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# On Three Pairs of Lucas and Fibonacci-type Combinatorial $p$-entities

R. Rangarajan, C. Goutham and P. Siva Kota Reddy

ABSTRACT: This work uses a general integer parameter $p$ to try to generalise 3-pairs of subsequences of Fibonacci numbers and Lucas numbers. In this study, we focus on constructing 3-pairs of generalized Lucas and Fibonacci-type combinatorial $p$-entities that satisfy a more general quadratic Diaphantine equation of the kind $x^2 - Ny^2 = \pm k^2$. Separate sections present a variety of Discrete Mathematical features of the combinatorial $p$-entities.

Key Words: Fibonacci sequence, Lucas sequence, generalized combinatorial entities, Discrete Mathematical properties.

## Contents



## 1. Introduction

Fibonnaci numbers $F_n = 0, 1, 1, 2, 3, 5, 8, \cdots$ and the Lucas numbers $L_n = 2, 1, 3, 4, 7, 11, 18, \cdots$ are naturally occurring sequence of positive integers reminding us again and again that the basic fact of the nature, namely what is natural mostly appear simple. Both the sequences play a central role in many problems of Combinatorial Number Theory and Discrete Mathematics [1,3,4,10]. Motivated by the simple and natural difference equations or 3-term recurrence relations satisfied by $L_n$ and $P_n$, quite often many researchers explore for more general ones. One such generalization available in the literature [10] is given below:

$$x_{n+1} = b\, x_n + c\, x_{n-1}; x_0 = 2, x_1 = b$$

$$y_{n+1} = b\, y_n + c\, y_{n-1}; y_0 = 0, y_1 = 1,$$

where $n \in 1, 2, 3 \ldots$. The solutions are expressed in the following Binet forms:

$$x_n = \left(\frac{b + \sqrt{b^2 + 4c}}{2}\right)^n + \left(\frac{b - \sqrt{b^2 + 4c}}{2}\right)^n$$

$$y_n = \frac{1}{\sqrt{b^2 + 4c}}\left[\left(\frac{b + \sqrt{b^2 + 4c}}{2}\right)^n - \left(\frac{b - \sqrt{b^2 + 4c}}{2}\right)^n\right]$$

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where $n = 1, 2, 3 \ldots.$

They satisfy the following quadratic Diophantine equation: $x_n^2 - (b^2 + 4c)\ y_n^2 = 4\ (-c)^n$. Since our main aim of the paper is to obtain a suitable generalization which satisfy $x^2 - Ny^2 = \pm k^2$, the above ones give no clue. The present paper demonstrates that subsequences of Lucas and Fibonacci numbers provide sufficient clues in this regard. Interested readers may refer to standard works on quadratic Diaphontine equations, recurrence relations and Binet forms in general context [1,3,4,5,6,11,12]. In [2], the D'Ocagne's identity for the generalized Fibonacci and Lucas sequences is established in terms of log convex identity of generalized Fibonacci and Lucas sequence by using mathematical induction. In [9], the authors constructed the sequences of Fibonacci and Lucas in any quadratic field $\mathbb{Q}(\sqrt{d})$ with $d > 0$ square free, noting that the general properties remain valid as those given by the classical sequences of Fibonacci and Lucas for the case $d = 5$, under the respective variants.

Let us consider the 3-pairs of Lucas and Fibonacci sequences:
$\{(L_{3n}, F_{3n}), (L_{3n+1}, F_{3n+1})\ and\ (L_{3n+2}, F_{3n+2})\ /\ n = 0, 1, 2 \ldots\}$. All the six sequences satisfy

$$x_{n+1} = 4\ x_n + x_{n-1}, n = 1, 2, 3, \ldots$$

with different values for $x_0$ and $x_1$. All of them have similar Binet forms of the form

$$c_1\ (2 + \sqrt{5})^n + c_2\ (2 - \sqrt{5})^n.$$

Put $\alpha = 2 + \sqrt{5}$ and $\beta = 2 - \sqrt{5}$ then one can show that

$$L_{3n} = \alpha^n + \beta^n, \quad F_{3n} = 2\ \left(\frac{\alpha^n - \beta^n}{\alpha - \beta};\right)$$

$$2L_{3n+k} = L_{3n}L_k + 5F_{3n}F_k;$$

$$2F_{3n+k} = L_{3n}F_k + F_{3n}L_k,$$

where $k = 1, 2;\quad n = 0, 1, 2 \ldots$

They satisfy

$$L_{3n+k}^2 - 5\ F_{3n+k}^2 = (-1)^{n+k}\ 4.$$

The above ideas provide a strong motivation to construct 3-pairs of Lucas and Fibonacci-type $p$-entities given by a generalized $p$-identity

$$x_{n+1}^{(p)} = 2\ (p+1)\ x_n^{(p)} + x_{n-1}^{(p)}$$

with appropriate two initial values. All of them will fit into a general Binet form

$$C_{1,p}\ \alpha_p^n + C_{2,p}\ \beta_p^n,$$

where $\alpha_p = (p+1) + \sqrt{(p+1)^2 + 1}$ and $\beta_p = (p+1) - \sqrt{(p+1)^2 + 1}$.

Note that $\alpha_1 = \alpha$ and $\beta_1 = \beta$.

We choose $C_{1,p}$ and $C_{2,p}$ in such a way that the each pair of generalized $p$-entity satisfies

$$x^2 - [(p+1)^2 + 1]\ y^2 = \pm\ (p+1)^2.$$

So, when $p = 1$, $x^2 - 5y^2 = \pm 4$ is satisfied by $(x, y) = (L_n, F_n)$. We further restrict $p$ to be a positive integer parameter such that $(p+1)^2 + 1 \neq m\ q^2$, where $m$ and $q$ are positive integers. Their Binet forms, recurrence relations, matrix power identities, Cassini-type identities, generating functions, summation identities, binomial identities and convolution identities are worked out in the coming sections.

## 2. The three pairs of Combinatorial $p$-entities

Through out this paper $p$ stands for a positive integer parameter such that $N_p = (p+1)^2 + 1$ is a square free positive integer. Put $\alpha_p = (p+1) + \sqrt{N_p}$ and $\beta_p = (p+1) - \sqrt{N_p}$. Note that $\alpha_p - \beta_p = 2\sqrt{N_p}$ and $\alpha_p \beta_p = -1$.

Definition:

$$(1)\ \lambda_{3n}^{(p)} = \frac{p+1}{2}\left(\alpha_p^n + \beta_p^n\right)$$

$$(2)\ \theta_{3n}^{(p)} = \frac{p+1}{\alpha_p - \beta_p}\left(\alpha_p^n - \beta_p^n\right)$$

$$(3)\ (p+1)\ \lambda_{3n+k}^{(p)} = \lambda_{3n}^{(p)}\lambda_k^{(p)} + N_p \theta_{3n}^{(p)}\theta_k^{(p)}$$

$$(4)\ (p+1)\ \theta_{3n+k}^{(p)} = \lambda_{3n}^{(p)}\theta_k^{(p)} + \theta_{3n}^{(p)}\lambda_k^{(p)}$$

$$\lambda_1^{(p)} = 1, \lambda_2^{(p)} = p^2 + p + 1, \theta_1^{(p)} = 1, \theta_2^{(p)} = p.$$

$$k = 1, 2; n = 0, 1, 2 \ldots.$$

Note that $[\lambda_k^{(p)}]^2 - N_p\ [\theta_k^{(p)}]^2 = (-1)^k (p+1)^2$, $k = 1, 2$. For the sake of convenience, here after words we write $\lambda_n := \lambda_n^{(p)}$, $\theta_n := \theta_n^{(p)}$ and $N := N_p$. Note that when $p = 1$, we get back $\lambda_{3n+k} = L_{3n+k}$; $\theta_{3n+k} = F_{3n+k}$, $k = 0, 1, 2$; $n = 0, 1, 2, \ldots$.

The first consequence of the definition is the following six 3-term recurrence relations:

$$(1)\ \lambda_{3n+6} = 2\ (p+1)\ \lambda_{3n+3} + \lambda_{3n} : \quad \lambda_0 = p+1,$$
$$\lambda_3 = (p+1)^2.$$

$$(2)\ \theta_{3n+6} = 2\ (p+1)\ \theta_{3n+3} + \theta_{3n} : \quad \theta_0 = 0,$$
$$\theta_3 = p+1.$$

$$(3)\ \lambda_{3n+7} = 2\ (p+1)\ \lambda_{3n+4} + \lambda_{3n+1} : \quad \lambda_1 = 1,$$
$$\lambda_4 = (p+1)^2 + (p+1) + 1.$$

$$(4)\ \theta_{3n+7} = 2\ (p+1)\ \theta_{3n+4} + \theta_{3n+1} : \quad \theta_1 = 1,$$
$$\theta_4 = p+2.$$

$$(5)\ \lambda_{3n+8} = 2\ (p+1)\ \lambda_{3n+5} + \lambda_{3n+2} : \quad \lambda_2 = p^2 + p + 1,$$
$$\lambda_5 = 2p^3 + 4p^2 + 4p + 1.$$

$$(6)\ \theta_{3n+8} = 2\ (p+1)\ \theta_{3n+5} + \theta_{3n+2} : \quad \theta_2 = p,$$

where $n = 0, 1, 2, \ldots$. $\quad \theta_5 = 2p^2 + 2p + 1,$

First three special cases are listed below:

$$
\begin{aligned}
&p = 1;\\
&\quad \lambda_n^{(1)} = L_n : \ 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199 \dots\\
&\quad \theta_n^{(1)} = F_n : \ 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \dots\\
&p = 2;\\
&\quad \lambda_n^{(2)} = 3, 1, 7, 9, 13, 41, 57, 79, 253, 351, 487, 1559 \dots\\
&\quad \theta_n^{(2)} = 0, 1, 2, 3, 4, 13, 18, 25, 80, 111, 154, 493 \dots\\
&p = 3;\\
&\quad \lambda_n^{(3)} = 4, 1, 13, 16, 21, 103, 132, 169, 837, 1072,\\
&\qquad\qquad 1373, 6799 \dots\\
&\quad \theta_n^{(3)} = 0, 1, 3, 4, 5, 25, 32, 41, 203, 260, 333, 1649 \dots
\end{aligned}
$$

**Main Result-1**

$$
\begin{aligned}
&\lambda_{3n+k}^2 - N\ \theta_{3n+k}^2 = (-1)^{n+k}\ (p+1)^2\\
&k = 0, 1, 2;\ \ n = 0, 1, 2 \dots .
\end{aligned}
$$

**Proof**
Case-1: $k = 0$

$$
\begin{aligned}
\lambda_{3n}^2 - N\ \theta_{3n}^2 &= \left(\frac{p+1}{2}\right)^2 \left[\left(\alpha_p^n + \beta_p^n\right)^2 - \left(\alpha_p^n - \beta_p^n\right)^2\right]\\
&= (-1)^n (p+1)^2.
\end{aligned}
$$

Case-2: $k = 1, 2$

$$
\begin{aligned}
&(p+1)^2\ [\lambda_k^{(p)}]^2 - N_p\ [\theta_k^{(p)}]^2\\
&\quad = (\lambda_{3n}\lambda_k + N\theta_{3n}\theta_k)^2 - N(\lambda_{3n}\theta_k + \theta_{3n}\lambda_k)^2\\
&\quad = (\lambda_{3n}^2 - N\ \theta_{3n})^2(\lambda_k^2 - N\ \theta_k)^2\\
&\quad = (-1)^{n+k}(p+1)^4. \qquad \square
\end{aligned}
$$

**Main Result-2**

$$
\begin{aligned}
&(1)\ \lambda_{3n+3} = (p+1)\ \lambda_{3n} + N\theta_{3n}\\
&(2)\ \theta_{3n+3} = \lambda_{3n} + (p+1)\ \theta_{3n}\\
&(3)\ (p+1)\ \lambda_{3n+2} = p\ \lambda_{3n+3} + \lambda_{3n}\\
&(4)\ (p+1)\ \theta_{3n+2} = p\ \theta_{3n+3} + \theta_{3n}\\
&(5)\ (p+1)\ \lambda_{3n+1} = \lambda_{3n+3} - p\ \lambda_{3n}\\
&(6)\ (p+1)\ \theta_{3n+1} = \theta_{3n+3} - p\ \theta_{3n}\\
&(7)\ (p+1)\ \lambda_{3n+4} = (p+2)\ \lambda_{3n+3} + \lambda_{3n}\\
&(8)\ (p+1)\ \theta_{3n+4} = (p+2)\ \theta_{3n+3} + \theta_{3n}\\
&(9)\ (p+1)\ \lambda_{3n-1} = \lambda_{3n+3} - (p+2)\ \lambda_{3n}\\
&(10)\ (p+1)\ \theta_{3n-1} = \theta_{3n+3} - (p+2)\ \theta_{3n},\\
&\quad \text{where } n = 0, 1, 2, \ldots
\end{aligned}
$$

The proof is by induction on $n$ with the application of three term recurrence relations.

**Main Result-3**

$$
\begin{aligned}
&(1)\ (p+1)\ \lambda_{3n+2} = p\ (p+1)\ \lambda_{3n+1} + (p^2+1)\ \lambda_{3n};\\
&\qquad \lambda_0 = p+1,\ \lambda_1 = 1.\\
&(2)\ (p^2+1)\ \lambda_{3n+3} = p\ (p+1)\ \lambda_{3n+2} + (p+1)\ \lambda_{3n+1};\\
&\qquad \lambda_1 = 1,\ \lambda_2 = p\ (p+1) + 1.\\
&(3)\ (p+1)\ \lambda_{3n+4} = 2\ \lambda_{3n+3} + (p+1)\ \lambda_{3n+2};\\
&\qquad \lambda_2 = p\ (p+1) + 1,\ \lambda_3 = (p+1)^2.\\
&(4)\ (p+1)\ \theta_{3n+2} = p\ (p+1)\ \theta_{3n+1} + (p^2+1)\ \theta_{3n};\\
&\qquad \theta_0 = 0,\ \theta_1 = 1.\\
&(5)\ (p^2+1)\ \theta_{3n+3} = p\ (p+1)\ \theta_{3n+2} + (p+1)\ \theta_{3n+1};\\
&\qquad \theta_1 = 1,\ \theta_2 = p.\\
&(6)\ (p+1)\ \theta_{3n+4} = 2\ \theta_{3n+3} + (p+1)\ \theta_{3n+2};\\
&\qquad \theta_2 = p,\ \theta_3 = p+1,\\
&\quad \text{where } n = 0, 1, 2, \ldots
\end{aligned}
$$

Again the proof of this result followed by using induction on $n$ and using three term recurrence relations. When $p = 1$, we get back the signature three term recurrence relation for $L_{3n+k}$ and $F_{3n+k}$, $k = 0, 1, 2, \ldots$; $n = 0, 1, 2, \ldots$. Hence we may regard $\{(\lambda_{3n+k}, \theta_{3n+k})\ /\ k = 0, 1, 2, \ldots; n = 0, 1, 2, \ldots\}$ as Lucas and Fibonacci-type Combinatorial $p$-entities.

## 3. Generating functions, Matrix power identities and Cassini determinant identities

### 3.1. Generating functions

$$(1)\ \sum_{n=0}^{\infty} L_{3n+k}\ x^n = \frac{L_k + (L_{k+3} - 4L_k)x}{1 - 4x - x^2}$$

$$(2)\ \sum_{n=0}^{\infty} F_{3n+k}\ x^n = \frac{F_k + (F_{k+3} - 4F_k)x}{1 - 4x - x^2}$$

$$(3)\ \sum_{n=0}^{\infty} \lambda_{3n+k}\ x^n = \frac{\lambda_k + [\lambda_{k+3} - 2(p+1)\lambda_k]x}{1 - 2(p+1)x - x^2}$$

$$(4)\ \sum_{n=0}^{\infty} \theta_{3n+k}\ x^n = \frac{\theta_k + [\theta_{k+3} - 2(p+1)\theta_k]x}{1 - 2(p+1)x - x^2},$$

where $k = 0, 1, 2.$

### 3.2. Matrix power identities

$$(1)\ \begin{bmatrix} F_{3n-3+k} & F_{3n+k} \\ F_{3n+k} & F_{3n+3+k} \end{bmatrix} = \begin{bmatrix} F_k & F_{k+3} \\ F_{k+3} & F_{k+6} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}^{n-1}$$

$$(2)\ \begin{bmatrix} L_{3n-3+k} & L_{3n+k} \\ L_{3n+k} & L_{3n+3+k} \end{bmatrix} = \begin{bmatrix} L_k & L_{k+3} \\ L_{k+3} & L_{k+6} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}^{n-1}$$

$$(3)\ \begin{bmatrix} L_{3n+k} & F_{3n+k} \\ 5F_{3n+k} & L_{3n+k} \end{bmatrix} = \begin{bmatrix} L_k & F_k \\ 5F_k & L_k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}^{n}$$

$$(4)\ \begin{bmatrix} \theta_{3n-3+k} & \theta_{3n+k} \\ \theta_{3n+k} & \theta_{3n+3+k} \end{bmatrix} = \begin{bmatrix} \theta_k & \theta_{k+3} \\ \theta_{k+3} & \theta_{k+6} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2(p+1) \end{bmatrix}^{n-1}$$

$$(5)\ \begin{bmatrix} \lambda_{3n-3+k} & \lambda_{3n+k} \\ \lambda_{3n+k} & \lambda_{3n+3+k} \end{bmatrix} = \begin{bmatrix} \lambda_k & \lambda_{k+3} \\ \lambda_{k+3} & \lambda_{k+6} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2(p+1) \end{bmatrix}^{n-1}$$

$$(6)\ \begin{bmatrix} \lambda_{3n+k} & \theta_{3n+k} \\ N\theta_{3n+k} & \lambda_{3n+k} \end{bmatrix} = \begin{bmatrix} \lambda_k & \theta_k \\ N\theta_k & \lambda_k \end{bmatrix} \begin{bmatrix} p+1 & 1 \\ N & p+1 \end{bmatrix}^{n},$$

where $k = 0, 1, 2$ and $n = 1, 2, 3, \ldots$

### 3.3. Cassini determinant identities

(1) $F_{3n-3+k}\ F_{3n+3+k} - F_{3n+k}^2 = (-1)^{n+k}\ 4$

(2) $L_{3n-3+k}\ L_{3n+3+k} - L_{3n+k}^2 = (-1)^{n-1+k}\ 20$

(3) $L_{3n+k}^2 - 5F_{3n+k}^2 = (-1)^{n+k}\ 4$

(4) $\theta_{3n-3+k}\ \theta_{3n+3+k} - \theta_{3n+k}^2 = (-1)^{n+k}\ (p+1)^2$

(5) $\lambda_{3n-3+k}\ \lambda_{3n+3+k} - \lambda_{3n+k}^2 = (-1)^{n-1+k}\ N(p+1)^2$

(6) $\lambda_{3n+k}^2 - N\theta_{3n+k}^2 = (-1)^{n+k}\ (p+1)^2,$

where $k = 0, 1, 2$ and $n = 1, 2, 3, \ldots$

The results in the above subsections can be checked by any reader using Mathematical induction and the results of the previous section.

## 4. Summation and Convolution identities

### 4.1. Summation identities

(1) $F_k + F_{3+k} + \cdots + F_{3n+k} = \dfrac{F_{3n+2+k} - F_{k-1}}{2}$

(2) $L_k + L_{3+k} + \cdots + L_{3n+k} = \dfrac{L_{3n+2+k} - L_{k-1}}{2}$

(3) $F_k^2 + F_{3+k}^2 + \cdots + F_{3n+k}^2$

$= \dfrac{F_{3n+k} + F_{3n+3+k} - F_k F_{k-3}}{4}$

(4) $L_k^2 + L_{3+k}^2 + \cdots + L_{3n+k}^2$

$= \dfrac{L_{3n+k} + L_{3n+3+k} - L_k L_{k-3}}{4},$

where $k = 0, 1, 2$ and $n = 1, 2, 3, \ldots$

### 4.2. Generalized Summation identities

(1) $\theta_k + \theta_{3+k} + \cdots + \theta_{3n+k}$

$= \dfrac{(\theta_{3n+k} + \theta_{3n+3+k}) - (\theta_k + \theta_{k-3})}{2(p+1)}$

(2) $\lambda_k + \lambda_{3+k} + \cdots + \lambda_{3n+k}$

$= \dfrac{(\lambda_{3n+k} + \lambda_{3n+3+k}) - (\lambda_k + \lambda_{k-3})}{2(p+1)}$

(3) $\theta_k^2 + \theta_{3+k}^2 + \cdots + \theta_{3n+k}^2 = \dfrac{\theta_{3n+k}\ \theta_{3n+3+k} - \theta_k\theta_{k-3}}{2(p+1)}$

(4) $\lambda_k^2 + \lambda_{3+k}^2 + \cdots + \lambda_{3n+k}^2 = \dfrac{\lambda_{3n+k}\ \lambda_{3n+3+k} - \lambda_k\lambda_{k-3}}{2(p+1)},$

where $k = 0, 1, 2$ and $n = 1, 2, 3, \ldots$

**4.3. Convolution Identities**

$$(1)\ \sum_{k=0}^{n} F_{3k+r}\ F_{3(n-k)+r}$$
$$= \frac{1}{5}\left[(n+1)L_{3n+2r} - (-1)^r F_{3n+3}\right]$$

$$(2)\ \sum_{k=0}^{n} L_{3k+r}\ L_{3(n-k)+r}$$
$$= \left[(n+1)L_{3n+2r} + (-1)^r F_{3n+3}\right]$$

$$(3)\ \sum_{k=0}^{n} L_{3k+r}\ F_{3(n-k)+r} = \sum_{k=0}^{n} F_{3k+r} L_{3(n-k)+r}$$
$$= (n+1)F_{3n+2r},$$

where $r = 0, 1, 2$ and $n = 1, 2, \dots$

**4.4. Generalized Convolution identities**

$$(1)\ \sum_{r=0}^{n} \theta_{3r+k}\ \theta_{3(n-r)+k}$$
$$= \left(\frac{p+1}{2N}\right)\left[(n+1)\left[\frac{c_k p^2+1}{c_k p+1}\right]\lambda_{3n+2k} - (-1)^k \theta_{3n+3}\right]$$
$$+ (-1)^k c_k \frac{(n+1)}{2N(p+1)}(p^3+p-2)\lambda_{3n+3}$$

$$(2)\ \sum_{r=0}^{n} \lambda_{3r+k}\ \lambda_{3(n-r)+k}$$
$$= \left(\frac{p+1}{2}\right)\left[(n+1)\left[\frac{c_k p^2+1}{c_k p+1}\right]\lambda_{3n+2k} + (-1)^k \theta_{3n+3}\right]$$
$$+ (-1)^k c_k \frac{(n+1)}{2(p+1)}(p^3+p-2)\lambda_{3n+3}$$

$$(3)\ \sum_{r=0}^{n} \theta_{3r+k}\ \lambda_{3(n-r)+k} = \sum_{r=0}^{n} \lambda_{3r+k}\ \theta_{3(n-r)+k}$$
$$= \left(\frac{p+1}{2}\right)(n+1)\left[\frac{c_k p^2+1}{c_k p+1}\right]\theta_{3n+2k}$$
$$+ (-1)^k c_k \frac{(n+1)}{2(p+1)}(p^3+p-2)\theta_{3n+3},$$

where $c_k = \dfrac{k(3-k)}{2}$ and $k = 0, 1, 2$.

The results in the above subsections can be checked by any reader using definitions and the results of the section-2.

## 5. Conclusion

Fibonacci numbers have many generalizations such as Tribonacci numbers, Multinacci numbers and many more. Keeping with specific purpose in aim, one can explore more general combinatorial entities corresponding to each such generalizations. Such general combinatorial entities find applications in tiling problems.

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*R. Rangaraja,*
*Department of Studies in Mathematics*
*University of Mysore, Mysuru-570 006, India.*
*E-mail address:* rajra63@gmail.com

*and*

*C. Goutham,*
*Department of Studies in Mathematics*
*University of Mysore, Mysuru-570 006, India.*
*E-mail address:* gautamc600@gmail.com

*and*

*Siva Kota Reddy (Corresponding author),*
*Department of Mathematics*
*JSS Science and Technology University*
*Mysuru-570 006, India.*

*and*

*Universidad Bernardo O'Higgins*
*Facultad de Ingeniería, Ciencia y Tecnología*
*Departamento de Formación y Desarrollo Científico en Ingeniería*
*Av. Viel 1497, Santiago, Chile.*
*E-mail address:* pskreddy@jssstuniv.in