Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–13.
ISSN-0037-8712
doi:10.5269/bspm.76889

Hermite-Hadamard Type Inequalities for Multiplicatively m-Convex Functions

Ashok Kumar Sahoo, Bibhakar Kodamasingh and Binod Chandra Tripathy

ABSTRACT: Convex functions play an important role in finding the inequalities, those help in finding the solutions of different types of equations and equations involving functions. In this article we have considered investigated on the m -convex functions in a normed linear space. We have established some results of Hermite-Hadamard like inequality. The results established will be useful for investigation in different branches of science and engineering for solving different problems.

Key Words: Integral inequality, m-Convex function, Hermite-Hadamard inequality, Harmonically convex function.

Contents

1. Introduction

The idea of convexity is thought to be improved upon by the Hermite-Hadamard inequality. This idea has been thoroughly studied by many researchers since it was separately discovered by in 1883 and Hadamard in 1896. Specifically, much work has been done over the last 20 years to establish new boundaries for the Hermite-Hadamard inequality's left and right sides. Numerous research works have suggested innovative methods to strengthen, expand, and enhance this disparity.

Convex functions can be used to derive some inequalities, according to numerous researcher. Among the most well-known inequalities relating to a convex function's integral mean HermiteHadamard inequality is what it is. The following is the statement of this double inequality (see [[1]-[3]]).

The following is the definition of classical convexity:
The function $g : [\hat{c}, \hat{d}] \sqsubseteq R \to R$ is considered convex in the standard way if

$$g(w\hat{x} + (1-w)\hat{y}) \leq wg(\hat{x}) + (1-w)g(\hat{y})$$

For all $\hat{x}, \hat{y} \in [\hat{c}, \hat{d}]$ and $w \in [0,1]$. If -g is convex, then the function g is said to be concave.

Theorem 1.1. Assume $g : I = [\hat{c}, \hat{d}] \sqsubseteq R \to R$ be a convex function that is integrable. Then

$$g\left(\frac{\hat{c}+\widehat{d}}{2}\right) \leq \frac{1}{\hat{d}-\hat{c}} \int_{\hat{c}}^{\widehat{d}} g(\hat{x}) d\widehat{x} \leq \frac{g(\hat{c})+g(\hat{d})}{2}$$

If g is concave, then both inequalities hold in the opposite direction.

Submitted May 18, 2025. Published July 13, 2025
2010 *Mathematics Subject Classification:* 26A51, 26B25, 26D15, 90C26.

2. Definitions and Preliminaries

Here are some fundamental concepts and findings in the case of intervals I and J.
It can established harmonically convexity and demonstrated the Hermite-Hadamard. There is an inequality for functions which are harmonically convex:

Definition 2.1. Let an interval be $I \subseteq R - \{0\}$. A $g : I \to R$ function is defined as Harmonically Convex function if

$$g\left(\frac{\hat{x}\hat{y}}{w\hat{x} + (1-w)\hat{y}}\right) \leq (1-w)g(\hat{x}) + wg(\hat{y})$$

For all $\hat{x}, \hat{y} \in [\hat{c}, \hat{d}]$ and $w \in [0, 1]$.

Theorem 2.1 Let $\hat{c}, \hat{d} \in I$ with $\hat{c} < \hat{d}$ and let $g : I \sqsubseteq R - \{0\} \to R$ be a harmonically convex function.

Inequalities hold if g is in $[\hat{c}, \hat{d}]$.

$$g\left(\frac{2\hat{c}\hat{d}}{\hat{c} + \hat{d}}\right) \leq \frac{\hat{d}\hat{c}}{\hat{d} - \hat{c}} \int_{\hat{c}}^{\hat{d}} \frac{g(\hat{x})}{\hat{x}^2} d\hat{x} \leq \frac{g(\hat{c}) + g(\hat{d})}{2} \tag{1}$$

A few new findings about Hermite-Hadamard inequality for convex functions of harmonic type as observed in [[5]- [8]].

Definition 2.2 ([9]) Let $m \in R - \{0\}$ and that $I \subset (0, \infty)$ is a real interval. Given a function $g : I \to R$, if a function is m-convex, then

$$g\left([w\hat{x}^m + (1-w)\hat{y}^m]^{1/m}\right) \leq wg(\hat{x}) + (1-w)g(\hat{y})$$

For each $\hat{x}, \hat{y} \in I$ and $w \in [0, 1]$.

It is clear from Definition (2.2) that for $m = 1$ and $m = -1$, it becomes clear that m convexity can be reduced to harmonic and classical convexity of functions defined on, respectively.

If we assume $I \subset (0, \infty), m \in R - \{0\}$, and $h(w) = w$ in Theorem 5 of [10], then the following Theorem holds.

Theorem 2.2 Let $g : I \subset (0, \infty) \to R$ be a m - convex function, $m \in R - \{0\}$ and $\hat{c}, \hat{d} \in I$
With $\hat{c} < \hat{d}$.If $g \in L[\hat{c}, \hat{d}]$,then

$$g\left(\left[\frac{\hat{c}^m + \hat{d}^m}{2}\right]^{\frac{1}{m}}\right) \leq \frac{m}{\hat{d}^m - \hat{c}^m} \int_{\hat{c}}^{\hat{d}} \left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right) d\hat{x} \leq \frac{g(\hat{c}) + g(\hat{d})}{2}$$

Definition 2.3 ([3]) If g: J $\to (0, \infty)$ is a logarithmic or multiplicatively convex function, then

$$g(w\hat{x} + (1-w)\hat{y}) \leq [g(\hat{x})]^w [g(\hat{y})]^{1-w}$$

For all $\hat{x}, \hat{y} \in J$ and $w \in [0, 1]$.

2.1 Multiplicative Calculus

Consider that the integral of multiplicative involves the product of terms raised to particular powers. a type of integral, is represented by $\int_{\hat{c}}^{\hat{d}} (g(\hat{x}))^{d\hat{x}}$, whereas the sum of terms is involved in the conventional integral, a form of integral, is usually indicated by $\int_{\hat{c}}^{\hat{c}} (g(\hat{x}))d\widehat{x}$.
It is easier to distinguish between these two kinds of integrals when different symbols are used.
The multiplicative integral and the Riemann integral have the following relationship [11].
Proposition 2.1 In this situation, g is multiplicatively integrable on $[\hat{c}, \hat{d}]$, and g is Riemann integrable on $[\hat{c}, \hat{d}]$.

$$\int_{\hat{c}}^{\hat{d}} (g(\hat{x}))^{d\hat{x}} = e^{\int_{\hat{c}}^{\hat{d}} \ln(g(\hat{x}))d\hat{x}}$$

Bashirov et al. demonstrate the following findings and notations for the multiplicative integral in [11]:

Proposition 2.2 On [cˆ,d], g is multiplicatively integrable if it is positive and Riemann integrable.

1. $\int_{\hat{c}}^{\hat{d}} ((g(\hat{x}))^m)^{d\hat{x}} = \int_{\hat{c}}^{\hat{d}} \left((g(\hat{x}))^{d\hat{x}}\right)^m$

2. $\int_{\hat{c}}^{\hat{d}} (g(\hat{x})h(\hat{x}))^{d\hat{x}} = \int_{\hat{c}}^{\hat{d}} (g(\hat{x}))^{d\hat{x}} \cdot \int_{\hat{c}}^{\hat{d}} (h(\hat{x}))^{d\hat{x}}$

3. $\int_{\hat{c}}^{\hat{d}} \left(\frac{g(\hat{x})}{h(\hat{x})}\right)^{d\hat{x}} = \frac{\int_{\hat{c}}^{\hat{d}} (g(\hat{x}))^{d\hat{x}}}{\int_{\hat{c}}^{\hat{d}} (h(\hat{x}))^{d\hat{x}}}$

4. $\int_{\hat{c}}^{\hat{d}} (g(\hat{x}))^{d\hat{x}} = \int_{\hat{c}}^{\theta} (g(\hat{x}))^{d\hat{x}} \cdot \int_{\theta}^{\hat{d}} (g(\hat{x}))^{d\hat{x}}, \hat{c} \leq \theta \leq \hat{d}$

5. $\int_{\hat{c}}^{\hat{d}} (g(\hat{x}))^{d\hat{x}} = 1$ and $\int_{\hat{c}}^{\hat{d}} (g(\hat{x}))^{d\hat{x}} = \left(\int_{\hat{d}}^{\hat{c}} (g(\hat{x}))^{d\hat{x}}\right)^{-1}$

In the context of multiplicative calculus. In [12], Ali et al. established the following Hermite-Hadamard inequality for multiplicatively convex functions:

Theorem 2.3 Let g be a positive function that is multiplicatively convex on $[\hat{c}, \hat{d}]$. Then

$$g\left(\frac{\hat{c}+\hat{d}}{2}\right) \leq \left(\int_{\hat{c}}^{\hat{d}} (g(\hat{x}))^{d\hat{x}}\right)^{\frac{1}{\hat{d}-\hat{c}}} \leq G(g(\hat{c}), g(\hat{d})),$$

where $G(.,.) is the geometric mean.$

In the 1970s, [13] conducted one of the earliest investigations on multiplicative calculus. Since then, a variety of intriguing outcomes have been attained as a result of its extensive use in numerous disciplines. For instance, complex multiplicative calculus was introduced by Baiskov and Riza in [14].Certain characteristics of stochastic multiplicative calculus have been examined in [15] and [16].Regarding certain uses and other facets of this field, please refer to [[17]- [22]] and the associated references.

3. Main Results

This section provides a new definition for the term "multicatively m-convex function" as well as various Hermite-Hadamard type integral inequalities for multiplicative m-convex and convex functions in the context of multiplicative calculus.

Definition 3.1 A function that is not negative If $g : I \to R$ is multiplicatively m-convex, then

$$g\left([w\hat{x}^m + (1-w)\hat{y}^m]^{1/m}\right) \leq [g(\hat{x})]^w [g(\hat{y})]^{1-w},$$

is valid for all $\hat{x}, \hat{y} \in I, w \in [0,1]$ and $m \in R - \{0\}$.

Theorem 3.1 Let g be a multiplicatively m - convex function on $[\hat{c}, \hat{d}]$.Then

$$g\left(\left[\frac{\hat{c}^m + \hat{d}^m}{2}\right]^{\frac{1}{m}}\right) \leq \left(\int_{\hat{c}}^{\hat{d}} \left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m}{\hat{d}^m - \hat{c}^m}} \leq G(g(\hat{c}), g(\hat{d}))$$

Proof: Note that

$$\text{lng}\left(\left[\frac{\hat{c}^m+\hat{d}^m}{2}\right]^{\frac{1}{m}}\right)=\text{lng}\left(\left[\frac{w\hat{c}^m+(1-w)\hat{d}^m+(1-w)\hat{c}^m+w\hat{d}^m}{2}\right]^{\frac{1}{m}}\right)$$
$$=\text{lng}\left(\left[\frac{w\hat{c}^m+(1-w)\hat{d}^m}{2}+\frac{(1-w)\hat{c}^m+w\hat{d}^m}{2}\right]^{\frac{1}{m}}\right)$$
$$\leq \ln\left[\left(g\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)^{\frac{1}{2}}\left(g\left((1-w)\hat{c}^m+w\hat{d}^m\right)^{\frac{1}{m}}\right)^{\frac{1}{2}}\right]$$
$$=\frac{1}{2}\,\text{lng}\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)+\frac{1}{2}\ln g\left(\left((1-w)\hat{c}^m+w\hat{d}^m\right)^{\frac{1}{m}}\right)$$

When we integrate the aforementioned inequality with regard to w on $[0,1]$, we obtain

$$\ln g\left(\left[\frac{\hat{c}^m+\hat{d}^m}{2}\right]^{\frac{1}{m}}\right)\leq\frac{1}{2}\int_0^1\ln g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)dw+\frac{1}{2}\int_0^1\ln g\,(((1-w)\hat{c}^m+$$
$$w\hat{d}^m\Big)^{\frac{1}{m}}\Big)\,dw$$
$$=\frac{m}{\hat{d}^m-\hat{c}^m}\int_{\hat{c}}^{\hat{d}}\ln\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)d\hat{x}.$$

Which implies,

$$\text{lng}\left(\left[\frac{\hat{c}^m+\hat{d}^m}{2}\right]^{\frac{1}{m}}\right)\leq\frac{m}{\hat{d}^m-\hat{c}^m}\int_{\hat{c}}^{\hat{d}}\ln\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)d\hat{x}$$

Thus, we have

$$g\left(\left[\frac{\hat{c}^m+\hat{a}^m}{2}\right]^{\frac{1}{m}}\right)\leq e^{\frac{m}{\hat{d}^m-\hat{c}^m}\int_{\hat{c}}^{\hat{\alpha}}\ln\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)d\hat{x}}$$
$$=\left(\int_{\hat{c}}^{\hat{a}}\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m}{\hat{a}^m-\hat{c}^m}}$$

It results in the initial inequality. Let us now examine the second inequality.

$$\left(\int_{\hat{c}}^{\hat{d}}\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m}{\hat{d}^m-\hat{c}^m}}$$
$$=e^{\left(\int_{\hat{c}}^{\hat{d}}\ln\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)d\hat{x}\right)^{\frac{m}{\hat{d}^m-\hat{c}^m}}}$$
$$=e^{\frac{m}{\hat{d}^m-\hat{c}^m}\int_{\hat{c}}^{\hat{a}}\ln\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)d\hat{x}}$$
$$=e^{\int_0^1\ln g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)dw}$$
$$\leq e^{\int_0^1\ln\left[g(\hat{c})^w g(\hat{d})^{1-w}\right]dw}$$
$$=e^{\int_0^1[w\ln g(\hat{c})+(1-w)\ln g(\hat{d})]dw}$$
$$=e^{\ln(g(\hat{c})g(\hat{d}))^{\int_0^1 wdw}}$$
$$=(g(\hat{c}),g(\hat{d}))$$

Thus, the proof is completed.

Remark 3.1 In Theorem 3.1, on taking $m=1$, we get Theorem 5 of Ali et al. [11].

Corollary 3.1 Assume g & h be m-convex functions that multiply on $[\hat{c}, \hat{d}]$. Then

$$\begin{aligned} & g\left(\left[\frac{\hat{c}^m+\hat{d}^m}{2}\right]^{\frac{1}{m}}\right) h\left(\left[\frac{\hat{c}^m+\hat{d}^m}{2}\right]^{\frac{1}{m}}\right) \\ & \leq\left(\int_{\hat{c}}^{\hat{d}}\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)^{d \hat{x}} \int_{\hat{c}}^{\hat{a}}\left(\frac{h(\hat{x})}{\hat{x}^{1-m}}\right)^{d \hat{x}}\right)^{\frac{m}{\hat{a}^m-\hat{c}^m}} \\ & \leq G(g(\hat{c}), g(\hat{d})) \cdot G(h(\hat{c}), h(\hat{d})). \end{aligned}$$

Proof: Multiactivity m-convex functions g and h imply that gh is also a multiplicatively mconvex function. Thus, we get the intended outcome if we apply Theorem 3.1 to the function gh.

Remark 3.2 In Corollary 3.1, if we select $m=1$, we obtain Theorem 7 in [12].

Corollary 3.2 Assume g & h be functions that multiply m-convex on $[\hat{c}, \hat{d}]$.Then

$$\frac{g\left(\left[\frac{\hat{c}^m+\widehat{d}^m}{2}\right]^{\frac{1}{m}}\right)}{h\left(\left[\frac{\hat{c}^m+\widehat{d}^m}{2}\right]^{\frac{1}{m}}\right)} \leq\left(\frac{\int_{\hat{c}}^{\widehat{d}}\left(\frac{g(\hat{x})}{\hat{\chi}^{1-m}}\right)^{d \widehat{x}}}{\int_{\hat{c}}^{\widehat{d}}\left(\frac{h(\widehat{x})}{\widehat{x}^{1-m}}\right)^{d \widehat{x}}}\right)^{\frac{m}{\widehat{d}^m-\hat{c}^m}} \leq \frac{G(g(\hat{c}), g(\hat{d}))}{G(h(\hat{c}), h(\hat{d}))}$$

Proof: Given that g and h are m -convex functions that multiply, then $\frac{g}{h}$ is a multiplicatively m convex

Thus, we obtain the intended result if we apply Theorem 3.1 to the function $\frac{g}{h}$.

Remark 3.3 Using corollary 3.2's choice of m = 1, we obtain Theorem 9 in [12].

Theorem 3.2 Assume that h is a multiplicatively m-convex function and g is a convex function. Then

$$\left(\frac{\int_{\hat{c}}^{\widehat{d}}\left(\frac{g(\hat{x})}{\hat{x}^{1-m})^{d \hat{x}}}\right.}{\int_{\hat{c}}^{\hat{d}}\left(\frac{h(\widehat{x})}{\widehat{x}^{1-m})^{d \hat{x}}}\right.}\right)^{\frac{m}{\widehat{d}^m-\hat{c}^m}} \leq \frac{\left(\frac{g(\widehat{d})^{g(\widehat{d})}}{g(\hat{c})^{g(\hat{c})})}\right)^{\frac{1}{g(\widehat{d})-g(\hat{c})}}}{G(h(\hat{c}), h(\hat{d})) \cdot e}$$

Proof: Note that

$$\begin{aligned} & \left(\frac{\int_{\hat{c}}^{\widehat{d}}\left(\frac{g(\hat{x})}{\widehat{x}^{1-m}}\right)^{d \widehat{x}}}{\int_{\hat{c}}^{\hat{d}}\left(\frac{h(\widehat{x})}{\widehat{x}^{1-m}}\right)^{d \widehat{x}}}\right)^{\frac{m}{\hat{d}^m-\hat{c}^m}} = \left(\frac{e^{\int_{\hat{c}}^{\hat{d}} \ln \left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right) d \widehat{x}}}{e^{\int_{\hat{c}}^{\hat{d}} \ln \left(\frac{h(\hat{x})}{\widehat{x}^{1-m}}\right) d \widehat{x}}}\right)^{\frac{m}{\bar{d}^m-\hat{c}^m}} \\ & =e^{\left(\int_{\hat{c}}^{\widehat{d}} \ln \left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right) d \hat{x}-\int_{\hat{c}}^{\widehat{d}} \ln \left(\frac{h(\hat{x})}{\hat{\hat{x}}^{1-m}}\right) d \hat{x}\right)^{\frac{m}{\hat{d}^m-\hat{c}^m}}} \\ & =e^{\left(\int_0^1 \ln g\left(\left(w \hat{c}^m+(1-w) \hat{d}^m\right)^{\frac{1}{m}}\right) d w-\int_0^1 \operatorname{lnh}\left(\left(w \hat{c}^m+(1-w) \hat{d}^m\right)^{\frac{1}{m}}\right) d w\right)} \\ & \leq e^{\left(\int_0^1 \ln (g(\hat{d})+w(g(\hat{c})-g(\hat{d}))) d w-\int_0^1 \ln \left((h(\hat{c}))^w(h(\hat{d}))^{(1-w)}\right) d w\right)} \\ & =e^{\ln \left(\left(\frac{g(\hat{d})^{g(\hat{d})}}{g(\hat{c})^{g(\hat{c})}}\right)^{\frac{1}{g(\hat{d})-g(\hat{c})}}\right)-1-\ln (h(\hat{c}) h(\hat{d}))^{\int_0^1 w d w}} \\ & =\frac{\left(\frac{g(\hat{d})^{g(\hat{d})}}{g(\hat{c})^{g(\hat{c})}}\right)^{\frac{1}{g(\hat{d})-g(\hat{c})}}}{G(h(\hat{c}), h(\hat{d})) \cdot e} \end{aligned}$$

The proof has been completed.

Remark 3.4 Applying $m=1$ in Theorem 3.2 allows us to get Theorem 11 in [12].

Theorem 3.3 Assume that h is a convex function and g is a multiplicatively m-convex function. Then

Proof: Note that

$$\left(\frac{\int_{\hat{c}}^{\widehat{d}}\left(\frac{g(\hat{x})}{\widehat{x}^{1-m}}\right)^{d\widehat{x}}}{\int_{\hat{c}}^{\hat{d}}\left(\frac{h(\widehat{x})}{\widehat{x}^{1-m}}\right)^{d\widehat{x}}}\right)^{\frac{m}{\hat{d}^m-\hat{c}^m}} = \left(\frac{e^{\int_{\hat{c}}^{\hat{d}}\ln\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)d\widehat{x}}}{e^{\int_{\hat{c}}^{\hat{d}}\ln\left(\frac{h(\hat{x})}{\hat{x}^{1-m}}\right)d\widehat{x}}}\right)^{\frac{m}{\bar{d}^m-\hat{c}^m}}$$

$$= e^{\left(\int_{\tilde{\imath}a}^{a}\ln\left(\frac{g(\hat{\chi})}{\hat{x}^{1-m}}\right)d\hat{x}-\int_{\tilde{\hat{c}}}^{\hat{a}}\ln\left(\frac{h(\hat{x})}{\hat{x}^{1-m}}\right)d\hat{x}\right)^{\frac{m}{\bar{m}^m-\hat{c}^m}}}$$

$$= e^{\left(\int_0^1 \ln g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)dw-\int_0^1 \ln\mathrm{h}\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)dw\right)}$$

$$\leq e^{\left(\int_0^1 \ln\left((g(\hat{c}))^w(g(\hat{d}))^{(1-w)}\right)dw-\int_0^1 \ln(h(\hat{d})+w(h(\hat{c})-h(\hat{d})))dw\right)}$$

$$= e^{\ln(g(\hat{c})g(\hat{d}))^{\int_0^1 w\,dw}-\ln\left(\left(\frac{g(\hat{d})^{g(\hat{d})}}{g(\hat{c})^{g(\hat{c})}}\right)^{\frac{1}{g(\hat{d})-g(\hat{c})}}\right)+1}$$

$$= \frac{G\left(g(\hat{c}),g(\hat{d})\right)\cdot e}{\left(\frac{h(\hat{(}d)^{h(\hat{(}d)}}{h(\hat{(}c)^{h(\hat{(}c)}}\right)^{\frac{1}{h(\hat{(}d)-h(\hat{(}c)}}}$$

The proof has been completed.

Remark 3.5 Applying $m=1$ in Theorem 3.3 allows us to reach Theorem 12 in [12].

Theorem 3.4 Assume that h is a multiplicatively m-convex function and g is a convex function. Then

$$\left(\int_{\hat{c}}^{\hat{d}}\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\cdot\int_{\hat{c}}^{\hat{d}}\left(\frac{h(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m}{\bar{d}^m-\hat{c}^m}} \leq \frac{\left(\frac{g(\hat{d})^{g(\hat{d})}}{g(\hat{c})^{g(\hat{c})}}\right)^{\frac{1}{g(\hat{d})-g(\hat{c})}}\cdot G(h(\hat{c}),h(\hat{d}))}{e}$$

Proof: Note that

$$\left(\int_{\hat{c}}^{\hat{d}}\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\cdot\int_{\hat{c}}^{\hat{d}}\left(\frac{h(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m}{\bar{d}^m-\hat{c}^m}}$$

$$= e^{\left(\int_{\hat{c}}^{\hat{d}}\ln\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)d\hat{x}+\int_{\hat{c}}^{\hat{d}}\ln\left(\frac{h(\hat{x})}{\hat{x}^{1-m}}\right)d\hat{x}\right)^{\frac{m}{\bar{d}^m-\hat{c}^m}}}$$

$$= e^{\left(\int_0^1 \ln g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)dw+\int_0^1 \ln\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)dw\right)}$$

$$\leq e^{\left(\int_0^1 \ln(g(\hat{d})+w(g(\hat{c})-g(\hat{d})))dw+\int_0^1 \ln\left((h(\hat{c}))^w(h(\hat{d}))^{(1-w)}\right)dw\right)}$$

$$= e^{\ln\left(\left(\frac{g(\hat{d})^{g(\hat{d})}}{g(\hat{c})^{g(\hat{c})}}\right)^{\frac{1}{g(\hat{d})-g(\hat{c})}}\right)-1+\ln(h(\hat{c})h(\hat{d}))^{\int_0^1 w\,dw}}$$

$$= \frac{\left(\frac{g(\hat{d})^{g(\hat{d})}}{g(\hat{c}g(\hat{c})}\right)^{\frac{1}{g(\hat{d})-g(\hat{c})}}\cdot G(h(\hat{c}),h(\hat{d}))}{e}.$$

The proof has been completed.

Remark 3.6 Applying $m=1$ in Theorem 3.4 allows us to arrive at Theorem 13 in [12].

Theorem 3.5 Let $g: I \to R$ be multiplicatively m - convex function, where $\hat{c},\hat{d} \in I$ and $\hat{c} < \hat{d}$.Then

$$\left(\int_{\hat{c}}^{\hat{d}}\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m}{\hat{a}^m-\hat{c}^m}} \leq \frac{g(\hat{c})+g(\hat{d})}{2}$$

Proof: Note that

$$\left(\int_{\hat{c}}^{\hat{d}}\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m}{\hat{d}^m-\hat{c}^m}}$$
$$= e^{\left(\int_{\hat{c}}^{\hat{d}}\ln\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)d\hat{x}\right)\frac{m}{\hat{d}^m-\hat{c}^m}}$$
$$= e^{\frac{m}{\hat{d}^m-\hat{c}^m}\int_{\hat{c}}^{\hat{d}}\ln\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)d\hat{x}}$$
$$= e^{\int_0^1 \ln\left(\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)dw\right.}$$

$$\leq \int_0^1 e^{\ln g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)}\mathrm{dw}$$
$$= \int_0^1 g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)dw$$
$$\leq \int_0^1 \left[(g(\hat{c}))^w(g(\hat{d}))^{(1-w)}\right]dw$$
$$= g(\hat{d})\int_0^1\left(\frac{g(\hat{c})}{g(\hat{d})}\right)^w dw$$
$$= \frac{g(\hat{c})-g(\hat{d})}{\log g(\hat{c})-\log g(\hat{d})}$$
$$\leq \frac{g(\hat{c})+g(\hat{d})}{2}$$

The proof has been completed.

Theorem 3.6 Let $g,h: I\to R$ be multiplicatively m - convex functions, where $\hat{c},\hat{d}\in I, \hat{c}<\hat{d}$. Then

$$\left(\int_{\hat{c}}^{\hat{d}}\left(\frac{g(\hat{\hat{x}})h(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m}{\hat{d}^m-\hat{c}^m}} \leq \frac{1}{4}\mu(\hat{c},\hat{d})$$

Where $\mu(\hat{c},\hat{d})=(g(\hat{c}))^2+(g(\hat{d}))^2+(h(\hat{c}))^2+(h(\hat{d}))^2$.

Proof: We note that

$$\left(\int_{\hat{c}}^{\hat{d}}\left(\frac{g(\hat{x})h(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m}{\hat{d}^m-\hat{c}^m}} = e^{\left(\int_{\hat{c}}^{\hat{d}}\ln\left(\frac{g(\hat{x})h(\hat{x})}{\hat{x}^{1-m}}\right)d\hat{x}\right)\frac{m}{\hat{d}^m-\hat{c}^m}}$$
$$= e^{\frac{m}{\hat{d}^m-\hat{c}^m}\int_{\hat{c}}^{\hat{d}}\ln\left(\frac{g(\hat{x})h(\hat{x})}{\hat{x}^{1-m}}\right)d\hat{x}}$$
$$= e^{\int_0^1\ln\left[g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\right]dw}$$
$$\leq \int_0^1 e^{\ln\left[g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\right]}\mathrm{dw}$$
$$= \int_0^1\left[g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\right]dw$$
$$\leq \int_0^1\left[(g(\hat{c}))^w(g(\hat{d}))^{(1-w)}(h(\hat{c}))^w(h(\hat{d}))^{(1-w)}\right]dw$$

$$= g(\hat{c})g(\hat{d})\int_0^1 \left(\frac{g(\hat{c})h(\hat{c})}{g(\hat{d})h(\hat{d})}\right)^w dw$$
$$= \frac{g(\hat{c})h(\hat{c}) - g(\hat{d})h(\hat{d})}{\log(g(\hat{c})h(\hat{c})) - \log(g(\hat{d})h(\hat{d}))}$$
$$\leq \frac{g(\hat{c})h(\hat{c}) + g(\hat{d})h(\hat{d})}{2}$$
$$\leq \frac{1}{2}\int_0^1 \left[\left(g\left(\left(w\hat{c}^m + (1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\right)^2 + \left(h\left(\left(w\hat{c}^m + (1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\right)^2\right] dw$$
$$\leq \frac{1}{2}\int_0^1 \left[\left((g(\hat{c}))^w (g(\hat{d}))^{(1-w)}\right)^2 + \left((h(\hat{c}))^w (h(\hat{d}))^{(1-w)}\right)^2\right] dw$$
$$= \frac{(g(\hat{d}))^2}{2}\int_0^1 \left(\frac{g(\hat{c})}{g(\hat{d})}\right)^{2w} dw + \frac{(h(\hat{d}))^2}{2}\int_0^1 \left(\frac{h(\hat{c})}{h(\hat{d})}\right)^{2w} dw$$
$$= \frac{1}{4}\frac{(g(\hat{c}))^2 - (g(\hat{d}))^2}{\log g(\hat{c}) - \log g(\hat{d})} + \frac{1}{4}\frac{(h(\hat{c}))^2 - (h(\hat{d}))^2}{\log h(\hat{c}) - \log h(\hat{d})}$$
$$\leq \frac{1}{2}\frac{g(\hat{c}) + g(\hat{d})}{2} \cdot \frac{g(\hat{c}) - g(\hat{d})}{\log g(\hat{c}) - \log g(\hat{d})} + \frac{1}{2}\frac{h(\hat{c}) + h(\hat{d})}{2} \cdot \frac{h(\hat{c}) - h(\hat{d})}{\log h(\hat{c}) - \log h(\hat{d})}$$
$$\leq \frac{1}{4}\left[(g(\hat{c}))^2 + (g(\hat{d}))^2 + (h(\hat{c}))^2 + (h(\hat{d}))^2\right]$$

The proof has been completed.

Theorem 3.7 Let $g, h : I \to R$ be multiplicatively m - convex functions, where $\hat{c}, \hat{d} \in I, \hat{c} < \hat{d}$. Then

$$\left(\int_{\hat{c}}^{\hat{d}} \left(\frac{g(\hat{x})h(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m}{\hat{d}^m - \hat{c}^m}} \leq \frac{1}{4}\mu(\hat{c}, \hat{d}) + \frac{1}{4}\vartheta(\hat{c}, \hat{d})$$

Where $\mu(\hat{c}, \hat{d}) = (g(\hat{c}))^2 + (g(\hat{d}))^2 + (h(\hat{c}))^2 + (h(\hat{d}))^2$

And $\vartheta(\hat{c}, \hat{d}) = g(\hat{c})h(\hat{c}) + g(\hat{d})h(\hat{d})$

Proof: Let g and h be multiplicatively m - convex functions. Using the inequality

$$\hat{c}\hat{d} \leq \frac{1}{4}(\hat{c} + \hat{d})^2, \text{ for all } \hat{c}, \hat{d} \in R$$

We have,

$$
\begin{aligned}
&\left(\int_{\hat{c}}^{\hat{d}}\left(\frac{g(\hat{x})h(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m}{\hat{d}^m-\hat{c}^m}}\\
&\leq \int_0^1\left[g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\right]\mathrm{d}w\\
&\leq \frac{1}{4}\int_0^1\left[g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)+h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\right]^2 dw\\
&\leq \frac{1}{4}\int_0^1\left[(g(\hat{c}))^w(g(\hat{d}))^{(1-w)}+(h(\hat{c}))^w(h(\hat{d}))^{(1-w)}\right]dw\\
&= \frac{1}{4}\int_0^1\left[g(\hat{d})\left(\frac{g(\hat{c})}{g(\hat{d})}\right)^w+h(\hat{d})\left(\frac{h(\hat{c})}{h(\hat{d})}\right)^w\right]^2 dw\\
&= \frac{(g(\hat{d}))^2}{8}\int_0^2\left(\frac{g(\hat{c})}{g(\hat{d})}\right)^\theta d\theta+\frac{(h(\hat{d}))^2}{8}\int_0^2\left(\frac{h(\hat{c})}{h(\hat{d})}\right)^\theta d\theta+\frac{g(\hat{d})h(\hat{d})}{2}\int_0^1\left(\frac{g(\hat{c})h(\hat{c})}{g(\hat{d})h(\hat{d})}\right)^w dw\\
&= \frac{1}{8}\frac{(g(\hat{c}))^2-(g(\hat{d}))^2}{\log g(\hat{c})-\log g(\hat{d})}+\frac{1}{8}\frac{(h(\hat{c}))^2-(h(\hat{d}))^2}{\log h(\hat{c})-\log h(\hat{d})}+\frac{1}{2}\frac{g(\hat{c})h(\hat{c})-g(\hat{d})h(\hat{d})}{\log(g(\hat{c})h(\hat{c}))-\log(g(\hat{d})h(\hat{d}))}\\
&= \frac{1}{4}\frac{g(\hat{c})+g(\hat{d})}{2}\cdot\frac{g(\hat{c})-g(\hat{d})}{\log g(\hat{c})-\log g(\hat{d})}+\frac{1}{4}\frac{h(\hat{c})+h(\hat{d})}{2}\cdot\frac{h(\hat{c})-h(\hat{d})}{\log h(\hat{c})-\log h(\hat{d})}+\\
&\leq \frac{1}{8}\left[(g(\hat{c}))^2+(g(\hat{d}))^2+(h(\hat{c}))^2+(h(\hat{d}))^2\right]+\frac{1}{4}[g(\hat{c})h(\hat{c})+g(\hat{d})h(\hat{d})]
\end{aligned}
$$

The proof has been completed.

Theorem 3.8 Let $g,h:I\to R$ be multiplicatively m - convex functions, where $\hat{c},\hat{d}\in I,\hat{c}<\hat{d}$. Then

$$
\left(\int_{\hat{c}}^{\hat{d}}\left(\frac{g(\hat{x})h(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m}{\hat{d}^m-\hat{c}^m}} \leq \gamma_1\frac{g(\hat{c})+g(\hat{d})}{2}\left[L_{\frac{1}{\gamma_1}-1}(g(\hat{c}),g(\hat{d}))\right]^{\frac{1}{\gamma_1}-1}+
\gamma_2\frac{h(\hat{c})+h(\hat{d})}{2}\left[L_{\frac{1}{\gamma_2}-1}(h(\hat{c}),h(\hat{d}))\right]^{\frac{1}{\gamma_2}-1}
$$

Proof: Let g and h be multicatively m - convex functions. Then using the inequality

$$
\hat{c}\hat{d}\leq\gamma_1\hat{c}^{\frac{1}{\gamma_1}}+\gamma_2\hat{d}^{\frac{1}{\gamma_2}},\quad \gamma_1,\gamma_2>0,\gamma_1+\gamma_2=1
$$

We have

$$\left(\int_{\hat{c}}^{\hat{a}}\left(\frac{g(\hat{x})h(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m}{\hat{d}^m-\hat{c}^m}}$$
$$\leq \int_0^1\left[g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\right]\mathrm{dw}$$
$$\leq \int_0^1\left[\gamma_1\left\{g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\right\}^{\frac{1}{\gamma_1}}+\gamma_2\left\{h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\right\}^{\frac{1}{\gamma_2}}\right]dw$$
$$\leq \int_0^1\left[\gamma_1\left\{(g(\hat{c}))^w(g(\hat{d}))^{(1-w)}\right\}^{\frac{1}{\gamma_1}}+\gamma_2\left\{(h(\hat{c}))^w(h(\hat{d}))^{(1-w)}\right\}^{\frac{1}{\gamma_2}}\right]dw$$
$$=\gamma_1(g(\hat{d}))^{\frac{1}{\gamma_1}}\int_0^1\left(\frac{g(\hat{c})}{g(\hat{d})}\right)^{\frac{1}{\gamma_1}}dw+\gamma_2(h(\hat{d}))^{\frac{1}{\gamma_2}}\int_0^1\left(\frac{h(\hat{c})}{h(\hat{d})}\right)^{\frac{1}{\gamma_2}}dw$$
$$=(\gamma_1)^2\left(g(\hat{d})\right)^{\frac{1}{\gamma_1}}\int_0^{\frac{1}{\gamma_1}}\left(\frac{g(\hat{c})}{g(\hat{d})}\right)^{\theta}d\theta+(\gamma_2)^2\left(h(\hat{d})\right)^{\frac{1}{\gamma_2}}\int_0^{\frac{1}{\gamma_2}}\left(\frac{h(\hat{c})}{h(\hat{d})}\right)^{\theta}d\theta$$
$$=(\gamma_1)^2\frac{(g(\hat{c}))^{\frac{1}{\gamma_1}}-(g(\hat{d}))^{\frac{1}{\gamma_1}}}{\log g(\hat{c})-\log g(\hat{d})}+(\gamma_2)^2\frac{(h(\hat{c}))^{\frac{1}{\gamma_2}}-(h(\hat{d}))^{\frac{1}{\gamma_2}}}{\log h(\hat{c})-\log h(\hat{d})}$$
$$=(\gamma_1)^2\frac{(g(\hat{c}))^{\frac{1}{\gamma_1}}-(g(\hat{d}))^{\frac{1}{\gamma_1}}}{g(\hat{c})-g(\hat{d})}L[g(\hat{c}),g(\hat{d})]+(\gamma_2)^2\frac{(h(\hat{c}))^{\frac{1}{\gamma_2}}-(h(\hat{d}))^{\frac{1}{\gamma_2}}}{h(\hat{c})-h(\hat{d})}L[h(\hat{c}),h(\hat{d})]$$
$$=\gamma_1\left[L_{\frac{1}{\gamma_1}-1}(g(\hat{c}),g(\hat{d}))\right]^{\frac{1}{\gamma_1}-1}L[g(\hat{c}),g(\hat{d})]+$$
$$\gamma_2\left[L_{\frac{1}{\gamma_2}-1}(h(\hat{c}),h(\hat{d}))\right]^{\frac{1}{\gamma_2}-1}L[h(\hat{c}),h(\hat{d})]$$
$$\leq \gamma_1\frac{g(\hat{c})+g(\hat{d})}{2}\left[L_{\frac{1}{\gamma_1}-1}(g(\hat{c}),g(\hat{d}))\right]^{\frac{1}{\gamma_1}-1}+\gamma_2\frac{h(\hat{c})+h(\hat{d})}{2}\left[L_{\frac{1}{\gamma_2}-1}(h(\hat{c}),h(\hat{d}))\right]^{\frac{1}{\gamma_2}-1}$$

The proof has been completed.

Theorem 3.9 Let $g,h:I\to R$ be increasing multiplicatively $m-$ convex functions, where $\hat{c},\hat{d}\in I,\hat{c}<\hat{d}$. Then

$$\left(\int_{\hat{c}}^{\hat{d}}\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m\cdot\ln G(h(\hat{c}),h(\hat{d}))}{\hat{d}^m-\hat{c}^m}}\cdot\left(\int_{\hat{c}}^{\hat{d}}\left(\frac{h(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m\cdot\ln G(g(\hat{c}),g(\hat{d}))}{\hat{d}^m-\hat{c}^m}}\leq 2L[g(\hat{c}),h(\hat{d}),g(\hat{d}),h(\hat{c})].$$

Proof: Assume that g and h are m-convex functions that increase multiplicatively. Then

$$g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\leq(g(\hat{c}))^w(g(\hat{d}))^{(1-w)}$$
$$h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\leq(h(\hat{c}))^w(h(\hat{d}))^{(1-w)}$$

Using $(\theta_1-\theta_2,\theta_3-\theta_4)\geq 0,\theta_1,\theta_2,\theta_3,\theta_4\in R$ and $\theta_1<\theta_2<\theta_3<\theta_4$, we have

$$g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)(h(\hat{c}))^w(h(\hat{d}))^{(1-w)}+h\left((w\hat{c}^m+(1-\right.$$
$$\left.w)\hat{d}^m\right)^{\frac{1}{m}}\Big)(g(\hat{c}))^w(g(\hat{d}))^{(1-w)}$$
$$\leq g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)$$
$$+(g(\hat{c}))^w(g(\hat{d}))^{(1-w)}(h(\hat{c}))^w(h(\hat{d}))^{(1-w)}$$

Using the logarithm, integrate the aforementioned inequalities on [0,1] with respect to w , we have

$$\begin{aligned}
&\int_0^1 \ln\left[g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)(h(\hat{c}))^w(h(\hat{d}))^{(1-w)}\right]dw\\
&+\int_0^1 \ln\left[h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)(g(\hat{c}))^w(g(\hat{d}))^{(1-w)}\right]dw\\
&\leq \int_0^1 \ln\left[g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)+\right.\\
&\left.(g(\hat{c}))^w(g(\hat{d}))^{(1-w)}(h(\hat{c}))^w(h(\hat{d}))^{(1-w)}\right]dw
\end{aligned}$$

Since g and h are increasing function, we have

$$\begin{aligned}
&\int_0^1 \ln\left[g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\right]dw\int_0^1 \ln\left[(h(\hat{c}))^w(h(\hat{d}))^{(1-w)}\right]dw\\
&+\int_0^1 \ln\left[h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\right]dw\int_0^1 \ln\left[(g(\hat{c}))^w(g(\hat{d}))^{(1-w)}\right]dw\\
&\leq \int_0^1 \ln\left[g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)+\right.\\
&\left.(g(\hat{c}))^w(g(\hat{d}))^{(1-w)}(h(\hat{c}))^w(h(\hat{d}))^{(1-w)}\right]dw
\end{aligned}$$

Which means,

$$\begin{aligned}
&\ln G(h(\hat{c}),h(\hat{d}))\int_0^1 \ln\left[g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\right]dw\\
&+\ln G(g(\hat{c}),g(\hat{d}))\int_0^1 \ln\left[h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)\right]dw\\
&\leq \int_0^1 \ln\left[g\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)h\left(\left(w\hat{c}^m+(1-w)\hat{d}^m\right)^{\frac{1}{m}}\right)+\right.\\
&\left.(g(\hat{c}))^w(g(\hat{d}))^{(1-w)}(h(\hat{c}))^w(h(\hat{d}))^{(1-w)}\right]dw
\end{aligned}$$

Taking the exponential on both sides, we get the result as follows.

$$\left(\int_{\hat{c}}^{\hat{d}}\left(\frac{g(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m\cdot\ln G(h(\hat{c}),h(\hat{d}))}{\hat{d}^m-\hat{c}^m}}\cdot\left(\int_{\hat{c}}^{\hat{d}}\left(\frac{h(\hat{x})}{\hat{x}^{1-m}}\right)^{d\hat{x}}\right)^{\frac{m\cdot\ln G(g(\hat{c}),g(\hat{d}))}{\hat{d}^m-\hat{c}^m}}\leq 2L[g(\hat{c}),h(\hat{d}),g(\hat{d}),h(\hat{c})].$$

4. Conclusion

In this study, we introduced the class of multiplicatively m-convex functions. We developed a novel variant of the Hermite-Hadamard type inequality using multiplicative calculus for convex functions and multiplicatively m-convex functions. Furthermore, we discovered several Hermite-Hadamard-type integral inequalities for the product and quotient of multiplicatively m-convex and convex functions. In recent years, Hadamard-Hermite inequality played a significant part in probability theory, optimization, mathematical analysis, and other mathematical fields. We assume that much of the study in this area of inequality and analysis will Centre on our recently established class of functions.

Conflict of Interest: The authors declare that they have no conflict of interest.

Authors Contribution: All the authors have equal contribution for the preparation of this article.

References

1. S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications* RGMIA Monographs.Victoria University (2000).

2. J. Hadamard, *L tude sur ies proprietesdes functions entries on particular d unc function considered par Riemann*, Journal of Mathematics Pure and Applied, 58, 171-215, (1893).

3. J. L. Precaric, L. Proschan and Y. L. Tong, *Convex functions*, Partial Ordering and Statistical Applications. Academic Press Boston, (1992).

4. L. Iscan, *Hermite-Hadamard type inequalities for harmonically convex functions.* Hacettepe Journal of Mathematics and Statistics, 43(6), 935-942, (2014).

5. S. l. Butt, S. Yousaf, K. A. Khan, R. M. Mabela, and A. M. Alshrif, *Fejer-pachpatte-Mercer type inequalities For harmonically convex functions involving exponential function in kernel.* Mathematical Problems in Engineering, 2022, 1-19, (2022).

6. S, I. Butt, A. O. AkdemIr, M. Nadeem, N. Mlaiki, I. Iscan and I. Abdeljawad, *(m,n)-Harmonically polynomial convex functions and some Hadamard type inequalities on the co-ordinates.* AIMS Mathematics, 6(5), 46-91. (2021).

7. M. A. Latif and T. Du, *Hermite-Hadamard type inequalities for harmonically convex functions Using fuzzy integrals.* Filomat, 36(12), 4099-4110, (2022).

8. S. Ozcan, *Some integral inequalities for harmonically* ($\propto, s$)*-convex functions.* Journal of Function Spaces, 2019, 1-8, (2019).

9. I. Iscan, *Ostrowski type inequalities for p-convex functions.* New Trends in Mathematical Sciences, 4(3), 140-150, (2016).

10. Z. B. Fang and R. Shi, *On the* (p, h)*-convex function and some integral inequalities.* Journal of Inequalities and Applications, 45, 1-16, (2014).

11. A. F. Bashirov, F. Kurpinar and A. Ozyapici, *Multiplicative calculus and applications.* Journal of Mathematical Analysis and Applications, 337(1), 36-48, (2008).

12. M. A. Ali, M. Abbas, Z. Zhang, I. B. Siaf and R. Arif, *On integral inequalities for product and Quotient of two multiplicatively convex functions.* Asian Research Journal in Mathematics, 12(3), 1-11, (2019).

13. M. Grossman and R. Kartz, *Non-Newtonian Calculus.* Lee. Press, Pigeon Cove. (1972).

14. A. E. Bashirov and M. Riza, *On Complex Multiplicative differentiation.* TWMS Journal of Applied and Engineering Mathematics, 1(1), 75-85, (2011).

15. Y. L. Daletskil and N. I. Teterina, *Multiplicative stochastic integrals*, Uspekhi Matematicheskikh Nauk, 27(2:164), 167-168, (1972).

16. R. I. Karandikar, *Multiplicative decomposition of non-singular matrix valued continuous Semimartingales.* Annals of Probability, 10(1), 1088-1091, (1982).

17. M. A. Ali, M. Abbas and A. A. Zafer, *On some Hermite-Hadamard Integral Inequalities in Multiplicative Calculus.* Journal of Inequalities and Special Functions, 10(1), 111-122, (2019).

18. M. A. Ali, H. Budak, M. Z. Sarikaya, Z. Zhang, *Ostrowski and Simpson type inequalities for multiplicative Integrals.* Proyeclones, 40(3), 743-763, (2021).

19. M. Kadakal, *Hermite-Hadamard and simpson type inequalities for multiplicatively harmonically p-functions.* Sigma Journal of Engineering and Natural Sciences, 37(4), 1315-1324, (2019).

20. S. Ozcan, *Hermite-Hadamard inequalities for multictively h-convex functions.* Konuralp Journal of Mathematics, 8(1), 158-164, (2020).

21. S. Ozcan, *Some integral inequalities of Hermite-Hadamard type for multiplicatively preinvex Functions.* AIMS Mathematics, 5(2), 1505-1518, (2020).

22. M. Riza, A. Ozyapici and E. Kurpinal, *Multiplicative finite difference methods.* Quarterly f Applied Mathematics, 67(4), 745-754, (2009).

23. A. K. Sahoo, B. Kodomsingh, P. K. Raut, and B. C. Tripathy, *Hermite-Hadarmad's inequalities for normed linear Convex function via fractional type inequalities.* Bull. Comput. Appl. Math., 13(1), 2025.

24. A. K. Sahoo, B. Kodomsingh, and B. C. Tripathy, *Some normed linear space and integral inequalities of composite convex functions.* South East Asian J. of Mathematics and Mathematical Sciences, 20(2), 133-142, (2024).

Ashok Kumar Sahoo,
Department of Mathematics,
Trident Academy of Technology, Bhubaneswar, 751024, Odisha, India.
and
Department of Mathematics,
Institute of Technical Education and Research, Sikhsha O Anusandhan University, Bhubaneswar 751030, Odisha, India.
E-mail address: aksmath2012@gmail.com, ashokbbs@yahoo.com

and

Bibhakar Kodamasingh ,
Department of Mathematics,
Institute of Technical Education and Research, Sikhsha O Anusandhan University, Bhubaneswar 751030, Odisha, India.
E-mail address: bibhakarkodamasingh@soa.ac.in

and

Binod Chandra Tripathy,
Department of Mathematics,
Tripura University, Agartala-799022, Tripura
India.
E-mail address: tripathybc@yahoo.com, tripathybc@gmail.com