



Bondage and Non-Bondage Number of Anti-Fuzzy Graph

Tahir Rasool and Aliya Fahmi

ABSTRACT: Many studies have focused on the development of fuzzy graph theory. Graph domination is a key area of study within fuzzy graph theory. It has been applied in numerous fields to address real-world issues such as multi-attribute, multi-object, multi-agent decision-making problems, neural networks, and other areas. This paper aims to define the bondage and non-bondage sets of anti-fuzzy graphs and analyze the bondage number and non-bondage number of anti-fuzzy graphs. The properties of the bondage and non-bondage numbers of anti-fuzzy graphs are examined. Several examples are used to illustrate the bondage and non-bondage numbers of anti-fuzzy graphs. Additionally, the bounds of bondage and non-bondage numbers are discussed, and some theorems related to bondage and non-bondage numbers are proven.

Keywords: Anti-fuzzy graph, bondage sets of anti-fuzzy graphs, bondage number of anti-fuzzy graphs, non-bondage sets of anti-fuzzy graphs, and non-bondage number of anti-fuzzy graph.

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1. Introduction

The notion of graph theory is entirely dependent on the Königsberg bridge problem. In 1736, Swiss mathematician Leonhard Euler simplified it. Fuzzy graph models are advantageous graphical structural tools for solving combinatorial problems in various domains, including algebra, optimization, social science, environmental science, topology, computer science, and operations research. Fuzzy structural models are superior to graphical models due to the natural existence of ambiguity and vagueness. Firstly, fuzzy set theory was needed to cope with many complex phenomena with incomplete information. A fuzzy set, as a superset of a crisp set expressed as the set of distinct things having a membership grade in $[0, 1]$, owes its origin to the work of L. A. Zadeh [1] in 1965. Zadeh’s remarkable development has led to many applications in numerous fields. In 1973, Kauffman [2] initiated his work on fuzzy graphs, a powerful tool of mathematics originating from Zadeh’s works on fuzzy relations [3]. It links with combinatorial issues of various analogs such as optimization, social science, topology, operations research, and environmental science. In 1975, Rosenfeld [4] introduced another definition of a fuzzy graph and extended his work to various fuzzy aspects of fuzzy graphs such as paths, cycles, and connectivity. Before that time, the idea of strong arcs was worthless in graphs, but it had great importance in fuzzy graphs. In 2008, Gani and Radha [5] worked on regular fuzzy graphs and regular fuzzy graphs. They give a comparative study of Regular fuzzy graphs and regular fuzzy graphs through several examples and provide an equivalency condition. Then Gani and Latha [6] initiated the concept of an irregular fuzzy graph and introduced the neighborly irregular fuzzy graph, neighborly total irregular fuzzy graph, highly irregular fuzzy graph, and highly total irregular fuzzy graph. They determined several results on neighborly irregular fuzzy graphs. In the model of fuzzy graphs, we discuss the uncertainties in the description of the objects and their

2020 *Mathematics Subject Classification:* 03E72, 94Dxx.

Submitted May 16, 2025. Published April 01, 2026

relations of minimum value among the objects, in a few situations, it may take to get the largest values. This type of situation constructs a new model as an Anti-fuzzy graph. In 2016, A new model of an anti-fuzzy graph was developed by Seethalakshmi and Gnanajothi [7]. Muthuraj and Sasireka [8] discussed some kinds of regular and irregular anti-fuzzy graphs. They express the idea of edge regular and edge regular on an anti-fuzzy graph. Moreover, they applied these ideas to several types of anti-fuzzy graphs and obtained the bounds on them. Muthuraj and Sasireka [9] examined the operations of anti-union, anti-join, anti-composition, anti-cartesian products and anti-complement of anti-fuzzy graphs. They get some results on various types of anti-fuzzy graphs. The theory of domination originated from a game of chess, based on the concept that the minimum number of queens needed to cover all squares on a chessboard is the domination number [10]. Somasundram et al. [11] explored domination using effective edges of fuzzy graphs in 1998. Initially considered insignificant, the concept of strong arcs in graphs has proven to be important in fuzzy graphs. Bhutani et al. [12] introduced the notion of strong arcs in fuzzy graphs in 2003, focusing on the strength of connectivity between nodes and introducing alpha-strong and delta-weak concepts. Gani et al. [13] categorized nodes based on domination properties and investigated the impact of removing nodes on domination numbers. Gani et al. [14] studied domination in fuzzy graphs using strong arcs, emphasizing the importance of dominating sets in wireless sensor networks and operational research. In 2015, Manjusha and Sunitha [15] introduced the concept of strong domination number in fuzzy graphs, considering membership values of strong arcs and evaluating this number for complete fuzzy graphs and complete bipartite fuzzy graphs, along with establishing bounds for the strong domination number. In 2018, Muthuraj and Sasireka [16] initiated the idea of domination on anti-fuzzy graph. They determine the domination set and domination number of anti-fuzzy graph and derived an algorithm for finding the minimal dominating set of an anti-fuzzy graph. Yousif et al. [17] analyze the idea of 2-anti fuzzy domination and determine the 2-anti fuzzy domination number for several classes of anti-fuzzy graphs. They make a comparison of the 2-anti-fuzzy domination number; vertex anti-fuzzy vertex covering number and anti-fuzzy domination and evaluate their results. Yousif and Omran [18] introduced the inverse 2-anti fuzzy domination set and calculated the inverse 2-anti fuzzy domination number. They work on the bounds of the inverse 2-anti fuzzy domination number and get relevant results. Manjusha [19] developed a new variation of domination called set domination using strong arcs, identifying characteristics of this new parameter about existing domination parameters. In 2023, Thota et al. [20] examined limitations of domination numbers of fuzzy graphs and anti-fuzzy graphs and they defined the fuzzy matrix on an anti-fuzzy graph and discuss the characteristics of the adjacency fuzzy matrix. The concept of the bondage number in graphs was introduced by Fink et al. [21] in 1990. They obtained sharp bounds for the bondage number and evaluated exact values for various classes of graphs. Later, in 1994, Hartnell et al. [22] worked on bounds for the bondage number and showed an infinite class of graphs, each with a bondage number greater than its order plus 1, disproving previously conjectured upper bounds. In 1996, Kulli and Janakiram [23] introduced the idea of the non-bondage number of graphs. Kulli [24] then surveyed the non-bondage number, presenting well-known expressions and issues, as well as other types of non-bondage numbers.

1.1. Literature Review

In 2015, Gani, Devi, and Akram [25] focused on the concept of bondage and non-bondage sets in fuzzy graphs. They worked on the bondage and non-bondage numbers of various classes of fuzzy graphs. Gani and Devi [26] analyzed the 2-bondage sets of fuzzy graphs and expressed the condition for a 2-bondage set to also be the bondage set. They also examined the exact values for some standard fuzzy graphs. In 2017, Hussain and Keyan [27] examined the co-bondage number for any fuzzy graphs and fuzzy strong line graphs. They defined neighborhood extension and analyzed its properties using the bondage arcs. In 2020, Shao et al. [28] worked on a new definition of strong arcs linked to the strength of connectivity in intuitionistic fuzzy graphs. They developed the concepts of the bondage set, bondage number, non-bondage sets, and non-bondage numbers in intuitionistic fuzzy graphs. Buvaneshwari and Umamaheswari [29] discussed the idea of strong edges in the domination set and studied the increasing and decreasing domination numbers using cardinality. They defined the bondage and non-bondage sets of regular intuitionistic fuzzy graphs and analyzed the characteristics of bondage and non-bondage numbers.

1.2. Importance

The bondage number is an essential factor in graphs that is based on a well-established domination number. It is a very effective tool for evaluating uncertainty and vagueness in a network or graph. Similarly, the bondage number of fuzzy graphs has been introduced to address real-world issues in various fields such as computer communication networks, radio stations, and land surveying. Recent research on the bondage number has been inspired by its importance in interconnection networks, capturing the interest of researchers. If we consider a fuzzy graph as a communication setup, then the bondage focused on domination number holds great value. Domination is a significant and more generalized tool that has been a highly discussed topic in fuzzy graphs. The wide range of applications and the variety of domination parameters that can be examined make domination theory research more interesting and attractive.

2. Background

Definition 2.1. [10] A simple graph $\hat{\mathcal{G}} = (V, E)$ is an ordered pair such that V is a set of nodes/vertices, E is a set of arcs/edges and (ν_1, ν_2) are arcs of $\hat{\mathcal{G}}$ formed by two nodes ν_1 and ν_2 . Then these nodes are adjacent to each other. A loop is formed when $\nu_1 = \nu_2$ in arcs (ν_1, ν_2) .

Definition 2.2. [25] A *bondage number* of $\hat{\mathcal{G}} = (V, E)$, denoted by $\mathcal{B}(\hat{\mathcal{G}})$, is the least cardinality among all sets of arcs $\hat{X} \subseteq E$ such that the domination number $\chi(\hat{\mathcal{G}} - \hat{X})$ is greater than the domination number of the graph, therefore

$$\chi(\hat{\mathcal{G}} - \hat{X}) > \chi(\hat{\mathcal{G}}).$$

The *non-bondage number*, denoted by $\mathcal{B}_n(\hat{\mathcal{G}})$, is the largest cardinality among all sets of arcs $\hat{X} \subseteq E$ such that the domination number $\chi(\hat{\mathcal{G}} - \hat{X})$ must be equal to the domination number of the graph, therefore

$$\chi(\hat{\mathcal{G}} - \hat{X}) = \chi(\hat{\mathcal{G}}).$$

Definition 2.3. [1] Let $\hat{\tau}$ and $\hat{\mu}$ be fuzzy subsets and a fuzzy relation on a nonempty set V and $V \times V$ respectively. They are defined as

$$\hat{\tau} : V \rightarrow [0, 1] \quad \text{and} \quad \hat{\mu} : V \times V \rightarrow [0, 1].$$

Definition 2.4. [25] A fuzzy graph $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$, depends on mappings expressed as

$$\hat{\mu}(\nu_1, \nu_2) \leq \hat{\tau}(\nu_1) \wedge \hat{\tau}(\nu_2), \quad \forall \nu_1, \nu_2 \in V.$$

An arc satisfying

$$\hat{\mu}(\nu_1, \nu_2) = \hat{\tau}(\nu_1) \wedge \hat{\tau}(\nu_2)$$

of \mathcal{G} is said to be an *effective arc*, and an arc having no effective arcs is said to be an *isolated edge*.

Definition 2.5. [10] A path of a fuzzy graph $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$, is an arrangement of nodes $\nu_1, \nu_2, \nu_3, \dots, \nu_n$, ($n \geq 2$), $\nu_1 \neq \nu_n$ such that

$$\hat{\mu}(\nu_i, \nu_{i+1}) > 0, \quad i = 1, 2, \dots, (n - 1).$$

Definition 2.6. [28] The path strength in $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$ is determined by the membership grade of the *weakest arc* (the arc with the least membership grade).

Definition 2.7. [28] For $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$, the strength of connectedness presented as

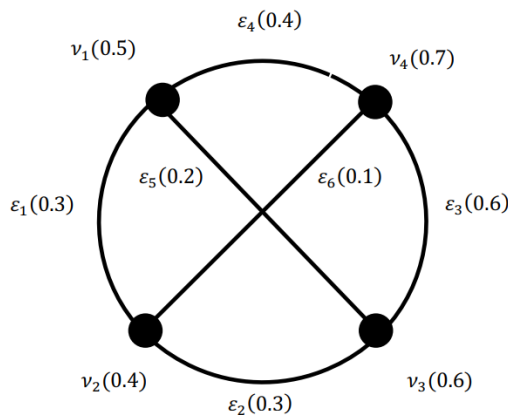
$$\text{CONN}_{\mathcal{G}}(\nu_1, \nu_2) / \hat{\mu}^{\infty}(\nu_1, \nu_2)$$

is the greatest strength of all paths that link two nodes ν_1 and ν_2 such that

$$\hat{\mu}(\nu_1, \nu_2) \geq \hat{\mu}^{\infty}(\nu_1, \nu_2).$$

Definition 2.8. [28] For $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$, an arc (ν_1, ν_2) is known as a *strong arc* if for all $\nu_1, \nu_2 \in V$, such that

$$\hat{\mu}(\nu_1, \nu_2) = \hat{\mu}^{\infty}(\nu_1, \nu_2).$$

Figure 2.1: Fuzzy Graph \mathcal{G}

Example 2.9. Consider the fuzzy graph $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$ under the simple graph $\hat{\mathcal{G}} = (V, E)$ such that

$$V = \{\nu_1, \nu_2, \nu_3, \nu_4\}, \quad E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\}$$

shown in Figure 2.1. In this graph, there are four paths between the nodes ν_1 and ν_2 . The strength of paths are computed as:

$$(\nu_1 - \nu_2) = 0.3, \quad (\nu_1 - \nu_3 - \nu_2) = \min(0.2, 0.3) = 0.2,$$

$$(\nu_1 - \nu_4 - \nu_2) = \min(0.4, 0.1) = 0.1, \quad (\nu_1 - \nu_4 - \nu_3 - \nu_2) = \min(0.4, 0.6, 0.3) = 0.3.$$

Hence, the strength of connectedness is

$$\hat{\mu}^\infty(\nu_1, \nu_2) = \max(0.3, 0.2, 0.1, 0.3) = 0.3.$$

Since $\hat{\mu}(\nu_1, \nu_2) = \hat{\mu}^\infty(\nu_1, \nu_2)$, arc (ν_1, ν_2) is a strong arc.

By the same computation, we obtain:

$$\hat{\mu}^\infty(\nu_2, \nu_3) = \hat{\mu}^\infty(\varepsilon_2) = \max(0.3, 0.1, 0.2, 0.3) = 0.3 = \hat{\mu}(\nu_2, \nu_3) = \hat{\mu}(\varepsilon_2),$$

$$\hat{\mu}^\infty(\nu_3, \nu_4) = \hat{\mu}^\infty(\varepsilon_3) = \max(0.6, 0.1, 0.2, 0.3) = 0.6 = \hat{\mu}(\nu_3, \nu_4) = \hat{\mu}(\varepsilon_3),$$

$$\hat{\mu}^\infty(\nu_4, \nu_1) = \hat{\mu}^\infty(\varepsilon_4) = \max(0.4, 0.2, 0.1, 0.3) = 0.4 = \hat{\mu}(\nu_4, \nu_1) = \hat{\mu}(\varepsilon_4),$$

$$\hat{\mu}^\infty(\nu_1, \nu_3) = \hat{\mu}^\infty(\varepsilon_5) = \max(0.2, 0.4, 0.3) = 0.4 \neq \hat{\mu}(\nu_1, \nu_3) = \hat{\mu}(\varepsilon_5),$$

$$\hat{\mu}^\infty(\nu_2, \nu_4) = \hat{\mu}^\infty(\varepsilon_6) = \max(0.1, 0.3, 0.3) = 0.3 \neq \hat{\mu}(\nu_2, \nu_4) = \hat{\mu}(\varepsilon_6).$$

Therefore, by the definition of strong arcs,

$$\hat{\mu}(\nu_1, \nu_2) = \hat{\mu}^\infty(\nu_1, \nu_2) \quad \forall \nu_1, \nu_2 \in V,$$

and thus $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are strong arcs.

Hence,

$$\hat{\mathcal{S}} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}.$$

Definition 2.10. [28] For two nodes ν_1 and ν_2 of V of fuzzy graph $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$, a vertex ν_1 *dominates* the vertex $\nu_2 \in V \setminus \{\nu_1\}$ if there is a strong arc between ν_1 and ν_2 .

In addition, an isolated node does not dominate any other node in \mathcal{G} .

Definition 2.11. [28] For a set $\hat{D} \subseteq V$ of $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$, it is said to be a *dominating set* if, for every

element $\nu_2 \in V \setminus \hat{D}$, there exists an element $\nu_1 \in \hat{D}$ such that ν_1 dominates ν_2 .

Definition 2.12. [28] The *domination number* of $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$ is the number of dominating sets with the smallest cardinality among all the dominating sets in \mathcal{G} , denoted as $\chi(\mathcal{G})$.

Definition 2.13. [28] Let $\hat{\mathcal{S}}$ be the set of all strong arcs of the fuzzy graph $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$ and let \hat{H} be a subset of $\hat{\mathcal{S}}$. If

$$\chi(\mathcal{G} - \hat{H}) > \chi(\mathcal{G}),$$

then \hat{H} is said to be a *bondage set* of \mathcal{G} .

Definition 2.14. [28] The *bondage number* $\mathcal{B}(\mathcal{G})$ in a fuzzy graph $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$ is the smallest cardinality among all bondage sets of \mathcal{G} .

Remark 2.15. [28] For any subset \hat{H} of the set of strong arcs $\hat{\mathcal{S}}$ in a fuzzy graph \mathcal{G} , if

$$\chi(\mathcal{G} - \hat{H}) \not> \chi(\mathcal{G}),$$

then we conclude that \mathcal{G} does not have a bondage set.

Example 2.16. Consider the fuzzy graph $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$ under a simple graph $\hat{\mathcal{G}} = (V, E)$ as shown in Figure 2.2. Then, the set of strong arcs is

$$\hat{\mathcal{S}} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\},$$

and the dominating sets of \mathcal{G} are

$$\{\{\nu_1, \nu_2\}, \{\nu_1, \nu_3\}, \{\nu_1, \nu_4\}, \{\nu_2, \nu_3\}, \{\nu_2, \nu_4\}, \{\nu_3, \nu_4\}, \{\nu_1, \nu_2, \nu_3\}, \{\nu_1, \nu_2, \nu_4\}, \{\nu_1, \nu_3, \nu_4\}, \{\nu_2, \nu_3, \nu_4\}\}.$$

The dominating set with the smallest cardinality is $\{\nu_1, \nu_2\}$. So, by Definition 2.12, $\chi(\mathcal{G})$ is obtained.

We now compute the bondage and non-bondage sets of \mathcal{G} . The bondage and non-bondage sets are subsets of the strong arcs.

Consider

$$\hat{\mathcal{K}}_1 = \{\varepsilon_1, \varepsilon_2\} \subseteq \hat{\mathcal{S}} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}.$$

For $\mathcal{G} - \hat{\mathcal{K}}_1$, we compute:

$$\hat{\mu}^\infty(\nu_3, \nu_4) = \hat{\mu}^\infty(\varepsilon_3) = \max(0.6, 0.2) = 0.6 = \hat{\mu}(\nu_3, \nu_4) = \hat{\mu}(\varepsilon_3),$$

$$\hat{\mu}^\infty(\nu_4, \nu_1) = \hat{\mu}^\infty(\varepsilon_4) = \max(0.4, 0.2) = 0.4 = \hat{\mu}(\nu_4, \nu_1) = \hat{\mu}(\varepsilon_4),$$

$$\hat{\mu}^\infty(\nu_1, \nu_3) = \hat{\mu}^\infty(\varepsilon_5) = \max(0.2, 0.4) = 0.4 \neq \hat{\mu}(\nu_1, \nu_3) = \hat{\mu}(\varepsilon_5),$$

$$\hat{\mu}^\infty(\nu_2, \nu_4) = \hat{\mu}^\infty(\varepsilon_6) = 0.1 = \hat{\mu}(\nu_2, \nu_4) = \hat{\mu}(\varepsilon_6).$$

Therefore, the set of strong arcs is

$$\{\varepsilon_3, \varepsilon_4, \varepsilon_6\},$$

and the dominating sets of $\mathcal{G} - \hat{\mathcal{K}}_1$ are

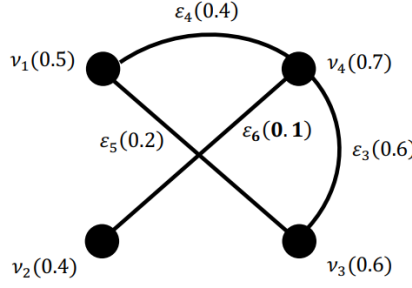
$$\{\{\nu_4\}, \{\nu_2, \nu_4\}, \{\nu_1, \nu_3, \nu_4\}, \{\nu_1, \nu_2, \nu_4\}\}.$$

The dominating set with the smallest cardinality is $\{\nu_4\}$. Thus, $\chi(\mathcal{G} - \hat{\mathcal{K}}_1) = 0.6 \not> \chi(\mathcal{G})$.

Hence, by Remark 2.15, for any set $\hat{\mathcal{K}} \subseteq \hat{\mathcal{S}}$, \mathcal{G} does not have a bondage set.

Definition 2.17. [28] Let $\hat{\mathcal{S}}$ be the set of all strong arcs of the fuzzy graph $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$ and let $\hat{\mathcal{K}}$ be a subset of $\hat{\mathcal{S}}$. If

$$\chi(\mathcal{G} - \hat{\mathcal{K}}) = \chi(\mathcal{G}),$$

Figure 2.2: Fuzzy Graph $\mathcal{G} - \hat{H}_1$

then $\hat{\mathcal{K}}$ is said to be a *non-bondage set* of \mathcal{G} .

Definition 2.18. The *non-bondage number* $\mathcal{B}_n(\mathcal{G})$ in a fuzzy graph \mathcal{G} is the maximal cardinality among all the non-bondage sets of \mathcal{G} .

Example 2.19. Consider the fuzzy graph $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$ under simple graph $\hat{\mathcal{G}} = (V, E)$ such that $V = \{\nu_1, \nu_2, \nu_3, \nu_4\}$ and $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ shown in Figure 2.3.

Therefore, the set of strong arcs is $\hat{\mathcal{S}} = \{\varepsilon_1, \varepsilon_3\}$ and the dominating sets of $\mathcal{G} - \hat{K}_1$ are

$$\{\{\nu_2, \nu_4\}, \{\nu_1, \nu_2, \nu_3\}, \{\nu_1, \nu_2, \nu_4\}, \{\nu_2, \nu_3, \nu_4\}\}$$

and the dominating set with the smallest cardinality is $\{\nu_2, \nu_4\}$.

First, we calculate the strong arcs of \mathcal{G} using the definition of strong arcs:

$$\hat{\mu}^\infty(\nu_1, \nu_2) = \hat{\mu}^\infty(\varepsilon_1) = (0.3 \vee 0.2) = 0.3 = \hat{\mu}(\varepsilon_1)$$

$$\hat{\mu}^\infty(\nu_2, \nu_3) = \hat{\mu}^\infty(\varepsilon_2) = (0.4 \vee 0.2) = 0.4 = \hat{\mu}(\varepsilon_2)$$

$$\hat{\mu}^\infty(\nu_3, \nu_4) = \hat{\mu}^\infty(\varepsilon_3) = (0.2 \vee 0.3) = 0.3 \neq \hat{\mu}(\varepsilon_3)$$

$$\hat{\mu}^\infty(\nu_4, \nu_1) = \hat{\mu}^\infty(\varepsilon_4) = (0.35 \vee 0.2) = 0.35 = \hat{\mu}(\varepsilon_4)$$

Therefore, the set of strong arcs is $\hat{\mathcal{S}} = \{\varepsilon_1, \varepsilon_2, \varepsilon_4\}$.

Secondly, we find all dominating sets of \mathcal{G} :

$$\{\{\nu_1, \nu_2\}, \{\nu_1, \nu_3\}, \{\nu_1, \nu_4\}, \{\nu_2, \nu_3\}, \{\nu_2, \nu_4\}, \{\nu_3, \nu_4\}\},$$

and the dominating set with the smallest cardinality is $\{\nu_1, \nu_2\}$. So,

$$\chi(\mathcal{G}) = |\{\nu_1, \nu_2\}| = 0.8$$

Third, we find the bondage sets, bondage number $\mathcal{B}(\mathcal{G})$, non-bondage sets and non-bondage number $\mathcal{B}_n(\mathcal{G})$.

Consider $\hat{K}_1 = \{\varepsilon_2, \varepsilon_4\} \subseteq \hat{\mathcal{S}} = \{\varepsilon_1, \varepsilon_2, \varepsilon_4\}$. Then:

$$\hat{\mu}^\infty(\nu_1, \nu_2) = \hat{\mu}^\infty(\varepsilon_1) = 0.3 = \hat{\mu}(\nu_1, \nu_2) = \hat{\mu}(\varepsilon_1)$$

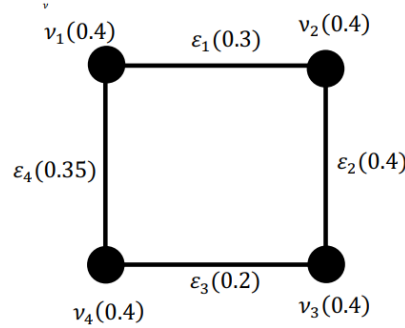
$$\hat{\mu}^\infty(\nu_3, \nu_4) = \hat{\mu}^\infty(\varepsilon_3) = 0.2 = \hat{\mu}(\nu_3, \nu_4) = \hat{\mu}(\varepsilon_3)$$

$$\chi(\mathcal{G} - \hat{K}_1) = |\{\nu_2, \nu_4\}| = 0.8 = \chi(\mathcal{G})$$

Since $\chi(\mathcal{G} - \hat{K}_1) = \chi(\mathcal{G})$, hence $\hat{K}_1 = \{\varepsilon_2\}$ is a non-bondage set.

For $\hat{K}_2 = \{\varepsilon_1, \varepsilon_2, \varepsilon_4\} \subseteq \hat{\mathcal{S}}$, we find

$$\hat{\mu}^\infty(\nu_3, \nu_4) = \hat{\mu}^\infty(\varepsilon_3) =$$

Figure 2.3: Fuzzy Graph \mathcal{G}

Example 2.19. Consider the fuzzy graph $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$ under simple graph $\hat{\mathcal{G}} = (V, E)$ such that $V = \{\nu_1, \nu_2, \nu_3, \nu_4\}$ and $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ shown in Figure 2.3. Therefore, the set of strong arcs is $\hat{\mathcal{S}} = \{\varepsilon_1, \varepsilon_3\}$ and the dominating sets of $\mathcal{G} - \hat{K}_1$ are

$$\{\{\nu_2, \nu_4\}, \{\nu_1, \nu_2, \nu_3\}, \{\nu_1, \nu_2, \nu_4\}, \{\nu_2, \nu_3, \nu_4\}\}$$

and the dominating set with the smallest cardinality is $\{\nu_2, \nu_4\}$.

First, we calculate the strong arcs of \mathcal{G} using the definition of strong arcs:

$$\begin{aligned} \hat{\mu}^\infty(\nu_1, \nu_2) &= \hat{\mu}^\infty(\varepsilon_1) = (0.3 \vee 0.2) = 0.3 = \hat{\mu}(\varepsilon_1) \\ \hat{\mu}^\infty(\nu_2, \nu_3) &= \hat{\mu}^\infty(\varepsilon_2) = (0.4 \vee 0.2) = 0.4 = \hat{\mu}(\varepsilon_2) \\ \hat{\mu}^\infty(\nu_3, \nu_4) &= \hat{\mu}^\infty(\varepsilon_3) = (0.2 \vee 0.3) = 0.3 \neq \hat{\mu}(\varepsilon_3) \\ \hat{\mu}^\infty(\nu_4, \nu_1) &= \hat{\mu}^\infty(\varepsilon_4) = (0.35 \vee 0.2) = 0.35 = \hat{\mu}(\varepsilon_4) \end{aligned}$$

Therefore, the set of strong arcs is $\hat{\mathcal{S}} = \{\varepsilon_1, \varepsilon_2, \varepsilon_4\}$.

Secondly, we find all dominating sets of \mathcal{G} :

$$\{\{\nu_1, \nu_2\}, \{\nu_1, \nu_3\}, \{\nu_1, \nu_4\}, \{\nu_2, \nu_3\}, \{\nu_2, \nu_4\}, \{\nu_3, \nu_4\}\},$$

and the dominating set with the smallest cardinality is $\{\nu_1, \nu_2\}$. So,

$$\chi(\mathcal{G}) = |\{\nu_1, \nu_2\}| = 0.8$$

Third, we find the bondage sets, bondage number $\mathcal{B}(\mathcal{G})$, non-bondage sets and non-bondage number $\mathcal{B}_n(\mathcal{G})$.

Consider $\hat{K}_1 = \{\varepsilon_2, \varepsilon_4\} \subseteq \hat{\mathcal{S}} = \{\varepsilon_1, \varepsilon_2, \varepsilon_4\}$. Then:

$$\begin{aligned} \hat{\mu}^\infty(\nu_1, \nu_2) &= \hat{\mu}^\infty(\varepsilon_1) = 0.3 = \hat{\mu}(\nu_1, \nu_2) = \hat{\mu}(\varepsilon_1) \\ \hat{\mu}^\infty(\nu_3, \nu_4) &= \hat{\mu}^\infty(\varepsilon_3) = 0.2 = \hat{\mu}(\nu_3, \nu_4) = \hat{\mu}(\varepsilon_3) \\ \chi(\mathcal{G} - \hat{K}_1) &= |\{\nu_2, \nu_4\}| = 0.8 = \chi(\mathcal{G}) \end{aligned}$$

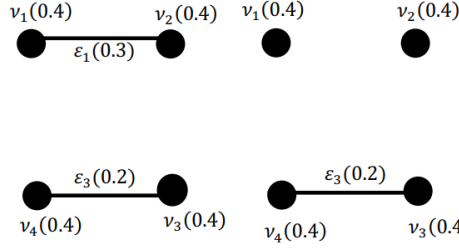
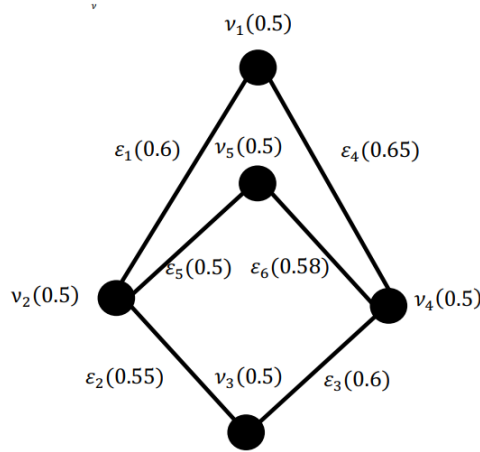
Since $\chi(\mathcal{G} - \hat{K}_1) = \chi(\mathcal{G})$, hence $\hat{K}_1 = \{\varepsilon_1\}$ is a non-bondage set.

For $\hat{K}_2 = \{\varepsilon_1, \varepsilon_2, \varepsilon_4\} \subseteq \hat{\mathcal{S}}$, we find

$$\hat{\mu}^\infty(\nu_3, \nu_4) = \hat{\mu}^\infty(\varepsilon_3) = 0.2 = \hat{\mu}(\nu_3, \nu_4) = \hat{\mu}(\varepsilon_3)$$

Therefore, the set of strong arcs is $\{\varepsilon_3\}$ and the dominating sets of $\mathcal{G} - \hat{K}_2$ are

$$\{\{\nu_1, \nu_2, \nu_3\}, \{\nu_1, \nu_2, \nu_4\}\}$$

Figure 2.3a: Fuzzy Graph
 $\mathcal{G} - \hat{\mathcal{K}}_1$ Figure 2.3b: Fuzzy graph
 $\mathcal{G} - \hat{\mathcal{K}}_2$ Figure 2.4: Anti-fuzzy Graph $\mathcal{G}_{AN} = (\hat{\tau}, \hat{\mu})$

and the dominating set with the smallest cardinality is $\{\nu_1, \nu_2, \nu_3\}$. Thus,

$$\chi(\mathcal{G} - \hat{K}_2) = |\{\nu_1, \nu_2, \nu_3\}| = 1.2 > \chi(\mathcal{G})$$

Hence $\hat{K}_2 = \{\varepsilon_1, \varepsilon_2, \varepsilon_4\}$ is a bondage set.

With similar computations, we find that there is only one bondage set $\{\varepsilon_1, \varepsilon_2, \varepsilon_4\}$ and the non-bondage sets are

$$\{\varepsilon_1\}, \{\varepsilon_2\}, \{\varepsilon_4\}, \{\varepsilon_1, \varepsilon_2\}, \{\varepsilon_1, \varepsilon_4\}, \{\varepsilon_2, \varepsilon_4\}.$$

Thus,

$$\mathcal{B}(\mathcal{G}) = |\{\varepsilon_1, \varepsilon_2, \varepsilon_4\}| = 3$$

$$\mathcal{B}_n(\mathcal{G}) = \max |\{\varepsilon_2, \varepsilon_4\}| = 2$$

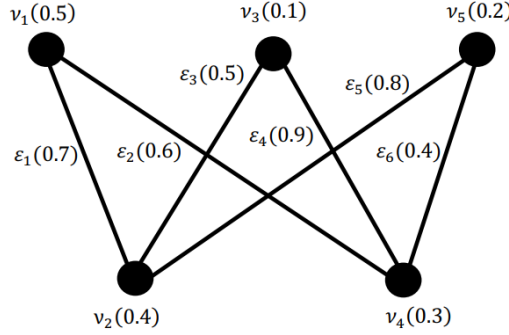
Definition 2.20. [7] A fuzzy graph $\mathcal{G} = (V, \hat{\tau}, \hat{\mu})$, with mappings expressed as $\hat{\tau} : V \rightarrow [0, 1]$ and $\hat{\mu} : V \times V \rightarrow [0, 1]$ is known as an *anti-fuzzy graph* such that

$$\hat{\mu}(\nu_1, \nu_2) \geq \hat{\tau}(\nu_1) \vee \hat{\tau}(\nu_2), \quad \forall \nu_1, \nu_2 \in V.$$

It is denoted as $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$, where $\hat{\mu}$ is reflexive and symmetric.

Example 2.21. Consider the anti-fuzzy graph $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ such that $V = \{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5\}$ and $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ shown in Figure 2.4.

Example 2.22. A path P_{an} of a $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ is an arrangement of nodes $\nu_1, \nu_2, \nu_3, \dots, \nu_n, (n \geq 2), \nu_1 \neq$


 Figure 3.1: Anti-fuzzy Graph $\mathcal{G}_{AN} = (\hat{\tau}, \hat{\mu})$

ν_n such that $\hat{\mu}(\nu_i, \nu_{i+1}) > 0$, $i = 1, 2, \dots, n$. The path strength in \mathcal{G}_{an} is determined by the membership grade of the weakest arc (the arc having the least membership grade).

Example 2.23. The strength of connectedness $\hat{\mu}_{an}^\infty(\nu_1, \nu_2)$ of a $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ is the greatest strength of all paths that link two nodes ν_1 and ν_2 . An arc (ν_1, ν_2) is the strong arc of a \mathcal{G}_{an} if for all $\nu_1, \nu_2 \in V$, such that

$$\hat{\mu}(\nu_1, \nu_2) = \hat{\mu}_{an}^\infty(\nu_1, \nu_2).$$

Definition 2.24. [16] A set $\hat{D} \subseteq V(\mathcal{G}_{an})$ is said to be a dominating set if, for every element $\nu_2 \in V(\mathcal{G}_{an}) \setminus \hat{D}$, there exists an element $\nu_1 \in \hat{D}$ such that ν_1 dominates ν_2 . A vertex ν_1 dominates the vertex $\nu_2 \in V \setminus \{\nu_1\}$ if there is a strong arc between ν_1 and ν_2 ; otherwise, it dominates itself.

Example 2.25. The domination number of $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$, denoted by $\chi(\mathcal{G}_{an})$, is the maximum cardinality among all minimal dominating sets. Equivalently, it is the size of a dominating set with the least number of nodes such that no proper subset of it is also a dominating set.

3. Bondage and Non-Bondage Number of an Anti-Fuzzy Graph

In this section, we focused on bondage set, bondage number, non-bondage set and non-bondage number on anti-fuzzy graph G_{an} . Some bounds on bondage and non-bondage numbers are described, and the relative results are expressed.

Definition 3.1. Let $G_{an} = (\hat{\tau}, \hat{\mu})$ be an anti-fuzzy graph and \hat{S} be the set of all strong arcs. For \hat{K} be a subset of \hat{S} , if

$$\chi(G_{an} - \hat{K}) > \chi(G_{an}),$$

then \hat{K} is said to be a *bondage set* of G_{an} .

Definition 3.2. The *bondage number* $B(G_{an})$ of $G_{an} = (\hat{\tau}, \hat{\mu})$ is the least cardinality among all bondage sets of G_{an} . Moreover,

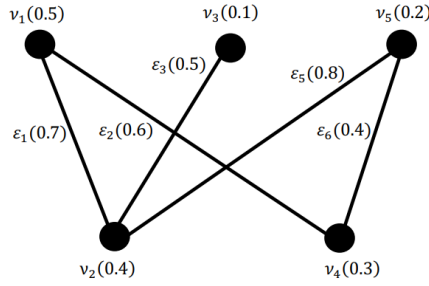
$$\chi(G_{an} - \hat{K}) \not\geq \chi(G_{an})$$

for any set $\hat{K} \subseteq \hat{S}$. Therefore, G_{an} does not have a bondage set.

Example 3.3. Consider the anti-fuzzy graph $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ such that

$$V = \{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5\}, \quad E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\},$$

as shown in Figure 3.1. First, we evaluate the strong arcs of \mathcal{G}_{an} using the definition of strong arcs.

Figure 3.1a: Anti-fuzzy Graph $\mathcal{G}_{AN} - \hat{\mathcal{K}}_1$

$$\begin{aligned}\hat{\mu}_{an}^{\infty}(\nu_1, \nu_2) &= \hat{\mu}_{an}^{\infty}(\varepsilon_1) = \max(0.7, 0.5, 0.4) = 0.7 = \hat{\mu}(\varepsilon_1), \\ \hat{\mu}_{an}^{\infty}(\nu_1, \nu_4) &= \hat{\mu}_{an}^{\infty}(\varepsilon_2) = \max(0.6, 0.5, 0.4) = 0.6 = \hat{\mu}(\varepsilon_2), \\ \hat{\mu}_{an}^{\infty}(\nu_2, \nu_3) &= \hat{\mu}_{an}^{\infty}(\varepsilon_3) = \max(0.5, 0.6, 0.4) = 0.6 \neq \hat{\mu}(\varepsilon_3), \\ \hat{\mu}_{an}^{\infty}(\nu_3, \nu_4) &= \hat{\mu}_{an}^{\infty}(\varepsilon_4) = \max(0.9, 0.4, 0.5) = 0.9 = \hat{\mu}(\varepsilon_4), \\ \hat{\mu}_{an}^{\infty}(\nu_2, \nu_5) &= \hat{\mu}_{an}^{\infty}(\varepsilon_5) = \max(0.8, 0.4, 0.4) = 0.8 = \hat{\mu}(\varepsilon_5), \\ \hat{\mu}_{an}^{\infty}(\nu_4, \nu_5) &= \hat{\mu}_{an}^{\infty}(\varepsilon_6) = \max(0.4, 0.5, 0.6) = 0.6 \neq \hat{\mu}(\varepsilon_6).\end{aligned}$$

Thus, the set of strong arcs is:

$$\hat{S} = \{\varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_5\}.$$

Secondly, the dominating sets of \mathcal{G}_{an} are:

$$\begin{aligned}\{\nu_1, \nu_3\}, \{\nu_2, \nu_4\}, \{\nu_4, \nu_5\}, \{\nu_1, \nu_2, \nu_3\}, \{\nu_1, \nu_2, \nu_4\}, \{\nu_1, \nu_3, \nu_4\}, \{\nu_1, \nu_3, \nu_5\}, \{\nu_1, \nu_4, \nu_5\}, \\ \{\nu_2, \nu_3, \nu_4\}, \{\nu_2, \nu_3, \nu_5\}, \{\nu_2, \nu_4, \nu_5\}, \{\nu_3, \nu_4, \nu_5\}, \{\nu_1, \nu_2, \nu_3, \nu_4\}, \{\nu_1, \nu_2, \nu_3, \nu_5\}, \\ \{\nu_1, \nu_2, \nu_4, \nu_5\}, \{\nu_2, \nu_3, \nu_4, \nu_5\}, \{\nu_1, \nu_3, \nu_4, \nu_5\}.\end{aligned}$$

The dominating set with the smallest cardinality is $\{\nu_1, \nu_3\}$. Hence, by Example 2.25:

$$\chi(\mathcal{G}_{an}) = |\{\nu_1, \nu_3\}| = 0.5.$$

Thirdly, we find the bondage sets and bondage number $\mathcal{B}(\mathcal{G}_{an})$. The bondage sets of \mathcal{G}_{an} are subsets of strong arcs in which elimination results in a greater $\chi(\mathcal{G}_{an})$.

- Let $\hat{K}_1 = \{\varepsilon_4\} \subseteq \hat{S}$. After recalculations (Figure 3.1(a)):

$$\chi(\mathcal{G}_{an} - \hat{K}_1) = |\{\nu_3, \nu_4, \nu_5\}| = 0.6 > \chi(\mathcal{G}_{an}) = 0.5.$$

Hence, \hat{K}_1 is a bondage set.

- Let $\hat{K}_2 = \{\varepsilon_1, \varepsilon_2\} \subseteq \hat{S}$. After recalculations (Figure 3.1(b)):

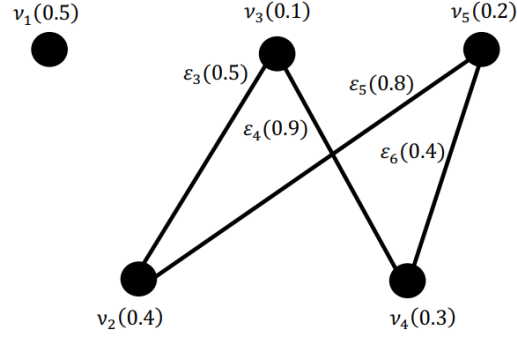
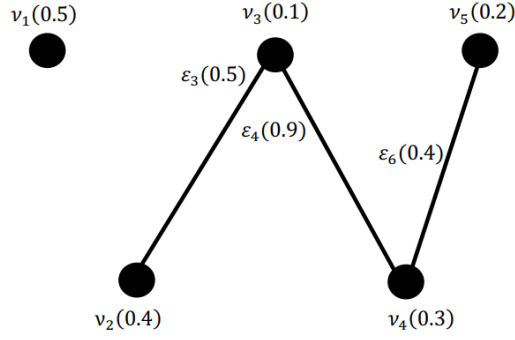
$$\chi(\mathcal{G}_{an} - \hat{K}_2) = |\{\nu_1, \nu_3, \nu_5\}| = 0.8 > \chi(\mathcal{G}_{an}) = 0.5.$$

Hence, \hat{K}_2 is a bondage set.

- Let $\hat{K}_3 = \{\varepsilon_1, \varepsilon_2, \varepsilon_4\} \subseteq \hat{S}$. After recalculations (Figure 3.1(c)):

$$\chi(\mathcal{G}_{an} - \hat{K}_3) = |\{\nu_1, \nu_3, \nu_5\}| = 0.8 > \chi(\mathcal{G}_{an}) = 0.5.$$

Hence, \hat{K}_3 is a bondage set.


 Figure 3.1b: Anti-fuzzy Graph $\mathcal{G}_{AN} - \hat{\mathcal{K}}_2$

 Figure 3.1c: Anti-fuzzy Graph $\mathcal{G}_{AN} - \hat{\mathcal{K}}_3$

Therefore, the bondage sets of \mathcal{G}_{an} are:

$$\{\varepsilon_4\}, \quad \{\varepsilon_1, \varepsilon_2\}, \quad \{\varepsilon_1, \varepsilon_2, \varepsilon_4\}, \quad \{\varepsilon_5\}.$$

Finally, the bondage number is:

$$\mathcal{B}(\mathcal{G}_{an}) = \min \{|\hat{K}|\} = |\{\varepsilon_5\}| = 0.8.$$

Theorem 3.4. Let $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ be an anti-fuzzy graph under a simple graph $\hat{\mathcal{G}} = (V, E)$. If \mathcal{G}_{an} has an isolated arc, then

$$\mathcal{B}(\mathcal{G}_{an}) = 1.$$

Proof: Let $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ be an anti-fuzzy graph and let $\varepsilon = (\nu_1, \nu_2)$ be an isolated arc of \mathcal{G}_{an} .

Obviously, (ν_1, ν_2) is a strong arc. Moreover, one of ν_1 or ν_2 (but not both) is a member of the minimal dominating set of \mathcal{G}_{an} .

After removing (ν_1, ν_2) , both ν_1 and ν_2 become isolated nodes. Therefore, ν_1 and ν_2 must both belong to each dominating set of $\mathcal{G}_{an} \setminus \{\varepsilon\}$.

Hence,

$$\chi(\mathcal{G}_{an} - \varepsilon) > \chi(\mathcal{G}_{an}),$$

which implies that $\varepsilon = (\nu_1, \nu_2)$ is a bondage set of \mathcal{G}_{an} .

Thus,

$$\mathcal{B}(\mathcal{G}_{an}) = 1.$$

Theorem 3.5. Let $\mathcal{G}_{an} = (V, E)$ be an anti-fuzzy graph under a simple graph $\hat{\mathcal{G}} = (V, E)$. If \mathcal{G}_{an} has a bondage set, then

$$\mathcal{B}(\mathcal{G}_{an}) \leq \min \left\{ |N_{\hat{S}}(\nu_1)| + |N_{\hat{S}}(\nu_2)| - 1 + |\hat{S}'| : \nu_1 \text{ and } \nu_2 \text{ are strong neighbours} \right\},$$

where $\hat{S}' = E - \hat{S}$ is the set of all non-strong arcs of \mathcal{G}_{an} . \square

Let $\mathcal{G}_{an} = (V, E)$ be an anti-fuzzy graph that has a bondage set. We prove the theorem by considering two cases.

Case (i): $\hat{S}' = \emptyset$. Then there are no non-strong arcs in \mathcal{G}_{an} . Let

$$\eta = \min \left\{ |N_{\hat{S}}(\nu_1)| + |N_{\hat{S}}(\nu_2)| - 1 + |\hat{S}'| : \nu_1, \nu_2 \text{ are strong neighbours} \right\}.$$

Suppose ν_1 and ν_2 are strong neighbours in \mathcal{G}_{an} such that

$$\eta = |N_{\hat{S}}(\nu_1)| + |N_{\hat{S}}(\nu_2)| - 1 + |\hat{S}'|.$$

Assume $\mathcal{B}(\mathcal{G}_{an}) > \eta$ and let \hat{S}_1 denote the set of strong arcs incident to either ν_1 or ν_2 . Then $|\hat{S}_1| = \eta$ and

$$\chi(\mathcal{G}_{an} - \hat{S}_1) = \chi(\mathcal{G}_{an}).$$

In $\mathcal{G}_{an} - \hat{S}_1$, no strong arc is incident on either ν_1 or ν_2 , so

$$\chi(\mathcal{G}_{an} - \nu_1 - \nu_2) = \chi(\mathcal{G}_{an}) - 2.$$

If $\mathcal{G}_{an} - \nu_1 - \nu_2$ has a minimum dominating set \hat{D} , then the cardinality of $\hat{D} - \{\nu_1\}$ is $\chi(\mathcal{G}_{an}) - 1$, which is a contradiction. Hence,

$$\mathcal{B}(\mathcal{G}_{an}) \leq \eta.$$

Case (ii): $\hat{S}' \neq \emptyset$. We prove this by induction on $|\hat{S}'|$.

If $|\hat{S}'| = 1$, then there is only one non-strong arc in \mathcal{G}_{an} , say ε . By Case (i), we have

$$\mathcal{B}(\mathcal{G}_{an} - \varepsilon) \leq \min \left\{ |N_{\hat{S}}(\nu_1)| + |N_{\hat{S}}(\nu_2)| - 1 + |\hat{S}'| : \nu_1, \nu_2 \text{ are strong neighbours} \right\}.$$

Let \hat{K} be a bondage set of $\mathcal{G}_{an} - \varepsilon$ such that $|\hat{K}| = \mathcal{B}(\mathcal{G}_{an} - \varepsilon)$. If ε is not a strong arc in $\mathcal{G}_{an} - \hat{K}$, then \hat{K} is also a bondage set of \mathcal{G}_{an} , and so

$$\mathcal{B}(\mathcal{G}_{an} - \varepsilon) = \mathcal{B}(\mathcal{G}_{an}).$$

If ε is a strong arc in $\mathcal{G}_{an} - \hat{K}$, then for some $\varepsilon_1 \in \hat{S} - \hat{K}$, the set $\hat{K} \cup \{\varepsilon_1\}$ is a bondage set of \mathcal{G}_{an} . Thus,

$$\mathcal{B}(\mathcal{G}_{an}) \leq \mathcal{B}(\mathcal{G}_{an} - \varepsilon) + 1.$$

Hence, the result holds for $|\hat{S}'| = 1$.

Now assume the result is true for all $|\hat{S}'| < h$. We show it holds for $|\hat{S}'| = h$.

Let $\varepsilon \in \hat{S}'$. Then $\mathcal{G}_{an} - \varepsilon$ has $h - 1$ non-strong arcs. By induction,

$$\mathcal{B}(\mathcal{G}_{an} - \varepsilon) \leq \min \left\{ |N_{\hat{S}}(\nu_1)| + |N_{\hat{S}}(\nu_2)| - 1 + |\hat{S}'| : \nu_1, \nu_2 \text{ are strong neighbours} \right\}.$$

If ε is not strong in $\mathcal{G}_{an} - \hat{D}_1$ for some bondage set \hat{D}_1 , then \hat{D}_1 is also a bondage set of \mathcal{G}_{an} , so

$$\mathcal{B}(\mathcal{G}_{an} - \varepsilon) = \mathcal{B}(\mathcal{G}_{an}).$$

Otherwise, if ε is strong, then for some $\varepsilon_1 \in \hat{S} - \hat{D}_1$, the set $\hat{D}_1 \cup \{\varepsilon_1\}$ is a bondage set of \mathcal{G}_{an} . Thus,

$$\mathcal{B}(\mathcal{G}_{an}) \leq \mathcal{B}(\mathcal{G}_{an} - \varepsilon) + 1.$$

Therefore,

$$\mathcal{B}(\mathcal{G}_{an}) \leq \min \left\{ |N_{\hat{S}}(\nu_1)| + |N_{\hat{S}}(\nu_2)| - 1 + |\hat{S}'| : \nu_1, \nu_2 \text{ are strong neighbours} \right\}.$$

This completes the proof.

Theorem 3.6. If $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ be an anti-fuzzy graph which is a complete anti-fuzzy graph under a simple graph $\hat{\mathcal{G}} = (V, E)$, then

$$\mathcal{B}(\mathcal{G}_{an}) = \begin{cases} \frac{m}{2}, & \text{if } m \text{ is even,} \\ \frac{m+1}{2}, & \text{if } m \text{ is odd.} \end{cases}$$

Proof: Let $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ be an anti-fuzzy graph with m nodes $(l_1, l_2, l_3, \dots, l_m)$. In \mathcal{G}_{an} each node dominates all other $(m-1)$ nodes. Hence

$$\{\{l_1\}, \{l_2\}, \{l_3\}, \dots, \{l_m\}\}$$

are all minimal dominating sets of \mathcal{G}_{an} and $\chi(\mathcal{G}_{an}) = 1$.

Now remove the arc $\varepsilon_1 = (\nu_1, \nu_2)$. Then ν_1 and ν_2 dominate all $(m-2)$ nodes except ν_2 and ν_1 respectively. Similarly, we remove the arcs

$$\varepsilon_1 = (\nu_1, \nu_2), \varepsilon_2 = (\nu_2, \nu_3), \varepsilon_3 = (\nu_3, \nu_4), \dots, \varepsilon_{m-1} = (\nu_{m-1}, \nu_m), \varepsilon_m = (\nu_m, \nu_{m-1}).$$

Further, we will discuss in two cases:

Case (i): If m is even, then we remove the arcs $\varepsilon_1, \varepsilon_3, \varepsilon_5, \dots$ and we get $m/2$ such arcs, forming a bondage set of \mathcal{G}_{an} . Thus $\mathcal{B}(\mathcal{G}_{an}) = m/2$.

Case (ii): If m is odd, then we remove the arcs $\varepsilon_2, \varepsilon_4, \varepsilon_6, \dots$ and we get $(m+1)/2$ such arcs, forming a bondage set of \mathcal{G}_{an} . Hence $\mathcal{B}(\mathcal{G}_{an}) = (m+1)/2$. \square

Theorem 3.7. If $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ is an anti-fuzzy graph under a star graph $\hat{\mathcal{G}} = (V, E)$, then $\mathcal{B}(\mathcal{G}_{an}) = 1$.

Proof:

Let $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ be an anti-fuzzy graph under a star graph $\hat{\mathcal{G}} = (V, E)$. In this anti-fuzzy graph \mathcal{G}_{an} , all arcs are strong arcs and the center node dominates all other nodes. So $\chi(\mathcal{G}_{an}) = 1$.

Removal of any arc $\varepsilon = (\nu_1, \nu_2)$ from \mathcal{G}_{an} results in $\chi(\mathcal{G}_{an} - \varepsilon) = 2 > \chi(\mathcal{G}_{an})$. Hence each arc will form a bondage set and the bondage number $\mathcal{B}(\mathcal{G}_{an}) = 1$. \square

Theorem 3.8. If $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ is an anti-fuzzy graph under a cycle graph $\hat{\mathcal{G}} = (V, E)$ with m nodes, then

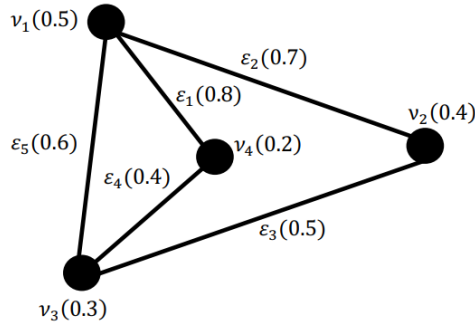
$$\mathcal{B}(\mathcal{G}_{an}) = \begin{cases} 3, & \text{if } m = 3n + 1, n = 1, 2, 3, \dots, \\ 2, & \text{otherwise.} \end{cases}$$

Proof: Let $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ be an anti-fuzzy graph under a cycle graph $\hat{\mathcal{G}} = (V, E)$ with m nodes. We prove by two cases:

Case (i): Suppose there are more than one weakest arcs in \mathcal{G}_{an} . Then all arcs of \mathcal{G}_{an} are strong arcs. If $m = 3n + 1$, then $\chi(\mathcal{G}_{an}) = m + 1$. The domination number increases only if we remove at least 3 strong arcs. Thus $\mathcal{B}(\mathcal{G}_{an}) = 3$.

If $m \neq 3n + 1$, then the domination number increases only if we remove at least 2 strong arcs adjacent to the same node. Thus $\mathcal{B}(\mathcal{G}_{an}) = 2$.

Case (ii): Suppose ε is the only weakest arc in \mathcal{G}_{an} . Then the strong arcs of \mathcal{G}_{an} are $(m-1)$. Removal of any one strong arc makes the weakest arc a strong arc. Clearly, $\mathcal{B}(\mathcal{G}_{an}) = 3$ if $m = 3n + 1$ and $\mathcal{B}(\mathcal{G}_{an}) = 2$ if $m \neq 3n + 1$, but no weakest arc belongs to any bondage set of \mathcal{G}_{an} . \square

Figure 3.2: Anti-fuzzy Graph $\mathcal{G}_{AN} = (\hat{\tau}, \hat{\mu})$

Definition 3.9. Let $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ be an anti-fuzzy graph and $\hat{\mathcal{S}}$ be the set of all strong arcs. For $\hat{\mathcal{K}} \subseteq \hat{\mathcal{S}}$, if $\chi(\mathcal{G}_{an} - \hat{\mathcal{K}}) = \chi(\mathcal{G}_{an})$, then $\hat{\mathcal{K}}$ is said to be a *non-bondage set* of \mathcal{G}_{an} .

Definition 3.10. The non-bondage number $\mathcal{B}_n(\mathcal{G}_{an})$ of $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ is the greatest cardinality among all non-bondage sets of \mathcal{G}_{an} .

Example 3.11. Consider the anti-fuzzy graph $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ such that $V = \{\nu_1, \nu_2, \nu_3, \nu_4\}$ and $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\}$ shown in Figure 3.2. First of all, we will evaluate the strong arcs of \mathcal{G}_{an} . Using the definition of strong arcs, we will find the strength of connectedness:

$$\hat{\mu}_{an}^{\infty}(\nu_1, \nu_4) = \hat{\mu}_{an}^{\infty}(\varepsilon_1) = \max(0.8, 0.4, 0.4) = 0.8 = \hat{\mu}(\varepsilon_1)$$

$$\hat{\mu}_{an}^{\infty}(\nu_1, \nu_2) = \hat{\mu}_{an}^{\infty}(\varepsilon_2) = \max(0.7, 0.5, 0.4) = 0.7 = \hat{\mu}(\varepsilon_2)$$

$$\hat{\mu}_{an}^{\infty}(\nu_2, \nu_3) = \hat{\mu}_{an}^{\infty}(\varepsilon_3) = \max(0.5, 0.6, 0.4) = 0.6 \neq \hat{\mu}(\varepsilon_3)$$

$$\hat{\mu}_{an}^{\infty}(\nu_3, \nu_4) = \hat{\mu}_{an}^{\infty}(\varepsilon_4) = \max(0.4, 0.5, 0.6) = 0.6 \neq \hat{\mu}(\varepsilon_4)$$

$$\hat{\mu}_{an}^{\infty}(\nu_1, \nu_3) = \hat{\mu}_{an}^{\infty}(\varepsilon_5) = \max(0.6, 0.5, 0.4) = 0.6 = \hat{\mu}(\varepsilon_5)$$

Then by definition of strong arcs we get:

$$\text{Set of strong arcs} = \hat{\mathcal{S}} = \{\varepsilon_1, \varepsilon_2, \varepsilon_5\}.$$

Secondly, the dominating sets of \mathcal{G}_{an} are:

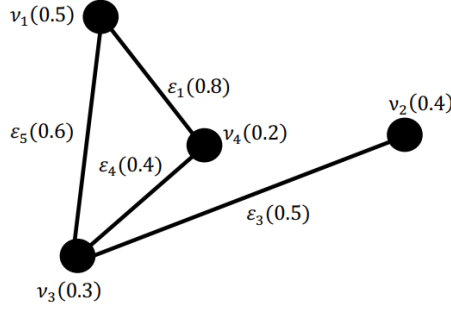
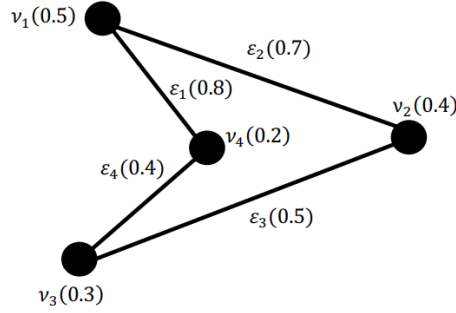
$$\{\{\nu_1\}, \{\nu_2\}, \{\nu_3\}, \{\nu_4\}, \{\nu_1, \nu_2\}, \{\nu_2, \nu_3\}, \{\nu_3, \nu_4\}, \{\nu_1, \nu_3\}, \{\nu_1, \nu_4\}\}$$

and then the dominating set with the smallest cardinality is $\{\nu_1\}$, so by Definition 2.25:

$$\chi(\mathcal{G}_{an}) = |\{\nu_1\}| = 0.5.$$

Thirdly, we find the bondage sets and bondage number $\mathbb{B}(\mathcal{G}_{an})$. The bondage sets of \mathcal{G}_{an} are subsets of strong arcs in which elimination from \mathcal{G}_{an} results in $\chi(\mathcal{G}_{an} - \hat{\mathcal{K}}) \geq \chi(\mathcal{G}_{an})$.

Consider $\hat{\mathcal{K}}_1 = \{\varepsilon_2\}$ as a subset of the strong arcs $\hat{\mathcal{S}} = \{\varepsilon_1, \varepsilon_2, \varepsilon_5\}$. Then we calculate the strong arcs of $\mathcal{G}_{an} - \hat{\mathcal{K}}_1$ (Figure 3.2(a)):


 Figure 3.2a: Anti-fuzzy Graph $\mathcal{G}_{AN} = (\hat{\tau}, \hat{\mu})$

 Figure 3.2b: Anti-fuzzy Graph $\mathcal{G}_{AN} = (\hat{\tau}, \hat{\mu})$

$$\hat{\mu}_{an}^{\infty}(\nu_1, \nu_4) = \hat{\mu}_{an}^{\infty}(\varepsilon_1) = \max(0.8, 0.4) = 0.8 = \hat{\mu}(\varepsilon_1)$$

$$\hat{\mu}_{an}^{\infty}(\nu_2, \nu_3) = \hat{\mu}_{an}^{\infty}(\varepsilon_3) = 0.5 = \hat{\mu}(\varepsilon_3)$$

$$\hat{\mu}_{an}^{\infty}(\nu_3, \nu_4) = \hat{\mu}_{an}^{\infty}(\varepsilon_4) = \max(0.4, 0.6) = 0.6 \neq \hat{\mu}(\varepsilon_4)$$

$$\hat{\mu}_{an}^{\infty}(\nu_1, \nu_3) = \hat{\mu}_{an}^{\infty}(\varepsilon_5) = \max(0.6, 0.4) = 0.6 = \hat{\mu}(\varepsilon_5)$$

Thus,

$$\hat{\mathcal{S}} = \{\varepsilon_1, \varepsilon_3, \varepsilon_5\}.$$

The dominating sets of $\mathcal{G}_{an} - \hat{\mathcal{K}}_1$ are

$$\{\{\nu_1, \nu_2\}, \{\nu_1, \nu_3\}, \{\nu_3, \nu_4\}, \{\nu_2, \nu_4\}\}.$$

The dominating set with smallest cardinality is $\{\nu_3, \nu_4\}$, hence

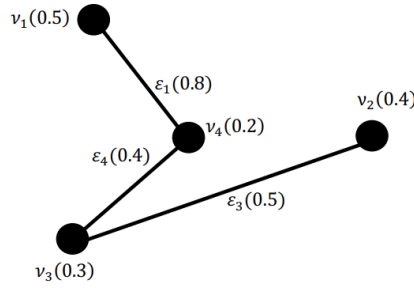
$$\chi(\mathcal{G}_{an} - \hat{\mathcal{K}}_1) = |\{\nu_3, \nu_4\}| = 0.5 = \chi(\mathcal{G}_{an}).$$

Thus $\hat{\mathcal{K}}_1 = \{\varepsilon_2\}$ is a non-bondage set. —

Similarly, consider $\hat{\mathcal{K}}_2 = \{\varepsilon_5\}$ and $\hat{\mathcal{K}}_3 = \{\varepsilon_2, \varepsilon_5\}$. After similar computations, we obtain that they are also non-bondage sets.

Hence, the non-bondage sets of \mathcal{G}_{an} are:

$$\{\{\varepsilon_2\}, \{\varepsilon_5\}, \{\varepsilon_2, \varepsilon_5\}\}.$$

Figure 3.2c: Anti-fuzzy Graph $\mathcal{G}_{AN} = (\hat{\tau}, \hat{\mu})$

Therefore, the non-bondage number is

$$\mathbb{B}_n(\mathcal{G}_{an}) = \max\{|\hat{\mathcal{K}}|\} = |\{\epsilon_2, \epsilon_5\}| = 0.13.$$

Theorem 3.12. If $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ is a complete anti-fuzzy graph with m nodes under a simple graph $\hat{\mathcal{G}} = (V, E)$, then

$$\mathbb{B}_n(\mathcal{G}_{an}) = \frac{(m-1)(m-2)}{2}.$$

Proof: Let $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ be an anti-fuzzy graph with m nodes $(l_1, l_2, l_3, \dots, l_m)$. In \mathcal{G}_{an} , there are all strong arcs. So, the total number of arcs in \mathcal{G}_{an} is $\frac{m(m-1)}{2}$.

Since $\chi(\mathcal{G}_{an}) = 1$, each node in \mathcal{G}_{an} dominates each other. So, we need at least $(m-1)$ arcs to keep $\chi(\mathcal{G}_{an}) = 1$.

Therefore, we can maximally remove $|\hat{\mathcal{S}}| - (m-1)$ arcs. Thus,

$$\mathbb{B}_n(\mathcal{G}_{an}) = |\hat{\mathcal{S}}| - (m-1) = \frac{m(m-1)}{2} - (m-1) = \frac{(m-1)(m-2)}{2}.$$

□

Theorem 3.13. If $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ is an anti-fuzzy graph under a star graph $\hat{\mathcal{G}} = (V, E)$, then

$$\mathbb{B}_n(\mathcal{G}_{an}) = 0.$$

Proof:

Let $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ be an anti-fuzzy graph under a star graph $\hat{\mathcal{G}} = (V, E)$. Then the domination number of \mathcal{G}_{an} is one.

Therefore, the node in the center dominates all other nodes. So, $\chi(\mathcal{G}_{an}) = 1$.

In \mathcal{G}_{an} , all arcs are strong arcs, and removal of any arc $\epsilon = (\nu_1, \nu_2)$ from \mathcal{G}_{an} results in $\chi(\mathcal{G}_{an}) = 2$.

Hence, we have no non-bondage set in $\mathcal{G}_{\hat{P}, V}$. Therefore, $\mathbb{B}_n(\mathcal{G}_{an}) = 0$. □

Theorem 3.14. If $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ is an anti-fuzzy graph, then

$$\mathbb{B}_n(\mathcal{G}_{an}) = |E| - |V| + \chi(\mathcal{G}_{an}).$$

Proof:

Let $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ be an anti-fuzzy graph. Let \hat{D} be a minimal dominating set of \mathcal{G}_{an} such that $\chi(\mathcal{G}_{an}) = |\hat{D}|$.

For each node $\nu_1 \in V - \hat{D}$, there is exactly one strong arc that is incident to ν_1 and to a node in \hat{D} . Let $\hat{\mathcal{S}}_1$ be the set of all such strong arcs.

Then obviously, $\hat{\mathcal{S}} - \hat{\mathcal{S}}_1$ is a non-bondage set of \mathcal{G}_{an} if there is no strong arc in \mathcal{G}_{an} .

Consider \mathcal{G}_{an} has non-strong arcs, then each such arc become strong by removing the corresponding strong arc in \mathcal{G}_{an} .

Therefore,

$$\mathbb{B}_n(\mathcal{G}_{an}) = |\hat{\mathcal{S}}| - (|V| - \chi(\mathcal{G}_{an})) + |\hat{\mathcal{S}}_1| = |\hat{\mathcal{S}}| + (|E - \hat{\mathcal{S}}| + \chi(\mathcal{G}_{an})) - |V| = |E| - |V| + \chi(\mathcal{G}_{an}).$$

□

Theorem 3.15. If $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ is an anti-fuzzy graph and \mathcal{G}_{an} does not have a bondage set, then

$$\mathbb{B}_n(\mathcal{G}_{an}) = |\hat{\mathcal{S}}|.$$

Proof:

Let $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ be an anti-fuzzy graph and \mathcal{G}_{an} does not have a bondage set such that there is no set $\hat{\mathcal{K}} \subseteq \hat{\mathcal{S}}$ with $\chi(\mathcal{G}_{an} - \hat{\mathcal{K}}) > \chi(\mathcal{G}_{an})$.

So, the domination number of \mathcal{G}_{an} does not increase if we remove all strong arcs of \mathcal{G}_{an} .

Now remove the set of all strong arcs, then the domination number becomes $\chi(\mathcal{G}_{an} - \hat{\mathcal{K}}) = \chi(\mathcal{G}_{an})$.

So,

$$\mathbb{B}_n(\mathcal{G}_{an}) = |\hat{\mathcal{S}}|.$$

□

Theorem 3.16. If $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ is an anti-fuzzy graph and \mathcal{G}_{an} has a bondage set, then

$$\mathbb{B}(\mathcal{G}_{an}) \leq \mathbb{B}_n(\mathcal{G}_{an}) + 1.$$

Proof: Let $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ be an anti-fuzzy graph and \mathcal{G}_{an} has a bondage set.

By definition, the non-bondage number set $\{\mathbb{B}_n\}$ is the maximum non-bondage set such that removal of each arc in a non-bondage number set results in $\chi(\mathcal{G}_{an} - \{\mathbb{B}_n\}) = \chi(\mathcal{G}_{an})$.

Therefore, removal of any strong arc $\varepsilon \notin \{\mathbb{B}_n\}$ with the arcs in the set $\{\mathbb{B}_n\}$ results in

$$\chi(\mathcal{G}_{an} - (\{\mathbb{B}_n\} \cup \varepsilon)) > \chi(\mathcal{G}_{an}),$$

which gives $\{\mathbb{B}_n\} \cup \varepsilon$ is a bondage set.

Thus,

$$\mathbb{B}(\mathcal{G}_{an}) \leq |\mathcal{G}_{an} - \{\mathbb{B}_n\}| = \mathbb{B}_n(\mathcal{G}_{an}) + 1.$$

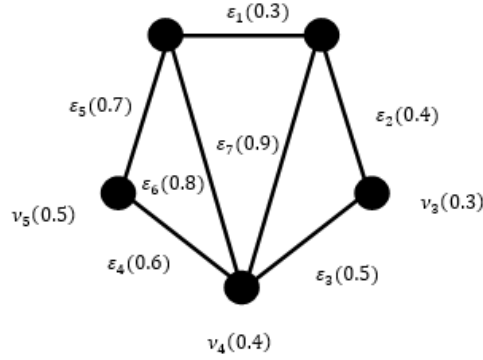
□

4. Comparison

Graphs are used for the representation of various issues and after presentation we become able to find more appropriate solution of that issue. In graph we use nodes and lines that join these nodes according to situation. From these nodes and connections, we demonstrate any kind of problem. While studying a fuzzy graph, the connection explains the least value among the nodes. Sometimes, we face complexity when it may lead to get maximum range. In such situations, the problem is constructed as an 'Anti-fuzzy Graph Model'. The study of the bondage numbers and non-bondage numbers of antifuzzy graphs are more reliable. These facts are significant for determining stability and are more generalized both theoretically and applicably compared to fuzzy graphs. Also, the anti-fuzzy graphs with a lot of variations and generalization are now well discussed. To make a perfectly comparison, we evaluate the bondage and non-bondage number of fuzzy graphs in example 2.19 as compared to calculate the bondage and non-bondage number of anti-fuzzy graph in Example 4.1.

Example 4.1. Consider the anti-fuzzy graph $\mathcal{G}_{an} = (\hat{\tau}, \hat{\mu})$ such that

$$V = \{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5\}, \quad E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$$

Figure 4.1: Anti-fuzzy Graph $\mathcal{G}_{AN} = (\hat{\mathcal{T}}, \hat{\mu})$

as shown in Figure 4.1.

Step 1: Strong arcs. Using the definition of strong arcs, we evaluate the strength of connectedness:

$$\begin{aligned} \hat{\mu}_{an}^{\infty}(\nu_1, \nu_2) &= \hat{\mu}_{an}^{\infty}(\varepsilon_1) = \max(0.3, 0.8, 0.4) = 0.8 \neq \hat{\mu}(\varepsilon_1), \\ \hat{\mu}_{an}^{\infty}(\nu_1, \nu_4) &= \hat{\mu}_{an}^{\infty}(\varepsilon_6) = \max(0.8, 0.3, 0.6) = 0.8 = \hat{\mu}(\varepsilon_6), \\ \hat{\mu}_{an}^{\infty}(\nu_1, \nu_5) &= \hat{\mu}_{an}^{\infty}(\varepsilon_5) = \max(0.7, 0.6, 0.3) = 0.7 = \hat{\mu}(\varepsilon_5), \\ \hat{\mu}_{an}^{\infty}(\nu_2, \nu_3) &= \hat{\mu}_{an}^{\infty}(\varepsilon_2) = \max(0.4, 0.5, 0.3) = 0.5 \neq \hat{\mu}(\varepsilon_2), \\ \hat{\mu}_{an}^{\infty}(\nu_3, \nu_4) &= \hat{\mu}_{an}^{\infty}(\varepsilon_3) = \max(0.5, 0.4, 0.3) = 0.5 = \hat{\mu}(\varepsilon_3), \\ \hat{\mu}_{an}^{\infty}(\nu_4, \nu_5) &= \hat{\mu}_{an}^{\infty}(\varepsilon_4) = \max(0.6, 0.7, 0.3) = 0.7 \neq \hat{\mu}(\varepsilon_4), \\ \hat{\mu}_{an}^{\infty}(\nu_2, \nu_4) &= \hat{\mu}_{an}^{\infty}(\varepsilon_7) = \max(0.9, 0.4, 0.3) = 0.9 = \hat{\mu}(\varepsilon_7). \end{aligned}$$

Hence, the set of strong arcs is:

$$\hat{\mathcal{S}} = \{\varepsilon_3, \varepsilon_5, \varepsilon_6, \varepsilon_7\}.$$

Step 2: Dominating sets. The dominating sets of \mathcal{G}_{an} are:

$$\begin{aligned} &\{\nu_1, \nu_4\}, \{\nu_4, \nu_5\}, \{\nu_1, \nu_2, \nu_4\}, \{\nu_1, \nu_3, \nu_4\}, \{\nu_1, \nu_4, \nu_5\}, \{\nu_2, \nu_3, \nu_5\}, \{\nu_2, \nu_4, \nu_5\}, \{\nu_3, \nu_4, \nu_5\}, \\ &\{\nu_1, \nu_2, \nu_3, \nu_4\}, \{\nu_1, \nu_2, \nu_3, \nu_5\}, \{\nu_1, \nu_2, \nu_4, \nu_5\}, \{\nu_2, \nu_3, \nu_4, \nu_5\}, \{\nu_1, \nu_3, \nu_4, \nu_5\}. \end{aligned}$$

The dominating set with the smallest cardinality is $\{\nu_1, \nu_4\}$. Thus, by Definition 2.25:

$$\chi(\mathcal{G}_{an}) = |\{\nu_1, \nu_4\}| = 0.5.$$

Step 3: Bondage sets. Consider $\hat{\mathcal{K}}_1 = \{\varepsilon_3, \varepsilon_5\} \subseteq \hat{\mathcal{S}}$. After elimination, the strong arcs become:

$$\hat{\mathcal{S}} = \{\varepsilon_2, \varepsilon_4, \varepsilon_6, \varepsilon_7\}.$$

The dominating set with smallest cardinality is $\{\nu_2, \nu_4\}$, giving

$$\chi(\mathcal{G}_{an} - \hat{\mathcal{K}}_1) = |\{\nu_2, \nu_4\}| = 0.6 > \chi(\mathcal{G}_{an}) = 0.5.$$

Hence, $\hat{\mathcal{K}}_1$ is a bondage set.

Now consider $\hat{\mathcal{K}}_2 = \{\varepsilon_5, \varepsilon_6\}$. The set of strong arcs becomes:

$$\hat{\mathcal{S}} = \{\varepsilon_1, \varepsilon_3, \varepsilon_4, \varepsilon_7\}.$$

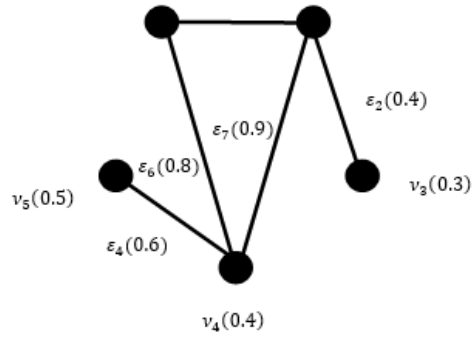


Figure 4.1(a): Anti-fuzzy Graph $\mathcal{G}_{AN} = (\hat{\tau}, \hat{\mu})$

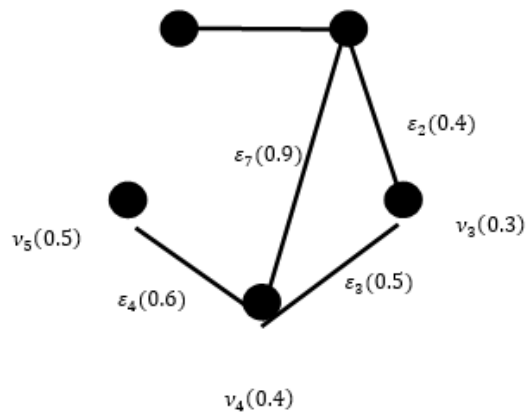


Figure 4.1(b): Anti-fuzzy Graph $\mathcal{G}_{AN} = (\hat{\tau}, \hat{\mu})$

The dominating set with smallest cardinality is $\{\nu_1, \nu_4\}$, giving

$$\chi(\mathcal{G}_{an} - \hat{\mathcal{K}}_2) = 0.5 = \chi(\mathcal{G}_{an}).$$

Hence, $\hat{\mathcal{K}}_2$ is a non-bondage set. **Conclusion.** By similar calculations, the bondage sets of \mathcal{G}_{an} are:

$$\{\varepsilon_3, \varepsilon_5\}, \{\varepsilon_5, \varepsilon_7\}, \{\varepsilon_3, \varepsilon_5, \varepsilon_6\}, \{\varepsilon_3, \varepsilon_5, \varepsilon_7\}, \{\varepsilon_5, \varepsilon_6, \varepsilon_7\}, \{\varepsilon_3, \varepsilon_6, \varepsilon_7\}, \{\varepsilon_3, \varepsilon_5, \varepsilon_6, \varepsilon_7\},$$

and the non-bondage sets are:

$$\{\varepsilon_6\}, \{\varepsilon_5, \varepsilon_6\}.$$

Therefore, the bondage number is:

$$\mathcal{B}(\mathcal{G}_{an}) = \min |\{\varepsilon_3, \varepsilon_5\}| = 1.2,$$

and the non-bondage number is:

$$\mathcal{B}_n(\mathcal{G}_{an}) = \max |\{\varepsilon_5, \varepsilon_6\}| = 1.5.$$

5. Conclusion

A real-life issue can be illuminated by representing the circumstance as a graphical demonstration. A graph is one of the most appropriate ways of representing the relationship between objects and the simplest way to arrive at a solution. In a graph, vertices and edges represent the objects and their connections. In some cases, there are uncertainties in the objects' representation, connections, or both, which need to be outlined as a fuzzy graph model. In a fuzzy graph, the connection represents only the minimum value among the objects. Sometimes, if complexity arises among the relations, it may lead to obtaining maximum values. In such cases, the problem is constructed as an 'Antifuzzy Graph Model'. The model of an antifuzzy graph is widely implemented and able to give compatibility, accuracy, and precision for the graphical structures. Antifuzzy graphs are more accurate than fuzzy graphs and have been applied in various fields of computer communications setups and engineering. In the present research, there is a brief description of results linked to extensions of several ideas of anti-fuzzy graphs. Also, bondage set, bondage number of anti-fuzzy graphs are introduced, and some results in this work are helpful to understand this new work. Further, non-bondage sets and non-bondage numbers are examined in antifuzzy graphs. These bondage and non-bondage numbers add vagueness and open the thoughts to new research. For future work, we plan to extend our research to find the bondage set, non-bondage set, bondage number, and non-bondage number for (I) Bipolar fuzzy graphs, (II) Intuitionistic fuzzy graphs, (III) Bipolar fuzzy hypergraphs, (IV) Rough fuzzy graphs, (V) Regular Intuitionistic fuzzy graphs (VI) Fuzzy soft graphs, and (VII) Fuzzy graph structures.

Funding: A Funding Declaration has been added to the manuscript. It states: "This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors."

Author Contribution: The Author Contribution section has been revised to include all authors. The updated statement now reads: "Muhammad Tahir Rasool contributed to the conceptualization and methodology. Aliya Fahmi supervised the research and contributed to the review and editing of the manuscript."

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