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Algorithmic solution for (3+1) Dimensional Klein-Gordon Equation using Non-Polynomial Cubic Spline Technique

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ABSTRACT: The algorithmic solution of problems based on the Klein-Gordon Equation in (3+1) dimensions has been discussed in this study. Time variable t is discretized by using the central difference formula. Cubic Spline function involving trigonometric functions is used for all three spacial variables and Suitable parametric values provide the accuracy of order $O\left(h^2+k^2+\sigma^2+\tau^2h^2+\tau^2k^2+\tau^2\sigma^2\right)$ in our proposed scheme. The stability of this scheme is also discussed in this paper and the truncation error too. This method elucidates numerical problems and compares them to the results obtained from the literature.

Key Words: Klein Gordon equation, three dimensional (3D) Wave equations, non-polynomial cubic spline function, stability analysis, truncation error.

Contents

1. Introduction

Numerous problems in the science research realm are articulated by Partial Differential Equations (PDE). Nonlinear phenomena are also significant in a variety of scientific disciplines, including engineering, applied mathematics, and physics. Nonlinear partial differential equations are employed to model numerous experiments [1]. The literature is replete with discussions of one-, two-, and three-dimensional space-dimensional models that are founded on PDEs. Klein Gordon Equation is a form of nonlinear PDEs that is extensively used to represent real-world scenarios like the wave behavior of particles while moving in plasma physics [2,3]. Abundant literature is available for 1-D and 2-D Linear and nonlinear Klein Gordon Equation [4,5]. However, fewer discussions are available for the 3-D version [6,8].

Particles such as mesons, which possess no spin, are characterized by the relativistic wave equation referred to as the three-dimensional Klein-Gordon equation. This equation can be used to study the propagation of particles in a field or a vacuum and to calculate their energy and momentum [9]. It is an important tool in particle physics and quantum field theory.

We consider a generalized three-space-dimensional Klein-Gordon equation.

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$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u - F(u), \quad t > 0 \tag{1.1}$$

where the solution domain is defined as

$$R = \{(x, y, z) \; ; \; (A_0, B_0, C_0) < (x, y, z) < (A_L, B_L, C_L)\}$$

along with the initial conditions:

$$\begin{cases} u(x, y, z, 0) = \varphi(x, y, z) \\ u_t(x, y, z, 0) = \psi(x, y, z) \end{cases} \tag{1.2}$$

subject to the boundary conditions:

$$\begin{cases} u(A_0, y, z, t) = h_{A_0}(y, z, t), \quad u(A_L, y, z, t) = h_{A_L}(y, z, t) \\ u(x, B_0, z, t) = h_{B_0}(x, z, t), \quad u(x, B_L, z, t) = h_{B_L}(x, z, t) \\ u(x, y, C_0, t) = h_{C_0}(x, y, t), \quad u(x, y, C_L, t) = h_{C_L}(x, y, t) \end{cases} \tag{1.3}$$

where $u(x, y, z, t)$ is the wave function and c^2 represents the wave speed. The functions φ and ψ, which appear in the initial conditions, are differentiable
Where F is a nonlinear smooth function on the real numbers with the following properties:

(i) $F(0) = 0$

(ii) $uF(u) \leq 0$

(iii) $|F'(u)| \leq \vartheta \left(1 + |u|^{p-1}\right)$, for some $\vartheta > 0$ and $p > 1$.

Hyperbolic PDEs have been widely discussed in the literature and developed numerous numerical algorithms that provide their solutions [10,11,12,13,14,15,16]. Numerous finite difference techniques for linear hyperbolic equations have been introduced in [11,12,13,14]. Ragget and Wilson [15] employed cubic spline and finite difference approximations to tackle these issues. Rashidinia *et al.* [16] and Ding *et al.* [17] derived numerical solutions for the Klein-Gordon equation utilizing a non-polynomial cubic spline approximation in the spatial domain and a finite difference method in the temporal domain. Zadvan H. and Rashidinia J. [14] devised two implicit three-time-level approaches for solving two-dimensional wave equations. Korkmaz *et al.* [18] also present a collocation method based on Trigonometric Cubic B-spline to bring the solution of the Klein-Gordon equation. In the literature, models with one, two, and three spatial dimensions are covered in great detail. Accurate numerical solutions to these issues are essential for their practical applications, but they present a number of difficulties, especially in three-dimensional scenarios. Furthermore, it is frequently challenging to discover accurate solutions for nonlinear partial differential equations (PDEs). Numerical techniques are therefore essential for resolving these kinds of problems. To guarantee that wave simulations are both accurate and useful, the right numerical approaches must be used.

Although accurate solutions offer a clear understanding of the physical events connected to the problem, numerical approaches are still indispensable. Many numerical methods have been established during the last three decades to solve the Klein-Gordon equation.

This study is arranged in the following manner. Section 2 introduces non-polynomial cubic spline functions for each space variable. Consistency relations are also provided in this section. The scheme for the sine Gordon equation given in eq (1) is developed here in Section 3. Sections 4 and 5 present the truncation error and stability analysis. Some numerical examples are discussed in Section 6. The final section contains conclusions.

2. Discretization of Variables

The domain set R consists of N subintervals in every spatial direction and J time steps. The spatial step sizes are defined as:

$$h = \frac{L_{1x} - L_{0x}}{N + 1}, \quad k = \frac{L_{1y} - L_{0y}}{N + 1}, \quad \sigma = \frac{L_{1z} - L_{0z}}{N + 1}$$

and τ is the time step size.
So that we got $(N+1)*(N+1)*(N+1)*J$ meshes over the whole domain and $N, J \in \mathbb{Z}^+$. Each grid point is represented as $(x_\wp, y_ß, z_\Upsilon, t_j)$

$$\begin{cases} x_\wp = A_0 + \wp h; & \wp = 0,1,\dots,N+1 \\ y_ß = B_0 + ßk; & ß = 0,1,\dots,N+1 \\ z_\Upsilon = C_0 + \Upsilon\sigma; & \Upsilon = 0,1,\dots,N+1 \end{cases} \quad \text{and} \quad t_j = j\tau; \quad 0 \le j \le J$$

For space variable:

For each subinterval in the x, y, and z-directions, let $s_{1ß,\Upsilon}(x)$, $s_{2\wp,\Upsilon}(y)$, and $s_{3\wp,ß}(z)$ denote the non-polynomial spline functions and be defined as follows:

$$s_{1ß,\Upsilon}(x) = a_{1\wp} + b_{1\wp}(x - x_\wp) + c_{1\wp}\sin(\lambda_1(x - x_\wp)) + d_{1\wp}\cos(\lambda_1(x - x_\wp)) \tag{2.1}$$

$$s_{2\wp,\Upsilon}(y) = a_{2ß} + b_{2ß}(y - y_ß) + c_{2ß}\sin(\lambda_2(y - y_ß)) + d_{2ß}\cos(\lambda_2(y - y_ß)) \tag{2.2}$$

$$s_{3\wp,ß}(z) = a_{3\Upsilon} + b_{3\Upsilon}(z - z_\Upsilon) + c_{3\Upsilon}\sin(\lambda_3(z - z_\Upsilon)) + d_{3\Upsilon}\cos(\lambda_3(z - z_\Upsilon)) \tag{2.3}$$

Where $a_{1\wp}$, $b_{1\wp}$, $c_{1\wp}$, $d_{1\wp}$, $a_{2ß}$, $b_{2ß}$, $c_{2ß}$, $d_{2ß}$, $a_{3\Upsilon}$, $b_{3\Upsilon}$, $c_{3\Upsilon}$, $d_{3\Upsilon}$ are undetermined coefficients, and λ_1, λ_2, λ_3 are arbitrary controlling parameters.

Using the following notations:

$$\begin{aligned} s_{1ß,\Upsilon}(x_\wp) &= u_{\wp,ß,\Upsilon}, & s''_{1ß,\Upsilon}(x_\wp) &= N_{1\wp,ß,\Upsilon}, \\ s_{2\wp,\Upsilon}(y_ß) &= u_{\wp,ß,\Upsilon}, & s''_{2\wp,\Upsilon}(y_ß) &= N_{2\wp,ß,\Upsilon}, \\ s_{3\wp,ß}(z_\Upsilon) &= u_{\wp,ß,\Upsilon}, & s''_{3\wp,ß}(z_\Upsilon) &= N_{3\wp,ß,\Upsilon} \end{aligned}$$

We find the following consistency relations by looking at the first and second derivatives of the cubic spline functions $s_{1ß,\Upsilon}(x)$, $s_{2\wp,\Upsilon}(y)$, and $s_{3\wp,ß}(z)$ at $(x_\wp, y_ß, z_\Upsilon, t_j)$:

$$u_{\wp-1,ß,\Upsilon} - 2u_{\wp,ß,\Upsilon} + u_{\wp+1,ß,\Upsilon} = h^2\left(\alpha_1 N_{1\wp-1,ß,\Upsilon} + 2\beta_1 N_{1\wp,ß,\Upsilon} + \alpha_1 N_{1\wp+1,ß,\Upsilon}\right) \tag{2.4}$$

$$u_{\wp,ß-1,\Upsilon} - 2u_{\wp,ß,\Upsilon} + u_{\wp,ß+1,\Upsilon} = k^2\left(\alpha_2 N_{2\wp,ß-1,\Upsilon} + 2\beta_2 N_{2\wp,ß,\Upsilon} + \alpha_2 N_{2\wp,ß+1,\Upsilon}\right) \tag{2.5}$$

$$u_{\wp,ß,\Upsilon-1} - 2u_{\wp,ß,\Upsilon} + u_{\wp,ß,\Upsilon+1} = \sigma^2\left(\alpha_3 N_{3\wp,ß,\Upsilon-1} + 2\beta_3 N_{3\wp,ß,\Upsilon} + \alpha_3 N_{3\wp,ß,\Upsilon+1}\right) \tag{2.6}$$

Where

$$\begin{aligned} \alpha_1 &= \frac{1}{h^2\lambda_1^2}\left(h\lambda_1\csc(h\lambda_1) - 1\right), \quad \alpha_2 = \frac{1}{k^2\lambda_2^2}\left(k\lambda_2\csc(k\lambda_2) - 1\right), \\ \alpha_3 &= \frac{1}{\sigma^2\lambda_3^2}\left(\sigma\lambda_3\csc(\sigma\lambda_3) - 1\right), \\ \beta_1 &= \frac{1}{h^2\lambda_1^2}\left(1 - h\lambda_1\cot(h\lambda_1)\right), \quad \beta_2 = \frac{1}{k^2\lambda_2^2}\left(1 - k\lambda_2\cot(k\lambda_2)\right), \\ \beta_3 &= \frac{1}{\sigma^2\lambda_3^2}\left(1 - \sigma\lambda_3\cot(\sigma\lambda_3)\right) \end{aligned}$$

For the time variable:

To discretize the terms in which time and its derivatives are involved, a finite difference approximation is used as

$$u_{tt}|^{j}_{(\wp,\text{ß},\Upsilon)} = \frac{u^{j-1}_{(\wp,\text{ß},\Upsilon)} - 2u^{j}_{(\wp,\text{ß},\Upsilon)} + u^{j+1}_{(\wp,\text{ß},\Upsilon)}}{\tau^2} + \mathcal{O}(\tau^2) \tag{2.7}$$

3. Spline Numerical Method

This section introduces an approximation approach for equation (1) that can be discretized at the grid point $(x_\wp, y_\text{ß}, z_\gamma, t_j)$ as

$$u_{tt}|^{j}_{(\wp,\text{ß},\Upsilon)} = c^2\left(u_{xx}|^{j}_{(\wp,\text{ß},\Upsilon)} + u_{yy}|^{j}_{(\wp,\text{ß},\Upsilon)} + u_{zz}|^{j}_{(\wp,\text{ß},\Upsilon)}\right) - F\left(u^{j}_{(\wp,\text{ß},\Upsilon)}\right) \tag{3.1}$$

Take

$$\begin{cases} u_{xx}|^{j}_{(\wp,\text{ß},\Upsilon)} = N^{j}_{(1\wp,\text{ß},\Upsilon)} + \mathcal{O}(h^2) \\ u_{yy}|^{j}_{(\wp,\text{ß},\Upsilon)} = N^{j}_{(2\wp,\text{ß},\Upsilon)} + \mathcal{O}(k^2) \\ u_{zz}|^{j}_{(\wp,\text{ß},\Upsilon)} = N^{j}_{(3\wp,\text{ß},\Upsilon)} + \mathcal{O}(\sigma^2) \end{cases} \tag{3.2}$$

By using equations (2.7) and (3.2), and ignoring truncation errors, equation (3.1) may be expressed as

$$\frac{u^{(j-1)}_{(\wp,\text{ß},\Upsilon)} - 2u^{j}_{(\wp,\text{ß},\Upsilon)} + u^{(j+1)}_{(\wp,\text{ß},\Upsilon)}}{\tau^2} = c^2\left(N^{j}_{(1\wp,\text{ß},\Upsilon)} + N^{j}_{(2\wp,\text{ß},\Upsilon)} + N^{j}_{(3\wp,\text{ß},\Upsilon)}\right) - F\left(u^{j}_{(\wp,\text{ß},\Upsilon)}\right) \tag{3.3}$$

or equivalently,

$$N^{j}_{(1\wp,\text{ß},\Upsilon)} + N^{j}_{(2\wp,\text{ß},\Upsilon)} + N^{j}_{(3\wp,\text{ß},\Upsilon)} = \frac{u^{(j-1)}_{(\wp,\text{ß},\Upsilon)} - 2u^{j}_{(\wp,\text{ß},\Upsilon)} + u^{(j+1)}_{(\wp,\text{ß},\Upsilon)}}{c^2\tau^2} + \frac{1}{c^2}F\left(u^{j}_{(\wp,\text{ß},\Upsilon)}\right) \tag{3.4}$$

Equation (3.4) describes the consistency relation for a grid point $(x_\wp, y_\text{ß}, z_\Upsilon, t_j)$. The complete domain in each direction may be represented as follows:

$$N^{j}_{1\wp\pm1,\text{ß},\Upsilon} + N^{j}_{2\wp\pm1,\text{ß},\Upsilon} + N^{j}_{3\wp\pm1,\text{ß},\Upsilon} = \frac{u^{j-1}_{\wp\pm1,\text{ß},\Upsilon} - 2u^{j}_{\wp\pm1,\text{ß},\Upsilon} + u^{j+1}_{\wp\pm1,\text{ß},\Upsilon}}{c^2\tau^2} + \frac{1}{c^2}F\left(u^{j}_{\wp\pm1,\text{ß},\Upsilon}\right) \tag{3.5}$$

$$N^{j}_{1\wp,\text{ß}\pm1,\Upsilon} + N^{j}_{2\wp,\text{ß}\pm1,\Upsilon} + N^{j}_{3\wp,\text{ß}\pm1,\Upsilon} = \frac{u^{j-1}_{\wp,\text{ß}\pm1,\Upsilon} - 2u^{j}_{\wp,\text{ß}\pm1,\Upsilon} + u^{j+1}_{\wp,\text{ß}\pm1,\Upsilon}}{c^2\tau^2} + \frac{1}{c^2}F\left(u^{j}_{\wp,\text{ß}\pm1,\Upsilon}\right) \tag{3.6}$$

$$N^{j}_{1\wp,\text{ß},\Upsilon\pm1} + N^{j}_{2\wp,\text{ß},\Upsilon\pm1} + N^{j}_{3\wp,\text{ß},\Upsilon\pm1} = \frac{u^{j-1}_{\wp,\text{ß},\Upsilon\pm1} - 2u^{j}_{\wp,\text{ß},\Upsilon\pm1} + u^{j+1}_{\wp,\text{ß},\Upsilon\pm1}}{c^2\tau^2} + \frac{1}{c^2}F\left(u^{j}_{\wp,\text{ß},\Upsilon\pm1}\right) \tag{3.7}$$

Nine equations are derived at each grid point within the entire domain from the consistency relations (2.4) and (3.5). In the same way, at each grid point, an additional set of nine equations is generated by relations (2.5) and (3.6). A third set of nine equations is derived from relations (2.6) and (3.7).

The following condensed equation is obtained by multiplying these equations by the appropriate coefficients and combining them:

$$
\begin{aligned}
&p_1u^{j+1}_{\wp-1,ß-1,\Upsilon-1}+p_2u^{j+1}_{\wp-1,ß-1,\Upsilon}+p_1u^{j+1}_{\wp-1,ß-1,\Upsilon+1}+p_3u^{j+1}_{\wp-1,ß,\Upsilon-1}+p_5u^{j+1}_{\wp-1,ß,\Upsilon}+p_3u^{j+1}_{\wp-1,ß,\Upsilon+1}\\
&+p_1u^{j+1}_{\wp-1,ß+1,\Upsilon-1}+p_2u^{j+1}_{\wp-1,ß+1,\Upsilon}+p_1u^{j+1}_{\wp-1,ß+1,\Upsilon+1}+p_4u^{j+1}_{\wp,ß-1,\Upsilon-1}+p_6u^{j+1}_{\wp,ß-1,\Upsilon}+p_4u^{j+1}_{\wp,ß-1,\Upsilon+1}\\
&+p_7u^{j+1}_{\wp,ß,\Upsilon-1}+p_8u^{j+1}_{\wp,ß,\Upsilon}+p_7u^{j+1}_{\wp,ß,\Upsilon+1}+p_4u^{j+1}_{\wp,ß+1,\Upsilon-1}+p_6u^{j+1}_{\wp,ß+1,\Upsilon}+p_4u^{j+1}_{\wp,ß+1,\Upsilon+1}\\
&+p_1u^{j+1}_{\wp+1,ß-1,\Upsilon-1}+p_2u^{j+1}_{\wp+1,ß-1,\Upsilon}+p_1u^{j+1}_{\wp+1,ß-1,\Upsilon+1}+p_3u^{j+1}_{\wp+1,ß,\Upsilon-1}+p_5u^{j+1}_{\wp+1,ß,\Upsilon}+p_3u^{j+1}_{\wp+1,ß,\Upsilon+1}\\
&+p_1u^{j+1}_{\wp+1,ß+1,\Upsilon-1}+p_2u^{j+1}_{\wp+1,ß+1,\Upsilon}+p_1u^{j+1}_{\wp+1,ß+1,\Upsilon+1}\\
&=-p_1u^{j-1}_{\wp-1,ß-1,\Upsilon-1}-p_2u^{j-1}_{\wp-1,ß-1,\Upsilon}-p_1u^{j-1}_{\wp-1,ß-1,\Upsilon+1}-p_3u^{j-1}_{\wp-1,ß,\Upsilon-1}-p_5u^{j-1}_{\wp-1,ß,\Upsilon}-p_3u^{j-1}_{\wp-1,ß,\Upsilon+1}\\
&-p_1u^{j-1}_{\wp-1,ß+1,\Upsilon-1}-p_2u^{j-1}_{\wp-1,ß+1,\Upsilon}-p_1u^{j-1}_{\wp-1,ß+1,\Upsilon+1}-p_4u^{j-1}_{\wp,ß-1,\Upsilon-1}-p_6u^{j-1}_{\wp,ß-1,\Upsilon}-p_4u^{j-1}_{\wp,ß-1,\Upsilon+1}\\
&-p_7u^{j-1}_{\wp,ß,\Upsilon-1}-p_8u^{j-1}_{\wp,ß,\Upsilon}-p_7u^{j-1}_{\wp,ß,\Upsilon+1}-p_4u^{j-1}_{\wp,ß+1,\Upsilon-1}-p_6u^{j-1}_{\wp,ß+1,\Upsilon}-p_4u^{j-1}_{\wp,ß+1,\Upsilon+1}\\
&-p_1u^{j-1}_{\wp+1,ß-1,\Upsilon-1}-p_2u^{j-1}_{\wp+1,ß-1,\Upsilon}-p_1u^{j-1}_{\wp+1,ß-1,\Upsilon+1}-p_3u^{j-1}_{\wp+1,ß,\Upsilon-1}-p_5u^{j-1}_{\wp+1,ß,\Upsilon}-p_3u^{j-1}_{\wp+1,ß,\Upsilon+1}\\
&-p_1u^{j-1}_{\wp+1,ß+1,\Upsilon-1}-p_2u^{j-1}_{\wp+1,ß+1,\Upsilon}-p_1u^{j-1}_{\wp+1,ß+1,\Upsilon+1}\\
&+p_{16}u^{j}_{\wp,ß,\Upsilon}+p_{13}u^{j}_{\wp+1,ß,\Upsilon}+p_{13}u^{j}_{\wp-1,ß,\Upsilon}+p_{14}u^{j}_{\wp,ß+1,\Upsilon}+p_{14}u^{j}_{\wp,ß-1,\Upsilon}\\
&+p_{15}u^{j}_{\wp,ß,\Upsilon+1}+p_{15}u^{j}_{\wp,ß,\Upsilon-1}+p_{10}u^{j}_{\wp+1,ß+1,\Upsilon}+p_{10}u^{j}_{\wp+1,ß-1,\Upsilon}+p_{10}u^{j}_{\wp-1,ß+1,\Upsilon}+p_{10}u^{j}_{\wp-1,ß-1,\Upsilon}\\
&+p_{12}u^{j}_{\wp,ß+1,\Upsilon+1}+p_{12}u^{j}_{\wp,ß+1,\Upsilon-1}+p_{12}u^{j}_{\wp,ß-1,\Upsilon+1}+p_{12}u^{j}_{\wp,ß-1,\Upsilon-1}\\
&+p_9u^{j}_{\wp+1,ß+1,\Upsilon+1}+p_9u^{j}_{\wp+1,ß+1,\Upsilon-1}+p_9u^{j}_{\wp+1,ß-1,\Upsilon+1}+p_9u^{j}_{\wp+1,ß-1,\Upsilon-1}\\
&+p_9u^{j}_{\wp-1,ß+1,\Upsilon+1}+p_9u^{j}_{\wp-1,ß+1,\Upsilon-1}+p_9u^{j}_{\wp-1,ß-1,\Upsilon+1}+p_9u^{j}_{\wp-1,ß-1,\Upsilon-1}\\
&+p_{11}u^{j}_{\wp+1,ß,\Upsilon+1}+p_{11}u^{j}_{\wp+1,ß,\Upsilon-1}+p_{11}u^{j}_{\wp-1,ß,\Upsilon+1}+p_{11}u^{j}_{\wp-1,ß,\Upsilon-1}\\
&+\tau^2\Big(p_1F^{j}_{\wp-1,ß-1,\Upsilon-1}+p_1F^{j}_{\wp-1,ß-1,\Upsilon+1}+p_1F^{j}_{\wp-1,ß+1,\Upsilon-1}+p_1F^{j}_{\wp-1,ß+1,\Upsilon+1}\\
&\quad+p_1F^{j}_{\wp+1,ß-1,\Upsilon-1}+p_1F^{j}_{\wp+1,ß-1,\Upsilon+1}+p_1F^{j}_{\wp+1,ß+1,\Upsilon-1}+p_1F^{j}_{\wp+1,ß+1,\Upsilon+1}\\
&\quad+p_2F^{j}_{\wp-1,ß-1,\Upsilon}+p_2F^{j}_{\wp-1,ß+1,\Upsilon}+p_2F^{j}_{\wp+1,ß-1,\Upsilon}+p_2F^{j}_{\wp+1,ß+1,\Upsilon}\\
&\quad+p_3F^{j}_{\wp-1,ß,\Upsilon-1}+p_3F^{j}_{\wp-1,ß,\Upsilon+1}+p_3F^{j}_{\wp+1,ß,\Upsilon-1}+p_3F^{j}_{\wp+1,ß,\Upsilon+1}\\
&\quad+p_4F^{j}_{\wp,ß-1,\Upsilon-1}+p_4F^{j}_{\wp,ß-1,\Upsilon+1}+p_4F^{j}_{\wp,ß+1,\Upsilon-1}+p_4F^{j}_{\wp,ß+1,\Upsilon+1}\\
&\quad+p_5F^{j}_{\wp-1,ß,\Upsilon}+p_5F^{j}_{\wp+1,ß,\Upsilon}+p_6F^{j}_{\wp,ß-1,\Upsilon}+p_6F^{j}_{\wp,ß+1,\Upsilon}\\
&\quad+p_7F^{j}_{\wp,ß,\Upsilon-1}+p_7F^{j}_{\wp,ß,\Upsilon+1}+p_8F^{j}_{\wp,ß,\Upsilon}\Big)
\end{aligned}
\tag{3.8}
$$

Where

$$
\begin{aligned}
&p_1=\alpha_1\alpha_2\alpha_3, && p_2=2\alpha_1\alpha_2\beta_3, && p_3=2\alpha_1\alpha_3\beta_2, && p_4=2\alpha_2\alpha_3\beta_1,\\
&p_5=4\alpha_1\beta_2\beta_3, && p_6=4\alpha_2\beta_1\beta_3, && p_7=4\alpha_3\beta_1\beta_2, && p_8=8\beta_1\beta_2\beta_3.
\end{aligned}
$$

$$
\begin{aligned}
p_9 &= \frac{c^2\tau^2\alpha_1\alpha_2}{\sigma^2} + \frac{c^2\tau^2\alpha_1\alpha_3}{k^2} + \frac{c^2\tau^2\alpha_2\alpha_3}{h^2} + 2\alpha_1\alpha_2\alpha_3,\\
p_{10} &= -\frac{2c^2\tau^2\alpha_1\alpha_2}{\sigma^2} + \frac{2c^2\tau^2\alpha_1\beta_3}{k^2} + \frac{2c^2\tau^2\alpha_2\beta_3}{h^2} + 4\alpha_1\alpha_2\beta_3,\\
p_{11} &= -\frac{2c^2\tau^2\alpha_1\alpha_3}{k^2} + \frac{2c^2\tau^2\alpha_1\beta_2}{\sigma^2} + \frac{2c^2\tau^2\alpha_3\beta_2}{h^2} + 4\alpha_1\alpha_3\beta_2,\\
p_{12} &= -\frac{2c^2\tau^2\alpha_2\alpha_3}{h^2} + \frac{2c^2\tau^2\alpha_2\beta_1}{\sigma^2} + \frac{2c^2\tau^2\alpha_3\beta_1}{k^2} + 4\alpha_2\alpha_3\beta_1,\\
p_{13} &= -\frac{4c^2\tau^2\alpha_1\beta_2}{\sigma^2} - \frac{4c^2\tau^2\alpha_1\beta_3}{k^2} + \frac{4c^2\tau^2\beta_2\beta_3}{h^2} + 8\alpha_1\beta_2\beta_3,\\
p_{14} &= -\frac{4c^2\tau^2\alpha_2\beta_1}{\sigma^2} - \frac{4c^2\tau^2\alpha_2\beta_3}{h^2} + \frac{4c^2\tau^2\beta_1\beta_3}{k^2} + 8\alpha_2\beta_1\beta_3,\\
p_{15} &= -\frac{4c^2\tau^2\alpha_3\beta_1}{k^2} - \frac{4c^2\tau^2\alpha_3\beta_2}{h^2} + \frac{4c^2\tau^2\beta_1\beta_2}{\sigma^2} + 8\alpha_3\beta_1\beta_2,\\
p_{16} &= -\frac{8c^2\tau^2\beta_1\beta_2}{\sigma^2} - \frac{8c^2\tau^2\beta_1\beta_3}{k^2} - \frac{8c^2\tau^2\beta_2\beta_3}{h^2} + 16\beta_1\beta_2\beta_3.
\end{aligned}
$$

The implicit three-time-level method is represented by the scheme (3.8). An explicit scheme of order $\mathcal{O}(\tau^3)$ is implemented to calculate the solution $u(x,y,z,t)$ at the initial time level, $t=\tau$. The third-order approximation of $u(x,y,z,t)$ at $t=\tau$ is calculated using the standard Taylor series expansion as follows:

$$u^1_{\wp,ß,\Upsilon} = u^0_{\wp,ß,\Upsilon} + \tau\ u_t|^0_{\wp,ß,\Upsilon} + \frac{\tau^2}{2!}\ u_{tt}|^0_{\wp,ß,\Upsilon} + \frac{\tau^3}{3!}\ u_{ttt}|^0_{\wp,ß,\Upsilon} + \mathcal{O}(\tau^4) \tag{3.9}$$

Since I.C.'s (1.2) and (3.4), we get

$$u|^0_{\wp,ß,\Upsilon} = \phi_{\wp,ß,\Upsilon}, \qquad u_t|^0_{\wp,ß,\Upsilon} = \psi_{\wp,ß,\Upsilon} \tag{3.10}$$

$$u_{tt}|^0_{\wp,ß,\Upsilon} = c^2\left(u_{xx}|^0_{\wp,ß,\Upsilon} + u_{yy}|^0_{\wp,ß,\Upsilon} + u_{zz}|^0_{\wp,ß,\Upsilon}\right) - F(u^0_{\wp,ß,\Upsilon}) \tag{3.11}$$

$$u_{ttt}|^0_{\wp,ß,\Upsilon} = c^2\left(u_{txx}|^0_{\wp,ß,\Upsilon} + u_{tyy}|^0_{\wp,ß,\Upsilon} + u_{tzz}|^0_{\wp,ß,\Upsilon}\right) - F'(u^j_{\wp,ß,\Upsilon})\cdot\psi|^0_{\wp,ß,\Upsilon} \tag{3.12}$$

Using (3.10)-(3.12) in (3.9), we obtain the approximate solution $u(x,y,z,t)$ at $t=\tau$ as follows:

$$
\begin{aligned}
u^1_{\wp,ß,\Upsilon} = \phi_{\wp,ß,\Upsilon} + \tau\psi_{\wp,ß,\Upsilon} &+ \frac{\tau^2}{2!}\left[c^2\left(\phi_{xx}|^0_{\wp,ß,\Upsilon} + \phi_{yy}|^0_{\wp,ß,\Upsilon} + \phi_{zz}|^0_{\wp,ß,\Upsilon}\right) - F(u^0_{\wp,ß,\Upsilon})\right]\\
&+ \frac{\tau^3}{3!}\left[c^2\left(\psi_{xx}|^0_{\wp,ß,\Upsilon} + \psi_{yy}|^0_{\wp,ß,\Upsilon} + \psi_{zz}|^0_{\wp,ß,\Upsilon}\right) - F'(u^j_{\wp,ß,\Upsilon})\cdot\psi|^0_{\wp,ß,\Upsilon}\right]
\end{aligned} \tag{3.13}
$$

The scheme given in equation (3.8) can be expressed in matrix form to get the answer at each time level:

$$AU^{(j+1)} = -AU^{(j-1)} + BU^j - \tau^2 AF(U^j), \quad j = 1,2,\ldots,J \tag{3.14}$$

Where U is the solution vector and A and B are blocked tridiagonal matrices of order N^3. A is a block tridiagonal matrix of order N in the form tri$[\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_1]$, where each of the block tridiagonal matrices $\mathcal{W}_1$ and $\mathcal{W}_2$ is of rank N and has the form:

$$\mathcal{W}_1 = \text{tri}[\varepsilon_1,\varepsilon_2,\varepsilon_1], \quad \mathcal{W}_2 = \text{tri}[\varepsilon_3,\varepsilon_4,\varepsilon_3]$$

with

$$\varepsilon_1 = \text{tri}[p_1, p_2, p_1], \quad \varepsilon_2 = \text{tri}[p_3, p_5, p_3], \quad \varepsilon_3 = \text{tri}[p_4, p_6, p_4], \quad \varepsilon_4 = \text{tri}[p_7, p_8, p_7]$$

Similarly, matrix B is a nested block tridiagonal matrix of order N^3, with each diagonal block being a block tridiagonal matrix of order N^2, structured as:

$$\text{tri}[\text{tri}[p_{12}, p_{14}, p_{12}], \text{tri}[p_{15}, p_{16}, p_{15}], \text{tri}[p_{12}, p_{14}, p_{12}]]$$

and off-diagonal blocks of order N^2 are:

$$\text{tri}[\text{tri}[p_9, p_{10}, p_9], \text{tri}[p_{11}, p_{13}, p_{11}], \text{tri}[p_9, p_{10}, p_9]]$$

4. Truncation error

From equation (3.4), we have

$$\sin\left(u^j_{\wp,ß,\Upsilon}\right) = u_{tt}|^j_{\wp,ß,\Upsilon} - c^2\left(u_{xx}|^j_{\wp,ß,\Upsilon} + u_{yy}|^j_{\wp,ß,\Upsilon} + u_{zz}|^j_{\wp,ß,\Upsilon}\right) \tag{4.1}$$

We have

$$F\left(u^j_{\wp+\rho,ß+\eta,\Upsilon+\gamma}\right) = u_{tt}|^j_{\wp+\rho,ß+\eta,\Upsilon+\gamma} - c^2\left(u_{xx}|^j_{\wp+\rho,ß+\eta,\Upsilon+\gamma} + u_{yy}|^j_{\wp+\rho,ß+\eta,\Upsilon+\gamma} + u_{zz}|^j_{\wp+\rho,ß+\eta,\Upsilon+\gamma}\right) \tag{4.2}$$

where $\rho, \eta, \gamma = 0, \pm 1$. By replacing these values in equation (3.8) and utilizing the Taylor series to expand both sides in terms of $u^j_{\wp,ß,\Upsilon}$, we can get the truncation error in the following manner

$$
\begin{aligned}
T^j_{\wp,\text{ß},\Upsilon} = \Bigg\{ & 4c^2\tau^2(-1+2\alpha_1+2\beta_1)(\alpha_2+\beta_2)(\alpha_3+\beta_3)\frac{\partial^2}{\partial x^2} \\
& + 4c^2\tau^2(\alpha_1+\beta_1)(-1+2\alpha_2+2\beta_2)(\alpha_3+\beta_3)\frac{\partial^2}{\partial y^2} \\
& + 4c^2\tau^2(\alpha_1+\beta_1)(\alpha_2+\beta_2)(-1+2\alpha_3+2\beta_3)\frac{\partial^2}{\partial z^2} \\
& + 2c^2\tau^2\left(k^2\alpha_2(-1+2\beta_1)+\alpha_1\left(2(h^2+k^2)\alpha_2+h^2(-1+2\beta_2)\right)\right)(\alpha_3+\beta_3)\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} \\
& + 2c^2\tau^2(\alpha_2+\beta_2)\left(\sigma^2\alpha_3(-1+2\beta_1)+\alpha_1\left(2(h^2+\sigma^2)\alpha_3+h^2(-1+2\beta_3)\right)\right)\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial z^2} \\
& + \frac{1}{3}c^2h^2\tau^2(-1+12\alpha_1)(\alpha_2+\beta_2)(\alpha_3+\beta_3)\frac{\partial^4}{\partial x^4} \\
& + \frac{1}{3}c^2k^2\tau^2(-1+12\alpha_2)(\alpha_1+\beta_1)(\alpha_3+\beta_3)\frac{\partial^4}{\partial y^4} \\
& + \frac{1}{3}c^2\sigma^2\tau^2(-1+12\alpha_3)(\alpha_1+\beta_1)(\alpha_2+\beta_2)\frac{\partial^4}{\partial z^4} \\
& + \frac{2}{3}\tau^4(\alpha_1+\beta_1)(\alpha_2+\beta_2)(\alpha_3+\beta_3)\frac{\partial^4}{\partial t^4} \\
& + \frac{1}{6}c^2h^2\tau^2(-k^2\alpha_2+\alpha_1(2(h^2+6k^2)\alpha_2+h^2(-1+2\beta_2)))(\alpha_3+\beta_3)\frac{\partial^4}{\partial x^4}\frac{\partial^2}{\partial y^2} \\
& + \frac{1}{6}c^2h^2\tau^2(\alpha_2+\beta_2)(-\sigma^2\alpha_3+\alpha_1(2(h^2+6\sigma^2)\alpha_3+h^2(-1+2\beta_3)))\frac{\partial^4}{\partial x^4}\frac{\partial^2}{\partial z^2} \\
& + \frac{1}{6}c^2k^2\tau^2\left(\alpha_1(-h^2+2(6h^2+k^2)\alpha_2)+k^2\alpha_2(-1+2\beta_1)\right)(\alpha_3+\beta_3)\frac{\partial^2}{\partial x^2}\frac{\partial^4}{\partial y^4} \\
& + \frac{1}{6}c^2\sigma^2\tau^2\left(\alpha_1(-h^2+2(6h^2+\sigma^2)\alpha_3)+\sigma^2\alpha_3(-1+2\beta_1)\right)(\alpha_2+\beta_2)\frac{\partial^2}{\partial x^2}\frac{\partial^4}{\partial z^4} \\
& + \frac{1}{3}h^2\tau^4\alpha_1(\alpha_2+\beta_2)(\alpha_3+\beta_3)\frac{\partial^2}{\partial x^2}\frac{\partial^4}{\partial t^4} \\
& + \frac{1}{6}c^2\sigma^2\tau^2(\alpha_1+\beta_1)\left(\alpha_2(-k^2+2(6k^2+\sigma^2)\alpha_3)+\sigma^2\alpha_3(-1+2\beta_2)\right)\frac{\partial^2}{\partial y^2}\frac{\partial^4}{\partial z^4} \\
& + \frac{1}{3}k^2\tau^4\alpha_2(\alpha_1+\beta_1)(\alpha_3+\beta_3)\frac{\partial^2}{\partial y^2}\frac{\partial^4}{\partial t^4} \\
& + \frac{1}{3}\sigma^2\tau^4\alpha_3(\alpha_1+\beta_1)(\alpha_2+\beta_2)\frac{\partial^2}{\partial z^2}\frac{\partial^4}{\partial t^4} \\
& + \frac{1}{90}c^2h^4\tau^2(-1+30\alpha_1)(\alpha_2+\beta_2)(\alpha_3+\beta_3)\frac{\partial^6}{\partial x^6} \\
& + \frac{1}{90}c^2k^4\tau^2(-1+30\alpha_2)(\alpha_1+\beta_1)(\alpha_3+\beta_3)\frac{\partial^6}{\partial y^6} \\
& + \frac{1}{6}c^2\sigma^4\tau^2\alpha_3\left(2\beta_1(\alpha_2+\beta_2)+\alpha_1(\alpha_2+2\beta_2)\right)\frac{\partial^6}{\partial z^6} \\
& + \frac{1}{180}c^2h^4k^2\tau^2(-1+30\alpha_1)\alpha_2(\alpha_3+\beta_3)\frac{\partial^6}{\partial x^6}\frac{\partial^2}{\partial y^2} \\
& + \frac{1}{180}c^2h^4\sigma^2\tau^2(-1+30\alpha_1)\alpha_3(\alpha_2+\beta_2)\frac{\partial^6}{\partial x^6}\frac{\partial^2}{\partial z^2} \\
& + \frac{1}{180}c^2k^4\sigma^2\tau^2(-1+30\alpha_2)\alpha_3(\alpha_1+\beta_1)\frac{\partial^6}{\partial y^6}\frac{\partial^2}{\partial z^2} + \cdots \Bigg\} u^j_{\wp,\text{ß},\Upsilon}
\end{aligned}
\tag{4.3}
$$

On selecting appropriate values of parameters $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$ and β_3, methods from various classes can be accessed. If we choose

$$\alpha_1 + \beta_1 = \frac{1}{2}, \quad \alpha_2 + \beta_2 = \frac{1}{2}, \quad \alpha_3 + \beta_3 = \frac{1}{2}. \tag{4.4}$$

(i) We obtain the schemes of order

$$O(h^2 + k^2 + \sigma^2 + \tau^2 h^2 + \tau^2 k^2 + \tau^2 \sigma^2).$$

Specifically, if we choose

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{6}, \quad \beta_1 = \beta_2 = \beta_3 = \frac{1}{3}.$$

(ii) A scheme of order

$$O(h^4 + k^4 + \sigma^4 + \tau^2 h^2 + \tau^2 k^2 + \tau^2 \sigma^2)$$

is obtained by choosing

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{12}, \quad \beta_1 = \beta_2 = \beta_3 = \frac{5}{12}.$$

(iii) A scheme of order

$$O(h^6 + k^6 + \sigma^6 + \tau^2 h^2 + \tau^2 k^2 + \tau^2 \sigma^2)$$

is obtained by choosing

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{30}, \quad \beta_1 = \beta_2 = \beta_3 = \frac{7}{15}.$$

(iv) A scheme is convergent of order

$$O(h^8 + k^8 + \sigma^8 + \tau^2 h^2 + \tau^2 k^2 + \tau^2 \sigma^2)$$

on choosing

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{56}, \quad \beta_1 = \beta_2 = \beta_3 = \frac{27}{56},$$

and so on.

5. Stability Analysis

Here we address the stability analysis of the system designed in (3.8). When we take into account the homogeneous portion of the scheme (3.8), we can have

$$\begin{aligned}
& C_1 Y^{(j+1)}_{\wp-1,ß-1,\Upsilon-1} + C_4 Y^{(j+1)}_{\wp,ß-1,\Upsilon-1} + C_1 Y^{(j+1)}_{\wp+1,ß-1,\Upsilon-1} + C_1 Y^{(j+1)}_{\wp-1,ß-1,\Upsilon+1} + C_4 Y^{(j+1)}_{\wp,ß-1,\Upsilon+1} + C_1 Y^{(j+1)}_{\wp+1,ß-1,\Upsilon+1} \\
& + C_1 Y^{(j+1)}_{\wp-1,ß+1,\Upsilon-1} + C_4 Y^{(j+1)}_{\wp,ß+1,\Upsilon-1} + C_1 Y^{(j+1)}_{\wp+1,ß+1,\Upsilon-1} + C_1 Y^{(j+1)}_{\wp-1,ß+1,\Upsilon+1} + C_4 Y^{(j+1)}_{\wp,ß+1,\Upsilon+1} + C_1 Y^{(j+1)}_{\wp+1,ß+1,\Upsilon+1} \\
& + C_2 Y^{(j+1)}_{\wp-1,ß-1,\Upsilon} + C_6 Y^{(j+1)}_{\wp,ß-1,\Upsilon} + C_2 Y^{(j+1)}_{\wp+1,ß-1,\Upsilon} + C_2 Y^{(j+1)}_{\wp-1,ß+1,\Upsilon} + C_6 Y^{(j+1)}_{\wp,ß+1,\Upsilon} + C_2 Y^{(j+1)}_{\wp+1,ß+1,\Upsilon} \\
& + C_3 Y^{(j+1)}_{\wp-1,ß,\Upsilon-1} + C_7 Y^{(j+1)}_{\wp,ß,\Upsilon-1} + C_3 Y^{(j+1)}_{\wp+1,ß,\Upsilon-1} + C_3 Y^{(j+1)}_{\wp-1,ß,\Upsilon+1} + C_7 Y^{(j+1)}_{\wp,ß,\Upsilon+1} + C_3 Y^{(j+1)}_{\wp+1,ß,\Upsilon+1} \\
& + C_5 Y^{(j+1)}_{\wp-1,ß,\Upsilon} + C_7 Y^{(j+1)}_{\wp,ß,\Upsilon} + C_5 Y^{(j+1)}_{\wp+1,ß,\Upsilon} \\
& = D_1 Y^{j}_{\wp-1,ß-1,\Upsilon-1} + D_4 Y^{j}_{\wp,ß-1,\Upsilon-1} + D_1 Y^{j}_{\wp+1,ß-1,\Upsilon-1} + D_1 Y^{j}_{\wp-1,ß-1,\Upsilon+1} + D_4 Y^{j}_{\wp,ß-1,\Upsilon+1} + D_1 Y^{j}_{\wp+1,ß-1,\Upsilon+1} \\
& + D_1 Y^{j}_{\wp-1,ß+1,\Upsilon-1} + D_4 Y^{j}_{\wp,ß+1,\Upsilon-1} + D_1 Y^{j}_{\wp+1,ß+1,\Upsilon-1} + D_1 Y^{j}_{\wp-1,ß+1,\Upsilon+1} + D_4 Y^{j}_{\wp,ß+1,\Upsilon+1} + D_1 Y^{j}_{\wp+1,ß+1,\Upsilon+1} \\
& + D_2 Y^{j}_{\wp-1,ß-1,\Upsilon} + D_6 Y^{j}_{\wp,ß-1,\Upsilon} + D_2 Y^{j}_{\wp+1,ß-1,\Upsilon} + D_2 Y^{j}_{\wp-1,ß+1,\Upsilon} + D_6 Y^{j}_{\wp,ß+1,\Upsilon} + D_2 Y^{j}_{\wp+1,ß+1,\Upsilon} \\
& + D_3 Y^{j}_{\wp-1,ß,\Upsilon-1} + D_7 Y^{j}_{\wp,ß,\Upsilon-1} + D_3 Y^{j}_{\wp+1,ß,\Upsilon-1} + D_3 Y^{j}_{\wp-1,ß,\Upsilon+1} + D_7 Y^{j}_{\wp,ß,\Upsilon+1} + D_3 Y^{j}_{\wp+1,ß,\Upsilon+1} \\
& + D_5 Y^{j}_{\wp-1,ß,\Upsilon} + D_8 Y^{j}_{\wp,ß,\Upsilon} + D_5 Y^{j}_{\wp+1,ß,\Upsilon}.
\end{aligned} \tag{5.1}$$

where $Y^j_{\wp,\text{ß},\Upsilon} = \begin{pmatrix} u^j_{\wp,\text{ß},\Upsilon} \\ v^j_{\wp,\text{ß},\Upsilon} \end{pmatrix}$ and $u^{j-1}_{\wp,\text{ß},\Upsilon} = v^j_{\wp,\text{ß},\Upsilon}$

$$C_1 = \begin{pmatrix} p_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} p_2 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} p_3 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} p_4 & 0 \\ 0 & 1 \end{pmatrix},$$
$$C_5 = \begin{pmatrix} p_5 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_6 = \begin{pmatrix} p_6 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_7 = \begin{pmatrix} p_7 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_8 = \begin{pmatrix} p_8 & 0 \\ 0 & 1 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} p_9 & -p_1 \\ 1 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} p_{10} & -p_2 \\ 1 & 0 \end{pmatrix}, \quad D_3 = \begin{pmatrix} p_{11} & -p_3 \\ 1 & 0 \end{pmatrix}, \quad D_4 = \begin{pmatrix} p_{12} & -p_4 \\ 1 & 0 \end{pmatrix},$$
$$D_5 = \begin{pmatrix} p_{13} & -p_5 \\ 1 & 0 \end{pmatrix}, \quad D_6 = \begin{pmatrix} p_{14} & -p_6 \\ 1 & 0 \end{pmatrix}, \quad D_7 = \begin{pmatrix} p_{15} & -p_7 \\ 1 & 0 \end{pmatrix}, \quad D_8 = \begin{pmatrix} p_{16} & -p_8 \\ 1 & 0 \end{pmatrix}.$$

Let $\bar{Y}^j_{\wp,\text{ß},\Upsilon}$ be the numerical value of $Y^j_{\wp,\text{ß},\Upsilon}$, then the error vector at the j-th time level is

$$\varepsilon^j_{\wp,\text{ß},\Upsilon} = Y^j_{\wp,\text{ß},\Upsilon} - \bar{Y}^j_{\wp,\text{ß},\Upsilon}.$$

Thus, from equation (5.1), we can write:

$$\begin{aligned}
&C_1\varepsilon^{j+1}_{\wp-1,\text{ß}-1,\Upsilon-1} + C_4\varepsilon^{j+1}_{\wp,\text{ß}-1,\Upsilon-1} + C_1\varepsilon^{j+1}_{\wp+1,\text{ß}-1,\Upsilon-1} + C_1\varepsilon^{j+1}_{\wp-1,\text{ß}-1,\Upsilon+1} \\
&+ C_4\varepsilon^{j+1}_{\wp,\text{ß}-1,\Upsilon+1} + C_1\varepsilon^{j+1}_{\wp+1,\text{ß}-1,\Upsilon+1} + C_1\varepsilon^{j+1}_{\wp-1,\text{ß}+1,\Upsilon-1} + C_4\varepsilon^{j+1}_{\wp,\text{ß}+1,\Upsilon-1} \\
&+ C_1\varepsilon^{j+1}_{\wp+1,\text{ß}+1,\Upsilon-1} + C_1\varepsilon^{j+1}_{\wp-1,\text{ß}+1,\Upsilon+1} + C_4\varepsilon^{j+1}_{\wp,\text{ß}+1,\Upsilon+1} + C_1\varepsilon^{j+1}_{\wp+1,\text{ß}+1,\Upsilon+1} \\
&+ C_2\varepsilon^{j+1}_{\wp-1,\text{ß}-1,\Upsilon} + C_6\varepsilon^{j+1}_{\wp,\text{ß}-1,\Upsilon} + C_2\varepsilon^{j+1}_{\wp+1,\text{ß}-1,\Upsilon} + C_2\varepsilon^{j+1}_{\wp-1,\text{ß}+1,\Upsilon} \\
&+ C_6\varepsilon^{j+1}_{\wp,\text{ß}+1,\Upsilon} + C_2\varepsilon^{j+1}_{\wp+1,\text{ß}+1,\Upsilon} + C_3\varepsilon^{j+1}_{\wp-1,\text{ß},\Upsilon-1} + C_7\varepsilon^{j+1}_{\wp,\text{ß},\Upsilon-1} \\
&+ C_3\varepsilon^{j+1}_{\wp+1,\text{ß},\Upsilon-1} + C_3\varepsilon^{j+1}_{\wp-1,\text{ß},\Upsilon+1} + C_7\varepsilon^{j+1}_{\wp,\text{ß},\Upsilon+1} + C_3\varepsilon^{j+1}_{\wp+1,\text{ß},\Upsilon+1} \\
&+ C_5\varepsilon^{j+1}_{\wp-1,\text{ß},\Upsilon} + C_7\varepsilon^{j+1}_{\wp,\text{ß},\Upsilon} + C_5\varepsilon^{j+1}_{\wp+1,\text{ß},\Upsilon} \\
&= D_1\varepsilon^{j}_{\wp-1,\text{ß}-1,\Upsilon-1} + D_4\varepsilon^{j}_{\wp,\text{ß}-1,\Upsilon-1} + D_1\varepsilon^{j}_{\wp+1,\text{ß}-1,\Upsilon-1} + D_1\varepsilon^{j}_{\wp-1,\text{ß}-1,\Upsilon+1} \\
&+ D_4\varepsilon^{j}_{\wp,\text{ß}-1,\Upsilon+1} + D_1\varepsilon^{j}_{\wp+1,\text{ß}-1,\Upsilon+1} + D_1\varepsilon^{j}_{\wp-1,\text{ß}+1,\Upsilon-1} + D_4\varepsilon^{j}_{\wp,\text{ß}+1,\Upsilon-1} \\
&+ D_1\varepsilon^{j}_{\wp+1,\text{ß}+1,\Upsilon-1} + D_1\varepsilon^{j}_{\wp-1,\text{ß}+1,\Upsilon+1} + D_4\varepsilon^{j}_{\wp,\text{ß}+1,\Upsilon+1} + D_1\varepsilon^{j}_{\wp+1,\text{ß}+1,\Upsilon+1} \\
&+ D_2\varepsilon^{j}_{\wp-1,\text{ß}-1,\Upsilon} + D_6\varepsilon^{j}_{\wp,\text{ß}-1,\Upsilon} + D_2\varepsilon^{j}_{\wp+1,\text{ß}-1,\Upsilon} + D_2\varepsilon^{j}_{\wp-1,\text{ß}+1,\Upsilon} \\
&+ D_6\varepsilon^{j}_{\wp,\text{ß}+1,\Upsilon} + D_2\varepsilon^{j}_{\wp+1,\text{ß}+1,\Upsilon} + D_3\varepsilon^{j}_{\wp-1,\text{ß},\Upsilon-1} + D_7\varepsilon^{j}_{\wp,\text{ß},\Upsilon-1} \\
&+ D_3\varepsilon^{j}_{\wp+1,\text{ß},\Upsilon-1} + D_3\varepsilon^{j}_{\wp-1,\text{ß},\Upsilon+1} + D_7\varepsilon^{j}_{\wp,\text{ß},\Upsilon+1} + D_3\varepsilon^{j}_{\wp+1,\text{ß},\Upsilon+1} \\
&+ D_5\varepsilon^{j}_{\wp-1,\text{ß},\Upsilon} + D_8\varepsilon^{j}_{\wp,\text{ß},\Upsilon} + D_5\varepsilon^{j}_{\wp+1,\text{ß},\Upsilon}
\end{aligned} \tag{5.2}$$

Let the solution of (5.2) at the grid point $(x_\wp, y_\text{ß}, z_\Upsilon, t_j)$ be of the form

$$\varepsilon^j_{\wp,\text{ß},\Upsilon} = \xi^j e^{i(\theta_1\wp+\theta_2\text{ß}+\theta_3\Upsilon)} \tag{30}$$

where $i = \sqrt{-1}$, $\theta_1, \theta_2, \theta_3$ are real phase angles, and ξ is in general complex. Substituting (5.1) into (4.4) and using Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$, after simplification, we get

$$\xi \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} = \begin{pmatrix} Q_3 & -Q_1 \\ Q_2 & 0 \end{pmatrix} \tag{5.3}$$

where

$$
\begin{aligned}
Q_1 &= 8(\alpha_1\cos\theta_1+\beta_1)(\alpha_2\cos\theta_2+\beta_2)(\alpha_3\cos\theta_3+\beta_3)\\
Q_2 &= (1+2\cos\theta_1)(1+2\cos\theta_2)(1+2\cos\theta_3)\\
Q_3 &= \frac{8}{h^2k^2\sigma^2}\Big(-c^2h^2\sigma^2\tau^2\alpha_3\beta_1\cos\theta_3+c^2h^2\sigma^2\tau^2\alpha_3\beta_1\cos\theta_2\cos\theta_3\\
&\quad -c^2k^2\sigma^2\tau^2\alpha_3\beta_2\cos\theta_3+c^2k^2\sigma^2\tau^2\alpha_3\beta_2\cos\theta_1\cos\theta_3-c^2h^2k^2\tau^2\beta_1\beta_2+c^2h^2k^2\tau^2\beta_1\beta_2\cos\theta_3\\
&\quad +2h^2k^2\sigma^2\alpha_3\beta_1\beta_2\cos\theta_3+\sigma^2\left(c^2k^2\tau^2\beta_2(-1+\cos\theta_1)+h^2\beta_1(c^2\tau^2(-1+\cos\theta_2)+2k^2\beta_2)\right)\beta_3\\
&\quad +k^2\alpha_2\cos\theta_2\left(\sigma^2\cos\theta_3\alpha_3(c^2\tau^2(-1+\cos\theta_1)+2h^2\beta_1)+c^2\sigma^2\tau^2(-1+\cos\theta_1)\beta_3\right.\\
&\quad \left.+h^2\beta_1(c^2\tau^2(-1+\cos\theta_3)+2\sigma^2\beta_3)\right)\\
&\quad +h^2\cos\theta_1\alpha_1\left(c^2k^2\tau^2(-1+\cos\theta_3)\beta_2+\sigma^2\alpha_3\cos\theta_3(c^2\tau^2(-1+\cos\theta_2)+2k^2\beta_2)\right.\\
&\quad \left.+\sigma^2(c^2\tau^2(-1+\cos\theta_2)+2k^2\beta_2)\beta_3+k^2\alpha_2\cos\theta_2(c^2\tau^2(-1+\cos\theta_3)+2\sigma^2(\cos\theta_3\alpha_3+\beta_3))\right)\Big)
\end{aligned}
$$

The amplification matrix for the technique proposed in (3.8) may now be found as follows.

$$
\begin{aligned}
G &= \left(\begin{pmatrix} Q_1 & 0\\ 0 & Q_2\end{pmatrix}\right)^{-1}\begin{pmatrix} Q_3 & -Q_1\\ Q_2 & 0\end{pmatrix}\\
&= \begin{pmatrix} 2-2c^2\tau^2\left(\frac{\sin^2(\theta_1/2)}{h^2(\alpha_1\cos\theta_1+\beta_1)}+\frac{\sin^2(\theta_2/2)}{k^2(\alpha_2\cos\theta_2+\beta_2)}+\frac{\sin^2(\theta_3/2)}{\sigma^2(\alpha_3\cos\theta_3+\beta_3)}\right) & -1\\ 1 & 0\end{pmatrix}
\end{aligned}
\tag{5.4}
$$

The following expression holds for the eigenvalues of the matrix G:

$$\lambda^2-2b\lambda+1=0 \tag{5.5}$$

Here,

$$b = 1-c^2\tau^2\left(\frac{\sin^2(\theta_1/2)}{h^2(\alpha_1\cos\theta_1+\beta_1)}+\frac{\sin^2(\theta_2/2)}{k^2(\alpha_2\cos\theta_2+\beta_2)}+\frac{\sin^2(\theta_3/2)}{\sigma^2(\alpha_3\cos\theta_3+\beta_3)}\right)$$

Using the transformation $\lambda=\frac{1+z}{1-z}$, equation (5.5) takes the form:

$$(2+b)z^2+(2-2b)=0 \tag{5.6}$$

When $|b|<1$, the stability criterion $|\lambda|<1$ will be satisfied, since the unit circle may be transformed to the left half of the complex plane using the transformation mentioned above.

6. Numerical Testing

This section contains error tables and graphs of numerical solutions for some Klein-Gordon Problems. Our proposed method is used to compute numerical values and tabulate L_2 ,and L_∞ errors. Additionally, the graphics display the surface plot of the numerical solution.

Problem 1: Consider a three-dimensional linear Klein-Gordon equation

$$u_{tt}=\frac{1}{3}\left(u_{xx}+u_{yy}+u_{zz}\right)-2\left(e^u-e^{-u}\right),\quad 1<x,y,z<3,\quad t>0 \tag{6.1}$$

Along with the initial conditions:

$$u(x,y,z,0)=\varphi(x,y,z)=\ln\left(\coth^2\left(K(x+y+z)\right)\right)$$

$$u_t(x,y,z,0)=\psi(x,y,z)=2\mu K\left(\coth\left(K(x+y+z)\right)-\tanh\left(K(x+y+z)\right)\right)$$

In order to determine the boundary conditions, the exact solution is provided as:

$$u(x,y,z,t) = \ln\left(\coth^2\left(K(x+y+z-\mu t)\right)\right)$$

The numerical solution to this problem is cracked by our proposed method. Table 1 represents the error estimation for different time levels with several time and space steps. It is observed that our proposed method provides healthier results with considerable accuracy. This provides the platform to discuss the Klein-Gordon equation in the three-dimensional space domain.

(h,k,σ)	Time (sec) j	$\tau \rightarrow$								
		0.01			0.005			0.001		
		L_2	L_∞	RMS	L_2	L_∞	RMS	L_2	L_∞	RMS
(0.125, 0.125, 0.125)	1	8.535(-6)	8.2475(-7)	8.2409(-7)	7.876(-5)	8.443(-6)	8.2409(-7)	7.845(-5)	8.369(-6)	8.2357(-7)
	2	3.2368(-5)	4.0412(-6)	5.5715(-7)	3.2368(-5)	4.0418(-6)	5.5737(-7)	3.2391(-5)	4.0422(-6)	5.5756(-7)
	4	5.8019(-5)	5.9633(-6)	9.9870(-7)	5.7999(-5)	5.9632(-6)	9.9836(-7)	5.7984(-5)	5.9632(-6)	9.9809(-7)
	6	8.6559(-5)	8.9716(-6)	1.4900(-6)	8.6564(-5)	8.9716(-6)	1.4901(-6)	8.6570(-5)	8.9722(-6)	1.4901(-6)
	8	1.2297(-4)	1.3269(-5)	2.1166(-6)	1.2296(-4)	1.3271(-5)	2.1165(-6)			
	10	1.8363(-4)	2.0114(-5)	3.1608(-6)	1.8361(-4)	2.0115(-5)	3.1605(-6)			
(0.1, 0.1, 0.1)	1	2.444(-5)	3.0743(-6)	5.1249(-7)	2.411(-5)	3.0704(-6)	5.1209(-7)	2.384(-5)	3.0674(-6)	5.1177(-7)
	2	2.8489(-5)	2.4306(-6)	3.4400(-7)	2.8500(-5)	2.4313(-6)	3.4413(-7)	2.8510(-5)	2.4320(-6)	3.4425(-7)
	4	5.1245(-5)	3.6072(-6)	6.1876(-7)	5.1227(-5)	3.6071(-6)	6.1855(-7)	1.4372(-4)	7.8234(-6)	1.2223(-6)
	6	7.6347(-5)	5.4008(-6)	9.2186(-7)	7.6351(-5)	5.4003(-6)	9.2190(-7)	7.6354(-5)	5.3998(-6)	9.2194(-7)
	8	1.0889(-4)	8.0459(-6)	1.3148(-6)	1.0889(-4)	8.0465(-6)	1.3148(-6)			
	10	1.6211(-4)	1.2116(-5)	1.9573(-6)						
(0.08, 0.08, 0.08)	1	3.7723(-5)	2.0132(-6)	3.2084(-7)	2.1(-3)	2.3277(-4)	1.7677(-5)	2.1(-3)	2.3273(-4)	1.7673(-5)
	2	3.8143(-5)	2.0989(-6)	3.4651(-7)	2.3(-3)	1.0355(-4)	1.9469(-5)	2.3(-3)	1.0351(-4)	1.9465(-5)
	4	3.9449(-5)	2.3312(-6)	3.8655(-7)						
	10	1.4372(-4)	7.8234(-6)	1.2223(-6)						

Table 1: L_2, L_∞ and RMS – errors in the numerical solution of problem 1 with $\mu = 0.1$ and over the domain $1 < x, y, z < 3$.

Problem 2: Consider another three-dimensional linear Klein-Gordon equation as

$$u_{tt} = \frac{a^2}{3}\left(u_{xx} + u_{yy} + u_{zz}\right) - au + bu^3, \qquad 0 < x, y, z < 1, \quad t > 0 \tag{6.2}$$

Along with the initial conditions:

$$u(x,y,z,0) = \varphi(x,y,z) = B\tanh\left(K(x+y+z)\right),$$
$$u_t(x,y,z,0) = \psi(x,y,z) = -BK\mu\,\mathrm{sech}^2\left(K(x+y+z)\right),$$

where

$$B = \sqrt{\frac{a}{b}}, \qquad K = \sqrt{\frac{a}{2(\mu^2 - a^2)}}.$$

Boundary conditions are established based on the subsequent exact solution for the addressed problem.

$$u(x,y,z,t) = B\tanh\left(K(x+y+z-\mu t)\right).$$

The above-developed method is used to compute the approximate numeric values for the problem. If parameters are chosen as $\mu = 0.3, a = 0.1, b = 1$, Tables 2 represent the error estimation for different times with several time and space steps. The error table shows considerable accuracy and our proposed method provides healthier results. This provides the platform to discuss the Klein-Gordon equation in three space-dimensional domains.

(h, k, σ)	Time (sec) j	$\tau \rightarrow$ 0.01 L_2	0.01 L_∞	0.01 RMS	0.05 L_2	0.05 L_∞	0.05 RMS	0.005 L_2	0.005 L_∞	0.005 RMS
(0.04, 0.04, 0.04)	1	1.652(-1)	2.9(-3)	1.4(-3)	1.585(-1)	2.8(-3)	1.3(-3)	1.99(-1)	1.55(-3)	1.5(-3)
	2	5.456(-1)	9.8(-3)	6(-3)	3.909(-1)	9.7(-3)	5(-3)	6.037(-1)	1.16(-3)	5(-3)
	3	9.753(-1)	1.79(-2)	8.3(-3)	9.628(-1)	1.785(-2)	8.2(-3)	8.351(-1)	1.74(-3)	9.1(-3)
	4	1.312(-1)	2.64(-2)	1.12(-2)	1.2998(-1)	2.62(-2)	1.11(-2)	9.944(-1)	2.99(-3)	1.24(-3)
(0.05, 0.05, 0.05)	1	1.19(-1)	2.7(-3)	1.5(-3)	1.142(-1)	2.6(-3)	1.4(-3)	1.572(-1)	2.7(-3)	1.5(-3)
	2	3.921(-1)	9.4(-3)	7(-3)	3.843(-1)	1.2(-3)	6(-3)	5.33(-1)	1.2(-3)	8(-3)
	3	7.0(-1)	1.81(-2)	8.2(-3)	6.911(-1)	1.785(-2)	8.3(-3)	6.92(-1)	1.81(-3)	8.9(-3)
	4	9.406(-1)	2.64(-2)	1.14(-2)	9.323(-1)	2.62(-2)	1.13(-2)	9.38(-1)	2.87(-3)	1.2(-3)
(0.06, 0.06, 0.06)	1	9.39(-2)	3.2(-3)	1.5(-3)	9.44(-2)	3.2(-3)	1.5(-3)	9.32(-2)	3.2(-3)	1.5(-3)
	2	3.099(-1)	1.03(-2)	6(-3)	3.106(-1)	1.03(-2)	6(-3)	3.1(-1)	1.03(-2)	9(-3)
	3	5.54(-1)	1.9(-2)	8.7(-3)	5.469(-1)	1.88(-3)	8.5(-3)	5.549(-1)	1.91(-2)	8.7(-3)
	4	7.465(-1)	2.73(-2)	1.17(-2)	7.399(-1)	2.7(-2)	1.16(-2)	7.474(-1)	2.73(-2)	1.17(-2)

Table 2: L_2, L_∞ and RMS – errors in the numerical solution of problem 2 with $\mu = 0.3$, $a = 0.1$, $b = 1$ and over the domain $0 < x, y, z < 1$.

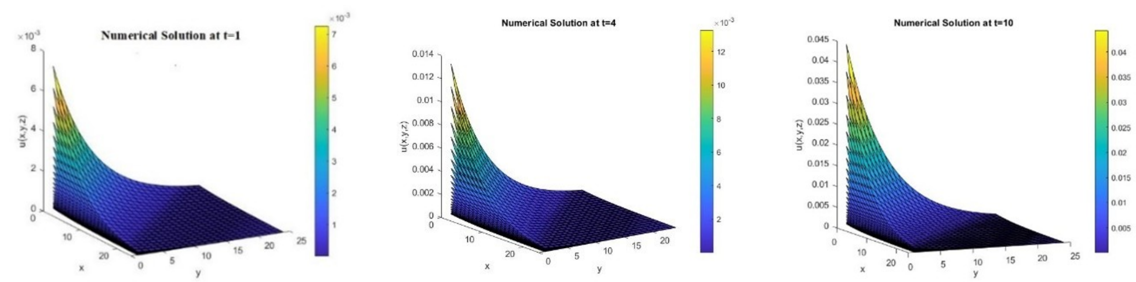

Figure 1: Surface plots of the numerical solution at three different time levels for $h = k = \sigma = 0.08$ and $\tau = 0.01$.

7. Figures and Their Significance

The numerical solutions for the Klein-Gordon Equation in (3+1) dimensions are illustrated in numerous figures in this paper using the Non-Polynomial Cubic Spline Technique. Combined, these figures illustrate the solution's behavior under a variety of parameter configurations, time steps, and discretization options, underscoring the precision, dependability, and resilience of the proposed approach. To create a baseline for comprehending the development of the solution over time, Figure 1 presents a surface plot of the numerical solution with parameters $h = k = \sigma = 0.08$ and a time step $\tau = 0.01$ at different dates. The surface plot at $\tau = 1$ is shown in Figure 2, which illustrates how different space step sizes affect solution accuracy, which is essential for accurate simulations. Figure 3 presents surface plots at t=6 for

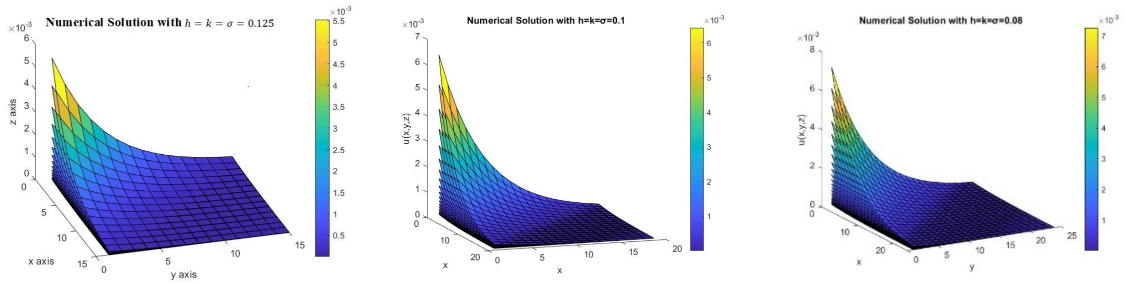

Figure 2: The surface plots for the numerical solution at $t = 1$ with time step $\tau = 0.01$ at different space step sizes.

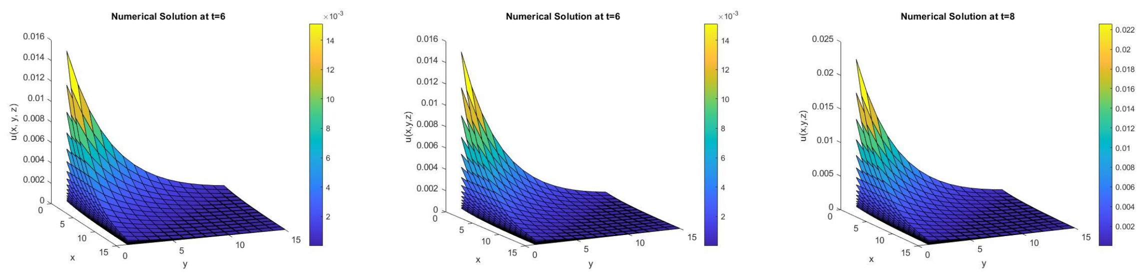

Figure 3: The surface plots for the numerical solution with $h = k = \sigma = 0.125$ at $t = 6$ with time step $\tau = 0.01, 0.005, 0.001$.

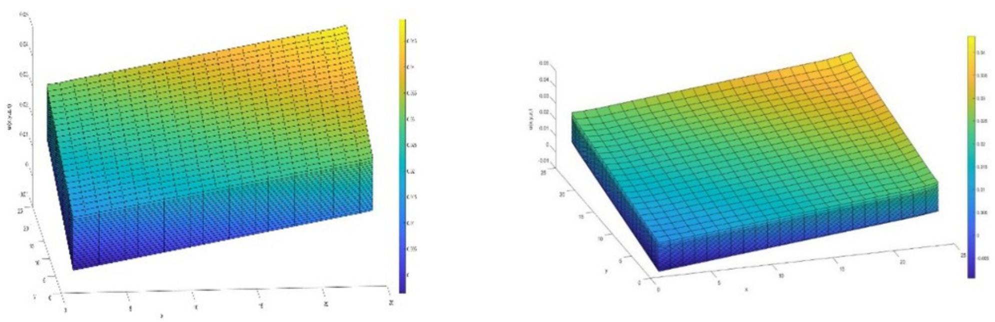

Figure 4: The surface plot for the numerical solution with $h = k = \sigma = 0.04$ and time steps $\tau = 0.01$ at time $t = 1, 2$.

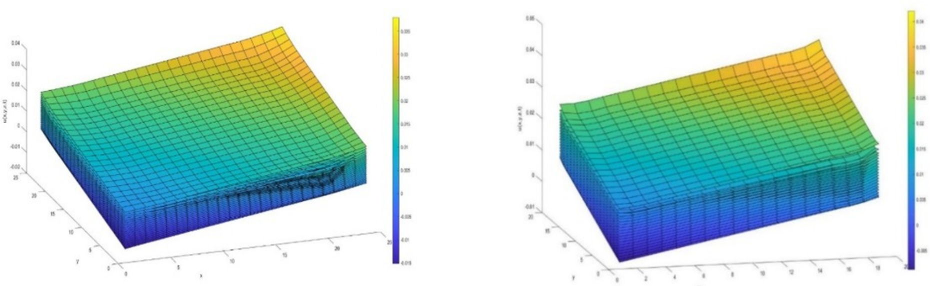

Figure 5: The surface plot for the numerical solution with $h = k = \sigma = 0.05$ and time steps $\tau = 0.05$ at time $t = 1, 2$

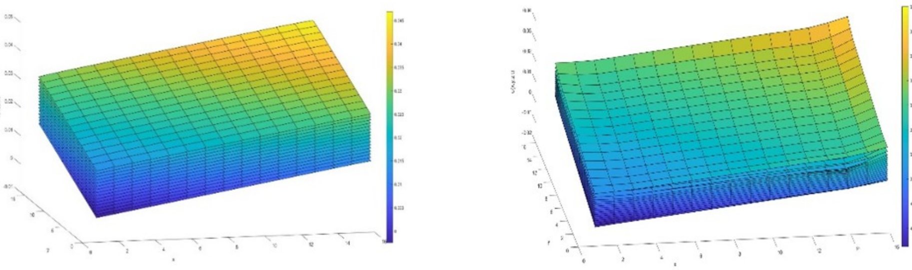

Figure 6: The surface plot for the numerical solution with $h = k = \sigma = 0.06$ and time steps $\tau = 0.05$ at time $t = 1, 2$.

various time steps $\tau = 0.01, 0.005, 0.001$ with parameters $h = k = \sigma = 0.125$ to investigate time sensitivity. This provides information on the stability and convergence of the numerical technique. Surface plots with increasingly smaller parameters $h = k = \sigma = 0.04, 0.05, 0.06$, and varying time are shown in Figures 4, 5, and 6 to provide a comparative picture of how parameter changes affect the solution. When taken as a whole, these figures show how the approach works in a variety of situations, how sensitive it is to discretization and time step selections, and how well it models the Klien-Gordon Equation's dynamics.

8. Conclusion

In this research paper, the proposed new scheme provides numerical solutions to the Klein-Gordon Equations. This method employs a non-polynomial cubic spline function to approximate terms using space variables. The term involving the second-order time derivative is estimated using the central difference method. Efficient numerical results are obtained by opting for appropriate parametric values with the order of convergence $O\left(h^2 + k^2 + \sigma^2 + \tau^2 h^2 + \tau^2 k^2 + \tau^2 \sigma^2\right)$. Numerical problems are resolved, and the results are compared with the corresponding precise solutions. The error analysis demonstrates that our suggested method yields harmonious outcomes and validates the correctness of our approach, which is both efficient and time-saving in implementation.

Compliance with ethical standards

Conflict of interest: The authors assert no conflict of interest.

Ethical Approval: None of the authors conducted any studies that involved human participants or animals for this article.

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