < t \\ -a_{e_{k_1}} a_{e_s} a_{e_{k_2}} - a_{e_{k_1}} a_{e_{k_2}} a_{e_t} & : 1 \leq k_1 < s < k_2 < t \\ -a_{e_{k_1}} a_{e_s} a_{e_{k_2}} + a_{e_{k_1}} a_{e_t} a_{e_{k_2}} & : 1 \leq k_1 < s < t < k_2 \leq \ell \\ a_{e_s} a_{e_{k_1}} a_{e_{k_2}} - a_{e_{k_1}} a_{e_{k_2}} a_{e_t} & : s < k_1 < k_2 < t \\ a_{e_s} a_{e_{k_1}} a_{e_{k_2}} + a_{e_{k_1}} a_{e_t} a_{e_{k_2}} & : s < k_1 < t < k_2 \leq \ell \\ a_{e_s} a_{e_{k_1}} a_{e_{k_2}} - a_{e_t} a_{e_{k_1}} a_{e_{k_2}} & : s < t < k_1 < k_2 \leq \ell \end{cases}$$

If G is tree or $c(G) > 4$, then there is no chordless 4-cycle, i.e. $NBC^3_{\trianglelefteq_G}(M_G) = S^3_{\trianglelefteq_G}\left(\Pi^G\right)$. It follows from the formula (3.1), $d_2\left(a_{e_s} a_{e_t}\right) = 0_{A_3(G)}$ and, for the other cases for $1 \leq k_1 < k_2 \leq \ell, d_2\left(a_{e_{k_1}} a_{e_{k_2}}\right) \neq 0_{A_3(G)}$, since it is expressed as a combination of NBCmonomials. Hence, $a_{e_s} a_{e_t} \in \ker d_2$. However, $a_{e_s} a_{e_t} \in \operatorname{Im} d_1$, so, $\ker d_2 = \operatorname{Im} d_1$. Furthermore, $\dim\left(\operatorname{Im} d_2\right) = \binom{\ell-2}{2} + (\ell - 2)$ is a straightforward result of the formula (3.1).

There are two outcomes for the consideration $c(G) = 4$:

1. If $m = 4$, then G is 4-generic and has just one chordless 4-cycle (i.e., $u_4(G) = 1$). So, $BC^3_{\trianglelefteq_G}(G) = \{\varepsilon - \{e_1\}\}$ and we have:

$$\partial_4^{A_*(G)}\left(a_\varepsilon\right) = \partial_4^{A_*(G)}\left(a_{e_1} a_{e_2} a_{e_3} a_{e_4}\right) = a_{e_2} a_{e_3} a_{e_4} - a_{e_1} a_{e_3} a_{e_4} + a_{e_1} a_{e_2} a_{e_4} - a_{e_1} a_{e_2} a_{e_3} = 0_{A_3(G)} \tag{3.2}$$
$$\Rightarrow a_{e_2} a_{e_3} a_{e_4} = a_{e_1} a_{e_3} a_{e_4} - a_{e_1} a_{e_2} a_{e_4} + a_{e_1} a_{e_2} a_{e_3}$$

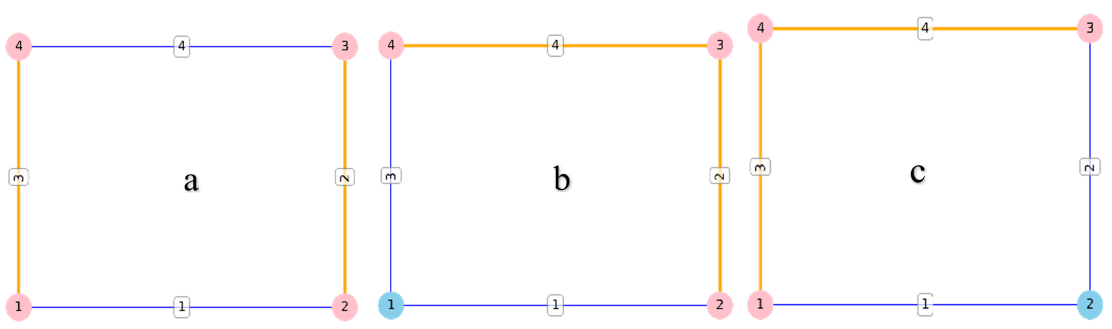

Figure 1: 4-generic graph with all values of $1 < s < t \leq 4$ can be chosen.

Thus, for $1 \leq k_1 < k_2 \leq 4$, below are three scenarios for values of s and t as shown in figure 1:
a. If $s = 2$ and $t = 3$, then:

$$d_2\left(a_{e_1}a_{e_2}\right)=-d_2\left(a_{e_1}a_{e_3}\right)=a_{e_1}a_{e_2}a_{e_3}, d_2\left(a_{e_1}a_{e_4}\right)=-a_{e_1}a_{e_2}a_{e_4}+a_{e_1}a_{e_3}a_{e_4}$$
$$d_2\left(a_{e_2}a_{e_3}\right)=0_{A_3(G)} \text{ and } d_2\left(a_{e_2}a_{e_4}\right)=d_2\left(a_{e_3}a_{e_4}\right)=d_2\left(a_{e_1}a_{e_2}\right)+d_2\left(a_{e_1}a_{e_4}\right)$$

b. If $s=2$ and $t=4$, then:

$$d_2\left(a_{e_1}a_{e_2}\right)=d_2\left(a_{e_1}a_{e_4}\right)=-a_{e_1}a_{e_2}a_{e_4}, d_2\left(a_{e_1}a_{e_3}\right)=-a_{e_1}a_{e_2}a_{e_3}-a_{e_1}a_{e_3}a_{e_4}$$
$$d_2\left(a_{e_2}a_{e_4}\right)=0_{\boldsymbol{A}_3(G)} \text{ and } d_2\left(a_{e_2}a_{e_3}\right)=-d_2\left(a_{e_3}a_{e_4}\right)=-d_2\left(a_{e_1}a_{e_2}\right)+d_2\left(a_{e_1}a_{e_3}\right)$$

c. If $s=3$ and $t=4$, then:

$$d_2\left(a_{e_1}a_{e_2}\right)=a_{e_1}a_{e_2}a_{e_3}-a_{e_1}a_{e_2}a_{e_4}, d_2\left(a_{e_1}a_{e_3}\right)=d_2\left(a_{e_1}a_{e_4}\right)=-a_{e_1}a_{e_3}a_{e_4}$$
$$d_2\left(a_{e_2}a_{e_3}\right)=d_2\left(a_{e_2}a_{e_4}\right)=-d_2\left(a_{e_1}a_{e_2}\right)+d_2\left(a_{e_1}a_{e_4}\right), \text{ and } d_2\left(a_{e_3}a_{e_4}\right)=0_{A_3}(G)$$

Therefore, $\dim\left(\operatorname{Im} d_2\right)=\binom{2}{2}+2-1=2, \ker d_2=\operatorname{Im} d_1$ and $H^1\left(A\left(\mathcal{A}_G\right);a\right)$ vanished.

2. If $m>4$, then G has at least five vertices. Let $c_4(G)=c_4$ be the number of all chordless 4cycle of G and assume that we have $u_4\geq 1$ of chordless 4-cycles that contain e_s and e_t. Thus, we have (c_4-u_4) of 4-cycles that does not simultaneously contain e_s and e_t. Let $S=\{e_{q_1},e_{q_2},e_{q_3},e_{q_4}\}$ represent a chordless 4-cycle. Subsequently, its related broken 4-cycle monomial can be represented as:

$$a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}}=a_{e_{q_1}}a_{e_{q_3}}a_{e_{q_4}}-a_{e_{q_1}}a_{e_{q_2}}a_{e_{q_4}}+a_{e_{q_1}}a_{e_{q_2}}a_{e_{q_3}} \tag{3.3}$$

Therefore, we have the following possible cases:

b.1. For (c_4-u_4) of 4-cycles that do not contain e_s and e_t at the same time, we have the following cases:

b.1.1. In the first case, assuming that $e_s,e_t\notin S$, then for $1\leq i<j\leq 4, d_2\left(a_{q_i}a_{q_j}\right)$ is written as a combination of NBC-monomials, as shown in formula (3.1).

b.1.2. Suppose either $e_s\in S$ or $e_t\in S$ for case two. If $e_s\in S$, for $1\leq i<j\leq 4$ and $i,j\neq s$ or t, we get $d_2\left(a_{e_s}a_{e_{q_i}}\right)=\mp d_2\left(a_{e_{q_i}}a_{e_t}\right)=\mp a_{e_s}a_{e_{q_i}}a_{e_t}$ and $d_2\left(a_{e_{q_i}}a_{e_{q_j}}\right)=a_{e_s}a_{e_{q_i}}a_{e_{q_j}}\mp a_{e_{q_i}}a_{e_{q_j}}a_{e_t}$. Conclude that either $d_2\left(a_{e_{q_i}}a_{e_{q_j}}\right)$ is a combination of NBC monomials if $a_{e_s}a_{e_{q_i}}a_{e_{q_j}}\neq\mp a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}}$ or if $a_{e_s}a_{e_{q_i}}a_{e_{q_j}}=\mp a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}}$ represent a broken 4-cycle monomial related to S, then by substitute formula (3.3) in $d_2\left(a_{e_{q_i}}a_{e_{q_j}}\right)$, it will be written as a combination of NBCmonomials. Thus, in this case we have $\binom{4-2}{2}+(4-2)=3$ monomials will be added to the $\operatorname{Im} d_2$'s basis. Similarly, if $e_t\in S$, one can deduce, three monomials will be added to the $\operatorname{Im} d_2$'s basis.

b.2. Assume S be one of the u_4, chordless 4 -cycle including e_s and e_t. As a next step, we will clarify how our selection of $q_1\leq s<t\leq q_4$ excludes a monomial from being an element of $\operatorname{Im} d_2$'s basis as follows:

b.2.1. If $s=q_1$ and $t=q_2: d_2\left(a_{e_{q_3}}a_{e_{q_4}}\right)=d_2\left(a_{e_{q_1}}a_{e_{q_3}}\right)-d_2\left(a_{e_{q_1}}a_{e_{q_4}}\right)$.

b.2.2. If $s=q_1$ and $t=q_3: d_2\left(a_{e_{q_2}}a_{e_{q_4}}\right)=d_2\left(a_{e_{q_1}}a_{e_{q_4}}\right)-d_2\left(a_{e_{q_1}}a_{e_{q_3}}\right)$.

b.2.3. If $s=q_1$ and $t=q_4: d_2\left(a_{e_{q_2}}a_{e_{q_3}}\right)=d_2\left(a_{e_{q_1}}a_{e_{q_2}}\right)-d_2\left(a_{e_{q_1}}a_{e_{q_3}}\right)$.

b.2.4. If $s=q_2$ and $t=q_3: d_2\left(a_{e_{q_2}}a_{e_{q_4}}\right)=d_2\left(a_{e_{q_1}}a_{e_{q_4}}\right)-d_2\left(a_{e_{q_1}}a_{e_{q_2}}\right)$.

b.2.5. If $s=q_2$ and $t=q_4$: $d_2\left(a_{e_{q_2}} a_{e_{q_3}}\right)=d_2\left(a_{e_{q_1}} a_{e_{q_3}}\right)-d_2\left(a_{e_{q_1}} a_{e_{q_2}}\right)$.

b.2.6. If $s=q_3$ and $t=q_4$: $d_2\left(a_{e_{q_2}} a_{e_{q_3}}\right)=d_2\left(a_{e_{q_1}} a_{e_{q_3}}\right)-d_2\left(a_{e_{q_1}} a_{e_{q_2}}\right)$.

In this case, the generators added to the $\operatorname{Im} d_2$'s basis, are exactly two elements calculated as: $\binom{4-2}{2}+(4-2)-1=2$.

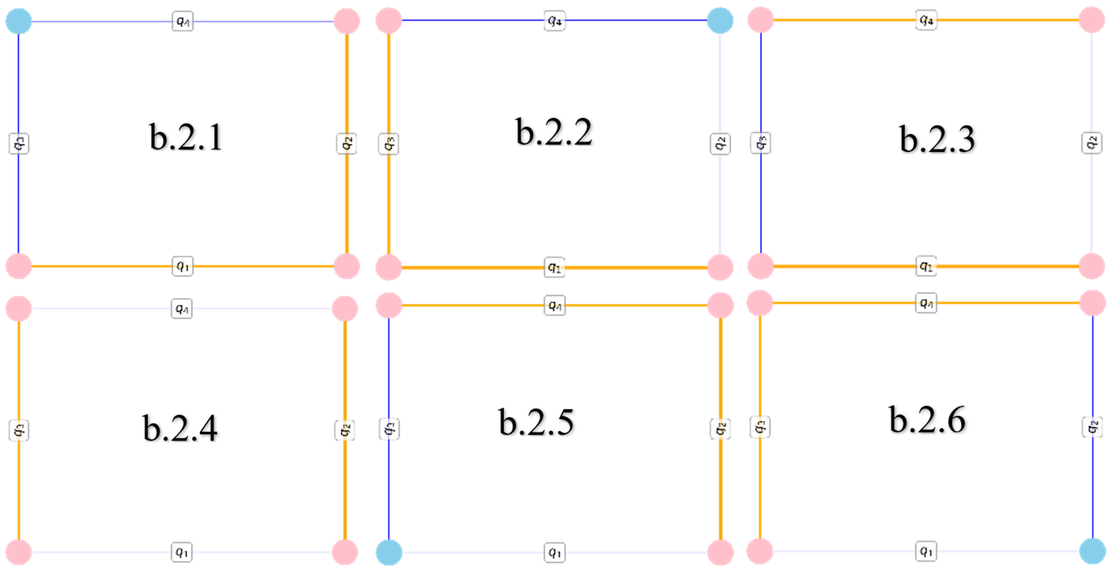

Figure 2: One of the u_4, chordless 4-cycle including e_s and e_t with all values of $q_1 \leq s<t \leq q_4$ can be chosen.

a. The number of generators that will be added to $\operatorname{Im} d_2$'s basis is $\binom{6-2}{2}+(6-2)-2=8$, if two chordless 4 -cycles intersected with e_s and e_t. All configurations for two chordless 4-cycles intersected by e_s and e_t are shown in figure 3, noting that three resultant 4-cycles are present.

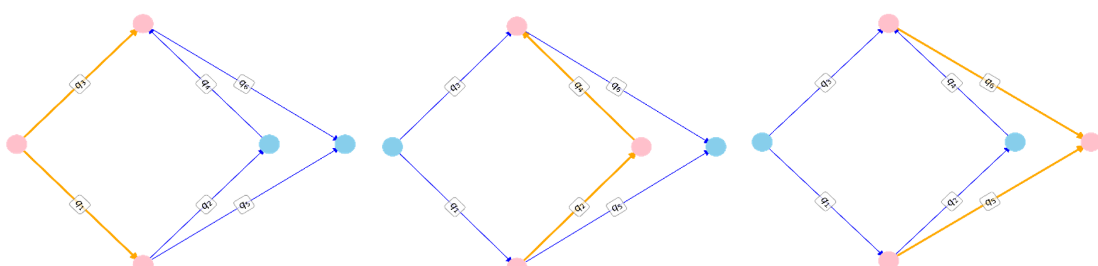

Figure 3: Two of the u_4, chordless 4-cycle intersected by e_s and e_t with all possible values of $q_1 \leq s<t \leq q_6$ can be chosen.

b. If two chordless 4-cycles intersected with e_s or e_t, the number of generators added to the basis of $\operatorname{Im} d_2$'s basis is either $\binom{7-2}{2}+(7-2)=15$ or $\binom{7-2}{2}+(7-2)-1=14$, depending on whether one of them contains e_s and e_t, as illustrated in figure 4.

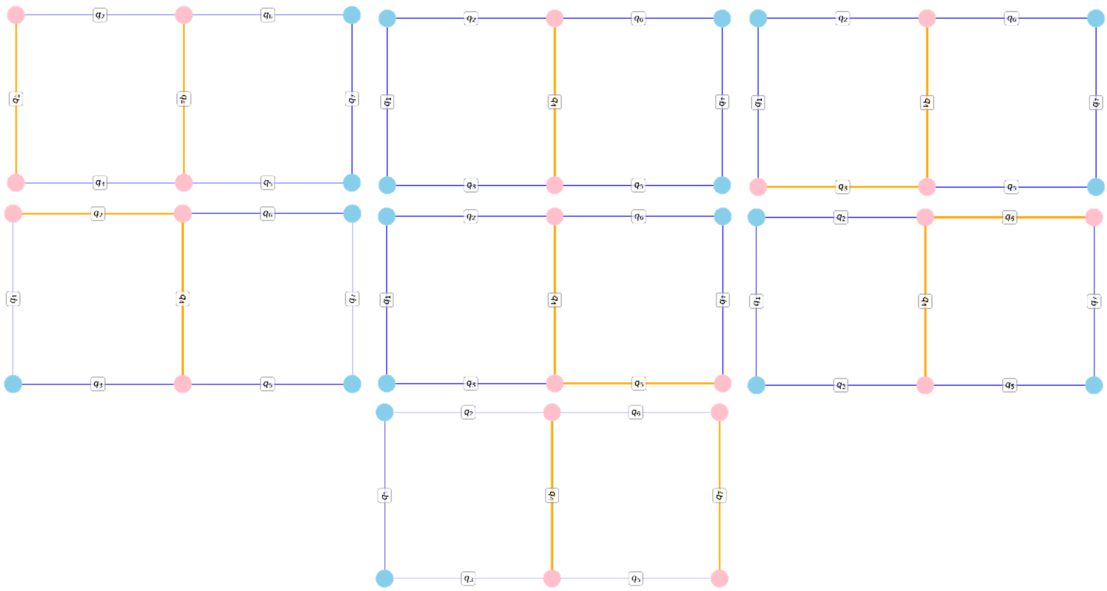

Figure 4: Two, chordless 4-cycle intersected by e_s or e_t with all values of $q_1 \leq s < t \leq q_7$ can be chosen.

Therefore, $\dim(\operatorname{Im} d_2) = \binom{\ell-2}{2} + (\ell-2) - u_4$. □

Theorem 3.4 *If $a = a_{e_s} - a_{e_t}$, for $2 \leq s < t \leq \ell$, then:*

1. *If G is tree or $c(G) > 4$, then $\ker d_3 = \operatorname{Im} d_2$ and $\dim(\operatorname{Im} d_3) = \binom{\ell-2}{3} + \binom{\ell-2}{2}$.*
2. *If $c(G) = 5$, then $\ker d_3 = \operatorname{Im} d_2$ and $\dim(\operatorname{Im} d_3) = \binom{\ell-2}{3} + \binom{\ell-2}{2} - u_5$.*
3. *If G is 4-generic graph, then $\ker d_3 = \boldsymbol{A}_3(G)$.*
4. *If $c(G) = 4$, then $\dim(\ker d_3) = \dim(\operatorname{Im} d_2) + (u_4^s(G) + 2(u_4(G) - u_4^s(G)))$ and $\dim(Imd_3) = \left(\binom{\ell-2}{3} + \binom{\ell-2}{2}\right) - \left((u_4(\ell-4) - b_4) + \left(v_4^s + v_4^t - v_4^{s,t}\right) + (u_4^s + \ 2(u_4 - u_4^s) + (c_4 + u_5)\right).$*

where c_4 be the number of chordless 4 -cycles, u_5 be the number of chordless 5 -cycles that includes e_s and e_t, u_4^s be the number of chordless 4-cycles that include each of e_s and e_t with e_s is the minimal edge via $\trianglelefteq_G, v_4^s$ be the number of broken cycles that related to chordless 4-cycle that contains e_S as not minimal edge via $\trianglelefteq_G, v_4^t$ be the number of broken cycles that related to chordless 4-cycle that contains e_t as not minimal edge via $\trianglelefteq_G$ and $v_4^{s,t}$ be the number of broken cycles B that related to chordless 4-cycle that contains e_S as not minimal edge via $\trianglelefteq_G$ such that $(B - \{e_s\}) \cup \{e_t\}$ is a broken cycles that related to chordless 4-cycle that contains e_t as not minimal edge via $\trianglelefteq_G$.

Proof: The structure of $\boldsymbol{A_4}(G)$ depends on the value of $c(G)$, as per construction (2.1). So, we have the following:

1. If G is tree or $c(G) > 5$, then $NBC^4_{\trianglelefteq_G}(G) = S^4_{\trianglelefteq_G}\left(\Pi^G\right)$ and $\boldsymbol{A_4}(G) \cong \bigoplus_{C \in S^4_{\trianglelefteq_G}}\left(\Pi^G\right) Ka_C$.
2. If $c(G) = 5$, then $NBC^4_{\trianglelefteq_G}(G) = S^4_{\trianglelefteq_G}\left(\Pi^G\right) - BC^4_{\trianglelefteq_G}(G)$ and;

$$\boldsymbol{A_4}(G) \cong \bigoplus_{C \in S^4_{\trianglelefteq_G}(\Pi^G) - BC^4_{\trianglelefteq_G}(G)} Ka_C$$

3. If $c(G) = 4$, then $NBC^4_{\trianglelefteq_G}(G) = S^4_{\trianglelefteq_G}\left(\Pi^G\right) - \left(BC^4_{\trianglelefteq_G}(G) \cup C^4_{\trianglelefteq_G}(G)\right)$ and;

$$\boldsymbol{A_4}(G) \cong \bigoplus_{C \in S^4_{\trianglelefteq G}} \left(\Pi^G\right) - \left(BC^4_{\trianglelefteq G}(G) \cup C^4_{\trianglelefteq G}(G)\right) K a_C$$

where $BC^4_{\trianglelefteq G}(G)$ includes the sections derived from the removal of the minimal edge via $\trianglelefteq_G$ from the chordless 5-cycles and $C^4_{\trianglelefteq G}(G)$ involves the sections related to chordless 4-cycles ordered via $\trianglelefteq_G$.

Based on the value of $c(G)$, we must now investigate the homomorphism $d_3 : \boldsymbol{A}_3(G) \xrightarrow{a} \boldsymbol{A}_4(G)$. Thus, for $1 \leq k_1 < k_2 < k_3 \leq \ell$, if $a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} \in NBC^3_{\trianglelefteq G}(G)$, we have the following:

$$d_3\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}\right) = a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} \left(a_{e_s} - a_{e_t}\right) \tag{3.4}$$

$$d_3\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}\right) = \begin{cases} 0_{A_4(G)} & : k_i = s \text{ and } k_j = t \text{ for some } 1 \leq i < j \leq 3 \end{cases}$$

$$d_3\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}\right) = \begin{cases}
-a_{e_s} a_{e_{k_2}} a_{e_{k_3}} a_{e_t} & : k_1 = s < k_2 < k_3 < t \leq \ell \\
a_{e_s} a_{e_{k_2}} a_{e_t} a_{e_{k_3}} & : k_1 = s < k_2 < t < k_3 \leq \ell \\
-a_{e_s} a_{e_t} a_{e_{k_2}} a_{e_{k_3}} & : k_1 = s < t < k_2 < k_3 \leq \ell \\
-a_{e_{k_1}} a_{e_s} a_{e_{k_3}} a_{e_t} & : 1 \leq k_1 < k_2 = s < k_3 < t \\
a_{e_{k_1}} a_{e_s} a_{e_t} a_{e_{k_3}} & : 1 \leq k_1 < k_2 = s < t < k_3 \leq \ell \\
-a_{e_{k_1}} a_{e_{k_2}} a_{e_s} a_{e_t} & : 1 \leq k_1 < k_2 < k_3 = s < t \\
-a_{e_s} a_{e_{e_t}} a_{e_{k_2}} a_{e_{k_3}} & : s < k_1 = t < k_2 < k_3 < t \\
-a_{e_s} a_{e_{k_1}} a_{e_{e_t}} a_{e_{k_3}} & : s < k_1 < k_2 = t < k_3 \leq \ell \\
a_{e_{k_1}} a_{e_s} a_{e_{e_t}} a_{e_{k_3}} & : 1 \leq k_1 < s < k_2 = t < k_3 \leq \ell \\
-a_{e_s} a_{e_{k_1}} a_{e_{k_2}} a_{e_t} & : s < k_1 < k_2 < k_3 = t \\
a_{e_{k_1}} a_{e_s} a_{e_{k_2}} a_{e_t} & : 1 \leq k_1 < s < k_2 < k_3 = t \\
-a_{e_{k_1}} a_{e_{k_2}} a_{e_s} a_{e_t} & : 1 \leq k_1 < k_2 < s < k_3 = t
\end{cases}$$

$$d_3\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}\right) = \begin{cases}
-a_{e_s} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} - a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} a_{e_t} & : s < k_1 < k_2 < k_3 < t \\
-a_{e_s} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} + a_{e_{k_1}} a_{e_{k_2}} a_{e_t} a_{e_{k_3}} & : s < k_1 < k_2 < t < k_3 \leq \ell \\
-a_{e_s} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} - a_{e_{k_1}} a_{e_t} a_{e_{k_2}} a_{e_{k_3}} & : s < k_1 < t < k_2 < k_3 \leq \ell \\
-a_{e_s} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} + a_{e_t} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} & : s < t < k_1 < k_2 < k_3 \leq \ell \\
a_{e_{k_1}} a_{e_s} a_{e_{k_2}} a_{e_{k_3}} - a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} a_{e_t} & : 1 \leq k_1 < s < k_2 < k_3 < t \\
a_{e_{k_1}} a_{e_s} a_{e_{k_2}} a_{e_{k_3}} + a_{e_{k_1}} a_{e_{k_2}} a_{e_t} a_{e_{k_3}} & : 1 \leq k_1 < s < k_2 < t < k_3 \leq \ell \\
a_{e_{k_1}} a_{e_s} a_{e_{k_2}} a_{e_{k_3}} - a_{e_{k_1}} a_{e_t} a_{e_{k_2}} a_{e_{k_3}} & : 1 \leq k_1 < s < t < k_2 < k_3 \leq \ell \\
-a_{e_{k_1}} a_{e_{k_2}} a_{e_s} a_{e_{k_3}} - a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} a_{e_t} & : 1 \leq k_1 < k_2 < s < k_3 < t \\
-a_{e_{k_1}} a_{e_{k_2}} a_{e_s} a_{e_{k_3}} + a_{e_{k_1}} a_{e_{k_2}} a_{e_t} a_{e_{k_3}} & : 1 \leq k_1 < k_2 < s < t < k_3 \leq \ell \\
a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} a_{e_s} - a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} a_{e_t} & : 1 \leq k_1 < k_2 < k_3 < s < t
\end{cases}$$

Per the type of graph G, we shall consider all conceivable possibilities as follows:

1. If G is a tree or $c(G) > 5$, hence $NBC^4_{\trianglelefteq G}(M_G) = S^4_{\Xi G}\left(\Pi^G\right)$. According formula (3.4), if $k_i = s$ and $k_j = t$ for some $1 \leq i < j \leq 3$, then $d_3\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}\right) = 0_{A_4(G)}$, and in all other cases, $d_3\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}\right)$ forms either NBC monomial or a combination of NBCmonomials. Hence, for any $1 < k \leq \ell$ and $k \neq s, t, a_{e_s} a_{e_t} a_{e_k} \in \ker d_3$ and due theorem $(2,5)$, $a_{e_s} a_{e_t} a_{e_k} \in \operatorname{Im} d_2$. Therefore, $\ker d_3 = \operatorname{Im} d_2$ and from formula (3.4), $\dim\left(\operatorname{Im} d_3\right) = \binom{\ell-2}{3} + \binom{\ell-2}{2}$.

2. If G has $c(G) = 5$, there are two possibilities:
   a. If $m = 5$, then G is 5 -generic. Assume:

$$\varepsilon = \left\{e_1 = [v_1, v_2], e_2 = [v_2, v_3], e_3 = [v_1, v_4], e_4 = [v_3, v_4], e_5 = [v_4, v_5]\right\}.$$

Thus, the broken 5 -cycle monomial expressed as:

$$a_{e_2}a_{e_3}a_{e_4}a_{e_5} = a_{e_1}a_{e_3}a_{e_4}a_{e_5} - a_{e_1}a_{e_2}a_{e_4}a_{e_5} + a_{e_1}a_{e_2}a_{e_3}a_{e_5} - a_{e_1}a_{e_2}a_{e_3}a_{e_4} \tag{3.5}$$

Below are four cases for values of s and t :

a.1. If $s = 2$ and $t = 3$, then:

$$\begin{gathered}
d_3\left(a_{e_1}a_{e_2}a_{e_3}\right) = d_3\left(a_{e_2}a_{e_3}a_{e_4}\right) = d_3\left(a_{e_2}a_{e_3}a_{e_5}\right) = 0_{A_4(G)},\\
d_3\left(a_{e_1}a_{e_2}a_{e_4}\right) = d_3\left(a_{e_1}a_{e_3}a_{e_4}\right) = a_{e_1}a_{e_2}a_{e_3}a_{e_4},\\
d_3\left(a_{e_1}a_{e_2}a_{e_5}\right) = d_3\left(a_{e_1}a_{e_3}a_{e_5}\right) = a_{e_1}a_{e_2}a_{e_3}a_{e_5},\\
d_3\left(a_{e_1}a_{e_4}a_{e_5}\right) = a_{e_1}a_{e_2}a_{e_4}a_{e_5} - a_{e_1}a_{e_3}a_{e_4}a_{e_5}, \text{ and ;}\\
d_3\left(a_{e_2}a_{e_4}a_{e_5}\right) = d_3\left(a_{e_3}a_{e_4}a_{e_5}\right) = d_3\left(a_{e_1}a_{e_2}a_{e_4}\right) - d_3\left(a_{e_1}a_{e_2}a_{e_5}\right) + d_3\left(a_{e_1}a_{e_4}a_{e_5}\right).
\end{gathered}$$

a.2. If $s = 2$ and $t = 4$, then:

$$\begin{gathered}
d_3\left(a_{e_1}a_{e_2}a_{e_4}\right) = d_3\left(a_{e_2}a_{e_3}a_{e_4}\right) = d_3\left(a_{e_2}a_{e_4}a_{e_5}\right) = 0_{A_4(G)},\\
d_3\left(a_{e_1}a_{e_2}a_{e_3}\right) = -d_3\left(a_{e_1}a_{e_3}a_{e_4}\right) = -a_{e_1}a_{e_2}a_{e_3}a_{e_4},\\
d_3\left(a_{e_1}a_{e_2}a_{e_5}\right) = d_3\left(a_{e_1}a_{e_4}a_{e_5}\right) = a_{e_1}a_{e_2}a_{e_4}a_{e_5},\\
d_3\left(a_{e_1}a_{e_3}a_{e_5}\right) = a_{e_1}a_{e_2}a_{e_3}a_{e_5} + a_{e_1}a_{e_3}a_{e_4}a_{e_5}, \text{ and ;}\\
d_3\left(a_{e_2}a_{e_3}a_{e_5}\right) = d_3\left(a_{e_3}a_{e_4}a_{e_5}\right) = d_3\left(a_{e_1}a_{e_2}a_{e_3}\right) - d_3\left(a_{e_1}a_{e_2}a_{e_5}\right) + d_3\left(a_{e_1}a_{e_3}a_{e_5}\right).
\end{gathered}$$

a.3. If $s = 2$ and $t = 5$, then:

$$\begin{gathered}
d_3\left(a_{e_1}a_{e_2}a_{e_5}\right) = d_3\left(a_{e_2}a_{e_3}a_{e_5}\right) = d_3\left(a_{e_2}a_{e_4}a_{e_5}\right) = 0_{A_4(G)},\\
d_3\left(a_{e_1}a_{e_2}a_{e_3}\right) = -d_3\left(a_{e_1}a_{e_3}a_{e_5}\right) = -a_{e_1}a_{e_2}a_{e_3}a_{e_5},\\
d_3\left(a_{e_1}a_{e_2}a_{e_4}\right) = -d_3\left(a_{e_1}a_{e_4}a_{e_5}\right) = -a_{e_1}a_{e_2}a_{e_4}a_{e_5},\\
d_3\left(a_{e_1}a_{e_3}a_{e_4}\right) = a_{e_1}a_{e_2}a_{e_3}a_{e_4} - a_{e_1}a_{e_3}a_{e_4}a_{e_5}, \text{ and ;}\\
d_3\left(a_{e_2}a_{e_3}a_{e_4}\right) = d_3\left(a_{e_3}a_{e_4}a_{e_5}\right) = d_3\left(a_{e_1}a_{e_2}a_{e_4}\right) - d_3\left(a_{e_1}a_{e_2}a_{e_3}\right) - d_3\left(a_{e_1}a_{e_3}a_{e_4}\right).
\end{gathered}$$

a.4. If $s = 3$ and $t = 4$, then:

$$\begin{gathered}
d_3\left(a_{e_1}a_{e_3}a_{e_4}\right) = d_3\left(a_{e_2}a_{e_3}a_{e_4}\right) = d_3\left(a_{e_3}a_{e_4}a_{e_5}\right) = 0_{A_4(G)},\\
d_3\left(a_{e_1}a_{e_2}a_{e_3}\right) = d_3\left(a_{e_1}a_{e_2}a_{e_4}\right) = -a_{e_1}a_{e_2}a_{e_3}a_{e_4},\\
d_3\left(a_{e_1}a_{e_3}a_{e_5}\right) = d_3\left(a_{e_1}a_{e_4}a_{e_5}\right) = a_{e_1}a_{e_3}a_{e_4}a_{e_5},\\
d_3\left(a_{e_1}a_{e_2}a_{e_5}\right) = -a_{e_1}a_{e_2}a_{e_3}a_{e_5} + a_{e_1}a_{e_2}a_{e_4}a_{e_5}, \text{ and;}\\
d_3\left(a_{e_2}a_{e_3}a_{e_5}\right) = d_3\left(a_{e_2}a_{e_4}a_{e_5}\right) = d_3\left(a_{e_1}a_{e_2}a_{e_3}\right) - d_3\left(a_{e_1}a_{e_2}a_{e_5}\right) + d_3\left(a_{e_1}a_{e_3}a_{e_5}\right).
\end{gathered}$$

a.5. If $s = 3$ and $t = 5$, then:

$$\begin{gathered}
d_3\left(a_{e_1}a_{e_3}a_{e_5}\right) = d_3\left(a_{e_2}a_{e_3}a_{e_5}\right) = d_3\left(a_{e_3}a_{e_4}a_{e_5}\right) = 0_{A_4(G)}\\
d_3\left(a_{e_1}a_{e_2}a_{e_3}\right) = d_3\left(a_{e_1}a_{e_2}a_{e_5}\right) = -a_{e_1}a_{e_2}a_{e_3}a_{e_5}\\
d_3\left(a_{e_1}a_{e_3}a_{e_4}\right) = -d_3\left(a_{e_1}a_{e_4}a_{e_5}\right) = -a_{e_1}a_{e_3}a_{e_4}a_{e_5},\\
d_3\left(a_{e_1}a_{e_2}a_{e_4}\right) = -a_{e_1}a_{e_2}a_{e_3}a_{e_4} - a_{e_1}a_{e_2}a_{e_4}a_{e_5}, \text{ and;}\\
d_3\left(a_{e_2}a_{e_3}a_{e_4}\right) = -d_3\left(a_{e_3}a_{e_4}a_{e_5}\right) = d_3\left(a_{e_1}a_{e_2}a_{e_3}\right) - d_3\left(a_{e_1}a_{e_2}a_{e_4}\right) + d_3\left(a_{e_1}a_{e_3}a_{e_4}\right).
\end{gathered}$$

a.6. If $s = 4$ and $t = 5$, then:

$$\begin{gathered}
d_3\left(a_{e_1}a_{e_4}a_{e_5}\right) = d_3\left(a_{e_2}a_{e_4}a_{e_5}\right) = d_3\left(a_{e_3}a_{e_4}a_{e_5}\right) = 0_{A_4(G)},\\
d_3\left(a_{e_1}a_{e_2}a_{e_4}\right) = d_3\left(a_{e_1}a_{e_2}a_{e_5}\right) = -a_{e_1}a_{e_2}a_{e_4}a_{e_5},\\
d_3\left(a_{e_1}a_{e_3}a_{e_4}\right) = -d_3\left(a_{e_1}a_{e_3}a_{e_5}\right) = -a_{e_1}a_{e_3}a_{e_4}a_{e_5},\\
d_3\left(a_{e_1}a_{e_2}a_{e_3}\right) = a_{e_1}a_{e_2}a_{e_3}a_{e_4} - a_{e_1}a_{e_2}a_{e_3}a_{e_5}, \text{ and ;}\\
d_3\left(a_{e_2}a_{e_3}a_{e_4}\right) = d_3\left(a_{e_2}a_{e_3}a_{e_5}\right) = d_3\left(a_{e_1}a_{e_2}a_{e_3}\right) - d_3\left(a_{e_1}a_{e_2}a_{e_4}\right) + d_3\left(a_{e_1}a_{e_3}a_{e_4}\right).
\end{gathered}$$

Therefore, for each $1 \leq k \leq \ell$ where $k \neq s, t, \mp a_{e_k} a_{e_s} a_{e_t} \in \ker d_3$, and according to theorem $(2,5)$, $a_{e_s} a_{e_t} a_{e_k} \in \operatorname{Im} d_2$. Thus, $\ker d_3 = \operatorname{Im} d_2$, and it is clear that $\dim(\operatorname{Im} d_3) = \binom{5-2}{3} + \binom{5-2}{2} - 1 = 3$ for this case.

b. If $m > 5$ and $c_5(G)$ be the number of chordless 5 -cycles then for this case,

$$\left|NBC^3_{\trianglelefteq_G}(G)\right| = \left|S^3_{\trianglelefteq_G}\left(\Pi^G\right)\right| = \binom{\ell}{3}, \left|BC^4_{\trianglelefteq_G}(G)\right| = c_5 \text{ and;}$$

$$\left|NBC^4_{\underline{\S}_G}(G)\right| = \left|S^4_{\underline{\trianglelefteq}_G}\left(\Pi^G\right)\right| - \left|BC^4_{\Xi_G}(G)\right| = \binom{\ell}{4} - c_5.$$

Let $S = \{e_{q_1}, e_{q_2}, e_{q_3}, e_{q_4}, e_{q_5}\}$ represent a chordless 5-cycle where $1 \leq q_1 < q_2 < q_3 < q_4 < q_5 \leq \ell$. Consequently, its broken cycle is $C = \{e_{q_2}, e_{q_3}, e_{q_4}, e_{q_5}\} \in BC^4_{-G}(G)$. It has been determined that every monomial of a broken 5 -circuit can be expressed as a combination of NBC monomials as:

$$a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_4}} a_{e_{q_5}} = a_{e_{q_1}} a_{e_{q_3}} a_{e_{q_4}} a_{e_{q_5}} - a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_4}} a_{e_{q_5}} + a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_5}} - a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_4}} \tag{3.6}$$

Therefore, we have the following possible cases:

b.1. If S is a chordless 5 -cycle that does not include e_1 and e_t at the same time, then:

b.1.1. If $e_s, e_t \notin S$, then formula (3.4) shows that $d_3\left(a_{e_{q_i}} a_{e_{q_j}} a_{e_{q_k}}\right)$ is written either as an NBC monomial or as a combination of NBC-monomials, for $1 \leq i < j < k \leq 5$.

b.1.2. For the second possible case assume either ($e_s \in S$ and $e_t \notin S$) or ($e_s \notin S$ and $e_t \in S$). If ($e_s \in S$ and $e_t \notin S$), then, $q_i = s$ for some $1 \leq i \leq 5$ and $q_j \neq t$. For all $1 \leq j \leq 5$. As a result to formula (3.4), if $q_1 = s < q_2 < q_3 < q_4 < q_5 < t$, hence $d_3\left(a_{e_s} a_{e_{q_i}} a_{e_{q_j}}\right) = d_3\left(a_{e_{q_i}} a_{e_{q_j}} a_{e_t}\right) = -a_{e_s} a_{e_{q_i}} a_{e_{q_j}} a_{e_t}$ is an NBC monomial and $d_2\left(a_{e_{q_i}} a_{e_{q_j}} a_{e_{q_k}}\right) = -a_{e_s} a_{e_{q_i}} a_{e_{q_j}} a_{e_{q_k}} - a_{e_{q_i}} a_{e_{q_j}} a_{e_{q_k}} a_{e_t}$, is written as a combination of an NBC monomials, for $1 < i < j < k \leq 5$. Therefore, the generators added to the $\operatorname{Im} d_3$'s basis, are exactly four elements calculated as: $\binom{5-2}{3} + \binom{5-2}{2} = 4$.

Similarly, as a result of formula (2.6.1) we can deduce that there are four generators that will be added to $\operatorname{Im} d_3$'s basis, for all the other possible choices of $q_i = s$ for some $2 \leq i \leq 5$ and if ($e_s \notin S$ and $e_t \in S$).

b.2. If S is a chordless 5-cycle, that including e_s and e_t, then we will prove that there is a 3NBC monomial image that cannot be added to $\operatorname{Im} d_2$'s basis, as follows:

b.2.1. If $s = q_1$ and $t = q_2$:

$$d_3\left(a_{e_{q_3}} a_{e_{q_4}} a_{e_{q_5}}\right) = d_3\left(a_{e_{q_1}} a_{e_{q_3}} a_{e_{q_4}}\right) - d_3\left(a_{e_{q_1}} a_{e_{q_3}} a_{e_{q_5}}\right) + d_3\left(a_{e_{q_1}} a_{e_{q_4}} a_{e_{q_5}}\right)$$

b.2.2. If $s = q_1$ and $t = q_3$:

$$d_3\left(a_{e_{q_2}} a_{e_{q_4}} a_{e_{q_5}}\right) = d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_4}}\right) - d_3\left(a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_4}}\right) + d_3\left(a_{e_{q_1}} a_{e_{q_4}} a_{e_{q_5}}\right)$$

b.2.3. If $s = q_1$ and $t = q_4$:

$$d_3\left(a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_5}}\right) = d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_3}}\right) - d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_5}}\right) + d_3\left(a_{e_{q_1}} a_{e_{q_3}} a_{e_{q_5}}\right).$$

b.2.4. If $s = q_1$ and $t = q_5$:

$$d_3\left(a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_4}}\right) = d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_3}}\right) - d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_4}}\right) + d_3\left(a_{e_{q_1}} a_{e_{q_3}} a_{e_{q_4}}\right).$$

b.2.5. If $s = q_2$ and $t = q_3$:

$$d_3\left(a_{e_{q_2}} a_{e_{q_4}} a_{e_{q_5}}\right) = d_3\left(a_{e_{q_3}} a_{e_{q_4}} a_{e_{q_5}}\right) = d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_4}}\right) - d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_5}}\right) + d_3\left(a_{e_{q_1}} a_{e_{q_4}} a_{e_{q_5}}\right).$$

b.2.6. If $s = q_2$ and $t = q_4$:

$$d_3\left(a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_5}}\right)=-d_3\left(a_{e_{q_3}} a_{e_{q_4}} a_{e_{q_5}}\right)=d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_3}}\right)-d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_5}}\right)+ d_3\left(a_{e_{q_1}} a_{e_{q_3}} a_{e_{q_5}}\right).$$

b.2.7. If $s=q_2$ and $t=q_5$:

$$d_3\left(a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_4}}\right)=d_3\left(a_{e_{q_3}} a_{e_{q_4}} a_{e_{q_5}}\right)=-d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_3}}\right)-d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_4}}\right)+ d_3\left(a_{e_{q_1}} a_{e_{q_3}} a_{e_{q_4}}\right)$$

b.2.8. If $s=q_3$ and $t=q_4$:

$$d_3\left(a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_5}}\right)=d_3\left(a_{e_{q_2}} a_{e_{q_4}} a_{e_{q_5}}\right)=d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_3}}\right)-d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_5}}\right)+ d_3\left(a_{e_{q_1}} a_{e_{q_3}} a_{e_{q_5}}\right)$$

b.2.9. If $s=q_3$ and $t=q_5$:

$$d_3\left(a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_4}}\right)=-d_3\left(a_{e_{q_2}} a_{e_{q_4}} a_{e_{q_5}}\right)=d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_3}}\right)-d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_4}}\right)- d_3\left(a_{e_{q_1}} a_{e_{q_3}} a_{e_{q_4}}\right).$$

b.2.10. If $s=q_4$ and $t=q_5$:

$$d_3\left(a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_4}}\right)=d_3\left(a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_5}}\right)=d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_3}}\right)-d_3\left(a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_4}}\right)- d_3\left(a_{e_{q_1}} a_{e_{q_3}} a_{e_{q_4}}\right).$$

For each case of (b.2.1-10), the generators added to the $\operatorname{Im} d_3$'s basis, are: $\binom{5-2}{3}+\binom{5-2}{2}-1=3$. For general case, the number of such generators is, $\binom{\ell-2}{3}+\binom{\ell-2}{2}-u_5$.
3. If $c(G)=4$, then,

$$\left|NBC_{\trianglelefteq_G}^3(G)\right|=\left|S_{\trianglelefteq_G}^3\left(\Pi^G\right)\right|-\left|BC_{\trianglelefteq_G}^3(G)\right|=\binom{\ell}{3}-c_4 \text{ and;}$$

$$\left|NBC_{\trianglelefteq_G}^4(G)\right|=\left|S_{\trianglelefteq_G}^4\left(\Pi^G\right)\right|-\left(\left|BC_{\trianglelefteq_G}^4(G)\right|+c_4(G)\right)=\binom{\ell}{4}-\left(c_5(G)+c_4(G)\right)$$

where c_4 and c_5 be the numbers of chordless 4 -cycles and chordless 5 -cycles, respectively. At this point, we need to realize wither G is generic or not:
a. If G is 4-generic, then $\boldsymbol{A_4}(G)=0$. Thus, $\ker d_3=\boldsymbol{A_3}(G)$, since $d_3: \boldsymbol{A_3}(G) \rightarrow 0$ is the zero homomorphism.
b. If G is not 4-generic (i.e., $c_4(G)>1$), then $NBC_{\trianglelefteq_G}^3(G)$ will partitioned into four parts say: $NBC_{\trianglelefteq_G}^3(G)=NBC_{\trianglelefteq_G}^{3,0}(G) \cup NBC_{\trianglelefteq_G}^{3,s}(G) \cup NBC_{\trianglelefteq_G}^{3,t}(G) \cup NBC_{\trianglelefteq_G}^{3,s,t}(G)$, where:
✓ $NBC_{\trianglelefteq_G}^{3,0}(G)$ be the set of all NBC bases that either (are not related to any chordless 4-cycles) or (related to chordless 4-cycle that are not contain e_s or e_t) or (related to chordless 4 cycle that contain e_s or e_t as minimal edge via $\trianglelefteq_G$).
✓ $NBC_{\trianglelefteq_G}^{3,S}(G)$ and $BC_{\trianglelefteq_G}^{3,S}(G)$ be the sets of all NBC bases and broken cycles that related to chordless 4-cycle that contains just e_S as not minimal edge via $\trianglelefteq_G$, respectively and let $v_4^s=\left|BC_{\trianglelefteq_G}^{3,s}(G)\right|$.
✓ $NBC_{\trianglelefteq_G}^{3,t}(G)$ and $BC_{\trianglelefteq_G}^{3,t}(G)$ be the sets of all NBC bases and broken circuits that related to chordless 4-cycle that contains just e_t as not minimal edge via $\trianglelefteq_G$, respectively and let $v_4^t=\left|BC_{\trianglelefteq_G}^{3,t}(G)\right|$.
✓ $NBC_{\trianglelefteq_G}^{3,s,t}(G)$ and $BC_{\trianglelefteq_G}^{3,s,t}(G)$ be the set of all NBC bases and broken circuits that related to chordless 4-cycles that contains each of e_s and e_t via $\trianglelefteq_G$, respectively, and $\left|BC_{\trianglelefteq_G}^{3,s,t}(G)\right|=u_4$.
✓ Let $v_4^{s,t}$ is the number of broken cycles $B \in BC_{\trianglelefteq_G}^{3,s}(G)$ and $\left(B-\left\{e_s\right\}\right) \cup\left\{e_t\right\} \in BC_{\trianglelefteq_G}^{3,t}(G)$.

So, we will investigate the behavior of d_3 as follows:
b.1. If $c_5(G)=0$, then every four edges of G, either they are in a chordless 4 -cycles or they cannot be broken circuit for a chordless 5-cycle. For this case $NBC_{\trianglelefteq_G}^4(G)=S_{\Xi_G}^4\left(\Pi^G\right)-C_{\Xi_G}^4(G)$, where $C_{\trianglelefteq_G}^4(G)$

is the set of all chordless 4-cycles. According to the formula (2.6.1), we have the following:
b.1.1. For $1 \leq k \leq \ell$ and $k \neq s,t$, $\{e_s, e_t, e_k\} \in NBC^{3,s,t}_{\trianglelefteq_G}(G)$ and $d_3\left(\pm a_{e_s} a_{e_t} a_{e_k}\right) = 0_{A_4(G)}$. These monomials part of $\mathrm{Im} d_2$'s basis and there are $(\ \ell - 2 - u_4\ )$ of them.
b.1.2. For every chordless 4-cycle $\{e_{k_1}, e_{k_2}, e_{k_3}, e_{k_4}\}$, there are three related NBCbase contained in $NBC^{3,s,t}_{\trianglelefteq_G}(G)$ each one of them include the minimal edge e_{k_1}. If $s = k_1$, we have just one NBC base in $NBC^{3,s,t}_{\trianglelefteq_G}(G)$ such that $d_3\left(a_{e_s} a_{e_{k_i}} a_{e_{k_j}}\right) = 0_{A_4(G)}$ with $2 \leq i < j \leq 4$ and $k_1, k_2 \neq t$. Assume that the number of such NBC bases is u_4^s and the number of other chordless 4cycles that contain each of e_s and e_t with e_s not the minimal edge is $(\ u_4 -\ u_4^s\ )$. In case $s = k_2, 3 \leq i \leq 4$ and $k_i \neq t$, we have, $d_3\left(a_{e_{k_1}} a_{e_s} a_{e_{k_i}}\right) = d_3\left(\pm a_{e_{k_1}} a_{e_t} a_{e_{k_i}}\right) = 0_{A_4(G)}$. Similarly, if $s = k_3$, we have, $d_3\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_s}\right) = d_3\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_t}\right) = 0_{A_4(G)}$. Thus, there exist $\left(u_4^s + 2\left(u_4 - u_4^s\right)\right)$ monomials of $NBC^{3,s,t}_{\trianglelefteq_G}(G)$ with this property, but Imd_2 's basis does not contain them.
b.1.3. If $\{e_{k_1}, e_{k_2}, e_{k_3}\} \in NBC^{3,0}_{\trianglelefteq_G}(M_G)$, such that $1 < k_1 < k_2 < k_3 \leq \ell$ and $k_1, k_2, k_3 \neq s,t$, as a direct result to formula (2.6.1), $d_3\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}\right)$ written as a combination of NBC monomials.
b.1.4. If $\{e_{k_1}, e_{k_2}, e_{k_3}\} \in NBC^{3,s}_{\trianglelefteq_G}(G) \cup NBC^{3,t}_{\trianglelefteq_G}(G)$, such that $1 < k_1 < k_2 < k_3 \leq \ell$ and $k_1, k_2, k_3 \neq s,t$, hence:
b.1.4.1. If $\{e_{k_1}, e_{k_2}, e_{k_3}\} \in NBC^{3,s}_{\trianglelefteq_G}(G)$, hence:

$$d_3\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}\right) = \pm a_{e_t} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} \tag{3.7}$$

either it is NBC monomial, or it contains a broken 4-cycle monomial. So, if $a_{e_t} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}$ is an NBC monomial, hence:

$$d_3\left(a_{e_t} a_{e_{k_2}} a_{e_{k_3}}\right) = \pm d_3\left(a_{e_s} a_{e_{k_1}} a_{e_{k_2}}\right) \pm d_3\left(a_{e_1} a_{e_{k_1}} a_{e_{k_3}}\right) \pm d_3\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}\right).$$

The number of such monomials is less than or equal to v_4^s.
b.1.4.2. If $\{e_{k_1}, e_{k_2}, e_{k_3}\} \in NBC^{3,t}_{\trianglelefteq_G}(G)$, hence:

$$d_3\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}\right) = \pm a_{e_s} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} \tag{3.8}$$

either it is NBC monomial, or it contains a broken 4-cycle monomial. So, if $a_{e_t} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}$ is an NBC monomial, hence:

$$d_3\left(a_{e_s} a_{e_{k_2}} a_{e_{k_3}}\right) = \pm d_3\left(a_{e_t} a_{e_{k_1}} a_{e_{k_3}}\right) \pm d_3\left(a_{e_t} a_{e_{k_1}} a_{e_{k_2}}\right) \pm d_3\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}\right).$$

The number of such monomials is less than or equal to v_4^t.
b.1.4.3. Here we will discuss whether each one of formula (3.7) and formula (3.8) contains broken 4-cycle monomial. Since $\{e_{k_1}, e_{k_2}, e_{k_3}\} \in$
$NBC^{3,s}_{\trianglelefteq_G}(G)$ with $k_1 < k_2 < k_3$ and $\pm a_{e_t} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}$ contains a broken 4-cycle, hence we have five vertices and six edges and without loss of generality, we can assume the graph given in figure 5 below as a configuration to our choice such that $\{e_s, e_{k_2}, e_{k_3}\} \in BC^{3,s}_{\trianglelefteq_G}(G)$, $\{e_t, e_{k_2}, e_{k_3}\} \in BC^{3,t}_{\trianglelefteq_G}(G)$, $\{k, e_s, e_t\} \in BC^{3,s,t}_{\trianglelefteq_G}(G)$ and the resulting three chordless 4-cycles $\{e_{k_1}, e_s, e_{k_2}, e_{k_3}\}$, $\{e_k, e_t, e_{k_2}, e_{k_3}\}$ and $\{e_{k_1}, e_k, e_s, e_t\}$ such that $\{e_k, e_{k_2}, e_{k_3}\} \in NBC^{3,t}_{\trianglelefteq_G}(G)$ that given in formula (3.8) that related to $\{e_{k_1}, e_{k_2}, e_{k_3}\} \in NBC^{3,s}_{\trianglelefteq_G}(G)$:

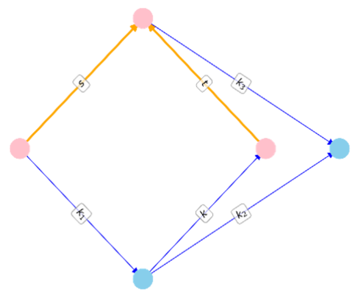

Figure 5: A configuration of three chordless 4-cycles with five vertices and six edges.

Therefore, for this case we have:

$$d_3\left(a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}\right) = d_3\left(a_{e_k}a_{e_{k_2}}a_{e_{k_3}}\right) - d_3\left(a_{e_{k_1}}a_{e_k}a_{e_{k_3}}\right) + d_3\left(a_{e_{k_1}}a_{e_k}a_{e_{k_3}}\right)$$

For $1 \leq q \leq \ell$ and $q \neq s, t, k, k_1, k_2, k_3$, we have:

$$\begin{aligned}
& d_3\left(a_{e_q}a_{e_{k_2}}a_{e_{k_3}}\right) = \pm a_{e_q}a_{e_s}a_{e_{k_2}}a_{e_{k_3}} \pm a_{e_q}a_{e_s}a_{e_{k_2}}a_{e_{k_3}} \\
& = \pm a_{e_q}\left(a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}} - a_{e_{k_1}}a_{e_s}a_{e_{k_3}} + a_{e_{k_1}}a_{e_s}a_{e_{k_2}}\right) \pm a_{e_q}\left(a_{e_k}a_{e_{k_2}}a_{e_{k_3}} - a_{e_k}a_{e_t}a_{e_{k_3}}\right. \\
& \left. + a_{e_k}a_{e_t}a_{e_{k_2}}\right) \\
& = \pm\left(a_{e_q}a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}} - a_{e_q}a_{e_{k_1}}a_{e_s}a_{e_{k_3}} + a_{e_q}a_{e_{k_1}}a_{e_s}a_{e_{k_2}}\right) \\
& \pm\left(a_{e_q}a_{e_k}a_{e_{k_2}}a_{e_{k_3}} - a_{e_q}a_{e_k}a_{e_t}a_{e_{k_3}} + a_{e_q}a_{e_k}a_{e_t}a_{e_{k_2}}\right)
\end{aligned}$$

We can deduce that $d_3\left(a_{e_q}a_{e_{k_2}}a_{e_{k_3}}\right)$ is written as a linear combination of NBC monomial. Similarly, the same result holds to all the other cases that are shown in figure 6:

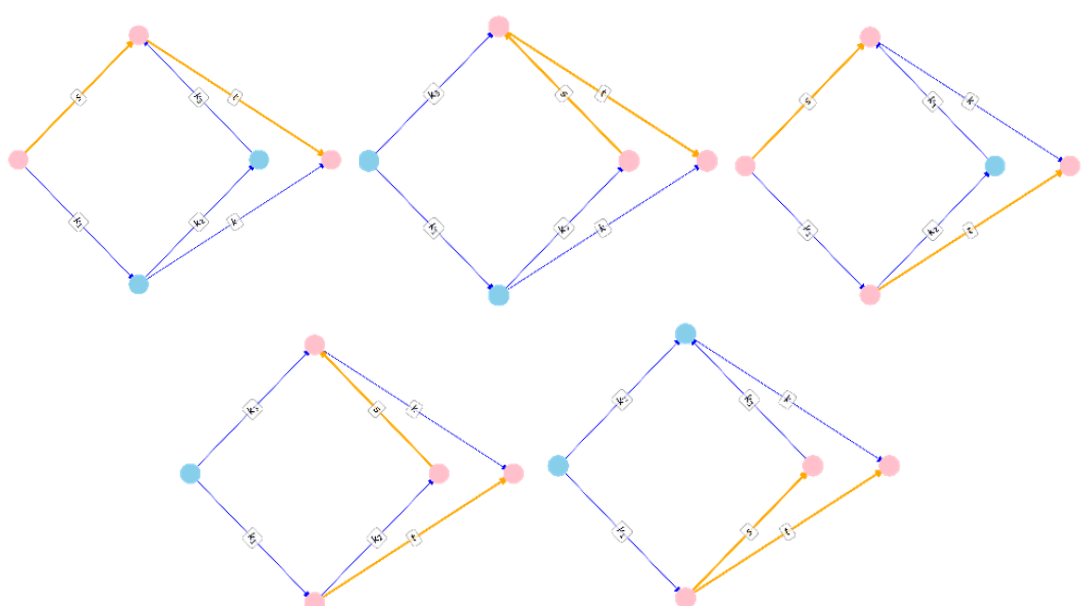

Figure 6: configurations of three chordless 4-cycles with five vertices and six edges with choices of s and t such that each one of formula (3.7) and formula (3.8) contains a broken 4-cycle monomial.

For this case, we have three chordless 4 -cycles and there are seventeen NBC bases related to them calculated by $\left(\binom{6-2}{3}-3\right)$. The number of non-zero images are $\left(\binom{6-2}{3}-1\right)+\left(\left(\binom{6-2}{2}-(3+2)\right)-3\right)$. The number of such monomials is $v_4^{s,t}$.
b.1.5. If $B=\{e_s,e_{k_1},e_{k_2}\}\in NBC^{3,s,t}_{\trianglelefteq_G}(G), 1<k_1<k_2\leq \ell$ and $k_1,k_2\neq t$, then for all $1\leq k\leq \ell$ with $k\neq k_1,k_2,s,t$ and $\{e_{k_1},e_{k_2},e_k\}$ not a broken 3-creuits;

$$d_3\left(a_{e_{k_1}}a_{e_{k_2}}a_{e_k}\right)=\pm d_3\left(a_{e_s}a_{e_{k_2}}a_{e_k}\right)\mp d_3\left(a_{e_s}a_{e_{k_1}}a_{e_k}\right)$$

The number of such monomials is ($u_4(\ell-4)-b_4$), where b_4 is the number of broken circuits that contains $B-\{e_s\}=\{e_{k_1},e_{k_2}\}$.

For this case, $\dim($ kerd $_3)=\dim($ Imd $_2)+(u_4^s+2\,(u_4-u_4^s))$ and:

$$\dim\left(Imd_3\right)=\left(\binom{\ell-2}{3}+\binom{\ell-2}{2}\right)-\left((u_4(\ell-4)-b_4)+\left(v_4^s+v_4^t-v_4^{s,t}\right)+(u_4^s+\right.$$
$$\left.2\,(u_4-u_4^s)+c_4\right)$$

b.2. The discussion of the case that $c_5(G)\neq 0$ is a mixture among all the cases that we analyzed in item $2(c(G)=5)$ and item b. $1\,(c(G)=4$ and $c_5(G)=0)$ above. Thus, $\dim($ kerd $_3)=\dim($ Imd $_2)+(u_4^s+2\,(u_4-u_4^s))$ and:

$$\dim\left(\mathrm{Imd}_3\right)=\left(\binom{\ell-2}{3}+\binom{\ell-2}{2}\right)-\left((u_4(\ell-4)-b_4)+\left(v_4^s+v_4^t-v_4^{s,t}\right)+(u_4^s+\right.$$
$$\left.2\,(u_4-u_4^s)+(c_4+u_5)\right)$$

□

Theorem 3.5 *If $a=a_{e_s}-a_{e_t}$, for $2\leq s<t\leq \ell$, then:*

1. *The Orlik-Solomon algebra $\boldsymbol{A}_*(G)$ has vanished first cohomological group, i.e., $H^1\left(\boldsymbol{A}_*(G);a\right)=0$.*
2. *The structure of $H^2\left(\boldsymbol{A}_*(G);a\right)$ depending on the value of $c(G)$, as follows:*

   i. *If G is tree or $c(G)\geq 5$, then $\boldsymbol{A}_*(G)$ has vanished $H^2\left(\boldsymbol{A}_*(G);a\right)$.*

   ii. *If $c(G)=4$, then the Orlik-Solomon algebra $\boldsymbol{A}_*(G)$ has non vanished second cohomological group with $\dim\left(H^2\left(\boldsymbol{A}_*(G);a\right)\right)=u_4^s+2\,(u_4-u_4^s)$, where u_4 be the number of chordless 4-cycles that includes e_s and e_t and u_4^s be the number of chordless 4-cycles that include each of e_s and e_t with e_s is the minimal edge via $\trianglelefteq_G$.*

Proof: This is a direct result of proposition (2.5) and theorem (2.6). □

4. Illustrations:

We will illustrate our results as follows:

4.1 Trees:

Each tree in figure (7) contains ten vertices and nine edges; also, all of them have a hypersolvable partition with an exponent vector $(1,1,\ldots,1)$. As a result of the theorem (2.7), the first and second cohomological groups of the Orlik-Solomon algebra have vanished:

Figure 7: Trees with ten vertices and nine edges.

4.2 Hypercubes:

Figure (8) illustrates hypercubes of dimensions 2, 3, and 4, each of which has a hypersolvable partition with an exponent vector of $(1, 1, \ldots, 1)$, as demonstrated in construction (2.1). The first cohomology of the Orlik-Solomon algebra has vanished because of the application of the theorem (2.7). The vanishing of the second cohomology of the Orlik-Solomon algebra is contingent upon the selection of $2 \leq t \leq \ell$, where $\ell = 4, 6, 24$ related to the dimension of the hypercube 2, 3, 4, as follows:

According to construction (2.1), hypercubes with 2, 3, and 4 dimensions each have a hypersolvable partition with an exponent vector of $(1, 1, \ldots, 1)$, as shown in Figure 8. Due to the application of theorem (2.7), the first cohomology of the Orlik-Solomon algebra has vanished. Selecting a value of $2 \leq s < t \leq \ell$, where $\ell = 4, 6$ and 24 , related to the dimension of the hypercube 2, 3, and 4 respectively, is necessary for the second cohomology of the OrlikSolomon algebra to vanish as follows:

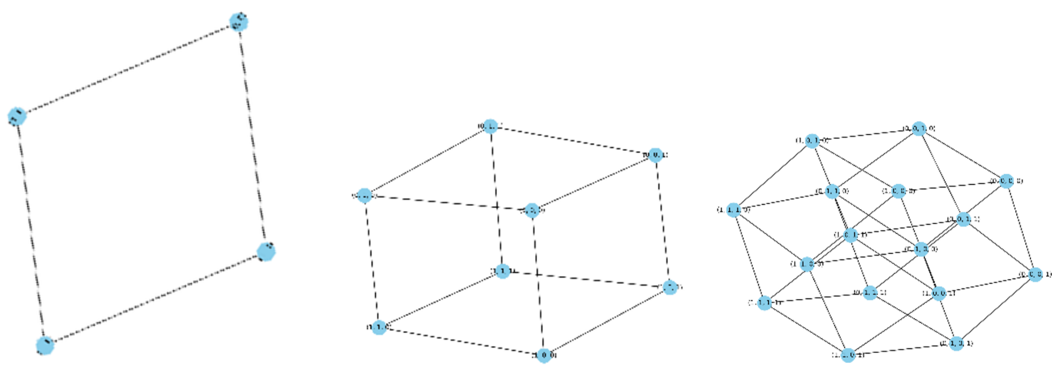

Figure 8: Hypercubes of dimension 2, 3 and 4, respectively.

1. The hypercube G of dimensions two has four edges and just one chordless 4-cycle, $\{e_1, e_2, e_3, e_4\}$ via the hypersolvable order. Thus, G is generic and $\dim\left(H^2\left(\boldsymbol{A}_*(G); a_{e_s} - a_{e_t}\right)\right) = 1$, for $2 \leq s < t \leq 4$.

2. If the dimension of the hypercube G is three, it has twelve edges and six chordless 4-cycles, say $\{e_1, e_2, e_4, e_5\}, \{e_1, e_3, e_6, e_7\}, \{e_2, e_3, e_8, e_9\}, \{e_4, e_6, e_{10}, e_{11}\}, \{e_5, e_8, e_{10}, e_{12}\}$ and $\{e_7, e_9, e_{11}, e_{12}\}$ via the hypersolvable order. For the following selecting values of s and t, ($s = 2$ and $t = 3, 4, 5, 8, 9$) ($s = 3$ and $t = 6, 7, 8, 9$) ($s = 4$ and $t = 5, 6, 10, 11$), ($s = 5$ and $t = 8, 10, 12$), ($s = 6$ and $t = 7, 10, 11$), ($s = 7$ and $t = 9, 11, 12$), ($s = 8$ and $t = 9, 10, 12$), ($s = 9$ and $t = 11, 12$), ($s = 10$ and $t = 11, 12$) and, ($s = 11$ and $t = 12$), there is just one chordless 4-cycle that can contain e_s and e_t, hence we have non-vanishing $H^2\left(\boldsymbol{A}_*(G); a_{e_s} - a_{e_t}\right)$ for these selections.

3. The four-dimensional hypercube G has thirty-two edges and twenty-four chordless 4-cycles, say $\{e_1,e_2,e_5,e_6\},\{e_1,e_3,e_7,e_8\},\{e_1,e_4,e_9,e_{10}\},\{e_2,e_3,e_{11},e_{12}\},\{e_2,e_4,e_{13},e_{14}\},\{e_3,e_4,e_{15},e_{16}\}$, $\{e_5,e_7,e_{17},e_{18}\},\{e_5,e_9,e_{20},e_{21}\},\{e_6,e_{11},e_{17},e_{19}\},\{e_6,e_{13},e_{20},e_{22}\}$, $\{e_7,e_9,e_{23},e_{24}\}$, $\{e_8,e_{12},e_{18},e_{19}\},\{e_8,e_{15},e_{23},e_{25}\},\{e_{10},e_{11},e_{21},e_{22}\},\{e_{10},e_{16},e_{24},e_{25}\}$, $\{e_{11},e_{13},e_{26},e_{27}\}$, $\{e_{12},e_{15},e_{26},e_{28}\}$, $\{e_{14},e_{16},e_{27},e_{28}\}$, $\{e_{17},e_{20},e_{29},e_{30}\}$, $\{e_{18},e_{23},e_{29},e_{31}\}$, $\{e_{19},e_{26},e_{29},e_{32}\}$, $\{e_{21},e_{24},e_{30},e_{31}\}$, $\{e_{22},e_{27},e_{30},e_{32}\}$ and $\{e_{25},e_{28},e_{31},e_{32}\}$ via the hypersolvable order $\trianglelefteq_G$. Clearly, $\dim\left(H^2\left(\boldsymbol{A}_*(G);a_{e_s}-a_{e_t}\right)\right)=1$, for specific selections of $2\leq s<t\leq 32$. For example, if $s=2$ and $t=3,4,5,6,11,12,13,14$, we have non vanished $H^2\left(\boldsymbol{A}_*(G);a_{e_s}-a_{e_t}\right)$ with dimension one and $H^2\left(\boldsymbol{A}_*(G);a_{e_s}-a_{e_t}\right)$ is vanished for the other selections of t.

4.3 Customized Graph:

Assume we have a graph G as shown in figure 9:

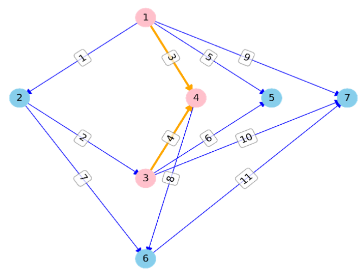

Figure 9: A free tringles graph G with seven vertices and eleven edges via the order $\trianglelefteq_G$.

G has 11 chordless 4-cycles, $\{e_1,e_2,e_3,e_4\},\{e_1,e_2,e_5,e_6\},\{e_1,e_3,e_7,e_8\},\{e_1,e_2,e_9,e_{10}\}$, $\{e_1,e_7,e_9,e_{11}\}$, $\{e_2,e_4,e_7,e_8\},\{e_2,e_7,e_{10},e_{11}\},\{e_3,e_4,e_5,e_6\},\{e_3,e_4,e_9,e_{10}\},\{e_3,e_8,e_9,e_{11}\}$, $\{e_5,e_6,e_9,e_{10}\}$ via the hypersolvable order $\trianglelefteq_G$. If $a=a_{e_3}-a_{e_4}$, then:

✓ $NBC^{3,3}_{\trianglelefteq_G}(G)=\{\{e_1,e_3,e_7\},\{e_1,e_3,e_8\},\{e_1,e_7,e_8\}\}$, and $BC^{3,3}_{\trianglelefteq_G}(G)=\{\{e_3,e_7,e_8\}\},\Big\}$ be the sets of all NBC bases and broken circuits that related to chordless 4 cycle that contains e_3 as not minimal edge via $\trianglelefteq_G$, respectively, and $v_4^3(G)=\left|BC^{3,3}_{\trianglelefteq_G}(G)\right|=1$.

✓ $NBC^{3,4}_{\trianglelefteq_G}(G)=\{\{e_2,e_4,e_7\},\{e_2,e_4,e_8\},\{e_2,e_7,e_8\}$ and $BC^{3,4}_{\trianglelefteq_G}(G)=\{\{e_4,e_7,e_8\}\}$, be the sets of all NBC bases and broken circuits that related to chordless 4-cycle that contains e_4 as not minimal edge via $\trianglelefteq_G$, respectively, and let $v_4^4(G)=\left|BC^{3,4}_{\trianglelefteq_G}(G)\right|=1$.

✓ $NBC^{3,3,4}_{\trianglelefteq_G}(G)=\{\{e_1,e_2,e_3\},\{e_1,e_2e_4\},\{e_1,e_3,e_4\},\{e_3,e_4,e_5\},\{e_3,e_4,e_6\},\{e_3,e_5,e_6\},$

$$\{e_3,e_4,e_9\},\{e_3,e_4,e_{10}\},\{e_3,e_9,e_{10}\}\}$$

and $BC^{3,3,4}_{\trianglelefteq_G}(G)=\{\{e_2,e_3,e_4\},\{e_4,e_5,e_6\},\{e_4,e_9,e_{10}\}\}$ be the set of all NBC bases and broken circuits that related to chordless 4-cycles that contains each of e_3 and e_4 via $\trianglelefteq_G$, respectively, and $v_4^{3,4}(G)=\left|BC^{3,3,4}_{\trianglelefteq_G}(G)\right|=3$.

According to theorem (2.7), G has nonvanishing second cohomology of the Orlik-Solomon algebra such that $u_4^s=2,u_4=3$ and $\dim\left(H^2\left(\boldsymbol{A}_*(G);a_{e_s}-a_{e_t}\right)\right)=u_4^s+2\left(u_4-u_4^s\right)=2+2(3-2)=4$.

5. Conclusion:

This work aims to investigate the first non-vanishing cohomological group $H^*\left(\boldsymbol{A}_*(G);a\right)$, of the Orlik-Solomon algebra $\boldsymbol{A}_*(G)$ for a free-tringles graph G such that $a=a_{e_s}-a_{e_t}$, $2\leq s<t\leq\ell$, specifically that has chordless 4 -cycles. The conclusion was based on the following results:

1. The vanishing of $H^1\left(\boldsymbol{A}_*(G);a\right)$, is attributed to the absence of triangles in G.

2. $H^2\left(\boldsymbol{A}_*(G);a\right)$ vanishined in certain cases as follows:

i. If G is tree.

ii. If G has no chordless 4 -cycles.

iii. If G has no chordless 4-cycles that includes e_s and e_t.

3. $H^2\left(\boldsymbol{A}_*(G); a\right)$ not vanished, if G has chordless 4 -cycles that includes e_s and e_t and:

$$\dim\left(H^2\left(\boldsymbol{A}_*(G); a\right)\right)=u_4^S+2\left(u_4-u_4^S\right)$$

where u_4 is the number of chordless 4-cycles that includes e_1 and e_t and u_4^s is the number of chordless 4-cycles that include each of e_s and e_t with e_s is the minimal edge via $\trianglelefteq_G$.

Our strategy generally reconstructs the Orlik-Solomon algebra by combining techniques from graph theory and arrangement theory. These findings not only improve our understanding of the underlying algebraic and combinatorial structures but also open opportunities for investigating the potential uses of these concepts in a variety of networks and associated mathematical fields.

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