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On the existence and uniqueness solution for a Fractional Benjamin-Ono equation for Conformable Fractional Derivative

Ravinder Kumar, Virendra Singh Chouhan*, Praveen Agarwal** and Shilpi Jain

ABSTRACT: In this paper, we discuss the existence and uniqueness solution of a conformable fractional derivative for the fractional Benjamin-Ono equation (FBOs), using the Sine-Gordon expansion method. Also, we get some exact solution to the fractional Benjamin-Ono equation and the graphical representation of the results. The results demonstrate how the current process is practically effective.

Key Words: Fractional Benjamin-Ono equation; Sine-Gordon expansion method; Conformable fractional derivative.

Contents

1. Introduction

Fractional partial differential equations (PDEs) are partial differential equations that involve fractional derivatives of unknown functions. Unlike classical partial differential equations, which involve integer-order derivatives, FPDEs include derivatives of fractional order. These equations have gained significant attention in recent years due to their ability to model various complex phenomena involving non-local interactions, anomalous diffusion, and long-range correlations. The fractional derivatives in these equations capture memory effects and can describe systems with fractal geometry or sub diffusive behavior.

A general form of a one-dimensional fractional partial differential equation is given by:

$$D^{\alpha}\Phi(\mu,\nu) = F(\mu,\nu,\Phi,\partial_{\mu}\Phi,...),$$

where D^{α} represents a fractional derivative operator of order α with respect to the spatial variable μ, and F is a nonlinear function of the variables μ,ν,Φ and its spatial derivatives. The parameter α can be any positive real number and determines the order of the fractional derivative [1-2]. There are several types of fractional derivatives, including the Riemann-Liouville fractional derivative, the Caputo fractional derivative, and the fractional Laplacian, each defined using different integral formulations. The choice of the fractional derivative operator depends on the specific physical problem being modeled and the initial or boundary conditions [3- 4]. FPDEs have applications in various fields, including physics, engineering, biology, and finance. They are used to describe phenomena such as anomalous diffusion in porous media, heat conduction in fractal media, viscoelasticity, and the behavior of complex systems with long-range interactions [5- 6]. Solving FPDEs can be challenging due to the non-local and non-integer nature of the fractional derivatives. Researchers employ a variety of techniques to study FPDEs, including numerical methods, analytical approaches based on Laplace transforms and integral transforms, and fractional

* *and** Corresponding authors.

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calculus techniques such as fractional Fourier transform and fractional Laplace transforms [7- 8]. The study of fractional partial differential equations continues to be an active area of research, with ongoing efforts focused on understanding their mathematical properties, developing efficient numerical methods, and exploring their applications in diverse scientific and engineering disciplines [9- 10]. The fractional Benjamin-Ono equation is a generalization of the classical Benjamin-Ono equation that includes fractional derivatives. Fractional derivatives are non-local operators that capture memory effects and long-range interactions, making them suitable for describing complex systems with anomalous diffusion or fractal-like behavior.
The fractional Benjamin-Ono equation is given by:(see ref.[9])

$$D^{\alpha}_{\nu\nu}\Phi + \beta(\Phi^2)_{\mu\mu} + \Phi_{\mu\mu\mu\mu} = 0, \quad 0 < \alpha \leq 1, \tag{1.1}$$

where $\Phi(\mu, \nu)$ represents the wave amplitude, ∂_ν, and ∂_μ are the partial derivatives with respect to time ν and space μ respectively, and $D^{\alpha}_{\nu\nu}$, represents the conformal fractional derivative $0 < \alpha \leq 1$. The fractional Benjamin-Ono equation incorporates the fractional Laplacian operator, which generalizes the second-order spatial derivative to a fractional order α. This fractional derivative operator captures long-range interactions, allowing the equation to model wave propagation in non-local and fractal-like media [10-13]. Due to the non-integer order derivatives they contain, fractional PDEs are extremely complex and challenging to study. Additionally, due to the non-locality of the fractional derivative, many methods that are frequently used to solve classical PDEs, such as variable separation, Laplace transforms, and Fourier analysis, might not be directly applicable to fractional PDEs [14- 16]. In the context of applied analysis, we look into two different kinds of solutions: analytical and numerical, for which various analytical and computational techniques are employed. Researchers are very interested in finding the analytical and numerical solutions for FDEs. Therefore, the researchers present various approaches to resolve FDEs [17- 20]. Furthermore, these equations' nonlinearity adds a further layer of complexity, making it difficult to find precise solutions—or even approximations—using numerical techniques. Overall, dealing with fractional nonlinear PDEs calls for sophisticated mathematical tools and computational methods, as well as a thorough understanding of both fractional calculus and nonlinear dynamics [21- 24]. Which focus on the fractional differential equations and multi-component and high-dimensional coupled nonlinear fractional partial differential equations,respectively [25- 27]. This manuscript is organized as follows: a few necessary definitions of fractional calculus theory which are required for establishing our results in section 2, the existence and Uniqueness are discussed in section 3, the Methodology in section 4, and the analytical solution of the Benjamin-Ono equation we present three exact solutions that show the efficiency of the methods in section 5. Finally, in the final section, the conclusion and applications will be drawn.

Basic definitions and tools :

A conformable derivative is given by K. Roshdi et. al. [19] Its definitions and some theorems are given as follows.

Definition 1. Given a function $h : [0, \infty) \longrightarrow R$. then the conformable derivative of h order θ is defined as

$$(T_\theta \Phi)(\nu) = \lim_{\varepsilon \to 0} \frac{\Phi(\nu + \varepsilon\nu^{1-\theta}) - \Phi(\nu)}{\varepsilon}$$

for all $\nu > 0$, $\theta \in (0, 1)$. If Φ is θ-differentiable in some $(0, a), a > 0$, and $\lim_{\nu \to 0^+} \Phi^{(\theta)}(\nu)$exists, then define $\Phi^{(\theta)}(0) = \lim_{\nu \to 0^+} \Phi^{(\theta)}(\nu)$.

Theorem 1 Let $\beta \in (0, 1]$ and Φ, j be β- differentiable at point $t > 0$.Then
1) $T_\beta(a\Phi + bj) = aT_\beta(\Phi) + bT_\beta(j)$, for all $a, b \in R$.
2) $T_\beta(\nu^p) = p\nu^{p-\beta}$ foll all $p \in R$.
3) $T_\beta(\chi) = 0$, for all constant functions $f(\nu) = \lambda$.
4) $T_\beta(\Phi j) = \Phi T_\beta(j) + jT_\beta(\Phi)$.
5) $T_\beta(\frac{\Phi}{j}) = \frac{\Phi T_\beta(\Phi) - \Phi T_\beta(j)}{j^2}$.
6) If h is differentiable, then $T_\beta(\Phi)(\nu) = \nu^{1-\theta} \frac{d\Phi}{dt}(\nu)$.

2. Existence and Uniqueness Results

In this section, we want to consider the existence and uniqueness of the conformable fractional Benjamin-Ono equation:

$$D^{\theta}_{\nu\nu}\Phi(\mu,\nu) = -\left(\beta\frac{\partial^2\Phi^2}{\partial\mu^2} + \gamma\frac{\partial^4\Phi}{\partial\mu^4}\right) \tag{2.1}$$

Imposing the conformable integral operator on both sides of Eq. (3.1)

$$\Phi(\mu,\nu) - \Phi(\mu,0) - \Phi_{\nu}(\mu,0) = -{}_t\mathbb{I}_{\theta}\left(\beta\frac{\partial^2\Phi^2}{\partial\mu^2} + \gamma\frac{\partial^4\Phi}{\partial\mu^4}\right) \tag{2.2}$$

With the notations:

$$\mathcal{K}_1\left(\mu,\nu,\Phi\right) = -\left(\beta\frac{\partial^2\Phi^2}{\partial\mu^2} + \gamma\frac{\partial^4\Phi}{\partial\mu^4}\right) \tag{2.3}$$

Eq.(2.3) become:

$$\Phi_m(\mu,\nu) - \Phi_m(\mu,0) - (\Phi_{\nu})_m(\mu,0) = {}_{\nu}\mathbb{I}_{\theta}\left[\mathcal{K}_m\left(\mu,\nu,\Phi_m\right)\right], m = 1,2 \tag{2.4}$$

for the conformable fractional Benjamin-Ono equation:, respectively. Now, it is required to show the Lipschitz condition for the operators $\mathcal{K}_1$ with respect to the third variable *i.* e.:

$$\left\|\mathcal{K}_m\left(\mu,\nu,\Phi_m\right) - \mathcal{K}_m\left(\mu,\nu,\phi_m\right)\right\| \leq \mathcal{H}_m\left\|\Phi_m - \phi_m\right\|, m = 1,2 \tag{2.5}$$

Here the used norm is defined:

$$\left\|\Phi_m(\mu,\nu)\right\| = \max_{(\mu,\nu)\in[a,b]\times[0,\infty)}\left|\Phi_m(\mu,\nu)\right|, m = 1,2 \tag{2.6}$$

The main part is finding the Lipschitz constants $\mathcal{H}_1$. Let us first consider the related operator of the conformable fractional Benjamin-Ono equation:

$$\begin{aligned}
&\left\|\mathcal{K}_1\left(\mu,\nu,\Phi\right) - \mathcal{K}_1\left(\mu,\nu,\phi\right)\right\| \\
&= \left\|-\beta\frac{\partial^2\Phi^2}{\partial\mu^2} - \gamma\frac{\partial^4\Phi}{\partial\mu^4} - \left(-\beta\frac{\partial^2\phi^2}{\partial\mu^2} - \gamma\frac{\partial^4\phi}{\partial\mu^4}\right)\right\| \\
&= \left\|-\beta\left(\frac{\partial^2\Phi^2}{\partial\mu^2} - \frac{\partial^2\phi^2}{\partial\mu^2}\right) - \gamma\left(\frac{\partial^4\Phi}{\partial\mu^4} - \frac{\partial^4\phi}{\partial\mu^4}\right)\right\| \\
&\leq \beta\left\|\frac{\partial^2\Phi^2}{\partial\mu^2} - \frac{\partial^2\phi^2}{\partial\mu^2}\right\| + \gamma\left\|\frac{\partial^4\Phi}{\partial\mu^2} - \frac{\partial^2\phi}{\partial\mu^4}\right\| \\
&\leq \beta\left\|\frac{\partial^2\left(\Phi^2 - \phi^2\right)}{\partial x^2}\right\| + \gamma\left\|\frac{\partial^4\left(\Phi - \phi\right)}{\partial x^4}\right\|
\end{aligned} \tag{2.7}$$

Let us to assume Φ and ϕ are bounded, *i.* e. there is a positive constant $\kappa_1 > 0$ such that $\max\left\{\|\Phi\|, \|\phi\|\right\} \leq \kappa_1$. Then, their second, and fourth-order derivative functions satisfy the Lipschitz condition and so, there is a constant $\delta \geq 0$.

$$\begin{aligned}
&\left\|\mathcal{K}_1\left(\mu,\nu,\Phi\right) - \mathcal{K}_1\left(\mu,\nu,\phi\right)\right\| \\
&\leq \beta\delta^2\left\|\Phi^2 - \phi^2\right\| + \gamma\delta^4\left\|\Phi - \phi\right\| \\
&\leq \beta\delta^2\left\|\Phi - \phi\right\|\left\|\Phi + \phi\right\| + \gamma\delta^4\left\|\Phi - \phi\right\| \\
&\leq \beta\delta^2\left(\|\Phi\| + \|\phi\|\right)\left\|\Phi - \phi\right\| + \gamma\delta^4\left\|\Phi - \phi\right\| \\
&\leq \left(2\kappa_1\gamma\delta^2 + \gamma\delta^4\right)\left\|\Phi - \phi\right\|
\end{aligned}$$

Therefore, we obtain the Lipschitz condition eq. (2.5) for eq. (2.1), provided that Φ, ϕ are bounded:

$$\mathcal{H}_1 = 2\kappa_1\gamma\delta^2 + \gamma\delta^4$$

Existence Solution

Here, we will use the notion of iterative formula to prove the existence of special solutions for conformable fractional Benjamin-Ono equation. An iterative formula can be immediately concluded from eq. (2.4):

$$\begin{cases} \Phi_{n+1,m}(\mu,\nu) = {}_{\nu}\mathbb{I}_{\alpha}\left[\mathcal{K}_m\left(\mu,\nu,\Phi_{n,m}\right)\right] \\ \Phi_{0,m}(\mu,\nu) = \Phi_m(\mu,0), \end{cases} \tag{2.8}$$

Additionally, we consider the notation:

$$\mathcal{G}_n^m(\mu,\nu) = \Phi_{n,m}(\mu,\nu) - \Phi_{n-1,m}(\mu,\nu), \quad m = 1, \tag{2.9}$$

for both of Eq (2.1). We emphasize:

$$\Phi_{n,m}(\mu,\nu) = \sum_{i=0}^{n} \mathcal{G}_i^m(\mu,\nu), \quad m = 1 \tag{2.10}$$

From eqs. (2.8) and (2.9) we can deduce:

$$\begin{aligned} \mathcal{G}_n^m(\mu,\nu) &= {}_{\nu}\mathbb{I}_{\theta}\left[\mathcal{K}_m\left(\mu,\nu,\Phi_{n-1,m}\right)\right] - \mathbb{I}_{\theta}\left[\mathcal{K}_m\left(\mu,\nu,\Phi_{n-2,m}\right)\right] \\ &= {}_{\nu}\mathbb{I}_{\alpha}\left[\mathcal{K}_m\left(\mu,\nu,\Phi_{n-1,m}\right) - \mathcal{K}_m\left(\mu,\nu,\Phi_{n-2,m}\right)\right] \\ &= \int_0^{\nu} s^{\theta-1}\left[\mathcal{K}_m\left(\mu,s,\Phi_{n-1,m}\right) - \mathcal{K}_m\left(\mu,s,\Phi_{n-2,m}\right)\right]\mathrm{d}s, \quad m = 1 \end{aligned} \tag{2.11}$$

Therefore:

$$\begin{aligned} \left\|\mathcal{G}_n^m(\mu,\nu)\right\| &= \left\|\int_0^{\nu} s^{\theta-1}\left[\mathcal{K}_m\left(\mu,s,\Phi_{n-1,m}\right) - \mathcal{K}_m\left(\mu,s,\Phi_{n-2,m}\right)\right]\mathrm{d}s\right\| \\ &\leq \int_0^{\nu} s^{\theta-1}\left\|\mathcal{K}_m\left(\mu,s,\Phi_{n-1,m}\right) - \mathcal{K}_m\left(\mu,s,Phi_{n-2,m}\right)\right\|\mathrm{d}s \\ \leq \int_0^{\nu} s^{\theta-1}\mathcal{H}_m\left\|\Phi_{n-1,m} - \Phi_{n-2,m}\right\|\mathrm{d}s &= \mathcal{H}_m\int_0^{\nu} s^{\theta-1}\left\|\mathcal{G}_{n-1}^m(\mu,s)\right\|\mathrm{d}s \\ &\leq \frac{\mathcal{H}_m\nu^{\alpha}}{\alpha}\left\|\mathcal{G}_{n-1}^m(\mu,\nu)\right\|, \quad m = 1 \end{aligned} \tag{2.12}$$

Theorem 2.1 The fractional Benjamin-Ono equation with time-conformable fractional derivatives has unique continuous solutions under the condition that we can find $\hat{\nu}$ satisfying [24]:

$$\mathcal{H}_m < \theta\hat{\nu}^{-\theta}, \quad m = 1,2 \tag{2.13}$$

Proof: We can write:

$$\left\|\mathcal{G}_n^m(\mu,\nu)\right\| \leq \left(\frac{\mathcal{H}_m\nu^{\theta}}{\theta}\right)^n \Phi_m(\mu,0), \quad m = 1,2 \tag{2.14}$$

If Eq. (2.13) is hold, then $\left[\left(\mathcal{H}_m\hat{\nu}^{\theta}\right)/\theta\right] < 1$ and therefore:

$$\lim_{n\to\infty}\left\|\mathcal{G}_n^m(\mu,\hat{\nu})\right\| = 0, \quad m = 1,2 \tag{2.15}$$

This fact shows:

$$\Phi_m(\mu,\nu) = \sum_{i=0}^{\infty} \mathcal{G}_i^m(\mu,\nu), \quad m = 1,2 \tag{2.16}$$

exist and are smooth functions for both of conformable fractional Benjamin-Ono equations. Now, we want to show that, obtained $\Phi(\mu,\nu)$ the solutions of conformable fractional Benjamin-Ono equation, respectively:

$$\mathcal{R}_n^m(\mu,\nu)=\Phi_m(\mu,\nu)-\Phi_{n,m}(\mu,\nu), \quad m=1,2 \tag{2.17}$$

where $\Phi_m(\mu,\nu)$ are obtained from eq. (2.16). It follows from eqs. (2.8) and (2.17):

$$\begin{aligned}\mathcal{R}_{n+1}^m(\mu,\nu)&={}_\nu\mathbb{I}_\theta\left[\mathcal{K}_m\left(\mu,\nu,\Phi_m\right)\right]-\mathbb{I}_\theta\left[\mathcal{K}_m\left(\mu,\nu,\Phi_{n,m}\right)\right]\\ &={}_t\mathbb{I}_\theta\left[\mathcal{K}_m\left(\mu,\nu,\Phi_m\right)-\mathcal{K}_m\left(\mu,\nu,\Phi_{n,m}\right)\right]\\ &=\int_0^\nu s^{\theta-1}\left[\mathcal{K}_m\left(\mu,s,\Phi_m\right)-\mathcal{K}_m\left(\mu,s,\Phi_{n,m}\right)\right]\mathrm{d}s, \quad m=1,2\end{aligned} \tag{2.18}$$

Hence:

$$\begin{aligned}\left\|\mathcal{R}_{n+1}^m(\mu,\nu)\right\|&=\left\|\int_0^\nu s^{\theta-1}\left[\mathcal{K}_m\left(\mu,s,\Phi_m\right)-\mathcal{K}_m\left(\mu,s,\Phi_{n,m}\right)\right]\mathrm{d}s\right\|\\ &\leq\int_0^\nu s^{\theta-1}\left\|\mathcal{K}_m\left(\mu,s,\Phi_m\right)-\mathcal{K}_m\left(\mu,s,\Phi_{n,m}\right)\right\|\mathrm{d}s\\ \leq\int_0^\nu s^{\theta-1}\mathcal{H}_m\left\|u_m-u_{n,m}\right\|\mathrm{d}s&=\mathcal{H}_m^t\int_0^t s^{\theta-1}\left\|\mathcal{R}_n^m(\mu,s)\right\|\mathrm{d}s\\ &\leq\frac{\mathcal{H}_m t^\theta}{\theta}\left\|\mathcal{R}_n^m(\mu,\nu)\right\|, \quad m=1,2\end{aligned} \tag{2.19}$$

Repeating this process recursively, yields:

$$\left\|\mathcal{R}_{n+1}^m(\mu,\nu)\right\|\leq\left(\frac{\mathcal{H}_m\nu^\theta}{\theta}\right)^{n+1}\left\|\Phi_m(\mu,0)\right\|, \quad m=1,2 \tag{2.20}$$

Then applying the infinity limit on both sides of eq. (3.20) and from eq. (3.13):

$$\lim_{n\to\infty}\left\|\mathcal{R}_n^m(\mu,\nu)\right\|=0, \quad m=1,2 \tag{2.21}$$

This completes the proof.

Uniqueness Solution

We can now proceed analogously, to show that the solutions of conformable fractional FBO equations are unique. To do this, we suppose that $\Phi_m,\phi_m, m=1,2$, are solutions for Eq.(2.1). Under the condition of Theorem 2.1:

$$\begin{aligned}\left\|\Phi_m-\phi_m\right\|&=\left\|\int_0^\nu s^{\theta-1}\left[\mathcal{K}_m\left(\mu,s,\Phi_m\right)-\mathcal{K}_m\left(\Phi,s,\phi_m\right)\right]\mathrm{d}s\right\|\\ &\leq\int_0^\nu s^{\theta-1}\left\|\mathcal{K}_m\left(\mu,s,\Phi_m\right)-\mathcal{K}_m\left(\mu,s,\phi_m\right)\right\|\mathrm{d}s\\ &\leq\int_0^\nu s^{\theta-1}\mathcal{H}_m\left\|\Phi_m-\phi_m\right\|\mathrm{d}s\\ &\leq\frac{\mathcal{H}_m\nu^\theta}{\theta}\left\|\Phi_m-\phi_m\right\|, \quad m=1,2\end{aligned} \tag{2.22}$$

Therefore:

$$\left\|\Phi_m-\phi_m\right\|\left(\frac{\mathcal{H}_m\nu^\theta}{\theta}-1\right)\geq 0, \quad m=1,2$$

From Theorem 2.1, we have $(\frac{\mathcal{H}_m\nu^\theta}{\theta})-1<0$, so $\left\|\Phi_m-\phi_m\right\|=0$, or equivalently $\Phi_m=\phi_m$ $m=1,2$

3. Methodology

Considering the Sine-Gordon equation [13]

$$u_{\mu\mu} - u_{\nu\nu} = r^2 \sin(u), \tag{3.1}$$

where $u = u(\mu, \nu)$ and r is a real const. Applying the wave transformation given as $u = u(x,t)= \Phi(\xi), \quad \xi = p(\mu - c\frac{\nu^\theta}{\theta})$ to equation (3.1)

$$\Phi^{''} = \frac{r^2}{p^2(1-c^2)} \sin(U), \tag{3.2}$$

where $\Phi = \Phi(\xi)$ and ξ, is the dimension of the travelling wave and c is the elan of the travelling wave. By applying some calculations, we reach

$$\left((\frac{\Phi}{2})^{'} \right)^2 = \frac{m^2}{p^2(1-c^2)} \sin^2(\frac{\Phi}{2}) + K, \tag{3.3}$$

where K is the integrating constant. Substituting $K = 0, w(\xi) = \frac{\Phi}{2}$ and $a^2 = \frac{r^2}{(p^2(1-c^2))}$, then equation (3.3) is converted to the following equation

$$w^{'} = a \sin(w) \tag{3.4}$$

Putting $a = 1$ into equation (3.4) gives

$$w^{'} = \sin(w). \tag{3.5}$$

Solving equation (3.5), we obtain the following two important properties as follows

$$\sin(w) = \sin(w(\xi)) = \frac{2pe^{\xi}}{q^2e^{2\xi}+1} \downarrow q_{=1} = Sech(\xi), \tag{3.6}$$

$$\cos(w) = \cos(w(\xi)) = \frac{2pe^{\xi}}{q^2e^{2\xi}+1} \downarrow q_{=1} = Tanh(\xi), \tag{3.7}$$

where q is the integral constant and nonzero. By considering these two properties, we can consider in general cases the following PDEs as

$$P(u, u_\mu, u_\nu, u_{\mu\mu}, u_{\nu\nu}, u_{\mu\nu}, u_{\mu\mu\mu}, \cdots) = 0. \tag{3.8}$$

With the help of $u = u(x,t)= U(\xi), \ \xi = \mu(x - c\frac{\nu^\theta}{\theta})$ into equation (3.8), we find the following ordinary differential equation

$$N(\Phi, \Phi', \Phi'', \Phi^2, \cdots) = 0.$$

In this equation, we suppose the following trial solution equation is defined by

$$\Phi(\xi) = \sum_{i=1}^{n} Tanh^{i-1}(\xi)[B_i Sech(\xi) + A_i Tanh(\xi)] + A_0. \tag{3.9}$$

Taking Eqs.(3.6, 3.7) into equation (3.9), we rewrite it as

$$\Phi(\omega) = \sum_{i=1}^{n} \cos^{i-1}(\omega)[B_i \sin(\omega) + A_i \cos(\omega)] + A_0. \tag{3.10}$$

We determine the value n via the balance principle. Taking the coefficients of $\sin^i(\omega)\cos^j(\omega)$ to be all zero, yields a system of equations. Solving this system by software computing program, the values of A_i, B_i, c and μ may be obtained. Then, we can find the wanted solutions for the governing model.

4. Application of SGEM to the Conformable Benjamin-Ono equation

Using the fractional derivative wave transformation and the previously stated definition and properties, we first translate FBO Eq.(1.1) into their conventional form. For this, we take into account the transformation [8-9]

$$\xi = x + \lambda \frac{t^{\alpha}}{\alpha}, \tag{4.1}$$

Substituting Eq.(1.1) into Eq.(4.1), the following NODES is obtained

$$\lambda^2 \Phi'' + \beta (\Phi^2)'' + \gamma \Phi'''' = 0 \tag{4.2}$$

Integrated Eq.(4.2) twice with respectability to ξ and getting to the zero for both integral constants, we get

$$\lambda^2 \Phi + \beta \Phi^2 + \gamma \Phi'' = 0 \tag{4.3}$$

With Balancing, it yields as $n = 2$, which produces for Eq.(4.3) which produces the following solution form in Eq.(3.10) given by

$$\Phi(w) = B_1 Sin(w) + A_1 Cos(w) + B_2 Cos(w) Sin(w) + A_2 Cos^2(w) + A_0. \tag{4.4}$$

For the second derivation of Eq.(4.4), it yields

$$\begin{aligned} \Phi''(w) = {} & B_1 Cos^2(w) Sin(w) - B_1 Sin^3(w) - 2A_1 Sin^2(w) Cos(w) + \\ & B_2 Cos^3 Sin(w) - 5B_2 Sin^3(w) Cos(w) - 4A_2 Cos^2(w) Sin^2(w) + 2A_2 Sin^4(w), \end{aligned} \tag{4.5}$$

where either A_2 or B_2 may be naught, but both A_2 and B_2 cannot be zero, simultaneously. By substituting Eq.(4.4) and (4.5) into Eq.(4.3) and using a few mathematical operations, we obtained the following results.

Case-1
If $A_0 = -A_2, A_1 = 0, B_1 = 0, B_2 = iA_2, \lambda = \frac{\beta\sqrt{A_2}}{\sqrt{3}}, \gamma = -\frac{1}{3}\beta^2 A_2$, the solutions in the first case becomes

$$\Phi(\mu,\nu) = -A_2 + i\text{sech}\left[\mu + \frac{\nu^{\theta}\beta\sqrt{A_2}}{\sqrt{3}\theta}\right] A_2\text{tanh}\left[\mu + \frac{\nu^{\theta}\beta\sqrt{A_2}}{\sqrt{3}\theta}\right] + A_2\text{tanh}\left[\mu + \frac{\nu^{\theta}\beta\sqrt{A_2}}{\sqrt{3}\theta}\right]^2. \tag{4.6}$$

It produces the following conjugate mixed dark-bright soliton.

Case-2
If $A_0 = -\frac{2A_2}{3}; A_1 = 0, B_1 = 0, B_2 = -iA_2, \lambda = -\frac{i\beta\sqrt{A_2}}{\sqrt{3}}, \gamma = -\frac{1}{3}\beta^2 A_2$, we have the solution sets,

$$\Phi(\mu,\nu) = -\frac{2A_2}{3} - i\text{sech}\left[\mu - \frac{i\nu^{\theta}\beta\sqrt{A_2}}{\sqrt{3}\theta}\right] A_2\text{tanh}\left[\mu - \frac{i\nu^{\theta}\beta\sqrt{A_2}}{\sqrt{3}\theta}\right] + A_2\text{tanh}\left[\mu - \frac{i\nu^{\theta}\beta\sqrt{A_2}}{\sqrt{3}\theta}\right]^2. \tag{4.7}$$

It produces the following new mixed dark-bright soliton

Case-3
If $A_0 = -\frac{A_2}{3}, A_1 = 0, B_1 = 0, B_2 = 0, \lambda = i\sqrt{\frac{2}{3}}\beta\sqrt{A_2}, \gamma = -\frac{1}{6}\beta^2 A_2$, we have the solution sets,

$$\Phi(\mu,\nu) = -\frac{A_2}{3} + A_2\text{tanh}\left[\mu + \frac{i\sqrt{\frac{2}{3}}\nu^{\theta}\beta\sqrt{A_2}}{\theta}\right]^2. \tag{4.8}$$

It presents a new dark soliton solution.

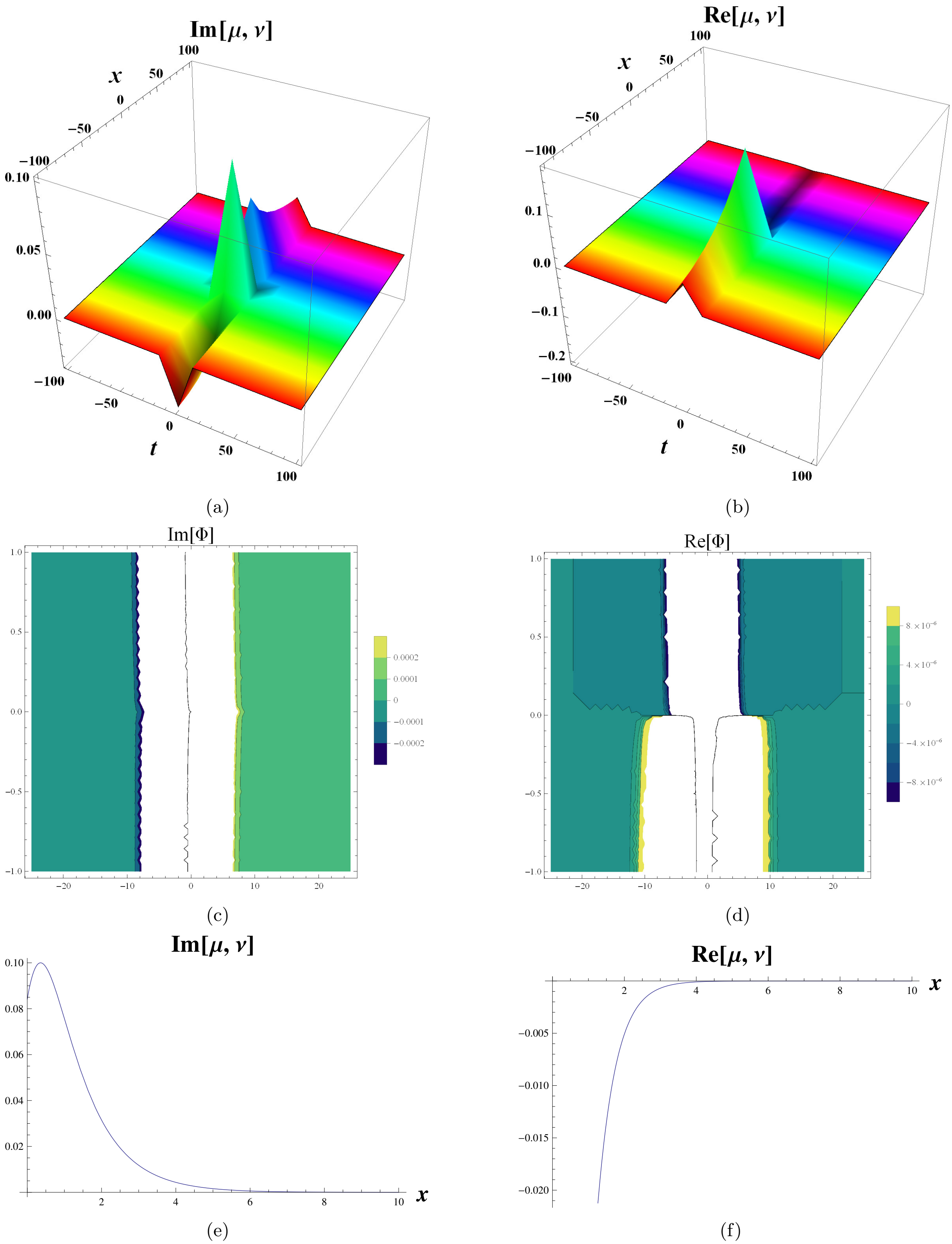

Figure 1: A 3D figure, and contour is a graphical representation for Eq.(4.6) substituting the values $\alpha = -0.3, \beta = -0.2, \gamma = 0.5, \delta = 0.27, y = -0.25, z = 0.9$, and $t = 1$ for the 2D graph within interval $-10 \leq x \leq 12,\ 10 \leq t \leq 15$.

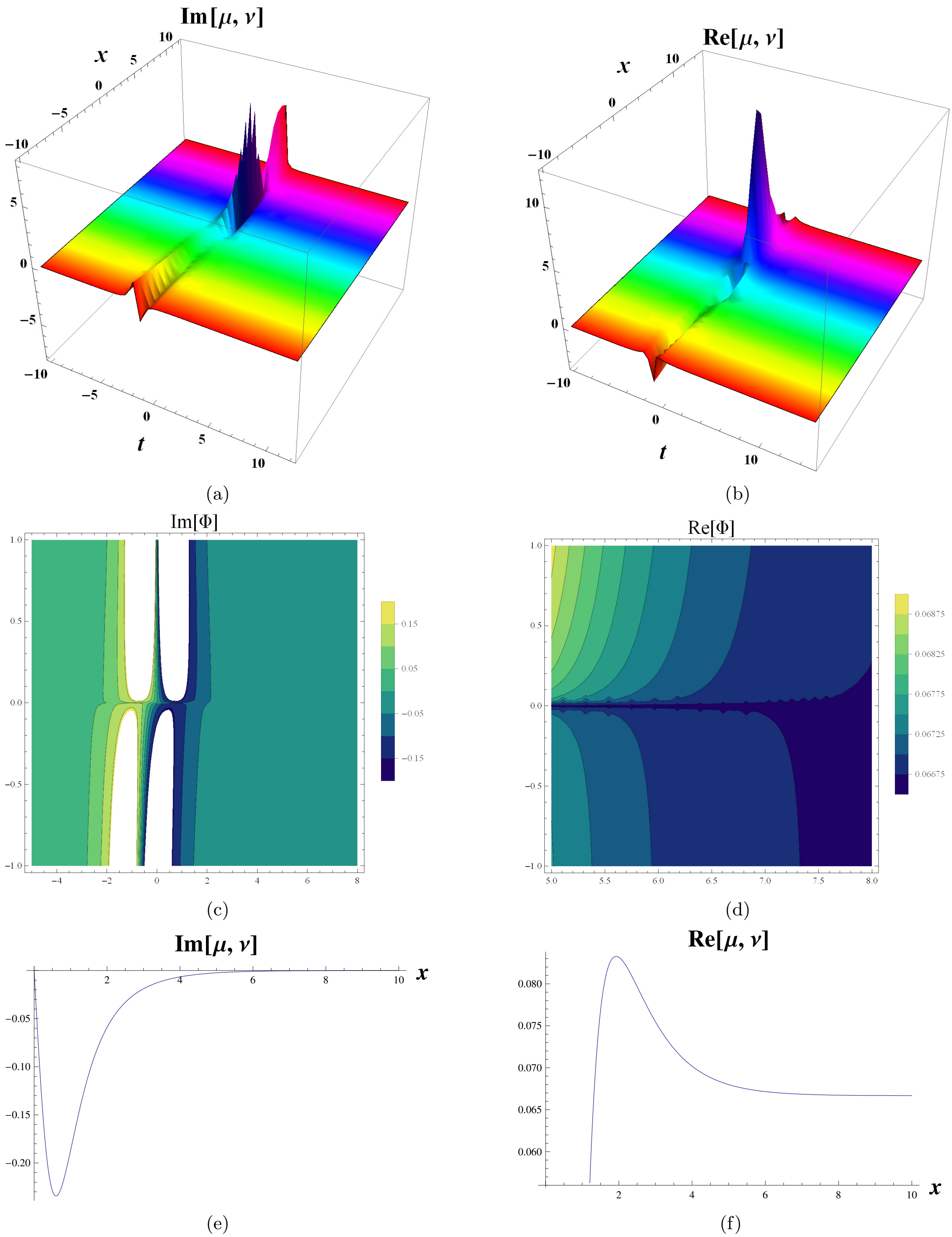

Figure 2: A 3D figure, and contour is a graphical representation for Eq.(4.7) substituting the values $\alpha = -0.3, \beta = -0.2, \gamma = 0.5, \delta = 0.27, y = -0.25, z = 0.9$, and $t = 1$ for the 2D graph within interval $-10 \le x \le 12,\ 10 \le t \le 15$.

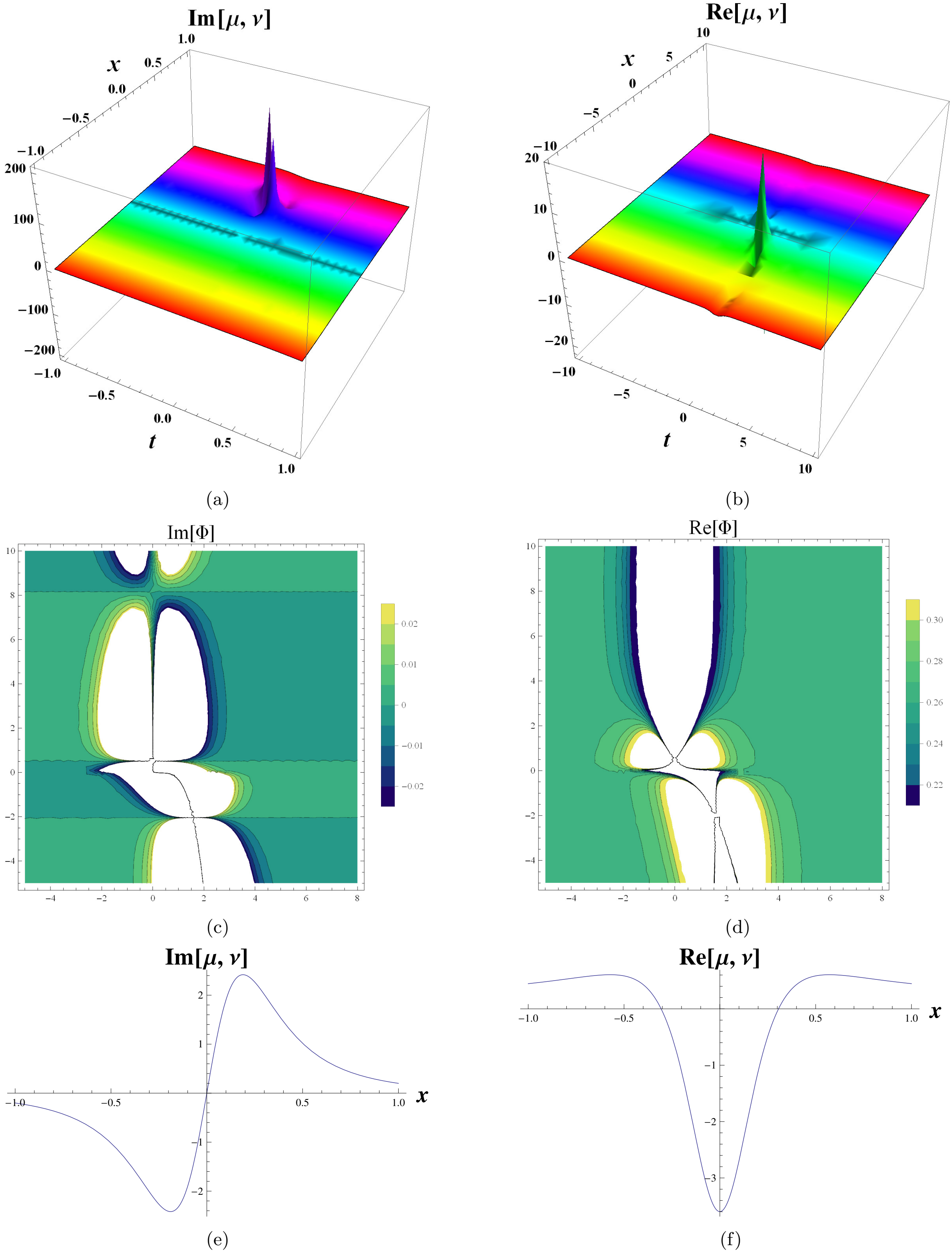

Figure 3: A 3D figure, and contour is a graphical representation for Eq.(4.8) substituting the values $\alpha = -0.3, \beta = -0.2, \gamma = 0.5, \delta = 0.27, y = -0.25, z = 0.9$, and $t = 1$ for the 2D graph within interval $-10 \leq x \leq 12,\ 10 \leq t \leq 15$.

5. Conclusion and Applications

In this article, the stability analysis (SA) of time-fractional Benney equations(TFBEs) using the FHPTM has been presented. the Caputo-Fabrizio fractional derivative(CFFD)using Laplace transform. Then, the FHPT technique is implemented for spatial discretization. Additionally, a thorough analysis of the proposed time-discrete scheme's convergence and stability is conducted. Several illustrated problems are used to carry out numerical experiments. It has been noted that the Kudryashov approach and the Tanh approach solutions exhibit excellent agreement with the results produced by the FHPTM. Additionally, it has been demonstrated that the suggested approach keeps the conservation constants (mass, momentum, and energy) over time. The wave structures of the resulting solutions are further described by plotting the 2D graphics and 3D surface solutions.

The fractional Benney equation has a wide range of applications in various fields. Here are some examples:

- Fluid mechanics: In fluid mechanics, the fractional Benney equation can be used to model waves in viscous fluids, such as lubricants and polymers. The equation is particularly useful for studying the behavior of long waves with small amplitudes.

- Oceanography: The fractional Benney equation is also used in oceanography to study wave propagation in shallow water. The equation can help predict the behavior of tsunamis, storm surges, and other large-scale water phenomena.

- Nonlinear optics: The fractional Benney equation can be used to study the propagation of light in nonlinear optical media, such as optical fibers. The equation describes the interaction between light waves and the refractive index of the medium, which depends on the intensity of the light.

- Materials science: The fractional Benney equation can also be used to study the dynamics of waves in materials with complex microstructures, such as composites and porous materials. The equation can help predict the properties of these materials, such as their thermal conductivity and acoustic response.

- Mathematical physics: The fractional Benney equation is also of interest in mathematical physics, where it plays an important role in the study of fractional differential equations. The equation is related to other models, such as the fractional Korteweg-de Vries equation and the fractional nonlinear Schrödinger equation, which describe different physical phenomena.

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Ravinder Kumar

Department of Mathematics and Statistics,

Manipal University jaipur, India.

E-mail address: `ravinder.202505032@muj.manipal.edu`

and

*Virendra Singh Chouhan (*Corresponding author)*

Department of Mathematics and Statistics,

Manipal University jaipur, India.

E-mail address: `virendrasingh.chouhan@jaipur.manipal.edu`

and

*Praveen Agarwal (**Corresponding author)*
Department of Mathematics, Saveetha School of Engineering, Chennai, Tamilnadu, 602105 India.

and

Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE.

and

Department of Mathematics, Anand International College of Engineering, Jaipur-303012 India.
E-mail address: goyal.praveen2011@gmail.com

and

Shilpi Jain
Department of Mathematics, Poornima College of Engineering, Jaipur-302022, India.
E-mail address: shilpi.jain1310@gmail.com