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Study of MHD flow of hybrid nanofluid (Graphene-Al_2O_3/PAO oil) with variable viscosity and thermal conductivity

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ABSTRACT: The investigation explores the heat transfer of (Graphene-Al_2O_3/Poly-alpha-olefin Oil) hybrid nanofluid with variable thermal conductivity and viscosity. The MHD hybrid nanofluid is considered in the unsteady radially stretching sheet. The present investigation explores the variation of external heat generation and internal friction on variable thermal conductivity ($k_{shnf}(T)$) and viscosity ($\mu_{shnf}(T)$) of the hybrid nanofluid. By utilizing suitable transformations, the governing PDEs are transformed into nonlinear ODEs. To solve the nonlinear ODEs in MATLAB, we utilize the bvp4c numerical technique. The variation of different nondimensional parameters on velocity, temperature, and heat transfer rate is discussed through graphs. Also, the Nusselt number and skin friction factor are discussed to measure the heat transfer rate and friction, as shown in the table. The velocity and temperature profiles degrade with an increase in variable viscosity, while a reverse trend is noticed in velocity and temperature with an increase in the thermal conductivity parameter. The temperature profile diminishes with an acceleration in the unsteadiness parameter and Prandtl number. The heat transfer rate increases with an increase in variable thermal conductivity and viscosity. Heat transfer enhancement increases with an increase in volume fractions of nanoparticles.

Key Words: Variable viscosity, stretching sheet, variable thermal conductivity, MHD flow, Graphene-Al_2O_3/Poly-alpha-olefin oil mixture, heat transfer.

Contents

List of symbols

t	time variable
c	positive constant
T_{ref}	reference temperature
T_∞	Ambient temperature
ν_{bf}	Kinematic viscosity
Q_0	coefficient of heat generation
B_0	magnetic field strength
ρ	fluid density
α	constant
k	thermal conductivity
$\frac{c}{(1-\alpha t)}$	stretching rate
c_p	specific heat
$f(\eta)$	dimensionless stream function
μ	viscosity coefficient
σ	electrical conductivity

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1. Introduction

The investigation about Heat transfer of MHD hybrid nanofluid with different geometry has many application in industrial area, heat exchanging system, chemical industry, gas turbine, nuclear reactors, biological sensing, solar system, textile industries, medical science and many technology. Firstly Choi and Eastmann [1], discovered about nanofluid and their thermal conductivity properties, and compare with normal fluid. In their study, they noticed that thermal conductivity of nanofluid is high. The study about MHD williamson nanofluid and thermal radiation effect through a stretching cylinder discovered by Bilal et al [2], take into consideration variable thermal conductivity. The investigation about heat transfer with boundary layer flow of $(TiO_2 - Al_2O_3)$/water hybrid nanofluid through a moving plate, studied by Aladdin et al [3]. The study about MHD williamson fluid with slip effect through a melting stretching surface, discussed by Gupta et al [4]. The variation of heat sink and thermal radiation on $\gamma - Al_2O_3$ nanofluid, investigated by Agrawal et al [5] with magneto marangoni flow, into a porous medium. The study about $(Al_2O_3 - Cu)$/water hybrid nanofluid with MHD flow, discovered by Jawed et al [6], with effect of melting heat transfer into stretching sheet. Gupta et al [7, 8], investigated about thermal radiation impact of MHD nanofluid, considering marangoni convection flow along a stretching surface.

The significant impact of thermal radiation on three different set of hybrid nanofluid, illustrated by Agrawal et al [9], with MHD flow into porous stretching surface. The heat transfer enhancement for different shape of Cu nanoparticles into ethylene glycol base fluid, studied by Iqbal et al [10] into radially stretching sheet. They noticed highest flow in platelet shape nanoparticles, and highest heat transfer rate in sphere shape nanoparticles. Kumar et al [11], studied about marangoni convection hybrid nanofluid flow considering radiation effect into a stretching surface. A study of heat transfer and mass transformation of $(ZnO + MoS_2)$/engine oil hybrid nanofluid, along a stretching wall surface with soret and thermo diffusion effect, investigated by Mediwal et al [12]. The heat transfer and thermal radiation impact of γ-Al_2O_3 nanofluids, discussed by Dadheech et al [13] in porous stretching surface, with magnetic field. By taking different shape of Cu nanoparticles, Saleem et al [14] analysed about heat transfer. The study about MHD slip flow and thermal radiation through a stretchable sheet, considering melting heat transfer, discovered by Kumar et al [15]. The study of heat transfer with partial slip and suction effect, for a different shape of $(Ag - Al_2O_3$/water) hybrid nanofluid with MHD flow into porous medium, discovered by Ragavi et al [16]. The analysis about heat transfer and entropy of MHD slip flow, for two different types of nanofluid in porous medium, illustrated by Dadheech et al [17].

In the flow of hybrid nanofluid, the viscosity and thermal conductivity are influence by internal friction and external heat, so viscosity and thermal conductivity may not remain constant, and these are dependent function of temperature. In view of natural convection with variable viscosity, Manjunatha et al [18] described about heat transfer for boundary layer hybrid nanofluid flow. In view of variable viscosity, Famakinwa et al [19] investigated about thermal radiation and viscous dissipation, of $(Al_2O_3 - CuO)$/water hybrid nanofluid, into two parallel plates. The impact of variable viscosity on heat transfer and flow structure, for nanolubricant(TiO_2/PAO oil) into the microchannel, examined by Makinde et al [20]. In view of variable thermal conductivity and viscosity, Ferdows et al [21] investigated about heat transfer of magnetohydrodynamics hybrid nanofluid, into a moving thin needle. In view of variable thermal conductivity and viscosity, Agrawal et al [22] analyzed about MHD casson fluid flow with porous medium into exponentially stretching surface. Subadra et al [23] investigated the impact of slip on Jeffrey fluid into a inclined tube. The study about casson nanofluid in vertical channel considering variable viscosity and thermal conductivity, discovered by Havaleppanavar et al [24]. The investigation about Jeffrey nanofluids into stretching surface using cattaneo-christov model, discussed by Khan et al [25].

In view of literature study, we noticed that their is a lake of research on heat transfer and flow properties of(Graphene-Al_2O_3/PAO oil) hybrid nanofluid, considering variable thermal conductivity($k_{shnf}(T)$) and viscosity($\mu_{shnf}(T)$), through a unsteady stretching sheet. The influence of external heat and internal friction on hybrid nanofluid are investigated. The governing PDEs are solved by bvp4c technique. The outcomes are discuss through graphs and tables. Also influence of different physical parameter namely prandtl number(Pr), viscosity parameter(R), External heat generation(Q), Unsteadiness parameter(J), viscous dissipation(E_k), magnetic parameter(M), thermal conductivity parameter(ϵ) on velocity and temperature profile, skin friction factor and heat transfer rate are investigated.

2. Mathematical formulation

In this investigation, two dimensional MHD hybrid nanofluid(Graphene-Al_2O_3/PAO engine oil) flow, with variable thermal conductivity ($k_{shnf}(T)$) and viscosity ($\mu_{shnf}(T)$), into unsteady stretching sheet are considered. The velocity of stretching sheet along radial direction is $V_w = \frac{cr}{(1-\alpha t)}$ [16]. The stretching sheet is considered into $z = 0$ plane with using cylindrical polar coordinates(r, θ, z). The surface temperature of MHD flow is $T_{sur} = T_\infty + T_{ref}(\frac{cr^2}{2\nu_{bf}})(1-\alpha t)^{-\frac{3}{2}}$ [16]. A time dependent magnetic field $B = \frac{B_0}{\sqrt{(1-\alpha t)}}$ [16] is applied in perpendicular direction of stretching sheet. The velocity profile is $w(r,z) = (u,0,v)$, where v(r,z) and u(r,z) denote the transverse and longitude components of sheet velocity across the axial and radial directions.
From these assumptions the governing PDEs can be express as follow [16] [18] [19] [21]

$$\frac{\partial v}{\partial z} + \frac{u}{r} + \frac{\partial u}{\partial r} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z} = \frac{1}{\rho_{shnf}}\frac{\partial}{\partial z}\left(\mu_{shnf}(T)\frac{\partial u}{\partial z}\right) - \frac{\sigma_{shnf}}{\rho_{shnf}}B^2(r,t)u, \tag{2.2}$$

$$\begin{aligned}\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial z} + u\frac{\partial T}{\partial r} &= \frac{1}{(\rho c_p)_{shnf}}\frac{\partial}{\partial z}\left(k_{shnf}(T)\frac{\partial T}{\partial z}\right) + \frac{\sigma_{shnf}}{(\rho c_p)_{shnf}}B^2(r,t)u^2 \\ &+ \frac{\mu_{shnf}(T)}{(\rho c_p)_{shnf}}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{Q_0}{(\rho c_p)_{shnf}}(T - T_\infty),\end{aligned} \tag{2.3}$$

The boundary conditions are express as follow [10] [16] [18]

$$\begin{aligned} \text{at z} = 0: \qquad & u = V_w, \\ & v = 0, \\ & -k_{shnf}(T)\frac{\partial T}{\partial z} = d_s(T_{sur} - T), \\ z \to \infty \qquad & u \to 0, \quad T \to T_\infty, \end{aligned} \tag{2.4}$$

The temperature dependent viscosity of hybrid nanofluid defined by reynold model [18]

$$\mu_{shnf}(T) = \mu_{shnf}e^{-R\theta(\eta)}, \tag{2.5}$$

Using maclaurin series, expanding equation (2.5) with omitting higher order terms, as we get

$$\mu_{shnf}(T) = \mu_{shnf}\left[1 - R\theta + \frac{R^2\theta^2}{2}\right], \tag{2.6}$$

The temperature dependent thermal conductivity is defined as follow [21]

$$k_{shnf}(T) = k_{shnf}(1 + \epsilon\theta(\eta)) \tag{2.7}$$

where $\epsilon > 0$ and $R > 0$ are thermal conductivity and viscosity parameter respectively.
Thermophysical characteristic of hybrid nanoparticle(Graphene-Al_2O_3) into PAO Oil base fluid are [16] [18]

$$\begin{aligned} \rho_{shnf} &= \rho_{n_1}\phi_1 + \rho_{n_2}\phi_2 + \rho_{bf}(1-\phi_1-\phi_2), \\ \mu_{shnf} &= \frac{\mu_{bf}}{(1-\phi_1)^{2.5}(1-\phi_2)^{2.5}}, \\ \sigma_{shnf} &= \sigma_{bf}\left(1 + \frac{3(\frac{\sigma_{n_2}}{\sigma_{bf}}-1)\phi_2}{(\frac{\sigma_{n_2}}{\sigma_{bf}}+2) - (\frac{\sigma_{n_2}}{\sigma_{bf}}-1)\phi_2}\right) \times \left(1 + \frac{3(\frac{\sigma_{n_1}}{\sigma_{bf}}-1)\phi_1}{(\frac{\sigma_{n_1}}{\sigma_{bf}}+2) - (\frac{\sigma_{n_1}}{\sigma_{bf}}-1)\phi_1}\right), \\ (\rho c_p)_{shnf} &= \phi_1(\rho c_p)_1 + \phi_2(\rho c_p)_2 + (1-\phi_1-\phi_2)(\rho c_p)_{bf}, \\ \frac{k_{shnf}}{k_{bf}} &= \frac{k_{n_2}+2k_{bf}+2(k_{n_2}-k_{bf})\phi_2}{k_{n_2}+2k_{bf}-(k_{n_2}-k_{bf})\phi_2} \times \frac{k_{n_1}+2k_{bf}+2(k_{n_1}-k_{bf})\phi_1}{k_{n_1}+2k_{bf}-(k_{n_1}-k_{bf})\phi_1}, \end{aligned} \tag{2.8}$$

The volume fractions of Graphene and Al_2O_3 nanoparticles denote by ϕ_1 and ϕ_2, and $(\rho c_p)_{shnf}$, μ_{shnf}, ρ_{shnf}, σ_{shnf}, k_{shnf} are heat capacity, dynamic Viscosity, Density, electrical Conductivity and thermal Conductivity of secondary hybrid nanofluid respectively. The subscripts $shnf$, bf, n_1, n_2 denote as Hybrid nanofluid, Base fluid, First nanoparticle (Graphene) and Second nanoparticle (Al_2O_3) respectively.

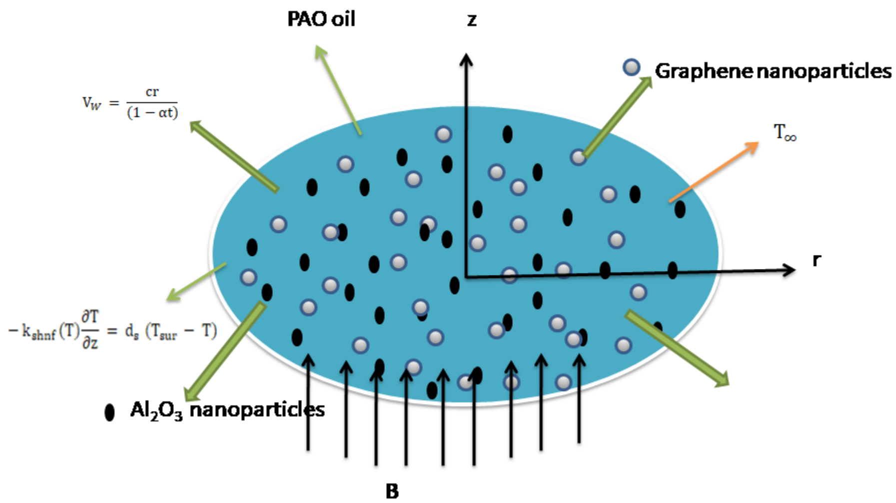

Figure 1: Physical representation of flow geometry in stretching sheet

Using suitable transformations and dimensionless variables as [16]

$$\begin{aligned}
\psi &= \frac{r^2 V_w f(\eta)}{\sqrt{Re}}, \\
Re &= \frac{r V_w}{\nu_{bf}}, \\
\theta(\eta) &= \frac{T - T_\infty}{T_{ref}\left(\frac{cr^2}{2\nu_{bf}}\right)(1-\alpha t)^{-\frac{3}{2}}}, \\
\eta &= \frac{z\sqrt{Re}}{r},
\end{aligned} \tag{2.9}$$

where Re and ψ denotes the reynolds number and stream function, and η is independent variable.

The longitude(u) and transverse(v) components of velocity are represent as [16]

$$\begin{aligned}
u &= -\frac{1}{r}\frac{\partial \psi}{\partial z} = V_w f'(\eta), \\
v &= \frac{1}{r}\frac{\partial \psi}{\partial r} = -2V_w Re^{\frac{-1}{2}} f(\eta),
\end{aligned} \tag{2.10}$$

From equations (2.9) and (2.10) we reduce equations (2.2) and (2.3) into non linear ODE, which are following

$$\begin{aligned}
\frac{B_1}{B_3}\left(1 - R\theta + \frac{R^2\theta^2}{2}\right)f''' + \frac{B_1}{B_3}R^2\theta\theta' f'' - \frac{B_1}{B_3}R\theta' f'' - M\frac{B_4}{B_3}f' \\
+2ff'' - J\left(f' + \frac{\eta}{2}f''\right) - (f')^2 = 0,
\end{aligned} \tag{2.11}$$

$$\frac{B_2(1+\epsilon\theta)}{B_5 Pr}\theta'' + \frac{B_2\epsilon}{B_5 Pr}(\theta')^2 - \left(\frac{J}{2}(3\theta+\eta\theta') + 2\theta f' - 2f\theta'\right)$$
$$+\frac{B_1}{B_5}E_k\left(1 - R\theta + \frac{R^2\theta^2}{2}\right)(f'')^2 + \frac{B_4}{B_5}ME_k(f')^2 + \frac{Q.\theta}{B_5} = 0, \tag{2.12}$$

The transform boundary conditions as follow

$$f(\eta) = 0, \quad \theta'(\eta) = -\frac{Bi}{B_2}\frac{(1-\theta(\eta))}{(1+\epsilon\theta(\eta))}, \quad f'(\eta) = 1 \quad \text{at } \eta = 0\ ;$$
$$f'(\eta) \to 0, \quad \theta(\eta) \to 0, \quad as\ \ \eta \to \infty; \tag{2.13}$$

The constants B_1, B_2, B_3, B_4 and B_5 are defined as

$$B_1 = \frac{\mu_{shnf}}{\mu_{bf}}, B_2 = \frac{k_{shnf}}{k_{bf}}, B_3 = \frac{\rho_{shnf}}{\rho_{bf}}, B_4 = \frac{\sigma_{shnf}}{\sigma_{bf}}, B_5 = \frac{(\rho c_p)_{shnf}}{(\rho c_p)_{bf}} \tag{2.14}$$

Several dimensionless parameter are as follow: $Bi = \frac{d_s}{k_{bf}}\sqrt{\frac{\nu_{bf}(1-\alpha t)}{c}}$ is biot number, $E_k = \frac{2\nu_{bf}c}{T_{ref}c_p\sqrt{(1-\alpha t)}}$ is Eckert number, $J = \frac{\alpha}{c}$ is the Unsteadiness parameter, $Pr = \frac{(\rho c_p)_{bf}\nu_{bf}}{k_{bf}}$ refers to Prandtl number, $M = \frac{B_0^2\sigma_{bf}}{c\rho_{bf}}$ is Magnetic parameter and $Q = \frac{Q_0(1-\alpha t)}{c(\rho c_p)_{bf}}$ is Heat generation coefficient.
The Skin friction coefficient(C_f) and nusselt number(Nu) are represent as [16]

$$C_f = \frac{\tau_w}{\rho_{bf}V_w^2}, \quad Nu = \frac{rq_w}{(T_{sur} - T_\infty)k_{bf}} \tag{2.15}$$

where $\tau_w = \mu_{shnf}(T)\left(\frac{\partial u}{\partial z}\right)_{z=0}$ and $q_w = -k_{shnf}(T)\left(\frac{\partial T}{\partial z}\right)_{z=0}$ denote as shear stress and wall heat flux over stretching sheet respectively and equation (2.15) can be transformed into the following form:

$$C_f Re^{\frac{1}{2}} = \frac{\mu_{shnf}}{\mu_{bf}}\left(1 - R\theta + \frac{R^2\theta^2}{2}\right)f''(0) = \frac{\left(1 - R\theta + \frac{R^2\theta^2}{2}\right)}{(1-\phi_2)^{2.5}(1-\phi_1)^{2.5}}f''(0)$$
$$NuRe^{\frac{-1}{2}} = -\frac{k_{shnf}}{k_{bf}}(1+\epsilon\theta)\theta'(0) \tag{2.16}$$

In hybrid nanofluid, the heat transfer enhancement is defined as follow [14]

$$E_R = \frac{NuRe^{\frac{-1}{2}}(Nanofluid) - NuRe^{\frac{-1}{2}}(Basefluid)}{NuRe^{\frac{-1}{2}}(Basefluid)} \times 100 \tag{2.17}$$

3. Solution Procedure

The transform non linear ODEs (2.11) and (2.12) together with boundary conditions (2.13) are solved using bvp4c numerical technique in MATLAB. For that we transform equations into first order ODEs and using following substitutions:

$$f = y_1,$$
$$f' = y_2,$$
$$f'' = y_3,$$
$$f''' = \frac{B_3}{B_1\left(1 - Ry_4 + \frac{R^2(y_4)^2}{2}\right)}\left(\frac{MB_4}{B_3}y_2 + \frac{RB_1}{B_3}y_3y_5 - \frac{R^2B_1}{B_3}y_3y_4y_5 + (y_2)^2 + J\big(y_2 + \frac{\eta}{2}y_3\big) - 2y_3y_1\right),$$
$$\theta = y_4,$$

$$\theta' = y_5,$$

$$\theta'' = \frac{B_5 Pr}{B_2(1+\epsilon y_4)}\left[\frac{J}{2}(\eta y_5 + 3y_4) + 2y_4y_2 - 2y_5y_1 - \frac{\epsilon B_2}{PrB_5}(y_5)^2 - \frac{B_4}{B_5}ME_k(y_2)^2 - \frac{B_1}{B_5}E_k\left(1 - Ry_4 + \frac{R^2(y_4)^2}{2}\right)(y_3)^2 - \frac{Qy_4}{B_5}\right],$$

and boundary conditions are defined as

$$y_1(0) = 0, \quad y_2(0) = 1, \quad y_5(0) = -\frac{Bi}{B_2}\frac{(1-y_4(0))}{(1+\epsilon y_2(0))}, \quad y_4(\infty) \to (0), \quad y_2(\infty) \to (0)$$

4. Analysis of results and discussion

The prime keynote is to analyzed about heat transfer and flow structure of hybrid nanofluid (Graphene-Al_2O_3/PAO oil) into a stretching sheet using variable thermal conductivity and viscosity. We are considering different non dimensional parameters such as Volume fraction(ϕ), prandtl number(Pr), Unsteadiness parameter(J), Magnetic parameter(M), Viscosity parameter(R), eckert number(E_k), Heat generation coefficient(Q), biot number(Bi) and Thermal conductivity parameter(ϵ). The outcomes are discuss through graphs and tables, and the variation of Nusselt Number and Skin Factor are discuss through tables. In entire research we are taking Pr=13, M=1, Q=0.5, E_k=0.4, J=0.4, R=1, ϵ=1, Bi=0.5, ϕ_1=ϕ_2=0.02 and varying all parameters for velocity and temperature.

The thermophysical values of hybrid nanoparticles(Graphene-Al_2O_3) and base fluid(PAO oil) are re-

Physical Characters	Graphene	Al_2O_3	PAO oil
Thermal Conductivity $K(\frac{W}{mK})$	2500	40	0.143
Specific heat $c_p(\frac{J}{Kg.K})$	2100	765	2303
Electrical Conductivity $\sigma(\frac{S}{m})$	1.09×10^7	35×10^6	1.0×10^{-8}
Density $\rho(\frac{Kg}{m^3})$	2250	3970	798

Table 1: Thermophysical values of hybrid nanoparticles(Graphene-Al_2O_3) and base fluid(PAO oil) [16] [20] [21]

flected by Table 1. The influence of magnetic parameter(M) on velocity, display by Figure 2. The decrement in velocity noticed, when increase the magnetic parameter from 1 to 4, because it generate resistance which is due to Lorentz force, and it opposes the fluid motion, and decrease velocity of hybrid nanofluid near a stretching sheet. The relation between unsteadiness parameter(J) and velocity, disclose by Figure 3. It indicates that velocity profile decrease with raising into unsteadiness parameter, this phenomenon occurs because of stretching of the sheet, and it diminish into the momentum boundary layer thickness.

The relation between viscosity parameter(R) and velocity, depicted by Figure 4. The velocity profile decay with increment in viscosity parameter, this phenomenon happened because viscosity resist the fluid flow due to internal friction. The relation between biot number(Bi) and velocity profile, disclose by Figure 5. The outcomes reveals that velocity is inversely proportional to biot number, it occurs because of loss in convective heat. The influence of eckert number(E_k) on velocity, display by Figure 6. The velocity profile decreased with increment in eckert number, it occurs because this parameter decay into the kinetic energy, and increase into thermal boundary layer thickness. The velocity decreased with rising into external heat generation parameter(Q), as display by Figure 7. This phenomenon arise

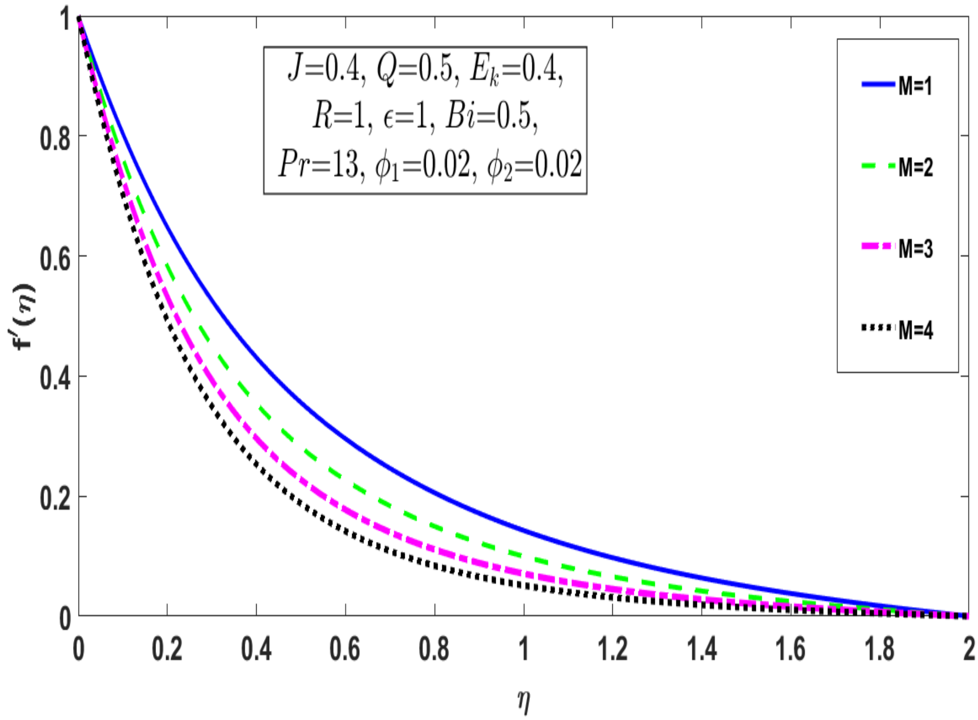

Figure 2: Influence in velocity by magnetic parameter(M)

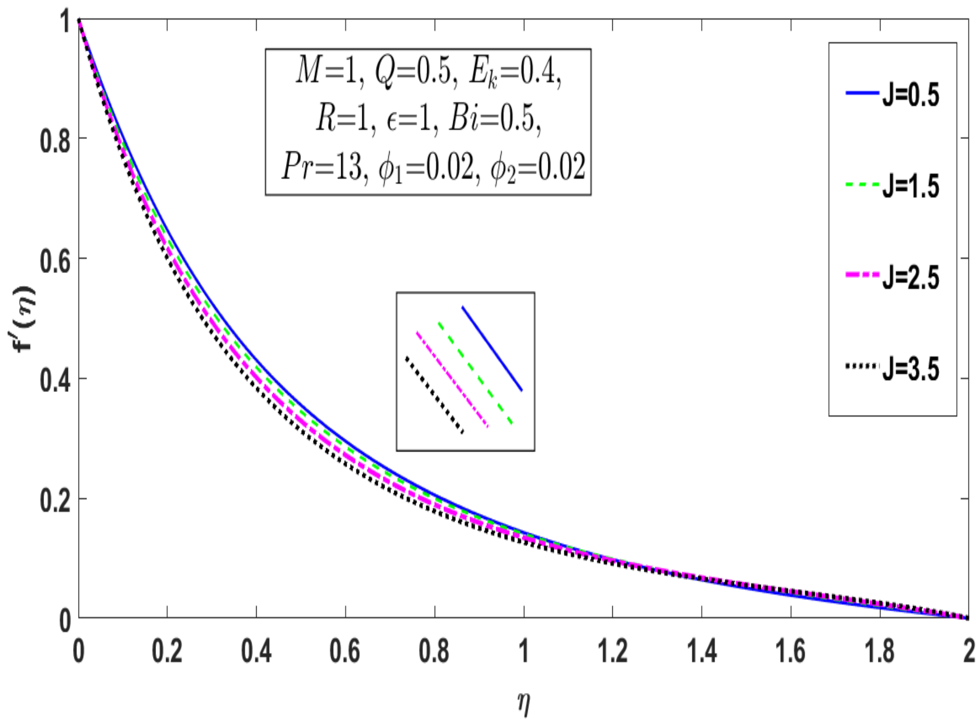

Figure 3: Influence in velocity by unsteadiness parameter(J)

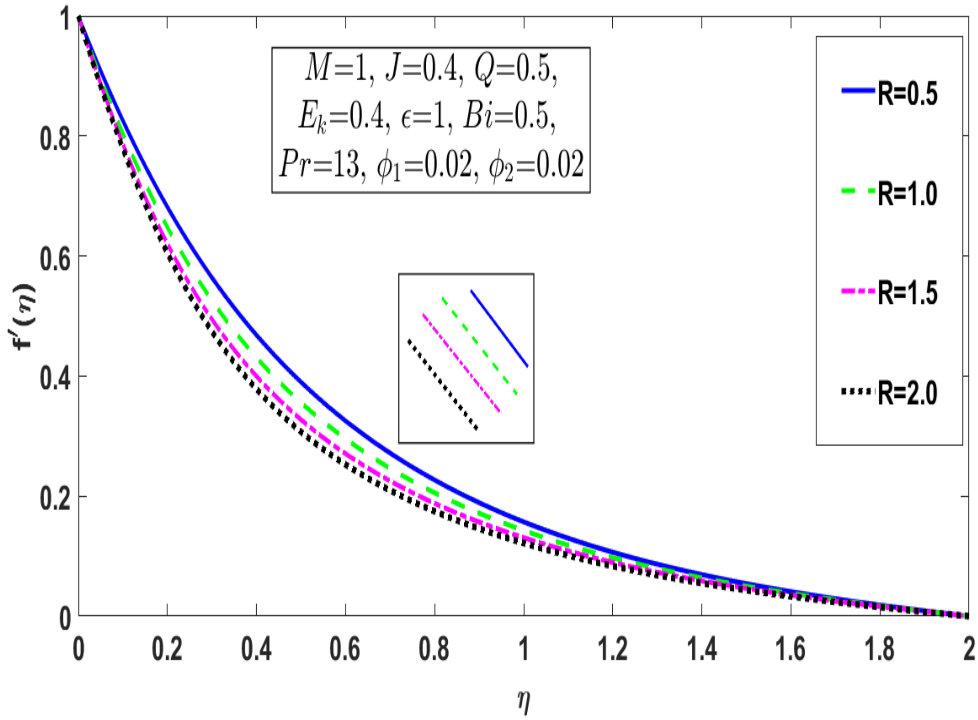

Figure 4: relation between viscosity parameter(R) with velocity

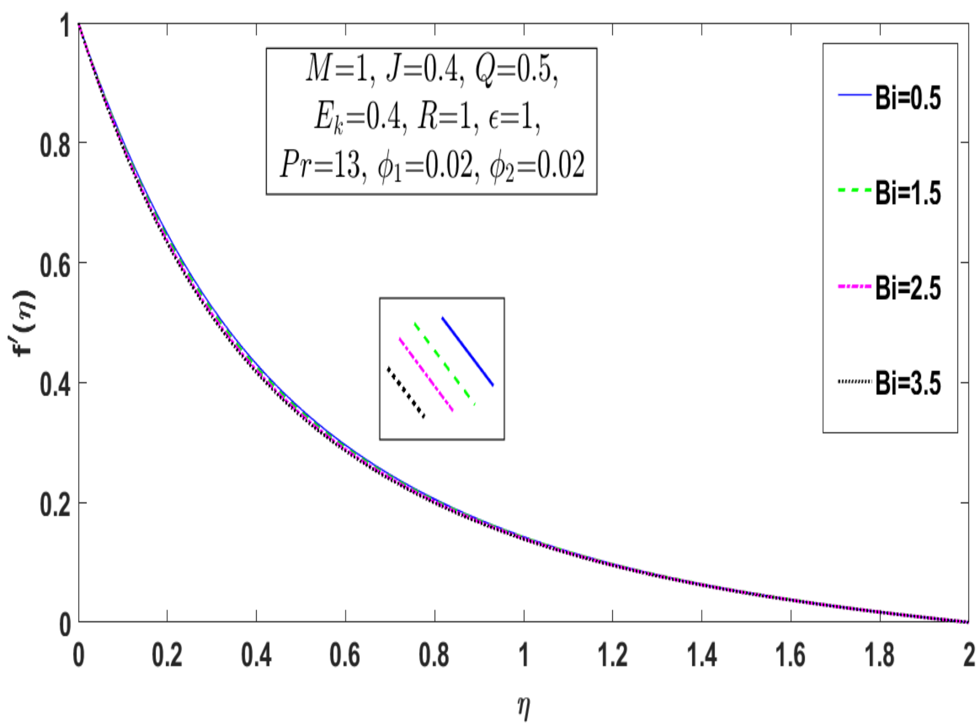

Figure 5: Velocity with Biot number(Bi)

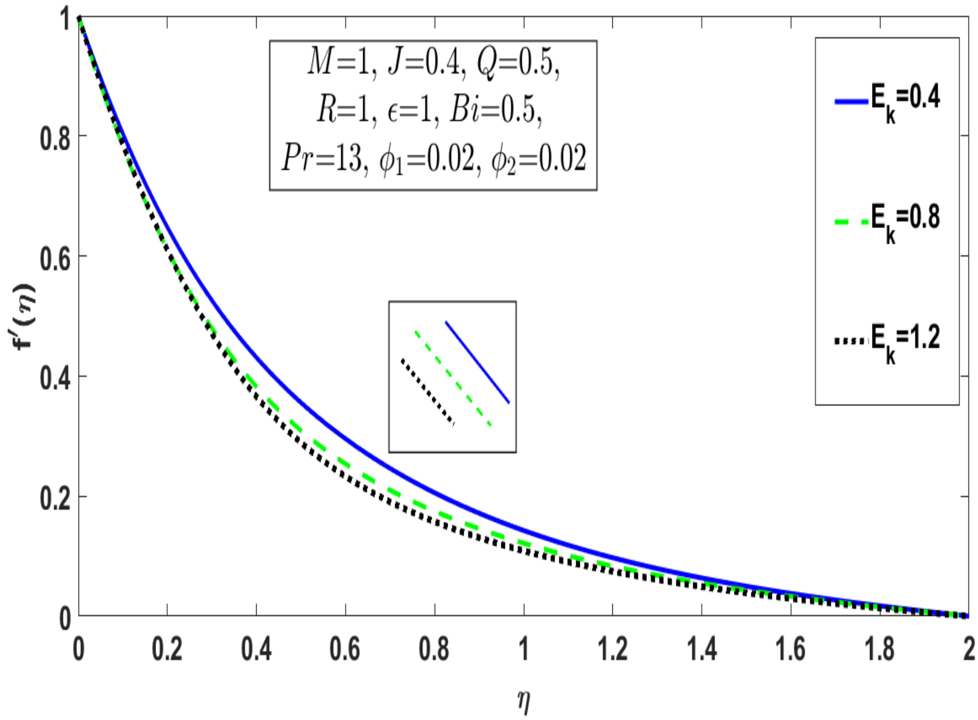

Figure 6: Influence in velocity by eckert number(E_k)

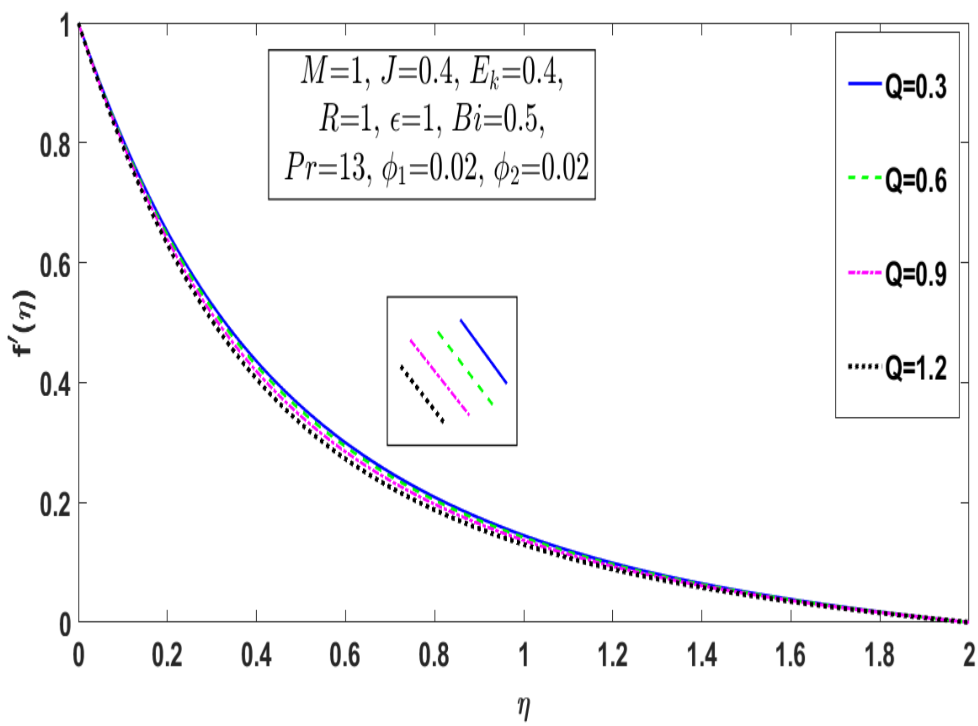

Figure 7: Impact on velocity by heat generation parameter(Q)

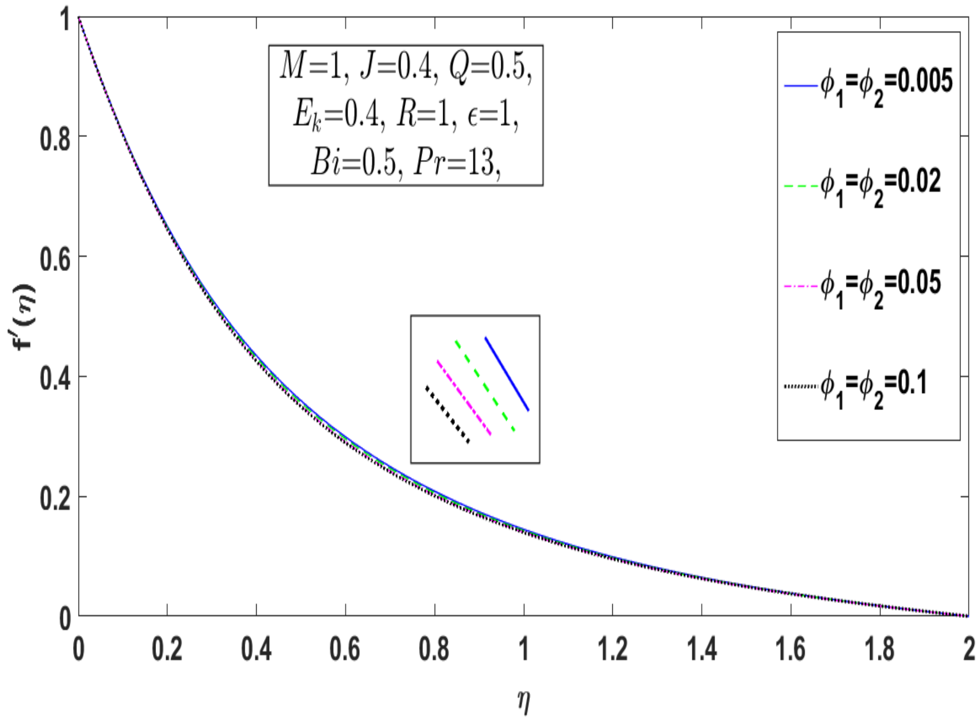

Figure 8: Velocity profile for volume fractions(ϕ_1 and ϕ_2)

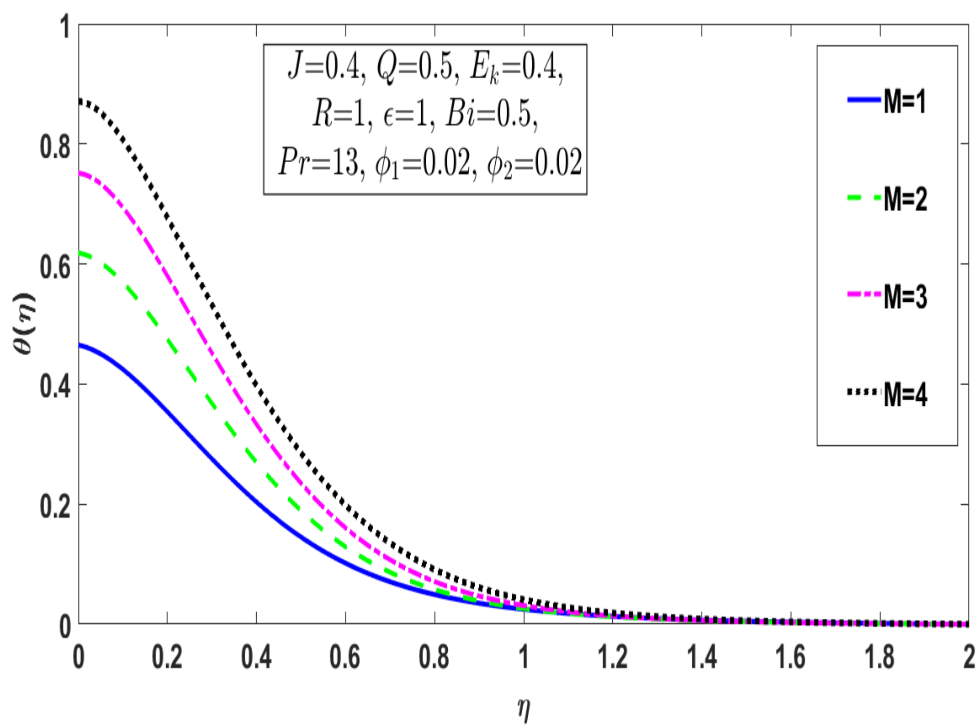

Figure 9: Influence in temperature by M

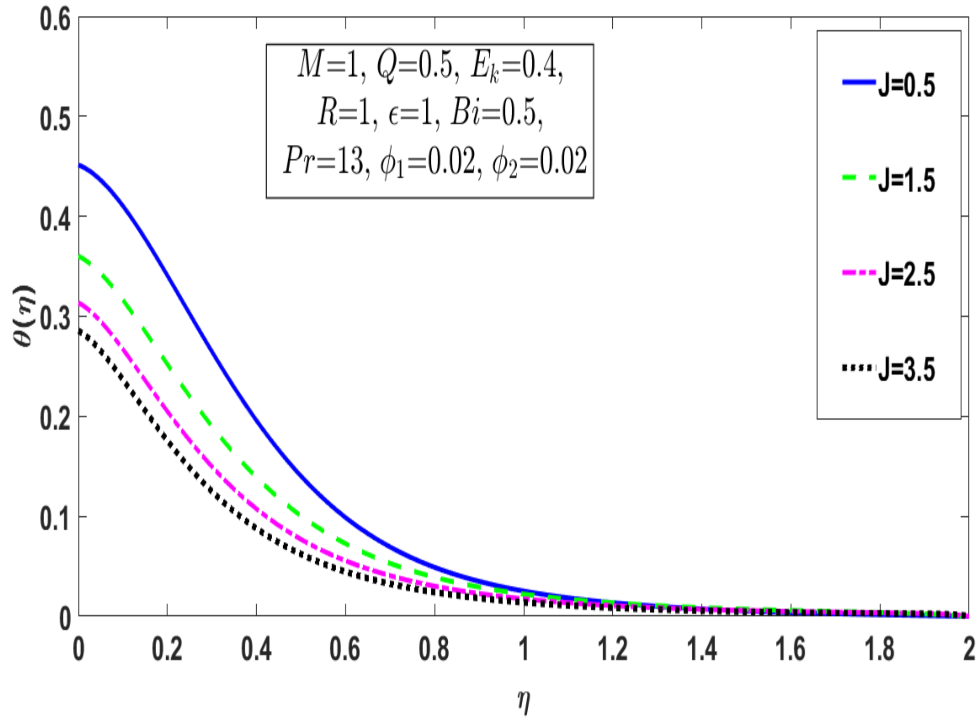

Figure 10: temperature for diverse value of J

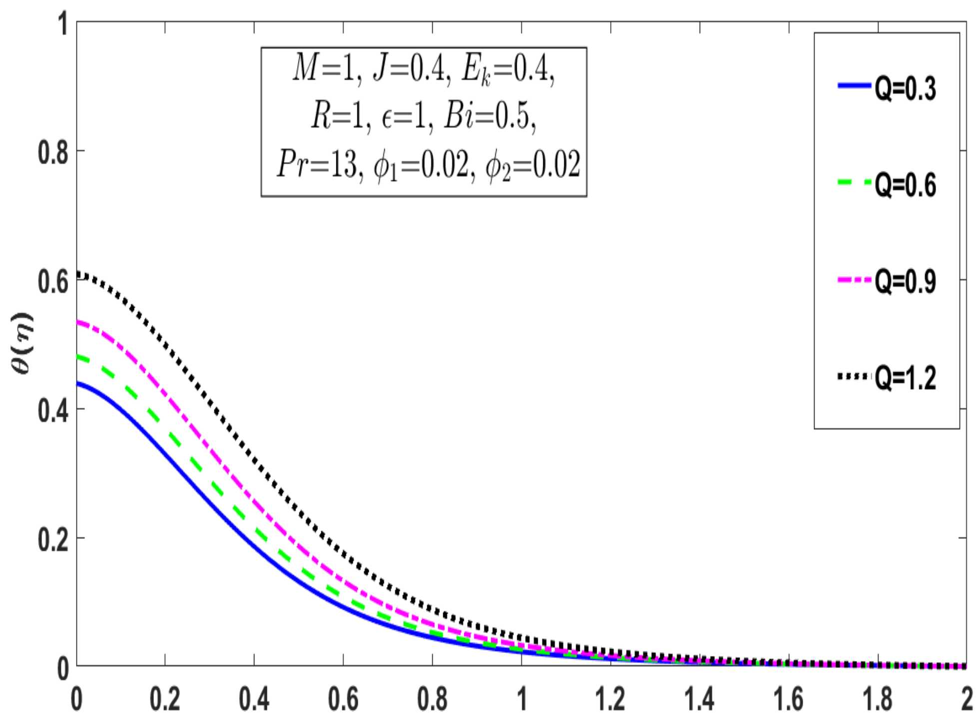

Figure 11: Temperature variation for heat generation parameter(Q)

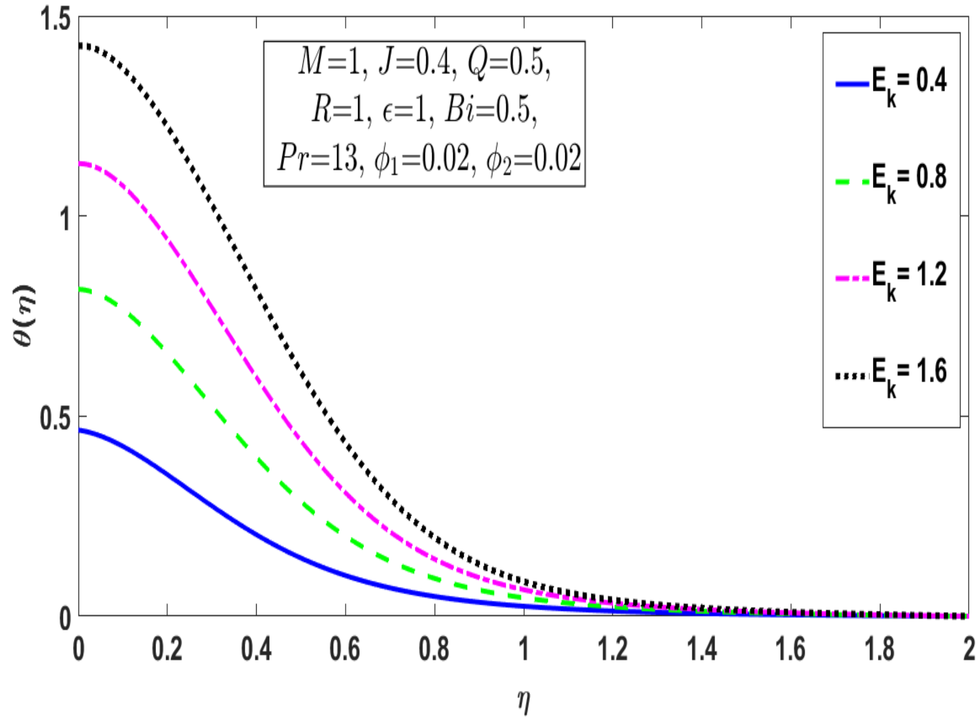

Figure 12: Significant impression of eckert number(E_k) on temperature

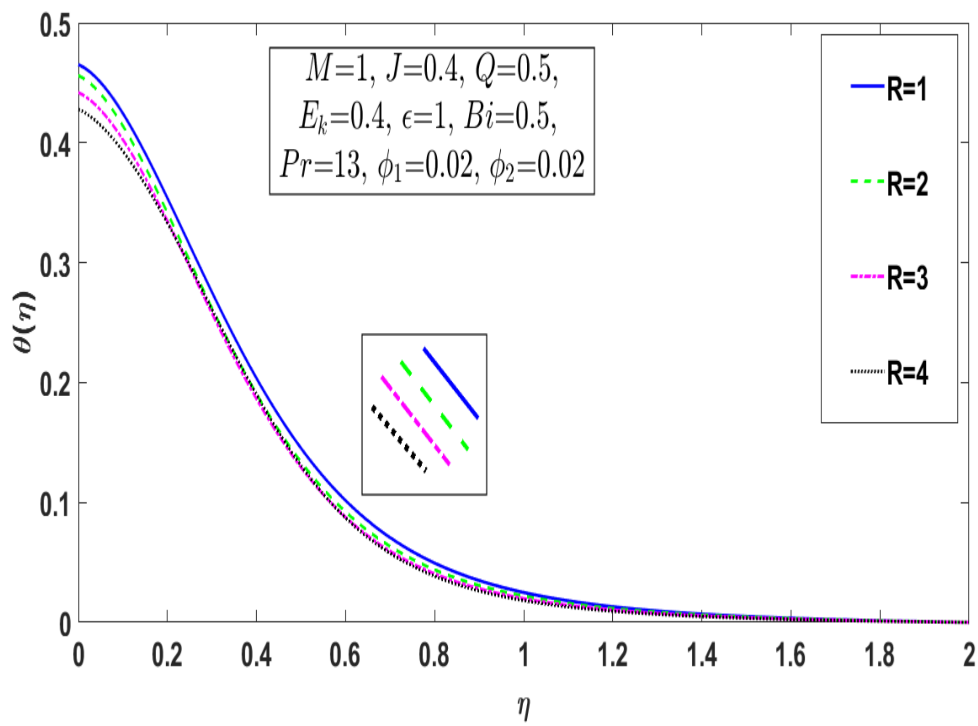

Figure 13: Temperature with viscosity parameter(R)

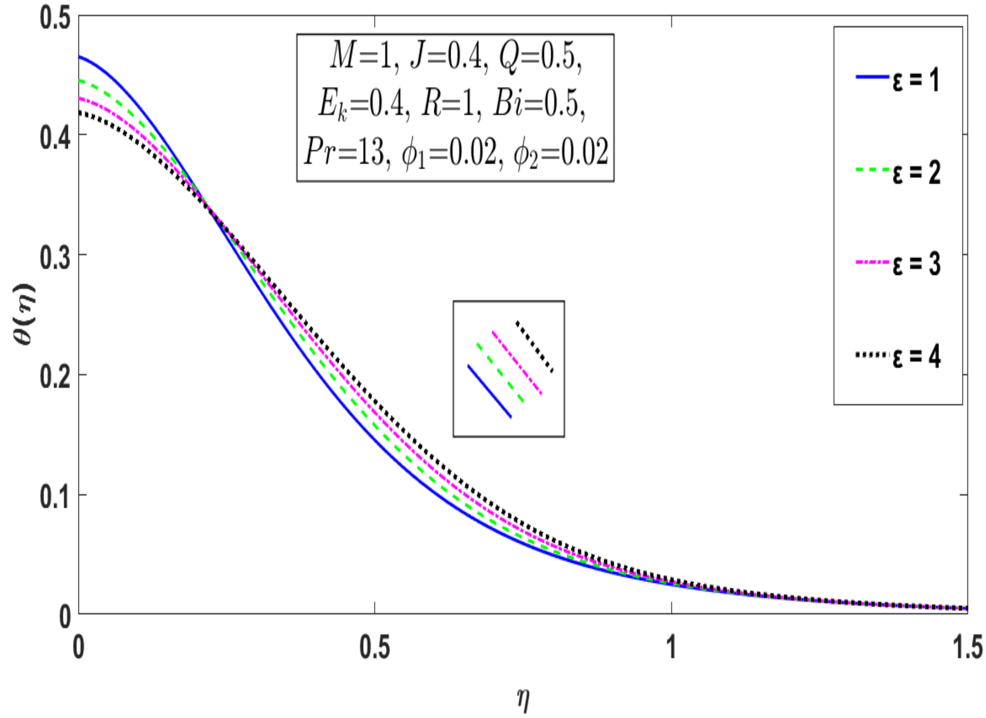

Figure 14: Temperature profile for diverse value of ϵ

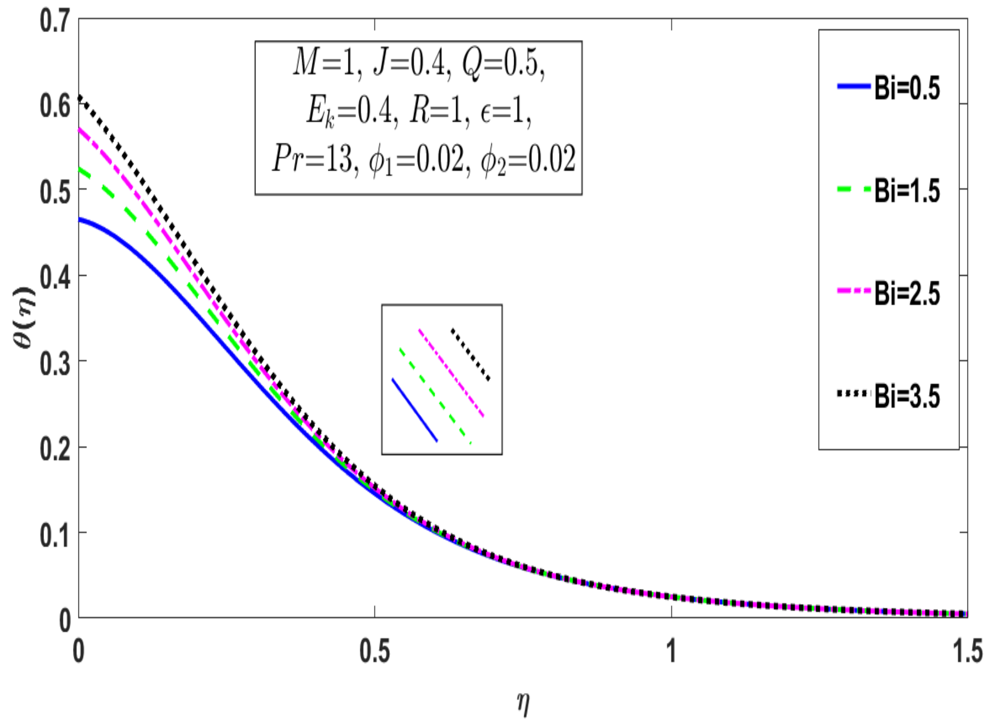

Figure 15: Temperature variation for Biot number(Bi)

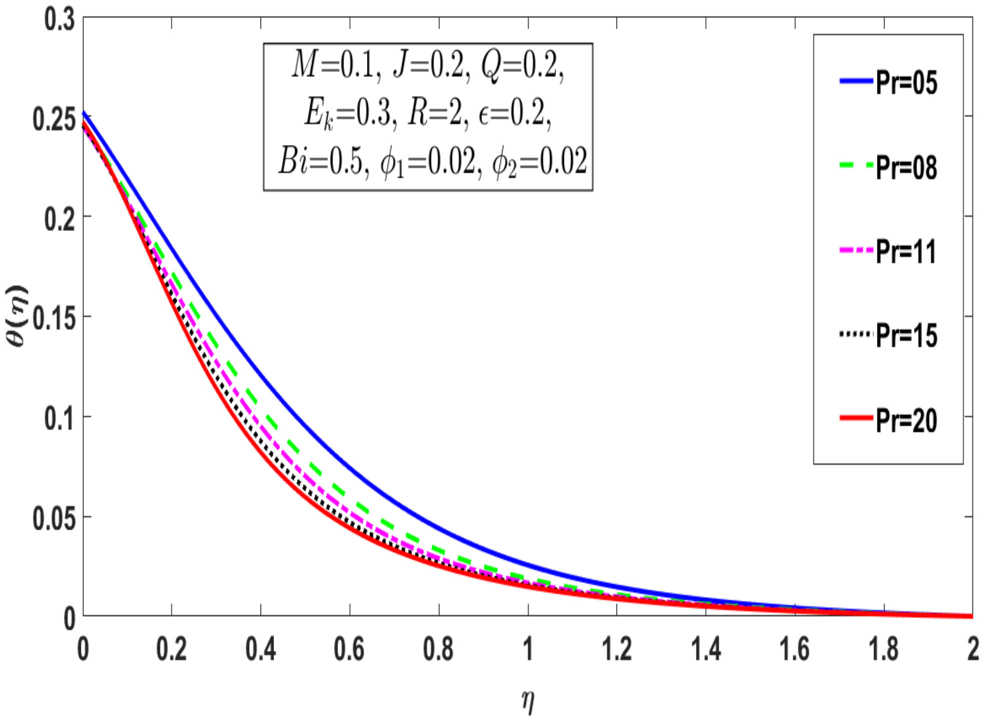

Figure 16: temperature variation effect by Pr

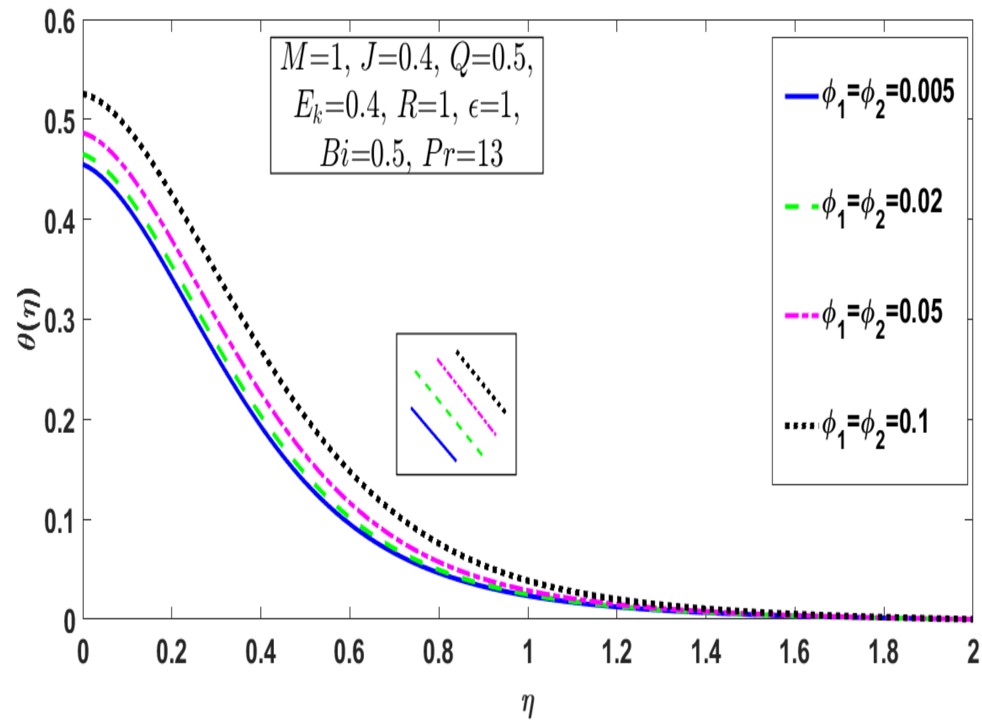

Figure 17: Impact of volume fractions ϕ_1 and ϕ_2 on temperature

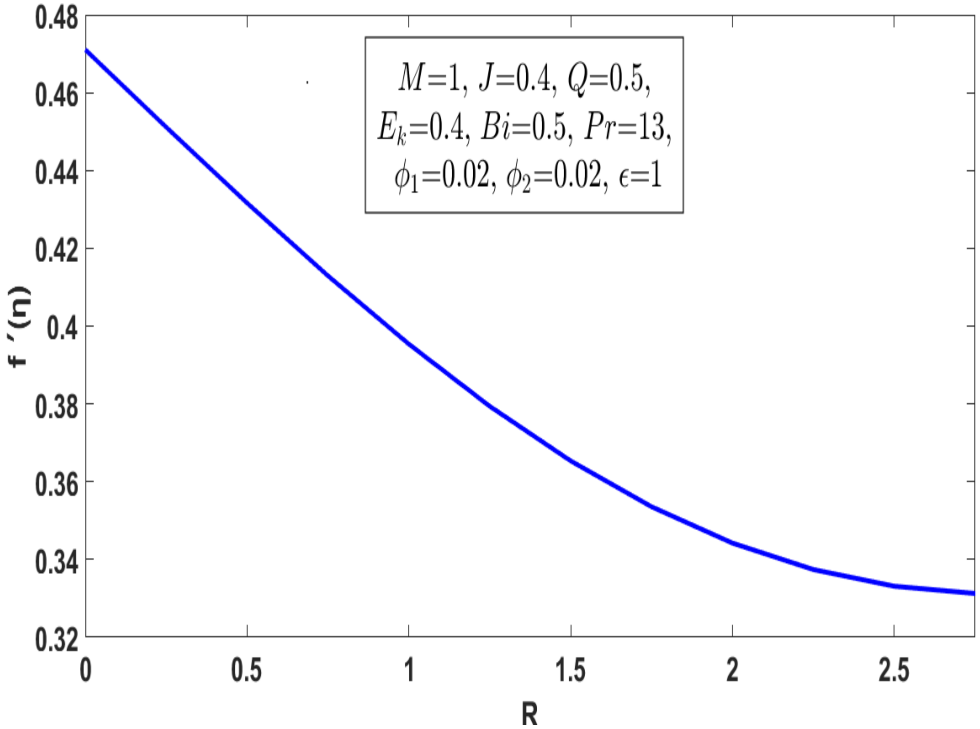

Figure 18: Graph between viscosity parameter(R) and velocity

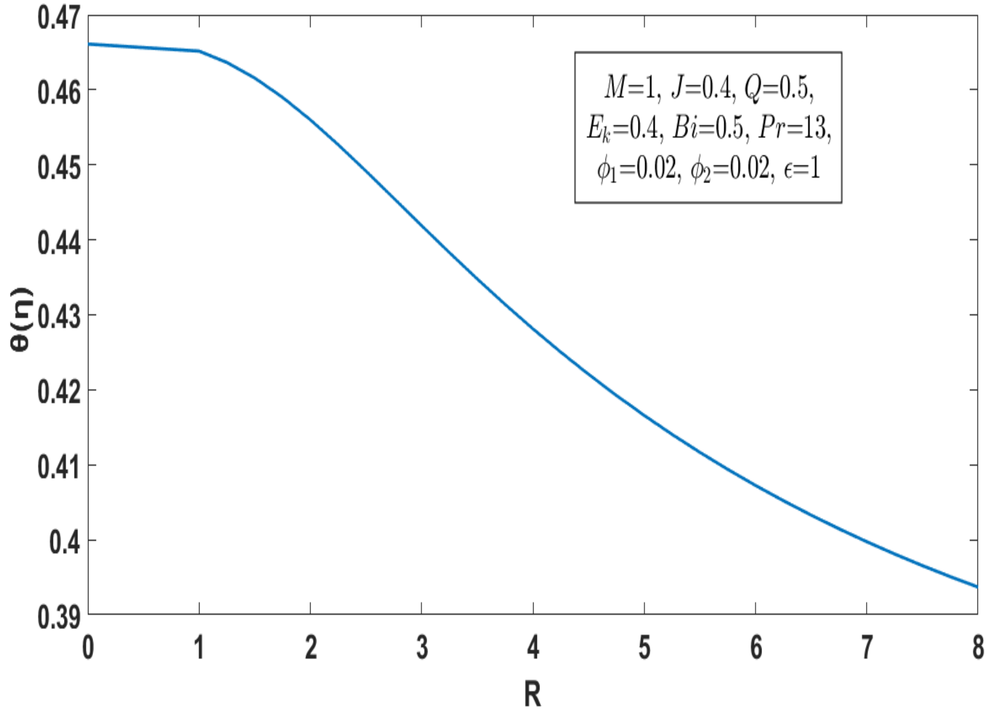

Figure 19: Relation between viscosity parameter(R) and temperature

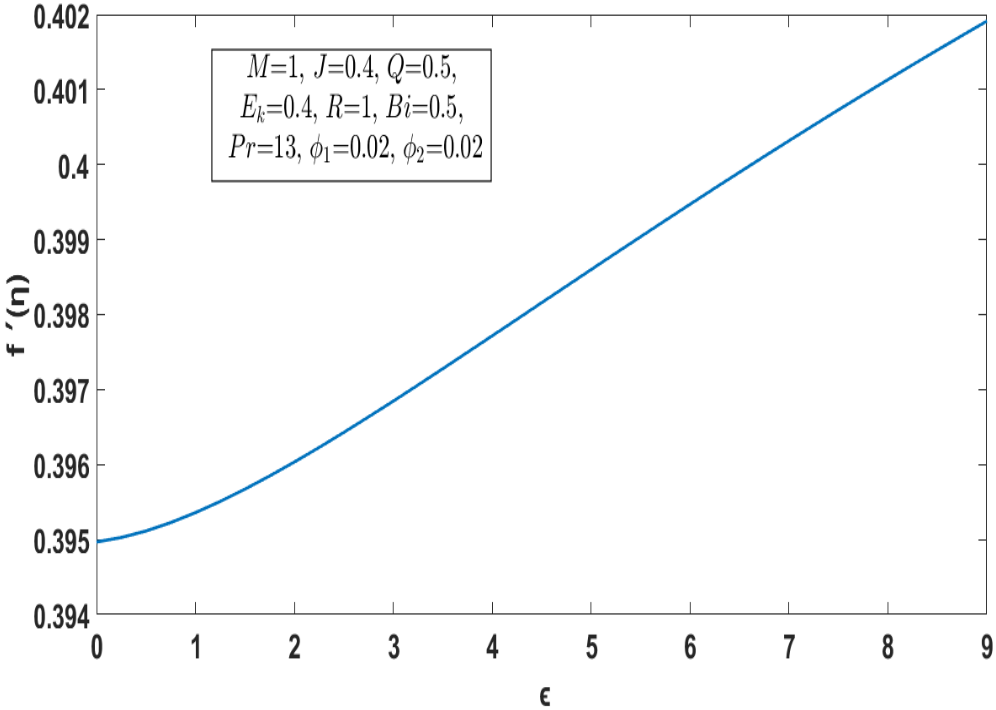

Figure 20: relation between velocity and ϵ

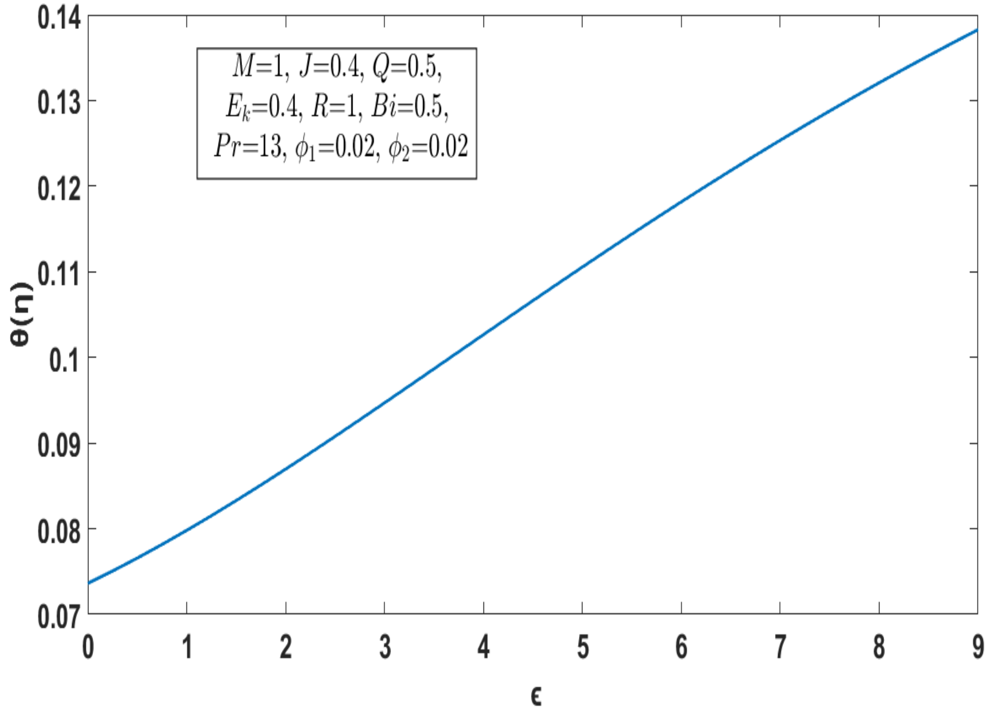

Figure 21: graph between temperature and ϵ

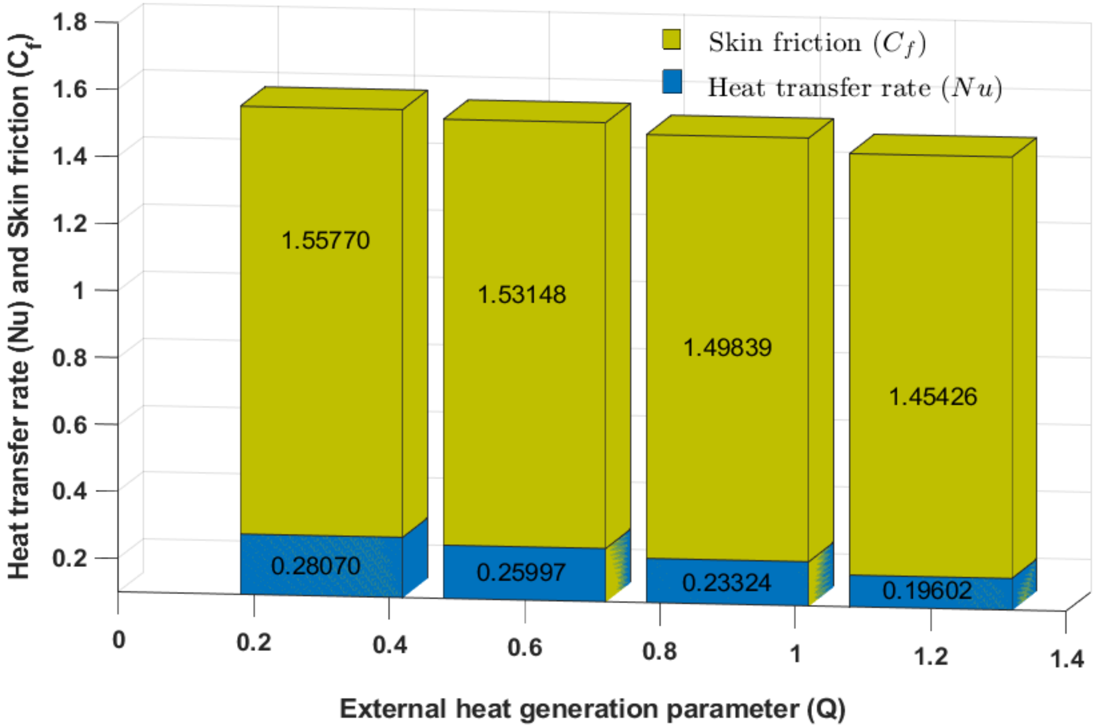

Figure 22: heat transfer rate & skin friction for heat generation parameter(Q)

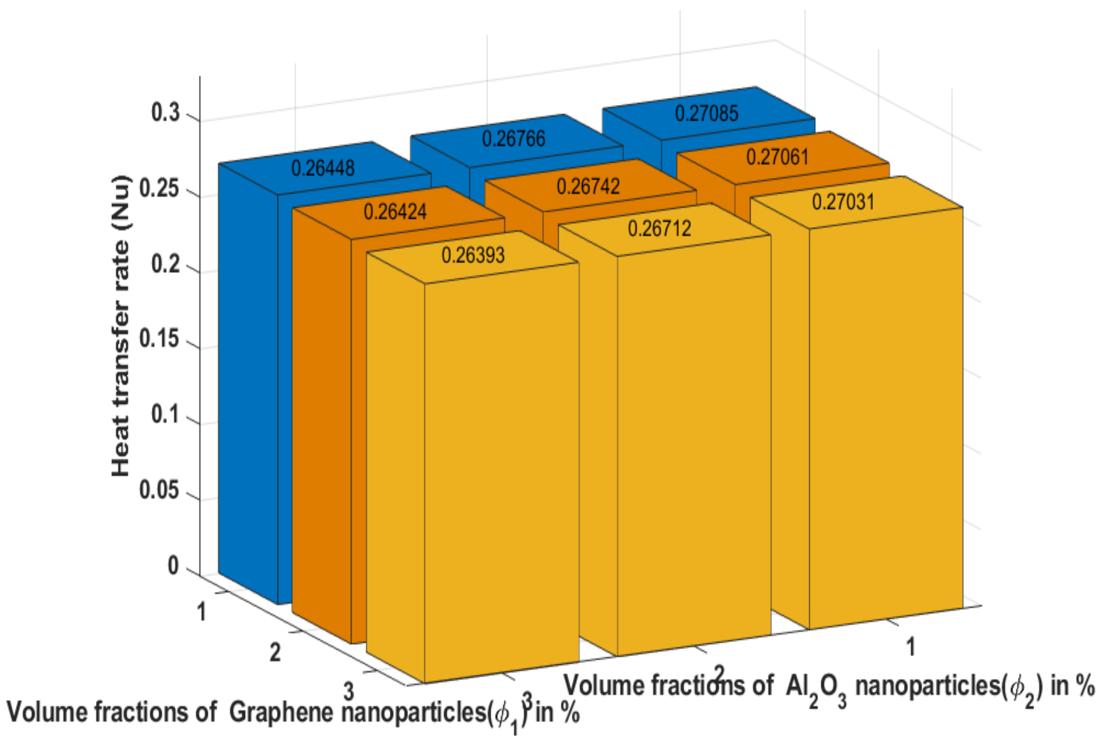

Figure 23: Heat transfer rate for volume fractions

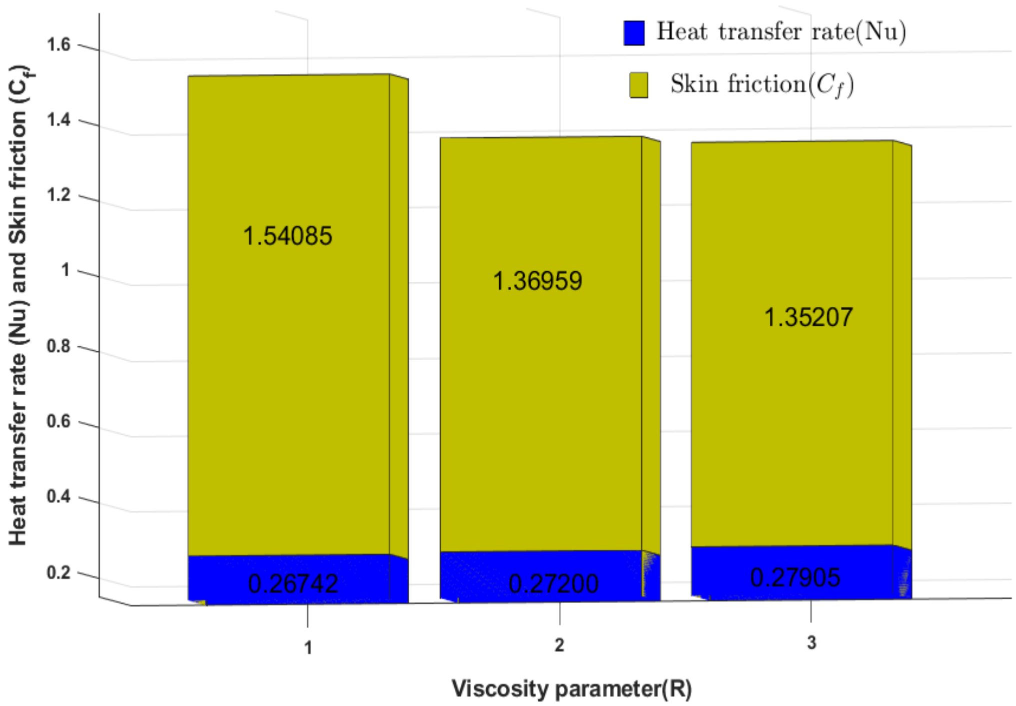

Figure 24: Influence of R on heat transfer rate & skin friction

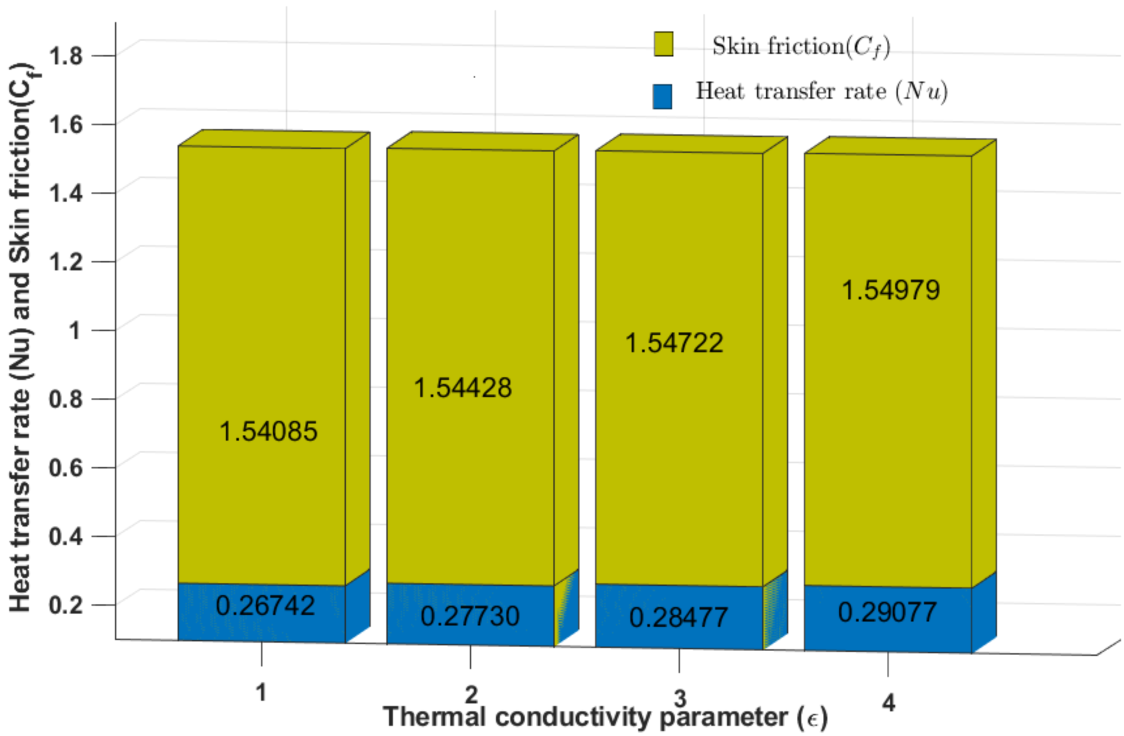

Figure 25: heat transfer rate & skin friction for ϵ

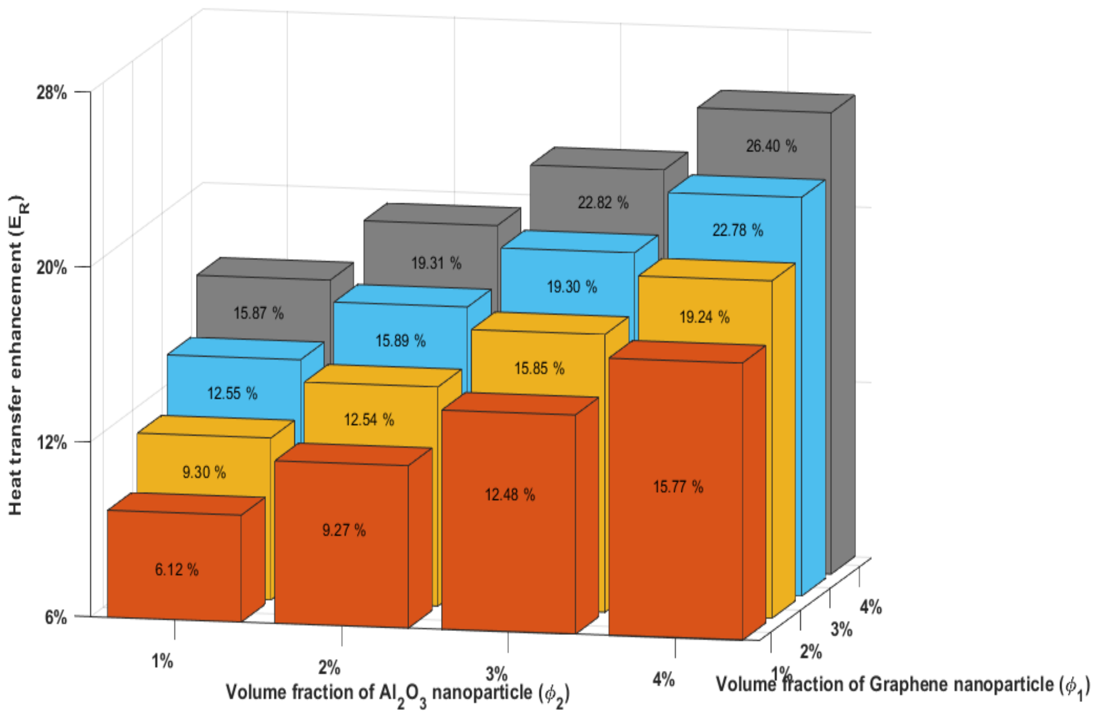

Figure 26: Heat transfer enhancement influence by volume fractions of nanoparticles

because of decrement in momentum boundary layer. The impact of solid volume fractions(ϕ_1, ϕ_2) on velocity, demonstrate by Figure 8. The result shows, velocity profile decreased with increment into volume fraction, because increase value of volume fractions leads to rise into viscosity of hybrid nanofluid, and nanoparticle interact for long time, and resist the fluid flow.

The relation between temperature and magnetic parameter(M), disclose by Figure 9. The temperature profile increased with raising into magnetic field, it happened because of Lorentz force which rise into temperature. The influence of unsteadiness parameter(J) on temperature, display by Figure 10. The result reveals that, temperature profile decreased with rising into unsteadiness parameter. This phenomenon arise because of heat loss into stretching surface, and less heat transmits from sheet to hybrid nanofluid, that's why we use unsteady flow in our study for cooling system. The variation in temperature influence by external heat generation parameter(Q), illustrates by Figure 11. We noticed that, temperature profile increased with increment into heat generation parameter, it occurs because increase value of Q store into fluid as temperature, and it increase the fluid temperature. The significant impression of eckert number(E_k) on temperature, display by Figure 12. The fluid temperature accelerate with raising into eckert number, this phenomenon occurs because it increase the kinetic energy and viscous dissipation of hybrid nanofluid, and increase the temperature of hybrid nanofluid.

The relation between viscosity parameter(R) and temperature, disclose by Figure 13. This figure display that, viscosity parameter inversely proportional to temperature, this phenomenon occurs because of increment in internal friction, which slow the fluid motion, and nanoparticles attract each other, and they reduce temperature profile. The relation between thermal conductivity parameter(ϵ) and temperature, shown by Figure 14. We noticed that fluid temperature decreased near stretching sheet and rise when we move away from sheet. This phenomenon occurs because heat transfer rate is high near a stretching sheet, and low when we move away from sheet surface. The significance impression of Biot number(Bi) on temperature, display by Figure 15. The temperature enhanced with raising into biot number, this happened because of more convection into fluid.

The relation between Prandtl number(Pr) and temperature, display by Figure 16. The acceleration

Parameters										$-C_f Re^{\frac{1}{2}}$	$Nu Re^{\frac{-1}{2}}$
M	J	R	Q	E_k	ϵ	Bi	Pr	ϕ_1	ϕ_2		
1	0.4	1	0.5	0.4	1	0.5	13	0.02	0.02	1.540849	0.267416
2										1.702023	0.190860
3										1.835215	0.124395
	0.5									1.568811	0.274308
	1.5									1.825337	0.319913
	2.5									2.049174	0.343207
		1								1.540849	0.267416
		2								1.369589	0.271998
		3								1.352071	0.279054
			0.3							1.557696	0.280697
			0.6							1.531477	0.259969
			0.9							1.498392	0.233238
				0.4						1.540849	0.267416
				0.8						1.379912	0.091561
				1.2						1.327777	0.065404
					1					1.540849	0.267416
					2					1.544277	0.277298
					3					1.547217	0.284775
						0.5				1.540849	0.267416
						1.5				1.520567	0.713862
						2.5				1.504815	1.073363
							10			1.542350	0.274588
							15			1.539862	0.263171
							20			1.537663	0.254031
								0.02	0.02	1.540849	0.267416
								0.05	0.05	1.782054	0.256772
								0.1	0.1	2.254657	0.237303

Table 2: Numerical observation for Skin friction & Nusselt number

into prandtl number diminish into temperature profile, because it enhance into viscosity and decay into thermal boundary layer. The impact of volume fractions(ϕ_1, ϕ_2) on temperature, disclose by Figure 17. The result reveals that temperature profile increase with enhancing into volume fractions, it occurs because they increase the thermal conductivity into hybrid nanofluid. Figure 18 and 19 indicates the influence in velocity and temperature profile affect by viscosity, both are decreased with increased in R. It happened because higher viscosity resist the fluid flow due to internal friction. Figure 20 and 21 illustrates about the influence in velocity and temperature profile affect by thermal conductivity parameter. Both are increased with increment in ϵ, this is because of increment in thermal boundary layer thickness.

The wall shear force and Heat transfer rate decreased with increment into heat generation parameter (Q), as display by Figure 22. The influence of solid volume fractions on heat transfer rate, disclose by Figure 23. The results reveals that, heat transfer rate decreased with rising into volume fractions of hybrid nanoparticles. Heat transfer rate is high when concentration of Graphene nanoparticles is high comparison to Al_2O_3 nanoparticles, into hybrid nanofluid. The increment in viscosity parameter(R), the heat transfer rate increased and reverse trend noticed for skin friction, as display by Figure 24. The heat transfer rate and wall shear force increased with rising into thermal conductivity parameter(ϵ), denoted by Figure 25. The relation between heat transfer enhancement with volume fraction of hybrid nanoparticles, display by Figure 26. The heat transfer enhancement increased with rising into volume fraction of hybrid nanoparticles, and no other physical parameter in our study influence the Heat transfer enhancement. Heat transfer enhancement is high, if we increase 1% volume fraction of Graphene nanoparticles in place of Al_2O_3 nanoparticle in hybrid nanoparticles.

Table 2 signifies the influence of various parameters like unsteadiness parameter(J), magnetic parameter (M), viscosity parameter(R), external heat generation coefficient(Q), eckert number(E_k), thermal conductivity parameter(ϵ), biot number(Bi), prandtl number(Pr) and volume fractions(ϕ_1,ϕ_2) on Skin friction & Nusselt number for hybrid nanoparticles over a stretching sheet. This analysis shows that an rising into parameter J, M, ϕ_1, ϕ_2, degrade the skin friction, and reverse trend noticed when rise into R, Q, Bi, E_k, Pr, ϵ. The consequence show that heat transfer rate accelerate when J, R, ϵ, Bi are increased and reverse trend noticed for rising into M, Pr, Q, E_k, ϕ_1, ϕ_2.

5. Conclusions

The work illustrate about variable thermal conductivity(k_{shnf}) and viscosity(μ_{shnf}) of (Graphene-Al_2O_3/PAO oil) hybrid nanofluid through a radially stretching sheet. Using suitable transformation, we reduce the governing PDEs into ODEs and solved it by numerically technique bvp4c. The outcomes disclosed by graphs and table. The following key outcomes are listed as below:

- The velocity and temperature profile diminish with increment into viscosity and unsteadiness parameters, and opposite trend noticed when we accelerate the thermal conductivity parameter.
- The velocity profile diminish, and temperature profile increased with raising into volume fraction of nanoparticles, eckert number, biot number, magnetic & heat generation parameters.
- The temperature profile decreased with rising into prandtl number.
- The Heat transfer rate increased with accelerating into variable viscosity and thermal conductivity, and reverse trend noticed when we rise the magnetic field, prandtl number and volume fractions of hybrid nanoparticles.
- The heat transfer enhancement increased with raising into volume fraction of hybrid nanoparticle, and no other physical parameters in our study influence the heat transfer enhancement.

Author contributions:
Ravi Gupta: validation, visualization, conceptualization, writing-review & editing, resources, supervision, methodology. **Bharat Kumar:** writing-original draft, problem formulation, data curation, software, investigation, visualization, formal analysis, Writing- Review & editing, methodology.
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Data availability statement: Whole data analyzed are included in this work
Conflict of interest: The authors have no any conflict of Interest.
Future scope: For future work different types of hybrid nanofluid with variable electrical conductivity can be investigate.

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