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Tubular surface and characterisations according to Sabban frame in Euclidean 3- Space

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ABSTRACT: Sabban frame is closely related to the study of the curvature of a surface. It provides a way to describe the normal and tangent directions of the surface, which are essential when studying the second fundamental form, Gaussian curvature, and mean curvature of a surface. The frame helps in expressing the curvature of the surface in terms of a local coordinate system and enables a more efficient calculation of these geometrical quantities. A tubular surface is a surface that is a neighbour of a curve and can be thought of as a 'tube' around the curve. Tubular surfaces are widely used to study surfaces deformed or displaced from a given curve in space. Since the Sabban framework has important applications here as well, in this work obtained the tubular surface and its characterisations defined according to the Sabban frame of the curve given on the unit sphere. Singularity, Gaussian curvature, mean curvature and basic forms of the tubular surface given according to this frame were calculated. In addition, necessary and sufficient conditions were given for the parameter curves of the tubular surface to be geodesic, asymptotic and line of curvature. Finally, the study was supported with various examples.

Key Words: Tubular surface, Sabban frame, geodesic curve, asymptotic curve, line of curvature, developable, parametre curves.

Contents

1. Introduction

Tubular surfaces are a special type of canal surfaces, first described by Gaspard Monge in 1850. In fact, it is the case that the radius in question on canal surfaces is constant. Tubular surfaces have an important role in engineering, the automotive industry, art and architecture, sports equipment, aircraft and spacecraft. In this sense, studies on tubular surfaces and their characterizations have been carried out in various spaces such as Euclidean, Minkowski and Galilean spaces [6, 8-14, 16]. An oriented surface is a surface for which a consistent orientation (or choice of "side" of the surface) is defined at every point. This means that at each point, there is a specified normal direction (in 3D) or an orientation of the surface, allowing for the definition of concepts like surface integrals, cross products, and more. In addition to this, tubular surfaces and oriented surfaces are related concepts in differential geometry and topology, particularly when considering their properties in the context of manifolds. Tubular surfaces can be oriented or non-oriented. When a tubular surface is constructed around an oriented submanifold (for instance, around an oriented curve or submanifold), the tubular surface inherits the orientation of the submanifold, making it an oriented tubular surface. In summary, a tubular surface can be an oriented surface if the submanifold around which it is constructed is oriented. The tubular surface provides a neighborhood or "tube" around the submanifold, which, if oriented, imparts an orientation to the tubular surface as well. Şahin, derived the intrinsic equations for a generalized relaxed elastic line on an oriented surface in the Galilean 3-dimensional space [18]. As it is known, there are various frames that can be installed on a curve, and the one that is most frequently studied is the Frenet frame. Although the

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Frenet frame is a frame that characterizes the curve, one of its disadvantages is that this frame cannot be defined if the curvature of the curve is zero. In 1975, Bishop eliminated this disadvantage and defined a new frame, the Bishop frame [5]. In addition, Sasai defined the modified orthogonal frame at points where the curvature is different from zero [17]. Akyiğit, Eren and Kosal gave tubular surfaces according to modified orthogonal frame with curvature in three dimensional Euclidean [1]. Atalay used this study to investigate the tubular surface and its characterisations with respect to the modified orthogonal frame with torsion [2]. Recently, Atalay studied Smarandache curved ruled surfaces and their characterizations in 3-dimensional Euclidean space according to the modified orthogonal frame [3].

Sabban frame was first defined by Koenderink and there are various studies on this frame [15]. Sabban frame plays a significant role in differential geometry, particularly in the study of surfaces and their geometry. It provides an elegant way of analyzing the geometry of submanifolds, especially in the context of tubular surfaces and other applications in differential geometry. In differential geometry, the Sabban frame is a special orthonormal frame that is used to study the intrinsic and extrinsic geometry of surfaces in Euclidean space. It helps in describing the structure of a surface, especially in the analysis of curvature and the relationship between the surface and the ambient space. When analyzing the geometry of tubular surfaces, the Sabban frame is particularly useful in understanding the structure of normal curvature and how the surface is embedded in Euclidean space. The frame allows for the analysis of the Gaussian curvature and mean curvature of these surfaces, which can be complex when the tube has a nontrivial shape or curvature. In the context of tubular surfaces, the Sabban frame helps in analysing submanifolds, especially in the study of deformations of surfaces and the way they curve within higher-dimensional spaces. This is particularly useful in physics and engineering, where tubular structures often model physical objects like pipes, tubes, and biological structures. Using the Sabban frame, Tasköprü and Tosun studied special Smarandache curves on S^2 [20]. Çalışkan and Şenyurt studied special Smarandache curves according to the Sabban frame with the help of a spherical indicatrix of a curve and they gave some characterisations of Smarandache curves [7]. Also, Şenyurt, Altun and Cevahir studied special Smarandache curves according to the Sabban frame which are drawn on the surface of a sphere by means of Darboux vectors of the involute and Bertrand partner curves [19]. Recently, Yakut and Kizilay investigated the evolution of the curve with respect to the modified orthogonal Sabban frame [21]. Also, Atalay studied the necessary and sufficient conditions for parameter curves to be geodesic, asymptotic and curvature lines by defining pairs of ruled surface with respect to the Sabban frame [4].

Considering these recent studies, in this study we have obtained the tubular surface and its characterisations defined according to the Sabban frame of the curve given on the unit sphere. Singularity, Gaussian curvature, mean curvature and basic forms of the tubular surface given according to this frame were calculated. In addition, necessary and sufficient conditions were given for the parameter curves of the tubular surface to be geodesic, asymptotic and line of curvature. Finally, the study was supported with various examples.

2. Preliminaries

Let $\alpha(s)$ be a C^3 space curve of arc-length parameter s in the Euclidean 3-space. Then, the Frenet frame $\{t(s), n(s), b(s)\}$ of the curve $\alpha(s)$ is given by

$$\begin{pmatrix} t'(s) \\ n'(s) \\ b'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix}$$

where t, n, b are tangent, princibal normal, binormal vectors and the function $\kappa(s) = \|\alpha''(s)\|$ and $\tau(s) = -\langle b'(s), n(s)\rangle$ are called the curvature and torsion of the curve $\alpha(s)$, respectively.
The sphere of radius $r = 1$ and with center in the origin in the space E^3 is defined by

$$S^2 = \{X = (x_1, x_2, x_3) : \langle X, X\rangle = 1\}.$$

Now, we give a new frame different from Frenet frame. Let γ be a unit speed spherical curve. We denote s as the arc-length parameter of γ. Let us denote $t(s) = \gamma'(s)$, and we call $t(s)$ a unit tangent vector of

γ . We now set a vector $d(s) = \gamma(s) \times t(s)$ along γ . This frame $\{\gamma(s), t(s), d(s)\}$ is called the Sabban frame of on S^2 . This gives us the following spherical Frenet formulae

$$\begin{cases} \gamma' = t, \\ t' = -\gamma + \kappa_g d, \\ d' = -\kappa_g t. \end{cases}$$

where is called the geodesic curvature of κ_g on S^2 and $\kappa_g = \langle t', d\rangle$ [20].

A surface in denotes by $P(s, v)$. The unit normal vector field of this surface is defined as follows:

$$U = \frac{P_s \times P_v}{\|P_s \times P_v\|}.$$

The coeeficents of the first end second fundamental form of this surface, respectively, is found by

$$\begin{aligned} E &= \|P_s\|^2, \quad F = \langle P_s, P_v\rangle, \quad G = \|P_v\|^2; \\ e &= \langle U, P_{ss}\rangle, \quad f = \langle U, P_{sv}\rangle, \quad g = \langle U, P_{w}\rangle. \end{aligned}$$

The Gaussian curvature and the mean curvature of the surface $P(s, v)$ are given as follows:

$$K = \frac{eg - f^2}{EG - F^2}, \quad H = \frac{eG - 2fF + gE}{2(EG - F^2)}.$$

3. Main Result

3.1. Characterizations of Tubular Surface According to Sabban Frame

A canal surface is termed as the encompassing boundary of a sphere in motion with a varying radius. When the radius remains constant, this surface is referred to as the tubular surface. The parametric equation delineating the tubular surface is presented below

$$P(s, v) = \gamma(s) + r\,[\cos(v)\, t(s) + \sin(v)\, d(s)]$$

where $v \in [0, 2\pi)$, r is the radius of the tubular surface and the curve $\gamma(s)$ is the center of the tubular surface. Since $\gamma = \gamma(s)$ is the curve defined on the unit sphere, $r = 1$. Also, the vectors t and d are perpendicular to the curve at the point of the curve γ. The deriatives according to parameters s and v of the tubular surface $P(s, v)$ are, respectively,

$$\begin{cases} P_s = -\cos(v)\, \gamma' + (1 - \kappa_g \sin(v))\, t + \kappa_g \cos(v)\, d, \\ P_v = -\sin(v)\, t + \cos(v)\, d. \end{cases} \tag{3.1}$$

From equations (3.1), the coeeficents of the first fundamental form are found as

$$\begin{cases} E = 1 + \cos^2(\nu) - 2\kappa_g \sin(\nu) + \kappa_g^2, \\ F = \kappa_g - \sin(\nu), \\ G = 1. \end{cases} \tag{3.2}$$

Morever, considering equations of (3.1), the unit normal vector field of the tubular surface $P(s, v)$ is obtained as

$$U = \frac{\cos(v)\, \gamma' + \cos^2(v)\, t + \sin(v)\cos(v)\, d}{A} \tag{3.3}$$

where $A = \sqrt{2}\cos(v) \neq 0$,and so, $\cos(v) \neq 0$.

If the unit normal vector at any point of a surface $P(s, v)$ vanishes, i.e. $P_s \times P_v = 0$ at any points, then these points are called the singular points of the surface. So the following result is obvius.

Corollary 1 *P(s,v) is a regular tubular surfaces if and only if* $v \neq \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$.

The second partial deriatives of tubular surface are found by

$$\begin{cases} P_s = (\kappa_g \sin(v) - 1)\gamma' + \big(- \cos(v) - \kappa_g' \sin(v) - \kappa_g^2 \cos(v)\big) t \\ \quad + \big(\kappa_g (1 - \kappa_g \sin(v)) + \kappa_g' \cos(v)\big) d, \\ P_v = \sin(v)\gamma' - \kappa_g \cos(v) t - \kappa_g \sin(v) d, \\ P_w = - \cos(v) t - \sin(v) d. \end{cases} \tag{3.4}$$

From the equations (3.3) and (3.4), the cooefficents of the second fundamental form are

$$\begin{cases} e = \frac{1}{\sqrt{2}} \left(-1 + 2\kappa_g \sin(v) - \kappa_g^2 - \cos^2(v)\right), \\ f = \frac{1}{\sqrt{2}} \left(-\kappa_g + \sin(v)\right), \\ g = \frac{1}{\sqrt{2}}. \end{cases} \tag{3.5}$$

The Gaussian and mean curvatures of the tubular surface $P(s, v)$ with help of equations (3.2) and (3.5) are obtained as

$$\begin{cases} K = \frac{2\kappa_g \sin(\nu) - \kappa_g^2 - 1}{2\cos^2(\nu)}, \\ H = \frac{\sqrt{2}(\kappa_g - \sin(\nu))^2}{4\cos^2(\nu)}. \end{cases} \tag{3.6}$$

respectively.

Theorem 1 *P(s,v) tubular surface*

i) The necessary and sufficient condition for it to be developable is that $\kappa_g^2 = 2\kappa_g \sin(v) + 1 = 0$, *where is* $\kappa_g \neq 0$ *and* $\kappa_g \neq \sin(v)$.

ii) minimal surface if and only if $\kappa_g = \sin(v)$.

Proof: From equation (3.6), it can tubular surface $P(s, v)$ is developable and minimal surface under the conditions stated in the hypothesis. □

As a result of this theory, we can state the following:

Corollary 2 *If the tubular surface* $P(s, v)$ *is minimal, it is definitely not developable.*

Let's give some theorems about geometric interpretation of parametric curves of the tubular surface $P(s, v)$.

Theorem 2 *i) The s-parameter curves of the tubular surface* $P(s, v)$ *are geodesic curves if and only if*

$$\kappa_g = \sin(v)$$

and

$$v = -\frac{\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}.$$

ii) The v-parameter curves of the tubular surface P(s,v) are non-geodesic.

Proof: i) For the s parameter curve of $P(s, v)$ to be geodesic curves, necessary and sufficient condition is that $U \times P_{ss} = 0$. In this case, from the equations (3.3) and (3.4), we obtain the following relations for the s-parameter curve.

$$\begin{aligned} U \times P_{ss} = & \left[-\frac{1}{\sqrt{2}} \left(\kappa_g \cos(\nu) + \kappa_g' \sin(\nu) \cos(\nu) \right) \right] \gamma' \\ & + \left[\frac{1}{\sqrt{2}} \left(\kappa_g - \kappa_g^2 \sin(\nu) + \kappa_g' \cos(\nu) + \sin(\nu) - \kappa_g \sin^2(\nu) \right) \right] t \\ & + \left[\frac{1}{\sqrt{2}} \left(\kappa_g \sin(\nu) \cos(\nu) + \kappa_g' \sin(\nu) + \kappa_g^2 \cos(\nu) \right) \right] d. \end{aligned}$$

Because of γ, t and d are linear independent, this means that $U \times P_{ss} = 0$ if and only if

$$\begin{cases} \kappa_g \cos(\nu) + \kappa_g' \sin(\nu)\cos(\nu) = 0, \\ \kappa_g - \kappa_g^2 \sin(\nu) + \kappa_g' \cos(\nu) + \sin(\nu) - \kappa_g \sin^2(\nu) = 0, \\ \kappa_g \sin(\nu)\cos(\nu) + \kappa_g' \sin(\nu) + \kappa_g^2 \cos(\nu) = 0. \end{cases}$$

If necessary operations are done from these equations, we get $\kappa_g = \sin(v)$ and $v = -\frac{\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$.

ii)From the equations (3.3) and (3.4) , we have $U \times P_{vv} = \frac{1}{\sqrt{2}}\left(-\sin(v)\,t + \cos(v)\,d\right)$. Since t and d are linearly independent, this means that $U \times P_{vv} = 0$ if and only if $\sin(v) = 0$ and $\cos(v) = 0$. This is not possible since $\cos(v)$ and $\sin(v) =$ can not be 0 at the same time. Therefore, the v-parameter curve can not be geodesic. □

Corollary 3 *From Theorem 1.ii) and Theorem 2.i), the surface $P(s,v)$ is minimal if the s-parameter curve is geodesic.*

Theorem 3 *i) The s-parameter curves of the tubular surface $P(s,v)$ are asymptotic curves if and only if*

$$2\kappa_g \sin(v) - \kappa_g^2 - \cos^2(v) = 1$$

where is $\kappa_g \neq 0$ and $\kappa_g \neq \sin(v)$.

ii) The v-parameter curves of the tubular surface $P(s,v)$ are not-asymptotic.

Proof: i) For the s parameter curve of $P(s,v)$ to be asymptotic curves, necessary and sufficient condition is that $e = 0$. From equatios (3.5), we have

$$e = \frac{1}{\sqrt{2}}\left(-1 + 2\kappa_g \sin(v) - \kappa_g^2 - \cos^2(v)\right)$$

s-parameter curves of the tubular surface are asymptotic curves if and only if $e = 0$. In this case, when necessary operations are taken in the equation above, we get

$$2\kappa_g \sin(v) - \kappa_g^2 - \cos^2(v) = 1$$

where is $\kappa_g \neq 0$ and $\kappa_g \neq \sin(v)$.

ii) From equatios (3.5), we know that

$$g = \frac{1}{\sqrt{2}}$$

v-parameter curves of the tubular surface are asymptotic curves if and only if $g = 0$. This is not possible since $g \neq 0$. Then the v-parameter curve can not be asymptotic. □

Corollary 4 *From Theorem 1.i) and Theorem 2.i), if the surface $P(s,v)$ is developable the s-parameter curve can not be asymptotic. Conversely, if the s-parameter curve is asymptotic, the surface $P(s,v)$ is not developable.*

Theorem 4 *The s and v-parameter curves of the tubular surface $P(s,v)$ is a line of curvature if and omly if $\kappa_g = \sin(v)$.*

Proof: If the parameter curves of surface are lines of curvature, then $F = f = 0$. In that case, from equations (3.2) and (3.5), we get $\kappa_g - \sin(v) = 0$. Thus, the proof is completed. □

Corollary 5 *From Theorem 1.ii) and Theorem 4, the surface $P(s,v)$ is minimal if and only if the s and v-parameter curves are line of curvature.*

3.2. Examples of Generating Tubular Surfaces According to Sabban Frame

Example 3.1 *Let us consider spherical curve* $\gamma(s) = (\cos s, \cos s \sin s, \sin^2 s)$. *It is obvious that the Sabban frame of* γ*:,*

$$\gamma(s) = (\cos s, \cos s \sin s, \sin^2 s),$$

$$t(s) = (-\sin s, -\cos 2s, \sin 2s),$$

$$d(s) = (\sin^2 s, -\sin s(\sin^2 s + 2\cos^2 s), \cos^3 s).$$

In view of the Sabban frame and the Sabban formulae, we have a geodesic curvature of γ*:*
$\kappa_g = -\cos s \sin^2 s + 2\sin^3 s \sin(2s) + 4\sin s \cos^2 s \sin(2s) + 2\cos^3 \cos(2s)$. *Now let's draw the figure of tubular surface whose equation is*

$$P(s,v) = \gamma(s) + [\cos(v)t(s) + \sin(v)d(s)]$$

according to the Sabban frame.

$$P(s,v) = \begin{pmatrix} \cos s - \cos v \sin s + \sin v \sin^2 s, \\ \sin s \cos s + \cos(2s)\cos v - \sin v \sin s(\sin^2 s + 2\cos^2 s), \\ sin^2 s + \cos(2s)\cos v + \sin v \cos^3 s \end{pmatrix}$$

The figure of this tubular surface is indicated in the Figure 1 for the values $0 \le s \le 2\pi,\ 0 \le v \le \pi/2$.

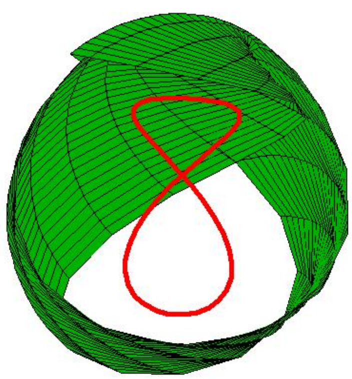

Figure 1: Tubular surface obtained by Sabban frame.

In particular, if $\kappa_g = \sin v$ *is taken here, the curves of the parameters s and v on the surface P(s,v) are lines of curvature and the surface is also minimal and is not developable. Thus, the equation of this minimal surface*

$$P(s,v) = \begin{pmatrix} \cos s - \cos v \sin s + \left(-\cos s \sin^2 s + 2\sin^3 s \sin(2s) + 4\sin s \cos^2 s \sin(2s) + 2\cos^3 s \cos(2s)\right)\sin^2 s, \\ \sin s \cos s + \cos(2s)\cos v \\ -(\sin^3 s + 2\sin s \cos^2 s)\cdot\left(-\cos s \sin^2 s + 2\sin^3 s \sin(2s) + 4\sin s \cos^2 s \sin(2s) + 2\cos^3 s \cos(2s)\right), \\ \sin^2 s + \cos(2s)\cos v + \left(-\cos s \sin^2 s + 2\sin^3 s \sin(2s) + 4\sin s \cos^2 s \sin(2s) + 2\cos^3 s \cos(2s)\right)\cos^3 s \end{pmatrix}$$

and for values $0 \le s \le 2\pi,\ 0 \le v \le \pi/2$*, the figure is as shown in Figure 2.*

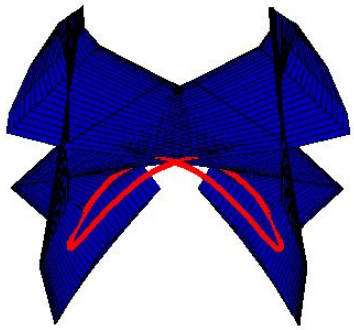

Figure 2: Minimal tubular surface obtained by Sabban frame.

Example 3.2 *Let's take spherical curve* $\gamma(s) = (-\frac{1}{\sqrt{2}}\cos s,\ -\frac{1}{\sqrt{2}}\sin s,\ \frac{1}{\sqrt{2}})$ *be a unit speed curve. It is obvious that the Sabban frame of* γ*:*

$$\gamma(s) = (-\tfrac{1}{\sqrt{2}}\cos s,\ -\tfrac{1}{\sqrt{2}}\sin s,\ \tfrac{1}{\sqrt{2}}),$$

$$t(s) = (-\tfrac{1}{\sqrt{2}}\sin s,\ -\tfrac{1}{\sqrt{2}}\cos s,\ 0),$$

$$d(s) = (\tfrac{1}{2}\cos s,\ \tfrac{1}{2}\sin s,\ \tfrac{1}{2}).$$

In view of the Sabban frame and the Sabban formulae, we have a geodesic curvature of γ*:*

$$\kappa_g = \frac{1}{2\sqrt{2}}.$$

Now let's draw the figure of tubular surface whose equation is

$$P(s,v) = \gamma(s) + [\cos(v)t(s) + \sin(v)d(s)]$$

according to the Sabban frame.

$$P(s,v) = (-\frac{1}{\sqrt{2}}\cos s+\frac{1}{\sqrt{2}}\cos v\sin s+\frac{1}{2}\sin v\cos s, -\frac{1}{\sqrt{2}}\sin s-\frac{1}{\sqrt{2}}\cos v\cos s+\frac{1}{2}\sin v\sin s,\ \frac{1}{\sqrt{2}}+\frac{1}{2}\sin v).$$

Since $\kappa_g \neq \pm\frac{1}{2}$ *, from Theorem 3.4.i), the s-parameter curve of the tubular surface P(s,v) is not geodesic on the surface. The figure of this tubular surface is indicated in the Figure 3 for the values* $0 \leq s \leq 2\pi,\ 0 \leq v \leq \pi/2$.

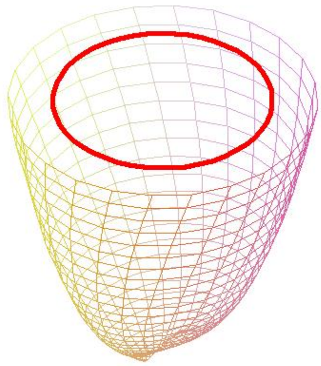

Figure 3: Tubular surface obtained by Sabban frame.

If $\kappa_g = \sin v$ *is chosen specially, the tubular surface P(s,v) is minimal and the s and v parameter curves are line of curvature. The equation of this minimal surface*

$$P(s,v) = (-\frac{1}{\sqrt{2}}\cos s + \frac{1}{\sqrt{2}}\cos v \sin s + \frac{1}{4\sqrt{2}}\cos s, -\frac{1}{\sqrt{2}}\sin s - \frac{1}{\sqrt{2}}\cos v \cos s + \frac{1}{4\sqrt{2}}\sin s,\ \frac{5}{4\sqrt{2}})$$

and its figure for values $0 \le s \le 2\pi,\ 0 \le v \le \pi/2$ *is as shown in Figure 4.*

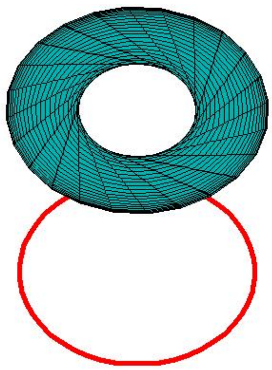

Figure 4: Minimal tubular surface obtained by Sabban frame.

Since $\kappa_g = \frac{1}{2\sqrt{2}}$ *, if* $\sin v = \frac{9\sqrt{2}}{8}$ *is chosen, from Theorem 3.4.i) and the corollary 3.7, the tubular surface P(s,v) is developable and the s-parameter curve on this surface is not asymptotic. The equation of this developable surface*

$$P(s,v) = (-\frac{1}{\sqrt{2}}\cos s + \frac{1}{\sqrt{2}}\cos v \cos s + \frac{9\sqrt{2}}{16}\cos s, -\frac{1}{\sqrt{2}}\sin s - \frac{1}{\sqrt{2}}\cos v \cos s + \frac{9\sqrt{2}}{16}\sin s,\ \frac{17}{8\sqrt{2}})$$

and its figure for values $0 \le s \le 2\pi,\ 0 \le v \le \pi/2$ *is as shown in Figure 5.*

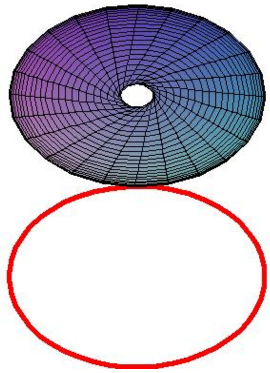

Figure 5: Developable tubular surface obtained by Sabban frame.

4. Conclusion

Sabban frame is a versatile tool in differential geometry, providing a coordinate-free way to express the geometry of surfaces. Its use in tubular surfaces aids in understanding their intrinsic curvature and how they are embedded in space. The frame also plays a crucial role in studying curvature properties, which is essential for many applications in physics, engineering, and computer graphics. The concept of a tubular neighborhood involves the idea of a surface that is a small neighborhood around a curve. The Sabban frame is used to study these neighborhoods by analyzing the geometry of the tube and understanding how the normal and tangent vectors behave in the local neighborhood. It provides a way of tracking how the curvature of the surface changes as one moves along the tubular surface. Thus, in view of the importance of both the Sabban roof and the Tubular surface for practical use, in this study, we describe the tubular surface of a given curve on the unit sphere with respect to the Sabban frame and obtain important results concerning the characterisation of the surface. In particular, we conclude that if the s-parameter curve of this surface is geodesic, the surface is also minimal; if the s-parameter curve is asymptotic, the surface can not be developable; the v-parameter curves can not be geodesic and asymptotic under any conditions; if the surface is minimal, the s and v-parameter curves are line of curvature. Finally, we give examples to support our work.

This work can be considered in Minkowski space, Galilean space or various higher dimensional spaces.

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References

1. Akyiğit, M., Eren, K., Kosal, H. H., *Tubular surfaces with modified orthogonal frame in Euclidean 3-space*, Honam Mathematical Journal. 43(3), 453–463, (2021).

2. Atalay, G. Ş., *A Study Of The Tubular Surfaces According To Modified Orthogonal Frame With Torsion*, Honam Mathematical J. 46(2), 279-290, (2024).

3. Atalay, G. Ş., *Smarandache Curved Ruled Surfaces and Their Characterizations According to Modified Orthogonal Frame in E^3*, Applications and Applied Mathematics: An International Journal (AAM). 20(1),19 pages, (2025).

4. Atalay, G. Ş.,*On the characterisations of ruled surface pairs according to the Sabban frame*, Filomat. 39(11), 3705-3717, (2025).

5. Bishop, R. L., *There is more than one way to frame a curve*, The American Mathematical Monthly. 82(3), 246–251, (1975).

6. Blaga, P. A., *On tubular surfaces in computer graphics*, Studia Universitatis Babeș-Bolyai Informatica. 50(2), 81-90, (2005).

7. Çalışkan, A., Şenyurt, S., *Smarandache curves in terms of Sabban frame of spherical indicatrix curves.*, General Mathematics Notes. 31(2), 1-15, (2015).

8. Doğan, F., Yaylı, Y., *On the curvatures of tubular surface with Bishop frame*, Communications Faculty of Sciences University of Ankara, Series A1: Mathematics and Statistics. 60(1), 59-69, (2011).

9. Doğan, F., Yaylı, Y., *Tubes with Darboux frame*, International Journal of Contemporary Mathematical Sciences. 7(16), 751-758, (2012).

10. Karacan, M. K., Tuncer, Y., *Tubular surfaces of Weingarten types in Galilean and pseudo-Galilean spaces*, Bulletin of Mathematical Analysis and Applications. 5(2), 87-100, (2013).

11. Karacan, M. K., Yaylı, Y., *On the geodesics of tubular surfaces in Minkowski 3-space*, Bulletin of the Malaysian Mathematical Sciences Society. 1, 1-10, (2008).

12. Karacan, M. K., Yoon, D. W., Tuncer, Y., *Weingarten and linear Weingarten type tubular surfaces in* E^3, Mathematical Problems in Engineering. 2011(13), 1-11, (2011).

13. Karacan, M. K., Yoon, D. W., Tuncer, Y., *Tubular surfaces of Weingarten types in Minkowski 3-space*, General Mathematics Notes., 22(1), 44-56, (2014).

14. Kızıltuğ, S., Kaya, S., Tarakcı, O., *Tube surfaces with type-2 Bishop frame of Weingarten types in* E^3, International Journal of Mathematical Analysis. 7(1), 9-18, (2013).

15. Koenderink, J. J., *Solid shape*, MIT Press. (1990).

16. Körpınar, T., Turhan, E., *Tubular surfaces around timelike biharmonic curves in Lorentzian Heisenberg group Heis*, Analele Ştiinţifice ale Universităţii Ovidius Constanţa. Seria Matematică. 20(1), 431-446, (2012).

17. Sasai, T., *The fundamental theorem of analytic space curves and apparent singularities of Fuchsian differential equations*, Tohoku Mathematical Journal. 36, 17-24, (1984).

18. Şahin, T., *Intrinsic equations for a generalized relaxed elastic line on an oriented surface in the Galilean space*, Acta Mathematica Scientia. 33B(3), 701-711, (2013).

19. Şenyurt, S., Altun, Y., Cevahir, C., *Smarandache curves according to Sabban frame of fixed pole curve belonging to the Bertrand curves pair*, AIP Conference Proceedings, 1726. https://doi.org/10.1063/1.4945871, (2016).

20. Taşköprü, K., Tosun, M., *Smarandache curves*, Boletim da Sociedade Paranaense de Matemática.32(1), 51-59, (2014).

21. Yakut, A. T., Kızılay, A., *On the curve evolution with a new modified orthogonal Sabban frame*, AIMS Mathematics. 9, 29573–29586, (2024).

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