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Solvability of a Class of Coupled Implicit Fractional Langevin Differential Equations by the Laplace transform

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ABSTRACT: This article is devoted to establishing the existence and unicity of the coupled implicit fractional Langevin differential equation with initial conditions, using Laplace transform method. At first, we obtained the existence and uniqueness results using the Banach contraction principle. Second, the existence result was obtained by applying Schaefer's fixed point theorem. To support our main results, we present an example.

Key Words: Caputo fractional differential equation, coupled implicit fractional Langevin systems, Laplace transform method, Banach contraction principle, Schaefer's fixed point theorem.

Contents

1. Introduction

The origins of fractional calculus trace back to the challenge of extending the concept of a derivative to a non-integer approximation, a question first raised by L'Hopital on September 30th, 1695.

Fractional calculus finds application in various engineering and research fields, including electromagnetics, viscoelasticity, fluid mechanics, electrochemistry, biological population models, optics, and signal processing [28,29,30,31]. Further insights into the classical and partial differential equations describing these real phenomena can be found in references [4,5,6,8,9,10,11,16,17,18]. It is adept at simulating technical and physical processes best represented by fractional differential equations. Models involving fractional derivatives are valuable for accurately representing damping in systems. Numerous analytical and numerical techniques have been developed in recent years across various disciplines, showcasing their applicability to new problems. These contributions to science and engineering are firmly rooted in mathematics.

The Langevin equation proposed by Paul Langevin in 1908 is among the finest illustrations of these fractional differential equations. Its objective is to provide descriptions of certain occurrences that professionals in medicine, engineering, economics, and other fields may utilize. Brownian motion, or the random movement of particles suspended in a liquid, was initially described using the Langevin equation. Brownian motion and stochastic differential equations are widely utilized in all sectors and are often used instruments in all scientific domains. [33,34,35,36,37,38].

Solving fractional differential equations can be difficult because fractional derivatives have non-local and non-standard characteristics. Numerous approaches, including numerical techniques, have been devised to solve these equations between them: Laplace transform. However, there is limited discussion on its validity when applied to such systems. This paper aims to justify the validity of applying the Laplace transform to fractional systems. It will demonstrate that the Laplace transform can be used for fractional systems under specific conditions. Various methods offer alternative solutions to fractional-order differential equations, distinct from the Laplace transform. These solutions are found to be of

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exponential order, which is a requirement for the application of the Laplace transform. Ultimately, the feasibility of the Laplace transform in solving fractional equations is demonstrated. [7,12,13,14,15].

Several researchers have used Shaefer's fixed point theorem to demonstrate the existence of solutions to a large number of fractional differential equations. Schaefer's fixed point theorem is a crucial tool since it can manage non-linearities and offer a structure for proving the existence of solutions in specific scenarios. [19,22,23] and [25,26,27].

Wang et al. explored the HU stability of the following linear fractional differential equation in [24] using Laplace transformation.

$$^{C}\mathfrak{D}^{\alpha}u(\sigma) - \xi^{C}\mathfrak{D}^{\beta}u(\sigma) = \phi(\sigma).$$

where $n-1<\alpha<n$, $m-1<\beta<m$ and $m<n$, $n,m \in \mathbb{R}$.

G. Jothilakshmi1 and B. Sundara Vadivoo1 in [21] studied the following the fractional linear delay system without impulse with the help of the Laplace transform method

$$\begin{cases} ^{C}\mathfrak{D}^{\varsigma}(^{C}\mathfrak{D}^{\varrho} - \mathcal{A})\mathcal{Y}(t) = \varphi(t) + \mathcal{C}\mathcal{U}(t), t \in [0,t], \\ \mathcal{Y}(0) = \mathcal{Y}_0, \quad ^{C}\mathfrak{D}^{\varrho}\mathcal{Y}(\iota)|_{t=0} = p_0, \end{cases}$$

with $\varsigma, \varrho \in (0,1]$ and $\varsigma + \varrho > 1$, $^{C}\mathfrak{D}^{\varsigma}, {}^{C}\mathfrak{D}^{\varrho}$ are the Caputo fractional derivative of order ς, ϱ respectively, $\mathcal{A}$ is a $n \times n$ real matrix, $\mathcal{C}$ is a $n \times m$ with $m<n$ real matrix, $\mathcal{Y}(t)$ is a state vector in $\mathbb{R}^n$ and $\mathcal{U}(t)$ is a control vector in $C_p([0,T],\mathbb{R}^n)$ which is a set of all piecewise continuous function from $[0,T] \to \mathbb{R}^n$

Motivated by these results, we will analyze the following coupled fractional differential equations using the Laplace transform method.

$$\begin{cases} ^{C}\mathfrak{D}^{p_1}(^{C}\mathfrak{D}^{q_1} - \xi_1)\chi(\iota) = \varpi(\iota, {}^{C}\mathfrak{D}^{\delta_1}\chi(\iota), {}^{C}\mathfrak{D}^{\delta_2}\zeta(\iota)), \iota \in \Lambda, \\ ^{C}\mathfrak{D}^{p_2}(^{C}\mathfrak{D}^{q_2} - \xi_2)\zeta(\iota) = \omega(\iota, {}^{C}\mathfrak{D}^{\delta_3}\chi(\iota), {}^{C}\mathfrak{D}^{\delta_4}\zeta(\iota)), \iota \in \Lambda, \\ \chi(0) = \chi_0, \quad ^{C}\mathfrak{D}^{q_1}\chi(\iota)|_{t=0} = \alpha_0, \\ \zeta(0) = \zeta_0, \quad ^{C}\mathfrak{D}^{q_2}\zeta(\iota)|_{t=0} = \beta_0, \end{cases} \tag{1.1}$$

where $\xi_1, \xi_2 \in \mathbb{R}$, $q_1, q_2, p_1, p_2 \in (0,1]$, and $p_i + q_i > 1$; for $i=1,2$. $0 < \delta_1, \delta_2, \delta_3, \delta_4 \leq 1$. The fractional derivative $\mathfrak{D}^q$ is the Caputo fractional derivative of order $q \in (0,1)$, χ_0, ζ_0, α_0, $\beta_0 \in \mathcal{X}$, and the real functions $\varpi : \Lambda \times \mathbb{R}^2 \to \mathbb{R}$ and $\omega : \Lambda \times \mathbb{R}^2 \to \mathbb{R}$ $\forall \iota \in [0,1] = \Lambda$,. and p_1, p_2, q_1, q_2 are the order of Caputo fractional derivative.

We can handle initial conditions in our system (1.1) with the aid of the Laplace transform because, upon applying the Laplace transform to a differential equation containing initial conditions, these conditions are incorporated into the transformed algebraic equation, facilitating the solution of the transformed function.

This paper is organized as follows: In Section 2, we introduce several lemmas and fundamental definitions for fractional calculus and derive the solution representation of the Langevin fractional differential equation using Laplace transform. To demonstrate the key results, we utilize the generalized Schaefer's and contraction mapping principles in Section 3. Finally in Section 4, an example is given to demonstrate the validity of the theoretical results.

2. Preliminaries

This section will introduce essential concepts and definitions that form the foundation for the main results discussed in the following sections. We proceed by setting $\Lambda = [0,1]$, also let $\mathcal{X} = C(\Lambda, \mathbb{R})$ a Banach space endowed with the norm $\|.\| = \sup\{|x(\iota)|, \iota \in \Lambda\}$. Then the product space $(\mathcal{C} = \mathcal{X} \times \mathcal{X}, \|(\chi,\zeta)\|_{\mathcal{X}}$ is also a Banach space equipped with the norm $\|(\chi,\zeta)\| = \|\chi\| + \|\zeta\|$ and $\|(\chi,\zeta)\| = \max\{\|\chi\|, \|\zeta\|\}$

Lemma 2.1 *Let $\mathcal{X}$ be a Banach space, $\mathcal{C} \subset \mathcal{X}$ be closed, and $\mathbb{K} : \mathcal{C} \to \mathcal{C}$ is a strict contraction,*

$$|\mathbb{K}\chi - \mathbb{K}\zeta| \leq \ell|\chi - \zeta|,$$

for some $\ell \in (0,1)$ and all $\chi, \zeta \in \mathcal{C}$. Then $\mathbb{K}$ has a unique fixed point in $\mathcal{C}$.

Lemma 2.2 (Arzela-Ascoli Theorem) *[32] A subset $\mathbb{K}$ in $C([a,b],R)$ is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 2.3 (Schaefer's Fixed Point Theorem) *[27] Let $\mathbb{K}:\mathcal{C}\to\mathcal{C}$ be a completely continuous (c.c) operator in the Banach space $\mathcal{C}$, and let the set $\phi=\{u\in\mathcal{C}|u=\mu\mathbb{K}u, 0<\mu<1\}$ be bounded. Then $\mathbb{K}$ has a fixed point in $\mathcal{C}$.*

Definition 2.1 [2,3] The fractional integral of order $q>0$ with the lower limit 0 for a function χ is defined for $\iota>0$, $q>0$ as

$$\mathfrak{I}^q\varpi(\iota)=\frac{1}{\Gamma(q)}\int_0^\iota(\iota-\kappa)^{q-1}\varpi(\kappa)d\kappa. \tag{2.1}$$

Definition 2.2 [2,3] The Caputo derivative of order $q>0$ with the lower limit 0 for a function χ is defined for $\iota>0$, $0\le n-1<q<n$ as

$${}^C\mathfrak{D}^q\varpi(\iota)=\frac{1}{\Gamma(n-q)}\int_0^\iota(\iota-\kappa)^{n-q-1}\varpi_{[n]}(\kappa)d\kappa, \tag{2.2}$$

A constant's Caputo derivative is equal to zero. The integrals that occur in Definitions 2.1 and 2.2 are taken in Bochner's meaning if χ is an abstract function with values in $\mathcal{X}$.

Lemma 2.4 *[2,3] Let $q>0$, and $\chi(0)\in AC^n[0,\infty)$ or $C^n[0,\infty)$. Then*

$$(\mathfrak{I}^C\mathfrak{D}^q\varpi)(\iota)=\varpi(\iota)-\sum_{k=0}^{n-1}\frac{\varpi^k(0)}{k!}\iota^k,\quad \iota>0,\quad n-1<q<n,$$

A function $\chi\in\mathcal{X}$ is said to have exponential order if there exist constants $a,b\in\mathbb{R}$ such that $|\chi(\iota)|\le ae^{b\iota}$ for all $\iota>0$. For every function χ that has exponential order, we can define the Laplace transform of $\chi(\iota)$.

$$\mathcal{L}\{\chi(\iota)\}(\varsigma):=\int_0^\infty e^{-\varsigma\iota}\chi(\varsigma)d\varsigma,\ \varsigma\in\mathbb{C}. \tag{2.3}$$

If the integral represented by (2.3) is convergent at $\varsigma_0\in\mathbb{C}$, then it converges absolutely for $\varsigma\in\mathbb{C}$ such that $R(\varsigma)>R(\varsigma_0)$. The convolution property of the Laplace transform can be expressed as

$$\mathcal{L}\{\chi(\iota)^\star\zeta(\iota)\}(\varsigma)=\mathcal{L}\{\chi(\iota)\}\mathcal{L}\{\zeta(\iota)\}(\varsigma), \tag{2.4}$$

where $\chi(\iota)^\star\zeta(\iota)=\int_0^\iota\chi(\iota-\kappa)\zeta(\kappa)d\kappa,$

Lemma 2.5 *[2] Let $p_1>0$, and $\alpha\in\mathbb{Z}^+$ be a positive integer such that $\alpha-1<p_1\le\alpha$. Suppose that χ and its derivatives up to order α have exponential order functions, the Laplace transforms of $\chi(\iota)$ and ${}^C\mathfrak{D}^\alpha\chi(\iota)$ exist, and $\lim_{\iota\to\infty}\mathfrak{D}^k\chi(\iota)=0$ for $k=0,...,\alpha-1$. Then the following relation holds:*

$$\mathcal{L}\{{}^C\mathfrak{D}^{p_1}\chi(\iota)\}(\varsigma)=\varsigma^{p_1}\mathcal{L}\{\chi(\iota)\}(\varsigma)-\sum_{k=0}^{\alpha-1}\varsigma^{p_1-k-1}(\mathfrak{D}^k\chi)(0). \tag{2.5}$$

If $\alpha=1$ then

$$\mathcal{L}\{{}^C\mathfrak{D}^{p_1}\chi(\iota)\}(\varsigma)=\varsigma^{p_1}\mathcal{L}\{\chi(\iota)\}(\varsigma)-\varsigma^{p_1-1}\chi(0).$$

Definition 2.3 [1] The Mittag–Leffler function E_{p_1,q_1} is defined as:

$$E_{p_1,q_1}(z)=\sum_{k=0}^\infty\frac{z^k}{\Gamma(p_1k+q_1)},\ z,q_1\in\mathbb{C},\ R(p_1)>0, \tag{2.6}$$

when $p_1=q_1=1$, we can see that $E_{1,1}(z)=e^z$.

Lemma 2.6 *[2] If $R(\varsigma) > 0$, $\xi_1 \in \mathbb{C}$, $|\xi_1 \varsigma^{-p_1}| < 1$, then*

$$\mathcal{L}\{\iota^{q_1-1}E_{p_1,q_1}(\xi_1\iota^{p_1})\}(\varsigma) = \frac{\varsigma^{p_1-q_1}}{\varsigma^{p_1}-\xi_1},$$

where $E_{p_1,q_1}(\xi_1\iota^{p_1}\}$ is the Mittag–Leffler function.

The next lemma focuses on exploring the initial version of the problem outlined previously. (1.1).

Lemma 2.7 *[21] Let ϖ be a continuous and nonlinear function. The solution of*

$$\begin{cases} {}^C\mathfrak{D}^p({}^C\mathfrak{D}^q - \xi)\chi(\iota) = \varpi(\iota), \iota \in \Lambda, \\ \chi(0) = \chi_0, \quad {}^C\mathfrak{D}^q\chi(\iota)|_{t=0} = \alpha_0, \end{cases} \tag{2.7}$$

is given as:

$$\chi(\iota) = \chi_0 + \int_0^\iota (\iota-\varsigma)^{q-1}E_{q,q}(\xi(\iota-\varsigma)^q)\alpha_0 d\varsigma + \int_0^\iota (\iota-\varsigma)^{p+q-1}E_{q,p}(\xi(\iota-\varsigma)^q)\varpi(\varsigma)d\varsigma, \tag{2.8}$$

where $\xi \in \mathbb{R}$, $p, q \in (0,1]$ and $p+q > 1$.

Proof. Applying the Laplace transform to Equation (2.7) and utilizing Lemma 2.5, we derive

$$\begin{aligned}\mathcal{L}\{\varpi(\iota)\}(\varsigma) &= \mathcal{L}\{{}^C\mathfrak{D}^p({}^C\mathfrak{D}^q - \xi)\chi(\iota)\}(\varsigma), \\ &= \varsigma^p(\varsigma^q\mathcal{L}\{\chi(\iota)\}(\varsigma) - \varsigma^{q-1}\chi_0 - \xi\mathcal{L}\{\chi(\iota)\}(\varsigma)) - \varsigma^{p-1}(\alpha_0 - \xi\chi_0),\end{aligned}$$

then

$$\begin{aligned}(\varsigma^q - \xi)\mathcal{L}\{\chi(\iota)\}(\varsigma) &= \varsigma^{q-1}\chi_0 + \varsigma^{-1}\alpha_0 - \varsigma^{-1}\xi\chi_0 + \varsigma^{-p}\mathcal{L}\{\varpi(\iota)\}(\varsigma), \\ &= \varsigma^q\frac{\chi_0}{\varsigma} + \frac{\alpha_0}{\varsigma} - \xi\frac{\chi_0}{\varsigma} + \varsigma^{-p}\mathcal{L}\{\varpi(\iota)\}(\varsigma), \\ &= \varsigma^q\mathcal{L}\{\chi_0\} + \mathcal{L}\{\alpha_0\} - \xi\mathcal{L}\{\chi_0\} + \varsigma^{-p}\mathcal{L}\{\varpi(\iota)\}(\varsigma),\end{aligned}$$

which means that

$$\mathcal{L}\{\chi(\iota)\}(\varsigma) = \frac{(\varsigma^q - \xi)\mathcal{L}\{\chi_0\} + \mathcal{L}\{\alpha_0\} + \varsigma^{-p}\mathcal{L}\{\varpi(\iota)\}(\varsigma)}{(\varsigma^q - \xi)}. \tag{2.9}$$

Now, by applying the inverse Laplace transform to equation (2.9), we can derive

$$\begin{aligned}\chi(\iota) &= \mathcal{L}^{-1}\left\{\frac{(\varsigma^q - \xi)\mathcal{L}\{\chi_0\} + \mathcal{L}\{\alpha_0\} + \varsigma^{-p}\mathcal{L}\{\varpi(\iota)\}(\varsigma)}{(\varsigma^q - \xi)}\right\}, \\ &= \chi_0 + \mathcal{L}^{-1}\left\{\frac{1}{(\varsigma^q - \xi)}\right\} \star \alpha_0 + \mathcal{L}^{-1}\left\{\frac{\varsigma^{-p}}{(\varsigma^q - \xi)}\right\} \star \varpi(\iota).\end{aligned} \tag{2.10}$$

Lemma 2.6 helps to find some inverse Laplace transforms.

$$\mathcal{L}^{-1}\left\{\frac{\varsigma^{-p}}{(\varsigma^q - \xi)}\right\} = \iota^{q+p-1}E_{q,q+p}(\xi\iota^q). \tag{2.11}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(\varsigma^q - \xi)}\right\} = \iota^{q-1}E_{q,q}(\xi\iota^q). \tag{2.12}$$

Substituting (2.11) and (2.12) in (2.10), we get

$$\chi(\iota) = \chi_0 + \int_0^\iota (\iota-\varsigma)^{q-1}E_{q,q}(\xi(\iota-\varsigma)^q)\alpha_0 d\varsigma + \int_0^\iota (\iota-\varsigma)^{q+p-1}E_{q,q+p}(\xi(\iota-\varsigma)^q)\varpi(\varsigma)d\varsigma,$$

3. Main results

From Lemma 2.7, we have shown that the pair $(\chi, \zeta) \in \mathcal{C}$ is the solution of system (1.1), where

$$\chi(\iota) = \chi_0 + \int_0^\iota (\iota - \varsigma)^{q_1-1} E_{q_1,q_1}(\xi_1(\iota-\varsigma)^{q_1})\alpha_0 d\varsigma + \int_0^\iota (\iota-\varsigma)^{q_1+p_1-1} E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\varpi(\varsigma, {}^C\mathfrak{D}^{\delta_1}\chi(\varsigma), {}^C\mathfrak{D}^{\delta_2}\zeta(\varsigma))d\varsigma, \tag{3.1}$$

and

$$\zeta(\iota) = \zeta_0 + \int_0^\iota (\iota - \varsigma)^{q_2-1} E_{q_2,q_2}(\xi_2(\iota-\varsigma)^{q_2})\beta_0 d\varsigma + \int_0^\iota (\iota-\varsigma)^{q_2+p_2-1} E_{q_2,q_2+p_2}(\xi_2(\iota-\varsigma)^{q_2})\omega(\varsigma, {}^C\mathfrak{D}^{\delta_3}\chi(\varsigma), {}^C\mathfrak{D}^{\delta_4}\zeta(\varsigma))d\varsigma, \tag{3.2}$$

In view of Lemma 2.7, We define the operator $\Omega : \mathcal{C} \to \mathcal{C}$ associated with system (1.1) as

$$\Omega(\chi, \zeta)(\iota) = (\Omega_1(\chi, \zeta)(\iota), \Omega_2(\chi, \zeta)(\iota)) \tag{3.3}$$

with

$$\Omega_1(\chi, \zeta)(\iota) = \chi_0 + \int_0^\iota (\iota - \varsigma)^{q_1-1} E_{q_1,q_1}(\xi_1(\iota-\varsigma)^{q_1})\alpha_0 d\varsigma + \int_0^\iota (\iota-\varsigma)^{q_1+p_1-1} E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\varpi(\varsigma, {}^C\mathfrak{D}^{\delta_1}\chi(\varsigma), {}^C\mathfrak{D}^{\delta_2}\zeta(\varsigma))d\varsigma, \tag{3.4}$$

and

$$\Omega_2(\chi, \zeta)(\iota) = \zeta_0 + \int_0^\iota (\iota - \varsigma)^{q_2-1} E_{q_2,q_2}(\xi_2(\iota-\varsigma)^{q_2})\beta_0 d\varsigma + \int_0^\iota (\iota-\varsigma)^{q_2+p_2-1} E_{q_2,q_2+p_2}(\xi_2(\iota-\varsigma)^{q_2})\omega(\varsigma, {}^C\mathfrak{D}^{\delta_3}\chi(\varsigma), {}^C\mathfrak{D}^{\delta_4}\zeta(\varsigma))d\varsigma, \tag{3.5}$$

First, we will proceed by creating a series of carefully constructed hypotheses to form the basis for demonstrating the main findings of our research. This step is vital for establishing the groundwork for the subsequent analysis and empirical validation of the core assertions of our study.

($\mathcal{H}_1$) Let $\varpi, \omega : \Lambda \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, then there exist constants L_{ϖ_1}, $L_{\omega_1} > 0$ and $0 < L_{\varpi_2}$, $L_{\omega_2} < 1$ such that $\iota \in \Lambda$ and $\forall \chi_1, \chi_2, \zeta_1, \zeta_2 \in \mathbb{R}$, the following holds

$$|\varpi(\iota, \chi_1, \zeta_1) - \varpi(\iota, \chi_2, \zeta_2)| \leq L_{\varpi_1}|\chi_1 - \chi_2| + L_{\varpi_2}|\zeta_1 - \zeta_2|,$$

and

$$|\omega(\iota, \chi_1, \zeta_1) - \omega(\iota, \chi_2, \zeta_2)| \leq L_{\omega_1}|\chi_1 - \chi_2| + L_{\omega_2}|\zeta_1 - \zeta_2|.$$

($\mathcal{H}_2$) With the given continuous functions k_{ϖ_1}, k_{ϖ_2} ,k_{ϖ_3} ,k_{ω_1} ,k_{ω_2} ,$k_{\omega_3} \in \mathcal{X}$ for $\chi, \zeta \in \mathbb{R}$, the nonlocal functions ϖ, ω satisfy the following growth conditions:

$$|\varpi(\iota, \chi, \zeta)| \leq k_{\varpi_1}(\iota) + k_{\varpi_2}(\iota)|\chi| + k_{\varpi_3}(\iota)|\zeta|$$

and

$$|\omega(\iota, \chi, \zeta)| \leq k_{\omega_1}(\iota) + k_{\omega_2}(\iota)|\chi| + k_{\omega_3}(\iota)|\zeta|,$$

with

$$k^*_{\varpi_1} = \sup_{\iota\in\Lambda} k_{\varpi_1}(\iota);\ k^*_{\varpi_2} = \sup_{\iota\in\Lambda} k_{\varpi_2}(\iota);\ k^*_{\varpi_3} = \sup_{\iota\in\Lambda} k_{\varpi_3}(\iota);\ k^*_{\omega_1} = \sup_{\iota\in\Lambda} k_{\omega_1}(\iota);$$
$$k^*_{\omega_2} = \sup_{\iota\in\Lambda} k_{\omega_2}(\iota);\ k^*_{\omega_3} = \sup_{\iota\in\Lambda} k_{\omega_3}(\iota);\ \textit{are positive constants.}$$

To improve the computational process efficiency, we introduce the following notation, simplifying complex calculations and facilitating streamlined analysis:

$$\mathfrak{W}_1 = \sup_{\iota\in\Lambda}\sum_{k=0}^{\infty}\frac{\xi_1^k \iota^{kq_1+q_1+p_1-\delta_1}}{\Gamma(kq_1+q_1+p_1)}B(2-\delta_1, kq_1+q_1+p_1);$$

$$\mathfrak{W}_2 = \sup_{\iota\in\Lambda}\sum_{k=0}^{\infty}\frac{\xi_1^k \iota^{kq_1+q_1+p_1-\delta_2}}{\Gamma(kq_1+q_1+p_1)}B(2-\delta_2, kq_1+q_1+p_1);$$

$$\mathfrak{W}_3 = \sup_{\iota\in\Lambda}\sum_{k=0}^{\infty}\frac{\xi_1^k \iota^{kq_1+q_1+p_1-\delta_3}}{\Gamma(kq_1+q_1+p_1)}B(2-\delta_3, kq_1+q_1+p_1);$$

$$\mathfrak{W}_4 = \sup_{\iota\in\Lambda}\sum_{k=0}^{\infty}\frac{\xi_1^k \iota^{kq_1+q_1+p_1-\delta_4}}{\Gamma(kq_1+q_1+p_1)}B(2-\delta_4, kq_1+q_1+p_1).$$

In the following theorem, we will present our initial finding, which leverages Banach's fixed point theorem, supporting a sole solution to equation (1.1).

Theorem 3.1 *Assuming conditions ($\mathcal{H}_1$) hold, there exists a unique solution to the coupled problem (1.1) if and only if*

$$\frac{L_{\varpi_1}\mathfrak{W}_1}{\Gamma(2-\delta_1)} + \frac{L_{\varpi_2}\mathfrak{W}_2}{\Gamma(2-\delta_2)} + \frac{L_{\omega_3}\mathfrak{W}_3}{\Gamma(2-\delta_3)} + \frac{L_{\omega_4}\mathfrak{W}_4}{\Gamma(2-\delta_4)} < 1. \tag{3.6}$$

Proof. Let $\chi, \overline{\chi}, \zeta, \overline{\zeta} \in \mathcal{C}$, by using the condition $\mathcal{H}_1$:

$$\begin{aligned}
&\left\|\Omega_1(\chi,\zeta) - \Omega_1(\overline{\chi},\overline{\zeta})\right\|_{\mathcal{X}} \\
&\le \left\|\int_0^{\iota}(\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})[\varpi(\varsigma, {}^{C}\mathfrak{D}^{\delta_1}\chi(\varsigma), {}^{C}\mathfrak{D}^{\delta_2}\zeta(\varsigma)) - \varpi(\varsigma, {}^{C}\mathfrak{D}^{\delta_1}\overline{\chi}(\varsigma), {}^{C}\mathfrak{D}^{\delta_2}\overline{\zeta}(\varsigma))]d\varsigma\right\|_{\mathcal{X}} \\
&\le \sup_{\iota\in\Lambda}\left|\int_0^{\iota}(\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\right||\varpi(\varsigma, {}^{C}\mathfrak{D}^{\delta_1}\chi(\varsigma), {}^{C}\mathfrak{D}^{\delta_2}\zeta(\varsigma)) - \varpi(\varsigma, {}^{C}\mathfrak{D}^{\delta_1}\overline{\chi}(\varsigma), {}^{C}\mathfrak{D}^{\delta_2}\overline{\zeta}(\varsigma))|d\varsigma \\
&\le \sup_{\iota\in\Lambda}\left|\int_0^{\iota}(\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\right|L_{\varpi_1}|{}^{C}\mathfrak{D}^{\delta_1}\chi - {}^{C}\mathfrak{D}^{\delta_1}\overline{\chi}|d\varsigma \\
&+ \sup_{\iota\in\Lambda}\left|\int_0^{\iota}(\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\right|L_{\varpi_2}|{}^{C}\mathfrak{D}^{\delta_2}\zeta - {}^{C}\mathfrak{D}^{\delta_2}\overline{\zeta}|d\varsigma \\
&\le \frac{L_{\varpi_1}}{\Gamma(2-\delta_1)}\sup_{\iota\in\Lambda}\int_0^{\iota}\varsigma^{1-\delta_1}\left|(\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\right|d\varsigma|\chi-\overline{\chi}| \\
&+ \frac{L_{\varpi_2}}{\Gamma(2-\delta_2)}\sup_{\iota\in\Lambda}\int_0^{\iota}\varsigma^{1-\delta_2}\left|(\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\right|d\varsigma|\zeta-\overline{\zeta}| \\
&\le \frac{L_{\varpi_1}\mathfrak{W}_1}{\Gamma(2-\delta_1)}\left\|\chi-\overline{\chi}\right\|_{\mathcal{X}} + \frac{L_{\varpi_2}\mathfrak{W}_2}{\Gamma(2-\delta_2)}\left\|\zeta-\overline{\zeta}\right\|_{\mathcal{X}} \\
&\le \left(\frac{L_{\varpi_1}\mathfrak{W}_1}{\Gamma(2-\delta_1)} + \frac{L_{\varpi_2}\mathfrak{W}_2}{\Gamma(2-\delta_2)}\right)\left\|(\chi,\zeta)-(\overline{\chi},\overline{\zeta})\right\|_{\mathcal{X}}.
\end{aligned}$$

We managed to produce an identical outcome using the very same approach.

$$\left\|\Omega_2(\chi,\zeta) - \Omega_2(\overline{\chi},\overline{\zeta})\right\|_{\mathcal{X}} \le \left(\frac{L_{\omega_1}\mathfrak{W}_3}{\Gamma(2-\delta_3)} + \frac{L_{\omega_2}\mathfrak{W}_4}{\Gamma(2-\delta_4)}\right)\left\|(\chi,\zeta)-(\overline{\chi},\overline{\zeta})\right\|_{\mathcal{X}}.$$

Consequently,

$$\begin{aligned}\left\|\Omega(\chi,\zeta)-\Omega(\overline{\chi},\overline{\zeta})\right\|_{\mathcal{X}} &\leq \left\|\Omega_1(\chi,\zeta)-\Omega_1(\overline{\chi},\overline{\zeta})\right\|_{\mathcal{X}}+\left\|\Omega_2(\chi,\zeta)-\Omega_2(\overline{\chi},\overline{\zeta})\right\|_{\mathcal{X}}\\ &\leq \left(\frac{L_{\varpi_1}\mathfrak{W}_1}{\Gamma(2-\delta_1)}+\frac{L_{\varpi_2}\mathfrak{W}_2}{\Gamma(2-\delta_2)}+\frac{L_{\omega_1}\mathfrak{W}_3}{\Gamma(2-\delta_3)}+\frac{L_{\omega_2}\mathfrak{W}_4}{\Gamma(2-\delta_4)}\right)\\ &\quad\times\left\|(\chi,\zeta)-(\overline{\chi},\overline{\zeta})\right\|,\end{aligned}$$

Assuming condition (3.6), it can be deduced that the operator Ω exhibits contraction characteristics. Consequently, by applying Banach's contraction mapping theorem, we can confirm the existence of a unique fixed point for Ω. This result unequivocally validates that system (1.1) has a unique solution.

Next, we'll delve into the outcomes of (1.1) and put Schaefer's fixed point theorem into action cited in [27].

Theorem 3.2 *Suppose ($\mathcal{H}_1$) and ($\mathcal{H}_2$) hold. Then the Coupled Implicit Fractional Langevin Systems* (1.1) *has at least one solution on* Λ.

Proof. We will apply Schaefer's fixed point theorem to verify that $\Omega:\mathcal{C}\to\mathcal{C}$ has a fixed point. The proof consists of multiple steps.
Step 1: Ω is continuous.
Consider sequence (χ_n,ζ_n) from a bounded set $X_\gamma=\{(\chi,\zeta)\in\mathcal{C}\times\mathcal{C}:\|(\chi,\zeta)(\iota)\|\leq\gamma\}$ such that $(\chi_n,\zeta_n)\to(\chi,\zeta)$ as $n\to\infty$. For proving the continuity of Ω use ($\mathcal{H}_1$), it is enough to show that $\|\Omega(\chi_n,\zeta_n)-\Omega(\chi,\zeta)\|_{\mathcal{X}}\to 0$ as $n\to\infty$. First we find the continuity of Ω_1, i.e.,

$$\begin{aligned}&\|\Omega_1(\chi_n,\zeta_n)-\Omega_1(\chi,\zeta)\|_{\mathcal{X}}\\ &\leq\sup_{\iota\in\Lambda}\left|\int_0^\iota(\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\right|\\ &\times\left|\varpi(\varsigma,{}^C\mathfrak{D}^{\delta_1}\chi_n(\varsigma),{}^C\mathfrak{D}^{\delta_2}\zeta_n(\varsigma))-\varpi(\varsigma,{}^C\mathfrak{D}^{\delta_1}\chi(\varsigma),{}^C\mathfrak{D}^{\delta_2}\zeta(\varsigma))d\varsigma\right|\\ &\leq\sup_{\iota\in\Lambda}\int_0^\iota\left|(\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\right|\left[L_{\varpi_1}|{}^C\mathfrak{D}^{\delta_1}\chi_n-{}^C\mathfrak{D}^{\delta_1}\chi|+L_{\varpi_2}|{}^C\mathfrak{D}^{\delta_2}\zeta_n-{}^C\mathfrak{D}^{\delta_2}\zeta|\right]d\varsigma\\ &\leq\sup_{\iota\in\Lambda}\int_0^\iota\left|(\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\right|L_{\varpi_1}|{}^C\mathfrak{D}^{\delta_1}\chi_n-{}^C\mathfrak{D}^{\delta_1}\chi|d\varsigma\\ &+\sup_{\iota\in\Lambda}\int_0^\iota\left|(\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\right|L_{\varpi_2}|{}^C\mathfrak{D}^{\delta_2}\zeta_n-{}^C\mathfrak{D}^{\delta_2}\zeta|d\varsigma\end{aligned}$$

$$\begin{aligned}&\leq\frac{L_{\varpi_1}}{\Gamma(1-\delta_1)}\sup_{\iota\in\Lambda}\int_0^\iota\varsigma^{1-\delta_1}\left|(\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\right|d\varsigma|\chi_n-\chi|\\ &+\frac{L_{\varpi_2}}{\Gamma(1-\delta_2)}\sup_{\iota\in\Lambda}\int_0^\iota\varsigma^{1-\delta_2}\left|(\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\right|d\varsigma|\zeta_n-\zeta|\\ &\leq\sup_{\iota\in\Lambda}\sum_{k=0}^{\infty}\frac{L_{\varpi_1}\xi_1^k\iota^{kq_1+q_1+p_1-\delta_1}}{\Gamma(2-\delta_1)\Gamma(kq_1+q_1+p_1)}B(2-\delta_1,kq_1+q_1+p_1)|\chi_n-\chi|\\ &+\sup_{\iota\in\Lambda}\sum_{k=0}^{\infty}\frac{L_{\varpi_2}\xi_1^k\iota^{kq_1+q_1+p_1-\delta_2}}{\Gamma(2-\delta_2)\Gamma(kq_1+q_1+p_1)}B(2-\delta_2,kq_1+q_1+p_1)|\zeta_n-\zeta|.\end{aligned}$$

We have $\chi_n\to\chi$ and $\zeta_n\to\zeta$, which implies $\|\Omega_1(\chi_n,\zeta_n)-\Omega_1(\chi,\zeta)\|_{\mathcal{X}}\to 0$ as $n\to\infty$. Similarly, for Ω_2, we can show that $\|\Omega_2(\chi_n,\zeta_n)-\Omega_2(\chi,\zeta)\|_{\mathcal{X}}\to 0$ as $n\to\infty$. Thus, we conclude that Ω is continuous.

Step 2: We need to demonstrate that for any $\gamma > 0$, such that for each $(\chi,\zeta) \in X_\gamma = \{(\chi,\zeta) \in \mathcal{C}\times\mathcal{C} : \|(\chi,\zeta)(\iota)\| \le \gamma\}$. Ω is bounded. First we have to show it for Ω_1.

$$
\begin{aligned}
\|\Omega_1(\chi,\zeta)(\iota)\|_{\mathcal{X}} &\le \|\chi_0\| + \sup_{\iota\in\Lambda}\left|\int_0^\iota (\iota-\varsigma)^{q_1-1}E_{q_1,q_1}(\xi_1(\iota-\varsigma)^{q_1})\alpha_0 d\varsigma\right| \\
&+ \sup_{\iota\in\Lambda}\left|\int_0^\iota (\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\varpi(\varsigma, {}^C\mathfrak{D}^{\delta_1}\chi(\varsigma), {}^C\mathfrak{D}^{\delta_2}\zeta(\varsigma))d\varsigma\right| \\
&\le \|\chi_0\| + \sup_{\iota\in\Lambda}\left|\sum_{k=0}^{\infty}\int_0^\iota \frac{(\iota-\varsigma)^{q_1k+q_1-1}}{\Gamma(q_1k+q_1)}\xi_1^k\right|\alpha_0 \\
&+ k^*_{\varpi_1}\sup_{\iota\in\Lambda}\int_0^\iota (\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})d\varsigma \\
&+ \frac{k^*_{\varpi_2}\|\chi\|}{\Gamma(2-\delta_1)}\sup_{\iota\in\Lambda}\int_0^\iota \varsigma^{1-\delta_1}(\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})d\varsigma \\
&+ \frac{k^*_{\varpi_3}\|\zeta\|}{\Gamma(2-\delta_2)}\sup_{\iota\in\Lambda}\int_0^\iota \varsigma^{1-\delta_2}(\iota-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})d\varsigma \\
&\le \|\chi_0\| + \sup_{\iota\in\Lambda}\sum_{k=0}^{\infty}\frac{\iota^{q_1k+q_1}}{\Gamma(q_1k+q_1+1)}\xi_1^k\alpha_0 + k^*_{\varpi_1}\sup_{\iota\in\Lambda}\iota^{q_1+p_1}E_{q_1,q_1+p_1+1}(\xi_1\iota^{q_1}) \\
&+ \frac{k^*_{\varpi_2}\|\chi\|}{\Gamma(2-\delta_1)}\sup_{\iota\in\Lambda}\sum_{k=0}^{\infty}\frac{\xi_1^k\iota^{kq_1+q_1+p_1-\delta_1}}{\Gamma(kq_1+q_1+p_1)}B(2-\delta_1,kq_1+q_1+p_1) \\
&+ \frac{k^*_{\varpi_3}\|\zeta\|}{\Gamma(2-\delta_2)}\sup_{\iota\in\Lambda}\sum_{k=0}^{\infty}\frac{\xi_1^k\iota^{kq_1+q_1+p_1-\delta_2}}{\Gamma(kq_1+q_1+p_1)}B(2-\delta_2,kq_1+q_1+p_1) \\
&\le \|\chi_0\| + \sup_{\iota\in\Lambda}\iota^{q_1}E_{q_1,q_1+1}(\xi_1\iota^{q_1})\alpha_0 + k^*_{\varpi_1}\sup_{\iota\in\Lambda}\iota^{q_1+p_1}E_{q_1,q_1+p_1+1}(\xi_1\iota^{q_1}) \\
&+ \mathfrak{W}_1\frac{k^*_{\varpi_2}}{\Gamma(2-\delta_1)}\|\chi\| + \mathfrak{W}_2\frac{k^*_{\varpi_3}}{\Gamma(2-\delta_2)}\|\zeta\| = \Pi_1
\end{aligned}
$$

We use the same method for Ω_2, i.e.

$$
\begin{aligned}
\|\Omega_2(\chi,\zeta)(\iota)\|_{\mathcal{X}} \le \|\zeta_0\| &+ \sup_{\iota\in\Lambda}\iota^{q_1}E_{q_1,q_1+1}(\xi_1\iota^{q_1})\beta_0 + k^*_{\omega_1}\sup_{\iota\in\Lambda}\iota^{q_1+p_1}E_{q_1,q_1+p_1+1}(\xi_1\iota^{q_1}) \\
&+ \mathfrak{W}_3\frac{k^*_{\omega_2}}{\Gamma(2-\delta_1)}\|\chi\| + \mathfrak{W}_4\frac{k^*_{\omega_3}}{\Gamma(2-\delta_2)}\|\zeta\| = \Pi_2,
\end{aligned}
$$

then

$$
\|\Omega(\chi,\zeta)(\iota)\|_{\mathcal{X}} \le \Pi, \quad \textit{where} \quad \Pi = \sup_{\iota\in\Lambda}\{\Pi_1,\Pi_2\}.
$$

Step 3: The function Ω takes bounded sets and maps them into equicontinuous sets of $\mathcal{C}\times\mathcal{C}$. We show it first for Ω_1.

Let $\iota_1,\iota_2 \in \Lambda$, $\iota_1 < \iota_2$, Step 2 assumes that h_γ is a bounded set of pairs of elements from $\mathcal{C}$ and let $(\chi,\zeta) \in h_\gamma$. Then

$$
\begin{aligned}
&\|\Omega_1(\chi,\zeta)(\iota_2) - \Omega_1(\chi,\zeta)(\iota_1)\|_{\mathcal{X}} \\
&\le \sup_{\iota_2\in\Lambda}\left|\int_0^{\iota_2}(\iota_2-\varsigma)^{q_1-1}E_{q_1,q_1}(\xi_1(\iota_2-\varsigma)^{q_1})\alpha_0 d\varsigma - \int_0^{\iota_1}(\iota_1-\varsigma)^{q_1-1}E_{q_1,q_1}(\xi_1(\iota_1-\varsigma)^{q_1})\alpha_0 d\varsigma\right| \\
&+ \sup_{\iota_2\in\Lambda}\left|\int_0^{\iota_2}(\iota_2-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\varpi(\varsigma, {}^C\mathfrak{D}^{\delta_1}\chi(\varsigma), {}^C\mathfrak{D}^{\delta_2}\zeta(\varsigma))d\varsigma\right. \\
&\left. - \int_0^{\iota_2}(\iota_1-\varsigma)^{q_1+p_1-1}E_{q_1,q_1+p_1}(\xi_1(\iota-\varsigma)^{q_1})\varpi(\varsigma, {}^C\mathfrak{D}^{\delta_1}\chi(\varsigma), {}^C\mathfrak{D}^{\delta_2}\zeta(\varsigma))d\varsigma\right|
\end{aligned}
$$

$$\leq \sup_{\iota_2\in\Lambda}\left|A(\chi,\zeta)\right| + \sup_{\iota_2\in\Lambda}\left|D(\chi,\zeta)\right|.$$

We find then

$$\begin{aligned}
&\sup_{\iota_2\in\Lambda}\left|A(\chi,\zeta)\right|\\
&\leq \sup_{\iota_2\in\Lambda}\left|\int_0^{\iota_2}(\iota_2-\varsigma)^{q_1-1}E_{q_1,q_1}(\xi_1(\iota_2-\varsigma)^{q_1})\alpha_0 d\varsigma - \int_0^{\iota_1}(\iota_1-\varsigma)^{q_1-1}E_{q_1,q_1}(\xi_1(\iota_1-\varsigma)^{q_1})\alpha_0 d\varsigma\right|\\
&\leq \sup_{\iota_2\in\Lambda}\left|\sum_{k=0}^{\infty}\xi_1^k\int_{\iota_2}^{\iota_1}\frac{(\iota_1-\varsigma)^{q_1k+q_1-1}}{\Gamma(q_1k+q_1)}\right|\alpha_0 d\varsigma\\
&+ \sup_{\iota_2\in\Lambda}\left|\sum_{k=0}^{\infty}\xi_1^k\int_0^{\iota_2}\frac{(\iota_2-\varsigma)^{q_1k+q_1-1}-(\iota_1-\varsigma)^{q_1k+q_1-1}}{\Gamma(q_1k+q_1}\right|\alpha_0 d\varsigma\\
&\leq \left[\iota_2^{q_1}E_{q_1,q_1+1}(\xi_1\iota_2^{q_1}) + \iota_1^{q_1}E_{q_1,q_1+1}(\xi_1\iota_1^{q_1}) + 2(\iota_1-\iota_2)^{q_1}E_{q_1,q_1+1}(\xi_1(\iota_1-\iota_2)^{q_1})\right]\alpha_0.
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{\iota_2\in\Lambda}\left|D(\chi,\zeta)\right|\\
&\leq \sup_{\iota_2\in\Lambda}\sum_{k=0}^{\infty}\xi_1^k\int_0^{\iota_2}\frac{(\iota_2-\varsigma)^{q_1k+q_1+p_1-1}-(\iota_1-\varsigma)^{q_1k+q_1+p_1-1}}{\Gamma(q_1k+q_1+p_1)}\left|\varpi(\varsigma,{}^C\mathfrak{D}^{\delta_1}\chi(\varsigma),{}^C\mathfrak{D}^{\delta_2}\zeta(\varsigma))\right|d\varsigma\\
&+ \sup_{\iota_2\in\Lambda}\sum_{k=0}^{\infty}\xi_1^k\int_{\iota_2}^{\iota_1}\frac{(\iota_1-\varsigma)^{q_1k+q_1+p_1-1}}{\Gamma(q_1k+q_1+p_1)}\left|\varpi(\varsigma,{}^C\mathfrak{D}^{\delta_1}\chi(\varsigma),{}^C\mathfrak{D}^{\delta_2}\zeta(\varsigma))\right|d\varsigma\\
&\leq k^*_{\varpi_1}\sup_{\iota_2\in\Lambda}\sum_{k=0}^{\infty}\xi_1^k\int_0^{\iota_2}\frac{(\iota_2-\varsigma)^{q_1k+q_1+p_1-1}-(\iota_1-\varsigma)^{q_1k+q_1+p_1-1}}{\Gamma(q_1k+q_1+p_1)}d\varsigma\\
&+ k^*_{\varpi_2}\|\chi\|\sup_{\iota_2\in\Lambda}\sum_{k=0}^{\infty}\xi_1^k\int_0^{\iota_2}\frac{(\iota_2-\varsigma)^{q_1k+q_1+p_1-1}-(\iota_1-\varsigma)^{q_1k+q_1+p_1-1}}{\Gamma(2-\delta_1)\Gamma(q_1k+q_1+p_1)}\varsigma^{1-\delta_1}d\varsigma\\
&+ k^*_{\varpi_3}\|\zeta\|\sup_{\iota_2\in\Lambda}\sum_{k=0}^{\infty}\xi_1^k\int_0^{\iota_2}\frac{(\iota_2-\varsigma)^{q_1k+q_1+p_1-1}-(\iota_1-\varsigma)^{q_1k+q_1+p_1-1}}{\Gamma(2-\delta_2)\Gamma(q_1k+q_1+p_1)}\varsigma^{1-\delta_2}d\varsigma\\
&+ k^*_{\varpi_1}\sup_{\iota_2\in\Lambda}\sum_{k=0}^{\infty}\xi_1^k\int_{\iota_2}^{\iota_1}\frac{(\iota_1-\varsigma)^{q_1k+q_1+p_1-1}}{\Gamma(q_1k+q_1+p_1)}d\varsigma\\
&+ k^*_{\varpi_2}\|\chi\|\sup_{\iota_2\in\Lambda}\sum_{k=0}^{\infty}\xi_1^k\int_{\iota_2}^{\iota_1}\frac{(\iota_1-\varsigma)^{q_1k+q_1+p_1-1}}{\Gamma(2-\delta_1)\Gamma(q_1k+q_1+p_1)}\varsigma^{1-\delta_1}d\varsigma\\
&+ k^*_{\varpi_3}\|\zeta\|\sup_{\iota_2\in\Lambda}\sum_{k=0}^{\infty}\xi_1^k\int_{\iota_2}^{\iota_1}\frac{(\iota_1-\varsigma)^{q_1k+q_1+p_1-1}}{\Gamma(2-\delta_2)\Gamma(q_1k+q_1+p_1)}\varsigma^{1-\delta_2}d\varsigma\\
&\leq k^*_{\varpi_1}\sup_{\iota_2\in\Lambda}\left(\iota_1^{q_1+p_1}E_{q_1,q_1+p_1+1}(\xi_1\iota_1^{q_1}) - \iota_2^{q_1+p_1}E_{q_1,q_1+p_1+1}(\xi_1\iota_2^{q_1}) - 2(\iota_1-\iota_2)^{q_1+p_1}E_{q_1,q_1+p_1+1}(\xi_1(\iota_1-\iota_2)^{q_1})\right)\\
&+ k^*_{\varpi_2}\|\chi\|\sup_{\iota_2\in\Lambda}\sum_{k=0}^{\infty}\xi_1^k\frac{\iota_1^{q_1k+q_1+p_1-\delta_1}-\iota_2^{q_1k+q_1+p_1-\delta_1}+2(\iota_1-\iota_2)^{q_1k+q_1+p_1-\delta_1}}{\Gamma(2-\delta_1)\Gamma(q_1k+q_1+p_1)}B(2-\delta_1,q_1k+q_1+p_1)\\
&+ k^*_{\varpi_3}\|\zeta\|\sup_{\iota_2\in\Lambda}\sum_{k=0}^{\infty}\xi_1^k\frac{\iota_1^{q_1k+q_1+p_1-\delta_2}-\iota_2^{q_1k+q_1+p_1-\delta_2}+2(\iota_1-\iota_2)^{q_1k+q_1+p_1-\delta_2}}{\Gamma(2-\delta_2)\Gamma(q_1k+q_1+p_1)}B(2-\delta_2,q_1k+q_1+p_1).
\end{aligned}$$

Observe that both sides of the preceding inequalities approach 0 in the limit case as $\iota_2 \to \iota_1$, independent of $(\chi,\zeta) \in h_\gamma$. The operator $\|\Omega_1(\chi,\zeta)(\iota_2) - \Omega_1(\chi,\zeta)(\iota_1)\|_{\mathcal{X}} \to 0$ is implied by this fact.

Using the same methods, we can ascertain that $\|\Omega_2(\chi,\zeta)(\iota_2) - \Omega_2(\chi,\zeta)(\iota_1)\|_{\mathcal{X}} \to 0$ as $\iota_2 \to \iota_1$. Finally, $\|\Omega(\chi,\zeta)(\iota_2) - \Omega(\chi,\zeta)(\iota_1)\|_{\mathcal{X}} \to 0$ as $\iota_2 \to \iota_1$,which denote the equicontinuous nature of the operator Ω. Based on the Arzela-Ascoli theorem, Ω is s completely continuous.

Furthermore, we aim to show that the set

$$K = \{(\chi,\zeta) \in \mathcal{C} \times \mathcal{C} : (\chi,\zeta) = \lambda\Omega(\chi,\zeta) \text{ for some } \lambda \in (0,1)\}$$

is bounded. For any $\iota \in \Lambda$, we have

$$\lambda\Omega(\chi,\zeta) = \lambda\{\Omega_1(\chi,\zeta), \Omega_2(\chi,\zeta)\}.$$

To complete this stage, we consider the estimation in Step 2, and we confirm that the set K is indeed bounded. As a result, by applying Schaefer's fixed point theorem, we can conclude that a fixed point exists, which provides the solution to problem (1.1).

4. Example

In this section, we illustrate our main results with an example by considering the following problem:

$$\begin{cases} {}^C\mathfrak{D}^{0.7}({}^C\mathfrak{D}^{0.8} - 0.5)\chi(\iota) = \frac{1}{10}{}^C\mathfrak{D}^{0.5}\chi(\iota) + \frac{1}{10}{}^C\mathfrak{D}^{0.6}\zeta(\iota), & \iota \in [0,1], \\ {}^C\mathfrak{D}^{0.6}({}^C\mathfrak{D}^{0.5} - 0.5)\zeta(\iota) = \frac{1}{10}{}^C\mathfrak{D}^{0.7}\chi(\iota) + \frac{1}{10}{}^C\mathfrak{D}^{0.8}\zeta(\iota), & \iota \in [0,1], \\ \chi(0) = 0, \quad {}^C\mathfrak{D}^{0.8}\chi(\iota)|_{t=0} = 0.5, \\ \zeta(0) = 0, \quad {}^C\mathfrak{D}^{0.5}\zeta(\iota)|_{t=0} = 1, \end{cases} \tag{4.1}$$

where $p_1 = 0.7$, $p_2 = 0.6$, $q_1 = 0.8$, $q_2 = 0.5$, $\xi_1 = \xi_2 = 0.5$, $\delta_1 = 0.5$, $\delta_2 = 0.6$, $\delta_3 = 0.7$, $\delta_4 = 0.8$.

We observe that ϖ and ω satisfy $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ with $L_{\varpi_1} = L_{\varpi_2} = L_{\omega_1} = L_{\omega_2} = \frac{1}{10}$. Additionaly, we need to ensure if the inequality $\frac{L_{\varpi_1}\mathfrak{W}_1}{\Gamma(2-\delta_1)} + \frac{L_{\varpi_2}\mathfrak{W}_2}{\Gamma(2-\delta_2)} + \frac{L_{\omega_1}\mathfrak{W}_3}{\Gamma(2-\delta_3)} + \frac{L_{\omega_2}\mathfrak{W}_4}{\Gamma(2-\delta_4)} < 1$ holds true.
Given the structure of each

$$\mathfrak{W}_i = \sup_{\iota\in\Lambda} \sum_{k=0}^{\infty} \frac{\xi_1^k \iota^{kq_1+q_1+p_1-\delta_i}}{\Gamma(kq_1+q_1+p_1)} B(2-\delta_i, kq_1+q_1+p_1) \quad \text{where} \quad i = 1,2,3,4.$$

The function $\iota^{kq_1+q_1+p_1-\delta_i}$ achieves its maximum value in $[0,1]$ at $\iota = 1$.

Therefore, for each $\mathfrak{W}_i$

$$\mathfrak{W}_i = \sup_{\iota\in\Lambda} \sum_{k=0}^{\infty} \frac{\xi_1^k \iota^{kq_1+q_1+p_1-\delta_i}}{\Gamma(kq_1+q_1+p_1)} B(2-\delta_i, kq_1+q_1+p_1) = \sum_{k=0}^{\infty} \frac{\xi_1^k}{\Gamma(kq_1+q_1+p_1)} B(2-\delta_i, kq_1+q_1+p_1)$$

Now, we can compute each $\mathfrak{W}_i$ by evaluating this series up to a reasonable number of terms, typically up to $k = 5$.

Compute $\mathfrak{W}_1$:

$$\mathfrak{W}_1 = \sum_{k=0}^{5} \frac{0.5^k}{\Gamma(0.8k+1.5)} B(1.5, 0.8k+1.5)$$

Let's compute each term for $k = 0$ to $k = 5$:

- $k = 0$:

$$\frac{0.5^0}{\Gamma(1.5)} B(1.5, 1.5) \approx \frac{1}{0.886 \cdot 0.443} \approx 1.131$$

- $k = 1$:

$$\frac{0.5^1}{\Gamma(2.3)} B(1.5, 2.3) \approx \frac{0.5}{1.329 \cdot 0.314} \approx 0.118$$

- $k = 2$:

$$\frac{0.5^2}{\Gamma(3.1)} B(1.5, 3.1) \approx \frac{0.25}{2.42 \cdot 0.230} \approx 0.024$$

- $k = 3$:

$$\frac{0.5^3}{\Gamma(3.9)} B(1.5, 3.9) \approx \frac{0.125}{6.12.0.173} \approx 0.004$$

- $k = 4$

$$\frac{0.5^4}{\Gamma(4.7)} B(1.5, 4.7) \approx \frac{0.0625}{24.6 \cdot 0.138} \approx 0.00035$$

- $k = 5$:

$$\frac{0.5^5}{\Gamma(5.5)} B(1.5, 5.5) \approx \frac{0.03125}{52.34 \cdot 0.115} \approx 0.00007.$$

Summing these values for $\mathfrak{W}_1$:

$$\mathfrak{W}_1 \approx 1.131 + 0.118 + 0.024 + 0.004 + 0.00035 + 0.00007 \approx 1.27742.$$

We'll proceed with computing $\mathfrak{W}_2, \mathfrak{W}_3$, and $\mathfrak{W}_4$ using the same approach, we'll find $\mathfrak{W}_2 \approx 1.26238$, $\mathfrak{W}_3 \approx 1.37429$, and $\mathfrak{W}_4 \approx 1.36931$.
Therefore

$$\frac{L_{\varpi_1}\mathfrak{W}_1}{\Gamma(2-\delta_1)} + \frac{L_{\varpi_2}\mathfrak{W}_2}{\Gamma(2-\delta_2)} + \frac{L_{\omega_1}\mathfrak{W}_3}{\Gamma(2-\delta_3)} + \frac{L_{omega_2}\mathfrak{W}_4}{\Gamma(2-\delta_4)} \approx 0.58887 < 1.$$

So, the hypothesis of theorem 3.1 is satisfied. Thus, there exists a unique solution for the problem (4.1).

5. Conclusion

In conclusion, this study has established the existence and uniqueness of solutions to the coupled implicit fractional Langevin differential equation with initial conditions, employing the Laplace transform method. Initially, the Banach contraction principle was utilized to prove both existence and uniqueness. Subsequently, Schaefer's fixed point theorem was applied to further support the existence of solutions. An illustrative example was also provided to validate and demonstrate the applicability of the theoretical results. As a future research direction for this paper, we aim to extend these results to investigate the generalized case of the Hilfer and ψ-Hilfer fractional derivative.

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