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Innovative Lucky-edge Odd Mean Labeling of Star-Related Graphs: Bridging Theory and Circuit Design

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ABSTRACT: This study explores the application of Lucky-edge Odd Mean Labeling (LEOML) to special classes of star-related graphs, including Middle, Total, and Central graphs. LEOML is an exclusive edge labeling technique in which the mean of edge labels incident to each vertex is controlled to be an odd number. The goal is to tie theoretical graph theory with practical applications in network and circuit design, providing intuitions into the structural and functional rewards of this labeling approach. We explore the theoretical implications of LEOML by computing the Lucky-edge Odd Mean Number (LEOMLN) for various special classes of star-related graphs. Furthermore, we examine the relationship between LEOMLN and the graph energy of circuit model representations. The energy of a graph, derived from its adjacency matrix and eigenvalues, offers intuitions into the graph's spectral properties, which are essential for understanding its performance in practical applications. The results reveal a significant correlation between LEOMLN and graph energy, opening up new opportunities for optimizing network and circuit performance based on theoretical graph properties. This study assists as a bridge between abstract graph theory and real-world applications, providing valuable knowledge to both theoretical research and practical design disciplines. The innovative application of LEOML has the potential to impact various fields, including network optimization, circuit design, and computational analysis.

Key Words: Central graph, circuit, middle graph, Lucky-edge odd mean labeling, total graph.

Contents

1. Introduction

In the mesmerizing topic of graph theory, graph labeling is the process of applying special rules to the edges or vertices of a graph to assign labels (usually represented by numbers). Rosa [24] was the one who initially presented the idea of graph labeling. The procedure of assigning numbers to vertices, edges, or both in some circumstances is known as graph labeling. The label is referred to as a vertex label (or edge label) if the list is a group of vertices (or edges). Over 1100 research papers have considered different labeling in the succeeding years [8,9,10,11,13,21,22,25,31,32,33].

In several disciplines, such as X-rays, crystallography, radar, astronomy, database administration, and coding theory, labeled graphs are used as models. Nellai Murugan A. *et al.* [19] projected the concept of lucky-edge labeling as a function φ which is a numerical assignment to the vertices such that $\varphi(u)+\varphi(v)$ is assigned to the edge uv and vw are adjacent edges. The least integer k for which a graph G has a

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lucky-edge labeling from the set $\{1, 2, ..., k\}$ is the lucky-edge number of G and is denoted by $\eta(G)$. A graph that discloses lucky-edge labeling is the lucky-edge labeled graph.

1.1. Review of Literature

Nellai Murugan A. *et al.* [19] have shown that path graphs, comb graphs, cycle graphs, and crown graphs are lucky-edge labeled graphs. Nellai Murugan A. *et al.* [15,16,17,18] have proved that Bistar graphs, wheel graphs, parachute graphs, complete graphs, complete bipartite graphs, fan graphs, spider graphs, twig graphs and H_n graphs for odd and even, and planar grid graphs are lucky-edge labeled graphs, Aishwarya [1,2,3] have proved that Ladder graphs, Shell graphs and Book with Triangular pages graphs, Helm graphs, Flower graphs, Double star graphs are lucky-edge labeled graphs and also Ramya N., and Shalini R. [23] have proved that fish tail graphs are lucky-edge labeled graphs.

Nellai Murugan A. *et al.* [14] have proved that Triangular snake graphs T_n Book with triangular Page graphs B_3^n, Triangular prism graphs $P_n \times C_3$ are lucky-edge labeled graphs, Sridevi R. and Ragavi S. [30] have proved that Complete graphs K_n, tadpole graphs $T_{m,n}$ and rectangular book graphs B_P^4 are lucky-edge labeled graphs and also Esakkiammal E. *et al.* [7] have proved that super subdivision of path graphs are lucky-edge labeled graphs.

Esakkiammal E. *et al.* [4,5,6] have proved that the H-super subdivision of path, cycle, corona of C_n graphs, the super subdivision of star and wheel graphs and super subdivision of planar grid graphs are lucky-edge labeled graphs.

Mariya Irudhaya A. *et al.* [20] have proved that $z - P_n$, Fish graphs $C_n@K_3$, Butterfly graphs B_3^2, Double Triangular snake graphs DT_n, Flower graphs fl_n, P_n^2 are lucky-edge labeled graphs. Nagarajan S. and Priyadharshini G. [12] have proved that car graphs, Lotus graphs, Prism graphs, $C_{2m}@P_t$ graphs are lucky-edge labeled graphs, Shalini Rajendra Babu. *et al.* [29] have proved that splitting graph of star graphs, alternative triangular snake graphs, alternative quadrilateral snake graphs and double alternative quadrilateral snake graphs are lucky-edge labeled graphs. Shalini Rajendra Babu. *et al.* [28,29] have proved that the H − graphs, n copies of H- graphs, Theta graphs, Path union of Theta graphs and Duplication of Theta graphs are lucky-edge labeled graphs. Senthil Amutha R. *et al.* [27] proposed the concept of lucky-edge geometric mean labeling and also, they have proved that path graphs, ladder graphs, open ladder graphs, slanting ladder graphs, star graphs, degree splitting graph of star graphs, bistar graphs, and barycentric edge subdivision of star graphs are lucky-edge geometric mean labeled graphs. Senthil Amutha R. *et al.* [26] proposed the concept of lucky-edge mean labeling and also, they have proved that Corona product of path graphs with path graphs, Corona product of path graphs with star graphs, Corona product of path graphs with Wheel graphs, and Corona product of path graphs with Complete graphs are lucky-edge mean labeled graphs. Inspired by these concepts, we have to introduced the concept of lucky-edge odd mean labeling. In this paper, we have to investigate the Middle graph of star graphs, the Total graph of star graphs, the Central graph of star graphs, the Middle graph of Bistar graphs, the Total graph of Bistar graphs and the Central graph of Bistar graphs admit lucky-edge odd mean labeling and also, we obtain the lucky-edge odd mean number of these graphs, in addition to that we compute the energy of the graph model representation of the circuit and find the relation between the energy of the graph and the lucky-edge odd mean labeling number.

Objectives of the Study

1. Investigate the mathematical properties and characteristics of LEOMLN in these graph classes.

2. Analyze the relationship between LEOMLN and other graph parameters, such as graph energy, particularly in the context of circuit model representations.

2. Materials and Methods

2.1. Materials

1. Focus on special classes of graphs, such as Middle, Total, and Central graphs derived from star-related graphs.
2. Access to academic papers, books, and journals on graph theory, especially those focusing on graph labeling, graph energy, and related areas.

2.2. Methods

1. **Graph Construction:** Begin by formally defining the graph classes of interest, such as Middle, Total, and Central graphs, derived from star graphs and Bistar graphs.
2. **Computing Lucky-edge Odd Mean Labeling (LEOML):**
   (a) Labeling odd integers to the vertices
   (b) Compute the edge labels by taking the ceiling or falling of the average of the incident vertex label
   (c) Stop the process if all the adjacent edge labels are distinct odd integers.
3. **Computing Lucky-edge Odd Mean Labeling Number (LEOMLN):** The least positive odd integer k for which a graph G has a lucky-edge odd mean labeling from the set $\{1, 3, 5, \ldots, k\}$ is the lucky-edge odd mean labeling number of G.
4. **Application Investigation:**
   (a) Consider the electric circuit
   (b) Draw the graph model representation of the circuit
   (c) Write the adjacency matrix of the graph
   (d) Compute the Energy of the graph, which is the sum of the Absolute values of Eigen values of the graph
5. **Validation and Testing:**

   Examine the relationship between LEOMLN and the graph energy of circuit model representations

3. Result analysis

Theorem 3.1: Middle graph of the star graph $\mathcal{M}[\mathcal{K}_{1,n}]$ is a lucky-edge odd mean graph
Proof: Let $V[\mathcal{M}[\mathcal{K}_{1,n}]] = \{u\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$

$$E[\mathcal{M}[\mathcal{K}_{1,n}]] = \{uv_1, uv_2, \ldots, uv_n\} \bigcup \{u_1v_1, u_2v_2, \ldots, u_nv_n\}$$

$$\bigcup \{v_1v_2, v_1v_3, \ldots, v_1v_n, vv_3, \ldots, vv_n, \ldots, v_{n-1}v_n\}$$

$$|V[\mathcal{M}[\mathcal{K}_{1,n}]]| = 2n + 1$$

$$|E[\mathcal{M}[\mathcal{K}_{1,n}]]| = \frac{n(n+3)}{2}$$

Express $\varphi : V[\mathcal{M}[\mathcal{K}_{1,n}]] \to \{1, 3, 5, \ldots\}$
The Vertex labels are expressed by

$$\varphi(u) = 5$$

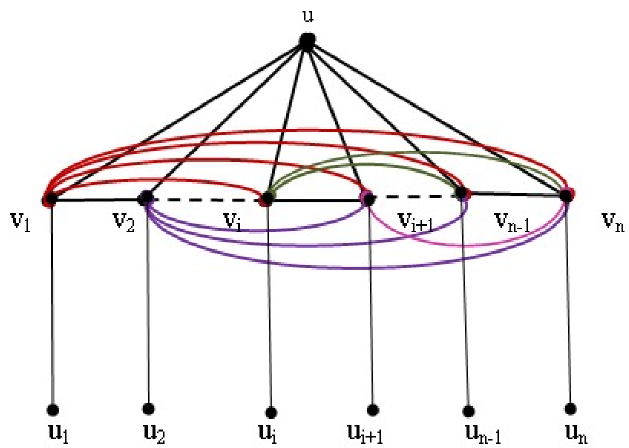

Figure 1: Labeling of Middle graph of Star $\mathcal{M}\left[\mathcal{K}_{1,n}\right]$

$$\varphi\left(u_1\right) = 1$$

$$\varphi\left(v_1\right) = 1$$

$$\varphi\left(u_\imath\right) = 4\imath + 1, \text{ for } \imath \text{ from 2 to n}$$

$$\varphi\left(v_\imath\right) = 4\imath + 1, \text{ for } \imath \text{ from 2 to n}$$

The function $\varphi^* : E\left[\mathcal{M}\left(\mathcal{K}_{1,n}\right)\right] \rightarrow \{1, 3, 5, \dots\}$ has perceptibly been induced, such that the proper coloring of G is characterized by the edge set $E\left[\mathcal{M}\left(\mathcal{K}_{1,n}\right)\right]$ and

$$\varphi^*(uv) = \left\lceil \frac{\varphi(u) + \varphi(v)}{2} \right\rceil,$$

i.e., $\varphi^*(uv) \neq \varphi^*(vw)$ for every pair of neighboring edges uv and vw.
In $\mathcal{M}\left(\mathcal{K}_{1,n}\right)$, the least positive odd integer $4n+1$ that represents the lucky-edge odd mean number of G is denoted by $\eta'_{OM}\left(\mathcal{M}\left(\mathcal{K}_{1,n}\right)\right)$:

$$\therefore \eta'_{OM}\left(\mathcal{M}\left(\mathcal{K}_{1,n}\right)\right) = 4n + 1$$

Hence, $\mathcal{M}\left(\mathcal{K}_{1,n}\right)$ is a lucky-edge odd mean graph.

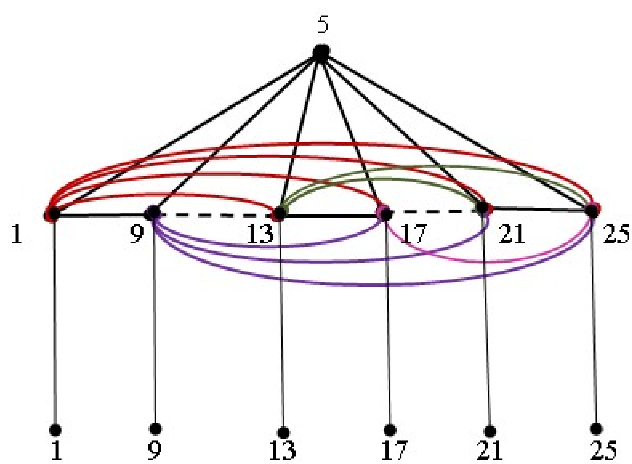

Figure 2: Lucky-edge odd mean labeling of $\mathcal{M}\left[\mathcal{K}_{1,6}\right]$

Theorem 3.2: Total graph of the Star graph $\mathcal{T}[\mathcal{K}_{1,n}]$ is a lucky-edge odd mean graph
Proof: Let $V[\mathcal{T}[\mathcal{K}_{1,n}]] = \{u\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$

$$E[\mathcal{T}[\mathcal{K}_{1,n}]] = \{uv_1,\ uv_2, \ldots, uv_n\}$$

$$\bigcup \{uu_1,\ uu_2, \ldots, uu_n\} \bigcup \{u_1v_1,\ u_2v_2, \ldots, u_nv_n\} \bigcup \{v_1v_2,\ v_1v_3,\ \ldots,\ v_1v_n,\ vv_3,\ \ldots,\ vv_n, \ldots, v_{n-1}v_n\}$$

$$|V[\mathcal{T}[\mathcal{K}_{1,n}]]| = 2n+1$$

$$|E[\mathcal{T}[\mathcal{K}_{1,n}]]| = \frac{n(n+5)}{2}.$$

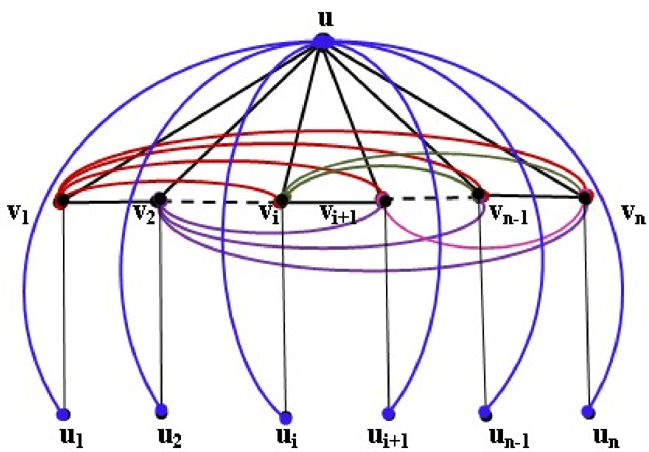

Figure 3: Labeling of Total graph of star $\mathcal{T}[\mathcal{K}_{1,n}]$

Express $\varphi : V[\mathcal{T}(\mathcal{K}_{1,n})] \to \{1, 3, 5, \ldots\}$
The Vertex labels are expressed by

$$\varphi(u) = 1$$

$$\varphi(u_\imath) = 8n - 4\imath + 5, \text{ for } \imath \text{ from } 1 \text{ to } n$$

$$\varphi(v_\imath) = 4\imath + 1, \text{ for } \imath \text{ from } 1 \text{ to } n$$

The function $\varphi^* : E[\mathcal{T}(\mathcal{K}_{1,n})] \to \{1, 3, 5, \ldots\}$ has perceptibly been induced, such that the proper coloring of G is characterized by the edge set $E[\mathcal{T}(\mathcal{K}_{1,n})]$ and

$$\varphi^*(uv) = \left\lceil \frac{\varphi(u) + \varphi(v)}{2} \right\rceil,$$

i.e., $\varphi^*(uv) \neq \varphi^*(vw)$ for every pair of neighboring edges uv and vw.
In $\mathcal{T}(\mathcal{K}_{1,n})$, the least positive odd integer $4n+3$ that represents the lucky-edge odd mean number of G is denoted by $\eta'_{OM}(\mathcal{T}(\mathcal{K}_{1,n}))$:

$$\therefore \eta'_{OM}(\mathcal{T}(\mathcal{K}_{1,n})) = 4n+3$$

Hence $\mathcal{T}[\mathcal{K}_{1,n}]$ is a lucky-edge odd mean graph.

Theorem 3.3: Central graph of Star graph $\mathcal{C}[\mathcal{K}_{1,n}]$ is a lucky-edge odd mean graph

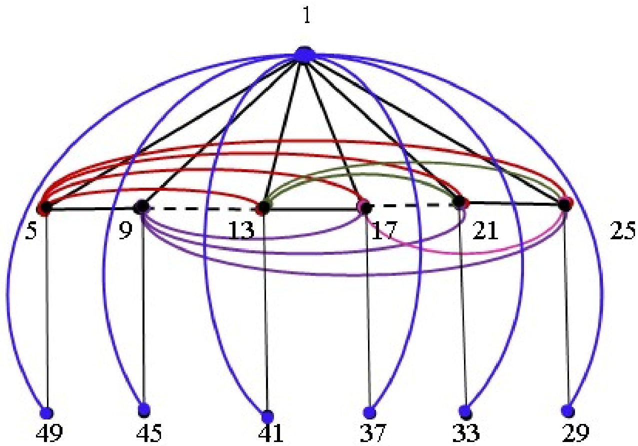

Figure 4: Lucky-edge odd mean labeling of $\mathcal{T}[\mathcal{K}_{1,6}]$

Proof: Let $V[\mathcal{C}[\mathcal{K}_{1,n}]] = \{u\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$

$$E[\mathcal{C}[\mathcal{K}_{1,n}]] = \{uv_1,\ uv_2, \ldots, uv_n\}$$

$$\bigcup\{u_1v_1,\ u_2v_2, \ldots, u_nv_n\}\bigcup\{v_1v_2,\ v_1v_3,\ \ldots,\ v_1v_n,\ v_2v_3,\ \ldots,\ v_2v_n, \ldots, v_{n-1}v_n\}$$

$$|V[\mathcal{C}[\mathcal{K}_{1,n}]]| = 2n+1$$

$$|E[\mathcal{C}[\mathcal{K}_{1,n}]]| = \frac{n(n+3)}{2}.$$

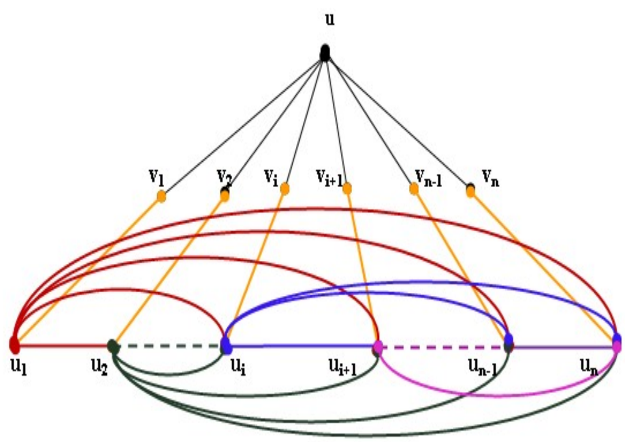

Figure 5: Labeling of Central Graph of Star $\boldsymbol{C}[\boldsymbol{K}_{\mathbf{1},\boldsymbol{n}}]$.

Express $\varphi : V[\mathcal{C}(\mathcal{K}_{1,n})] \to \{1, 3, 5, \ldots\}$
The Vertex labels are expressed by

$$\varphi(u) = 1$$

$$\varphi(u_\imath) = 4\imath + 1,\ \text{for } \imath \text{ from } 1 \text{ to } n$$

$$\varphi(v_\imath) = 4\imath + 1, \text{ for } \imath \text{ from } 1 \text{ to } n$$

The function $\varphi^* : E[\mathcal{C}(\mathcal{K}_{1,n})] \to \{1,3,5,\ldots\}$ has perceptibly been induced, such that the proper coloring of G is characterized by the edge set $E[\mathcal{C}(\mathcal{K}_{1,n})]$ and

$$\varphi^*(uv) = \left\lceil \frac{\varphi(u)+\varphi(v)}{2} \right\rceil,$$

i.e., $\varphi^*(uv) \neq \varphi^*(vw)$ for every pair of neighboring edges uv and vw.
In $\mathcal{C}(\mathcal{K}_{1,n})$, the least positive odd integer $4n+1$ that represents the lucky-edge odd mean number of G is denoted by $\eta'_{OM}(\mathcal{C}(\mathcal{K}_{1,n}))$:

$$\therefore \eta'_{OM}(\mathcal{C}(\mathcal{K}_{1,n})) = 4n+1$$

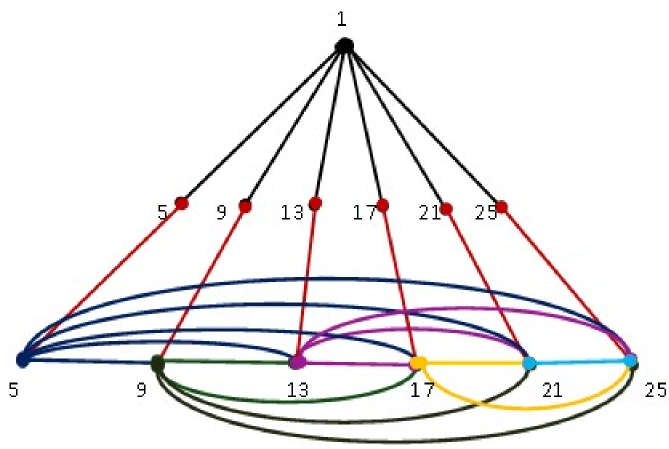

Figure 6: Lucky-edge odd mean labeling of $\mathcal{C}[\mathcal{K}_{1,6}]$

Hence $\mathcal{C}[\mathcal{K}_{1,n}]$ is a lucky-edge odd mean graph.

Theorem 3.4: Middle graph of Bistar graph $\mathcal{M}[\mathcal{B}_{n,n}]$ is a lucky-edge odd mean graph.
Proof: Let

$$V[\mathcal{M}(\mathcal{B}_{n,n})] = \{u\} \cup \{u_1,u_2,\ldots,u_n\} \cup \{v\} \cup \{v_1,v_2,\ldots,v_n\} \cup \{w\} \cup \{u'_1,u'_2,\ldots,u'_n\} \cup \{v'_1,v'_2,\ldots,v'_n\}$$

$$\begin{aligned} E[\mathcal{M}(\mathcal{B}_{n,n})] = \{wu'_1,wu'_2,\ldots,wu'_n\} \cup \{wv'_1,wv'_2,\ldots,wv'_n\} \cup \{uu'_1,uu'_2,\ldots,uu'_n\} \cup \{vv'_1,vv'_2,\ldots,vv'_n\} \\ \cup\{u_1u'_1,u_2u'_2,\ldots,u_nu'_n\} \cup \{v_1v'_1,v_2v'_2,\ldots,v_nv'_n\} \cup \{uw,wv\} \\ \cup\{u'_1u'_2,u'_1u'_3,\ldots,u'_1u'_n,u'_2u'_3,\ldots,u'_{n-1}u'_n\} \\ \cup\{v'_1v'_2,v'_1v'_3,\ldots,v'_1v'_n,v'_2v'_3,\ldots,v'_{n-1}v'_n\} \end{aligned}$$

$$|V[\mathcal{M}(\mathcal{B}_{n,n})]| = 4n+3$$

$$|E[\mathcal{M}(\mathcal{B}_{n,n})]| = n^2+5n+2$$

Express $\varphi : V[\mathcal{M}(\mathcal{B}_{n,n})] \to \{1,3,5,\ldots\}$

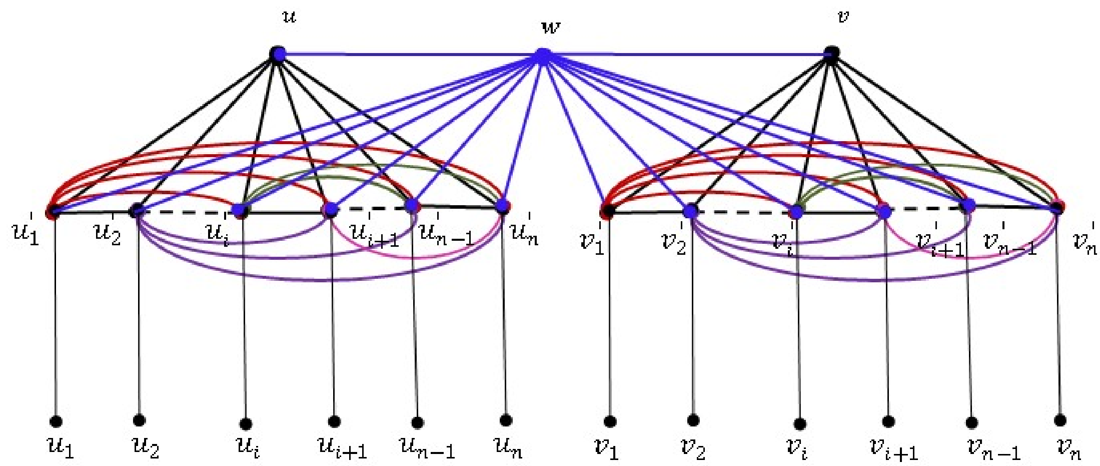

Figure 7: Labeling of the Middle Graph of Bistar $\mathcal{M}\left[\mathcal{B}_{n,n}\right]$

The vertex labels are expressed by:

$$\begin{aligned}
\varphi(u) &= 5\\
\varphi(v) &= 4n+9\\
\varphi(w) &= 8n+9\\
\varphi(u_1) &= 1\\
\varphi(v_1) &= 4n+5\\
\varphi(u_1') &= 1\\
\varphi(v_1') &= 4n+5\\
\varphi(u_i) &= 4i+1, \quad \text{for } i = 2,3,\ldots,n
\end{aligned}$$

$$\varphi\left(v_\imath\right) = 4\imath + 4n + 5, \text{ for } i = 2,3,\ldots,n$$

$$\varphi\left(u_i'\right) = 4i + 1, \text{ for } i = 2,3,\ldots,n$$

$$\varphi\left(v_i'\right) = 4i + 4n + 5, \text{ for } i = 2,3,\ldots,n$$

The function $\varphi^* : E\left[\mathcal{M}\left(\mathcal{B}_{n,n}\right)\right] \rightarrow \{1,3,5,\ldots\}$ has perceptibly been induced, such that the proper coloring of G is characterized by the edge set $E\left[\mathcal{M}\left(\mathcal{B}_{n,n}\right)\right]$, and

$$\varphi^*(uv) = \left\lceil \frac{\varphi(u)+\varphi(v)}{2} \right\rceil,$$

i.e., $\varphi^*(uv) \neq \varphi^*(vw)$ for every pair of neighboring edges uv and vw.
In $\mathcal{M}\left(\mathcal{B}_{n,n}\right)$, the least positive odd integer $8n+5$ that represents the lucky-edge odd mean number of G is denoted by $\eta_{OM}'\left(\mathcal{M}\left(\mathcal{B}_{n,n}\right)\right)$:

$$\therefore \eta_{OM}'\left(\mathcal{M}\left(\mathcal{B}_{n,n}\right)\right) = 8n+5$$

Hence $\mathcal{M}\left[\mathcal{B}_{n,n}\right]$ is a lucky-edge odd mean graph.

Theorem 3.5: Total graph of a Bistar graph $\mathcal{T}\left[\mathcal{B}_{n,n}\right]$ is a lucky-edge odd mean graph.
Proof: Let

$$V\left[\mathcal{T}\left(\mathcal{B}_{n,n}\right)\right] = \{u\} \cup \{u_1,u_2,\ldots,u_n\} \cup \{v\} \cup \{v_1,v_2,\ldots,v_n\} \cup \{w\} \cup \{u_1',u_2',\ldots,u_n'\} \cup \{v_1',v_2',\ldots,v_n'\}$$

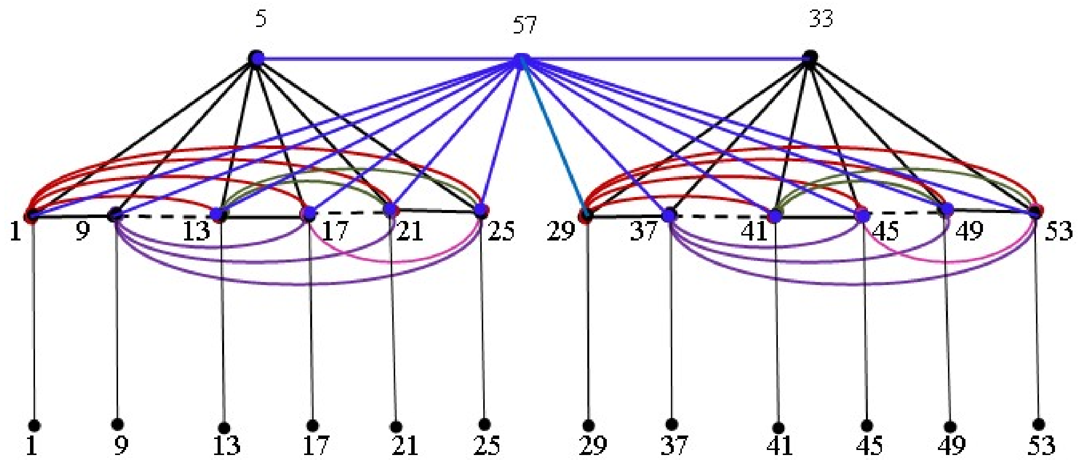

Figure 8: Lucky-edge odd Mean Labeling of $\mathcal{M}\left[\mathcal{B}_{6,\ 6}\right]$.

$$E\left[\mathcal{T}\left(\mathcal{B}_{n,n}\right)\right] = \{wu_1', wu_2', \ldots, wu_n'\} \cup \{wv_1', wv_2', \ldots, wv_n'\} \cup \{uu_1, uu_2, \ldots, uu_n\} \cup \{vv_1, vv_2, \ldots, vv_n\}$$
$$\cup\{uu_1', uu_2', \ldots, uu_n'\} \cup \{vv_1', vv_2', \ldots, vv_n'\} \cup \{u_1u_1', u_2u_2', \ldots, u_nu_n'\} \cup \{v_1v_1', v_2v_2', \ldots, v_nv_n'\}$$
$$\cup\{uw, wv, uv\} \cup \{u_1'u_2', u_1'u_3', \ldots, u_1'u_n', u_2'u_3', \ldots, u_{n-1}'u_n'\}$$
$$\cup\{v_1'v_2', v_1'v_3', \ldots, v_1'v_n', v_2'v_3', \ldots, v_{n-1}'v_n'\}$$

$$\left|V\left[\mathcal{T}\left(\mathcal{B}_{n,n}\right)\right]\right| = 4n+3$$

$$\left|E\left[\mathcal{T}\left(\mathcal{B}_{n,n}\right)\right]\right| = n^2+7n+2$$

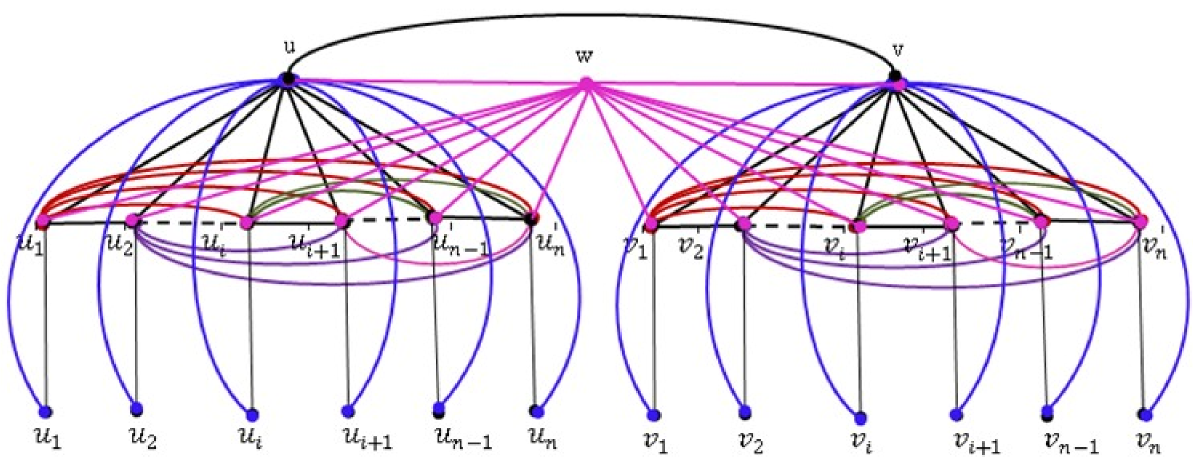

Figure 9: Labeling of Total Graph of Bistar $T\left[B_{n,n}\right]$

Express $\varphi : V\left[\mathcal{T}\left(\mathcal{B}_{n,n}\right)\right] \rightarrow \{1, 3, 5, \ldots\}$

The vertex labels are expressed by:

$$\varphi(u) = 5$$
$$\varphi(v) = 4n + 9$$
$$\varphi(w) = 8n + 9$$
$$\varphi(u'_1) = 1$$
$$\varphi(v'_1) = 4n + 5$$
$$\varphi(u_i) = 8n + 4i + 9, \quad \text{for } i = 1, 2, \ldots, n$$
$$\varphi(v_\imath) = 8n + 4\imath + 9, \quad \text{for } i = 1, 2, \ldots, n$$
$$\varphi(u'_i) = 4i + 1, \quad \text{for } i = 2, 3, \ldots, n$$
$$\varphi(v'_i) = 4i + 4n + 5, \quad \text{for } i = 2, 3, \ldots, n$$

The function $\varphi^* : E[\mathcal{T}(\mathcal{B}_{n,n})] \to \{1, 3, 5, \ldots\}$ has perceptibly been induced, such that the proper coloring of G is characterized by the edge set $E[\mathcal{T}(\mathcal{B}_{n,n})]$, and

$$\varphi^*(uv) = \left\lceil \frac{\varphi(u) + \varphi(v)}{2} \right\rceil$$

That is, $\varphi^*(uv) \neq \varphi^*(vw)$ for every pair of neighboring edges uv and vw.
In $\mathcal{T}(\mathcal{B}_{n,n})$, the least positive odd integer $10n + 7$ that represents the lucky-edge odd mean number of G is denoted by $\eta'_{OM}(\mathcal{T}(\mathcal{B}_{n,n}))$:

$$\therefore \eta'_{OM}(\mathcal{T}(\mathcal{B}_{n,n})) = 10n + 7$$

Hence $\mathcal{T}[\mathcal{B}_{n,n}]$ is a lucky-edge odd mean graph.

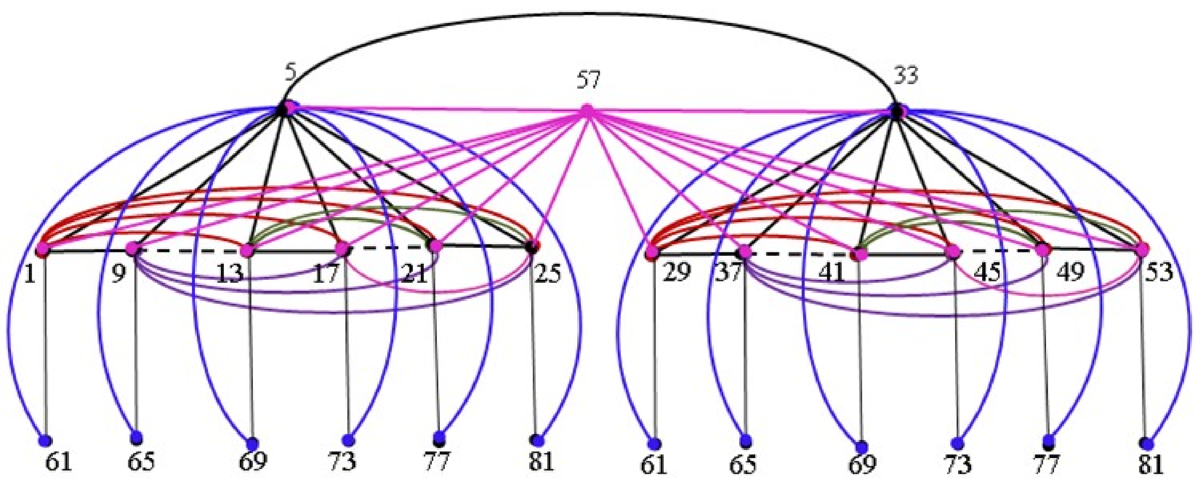

Figure 10: Lucky-edge odd mean Labeling of Total Graph of Bistar $\mathcal{T}[\mathcal{B}_{\mathbf{6,\ 6}}]$.

Theorem 3.6: Central graph of a Bistar graph $\mathcal{C}[\mathcal{B}_{n,n}]$ is a lucky-edge odd mean graph
Proof: Let

$$V(\mathcal{C}(\mathcal{B}_{n,n})) = \{u\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{v\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{w\} \cup \{u'_1, u'_2, \ldots, u'_n\} \cup \{v'_1, v'_2, \ldots, v'_n\}$$

$$E(\mathcal{C}(\mathcal{B}_{n,n})) = \{uu'_1, uu'_2, \ldots, uu'_n\} \cup \{uv_1, uv_2, \ldots, uv_n\} \cup \{vu_1, vu_2, \ldots, vu_n\} \cup \{vv'_1, vv'_2, \ldots, vv'_n\}$$

$$\cup\{u_1u_1', u_2u_2', \ldots, u_nu_n'\} \cup \{v_1v_1', v_2v_2', \ldots, v_nv_n'\} \cup \{uw, wv\}$$

$$\cup\{u_1u_2, u_1u_3, \ldots, u_1u_n, u_2u_3, \ldots, u_2u_n, \ldots, u_{n-1}u_n\}$$

$$\cup\{v_1v_2, v_1v_3, \ldots, v_1v_n, v_2v_3, \ldots, v_2v_n, \ldots, v_{n-1}v_n\}$$

$$\cup\{u_1v_1, u_1v_2, \ldots, u_1v_n, u_2v_1, u_2v_2, \ldots, u_2v_n, \ldots, u_nv_1, u_nv_2, \ldots, u_nv_n\}$$

$$|V(\mathcal{C}(\mathcal{B}_{n,n}))| = 4n+3 \quad \text{and} \quad |E(\mathcal{C}(\mathcal{B}_{n,n}))| = 2n^2+5n+2$$

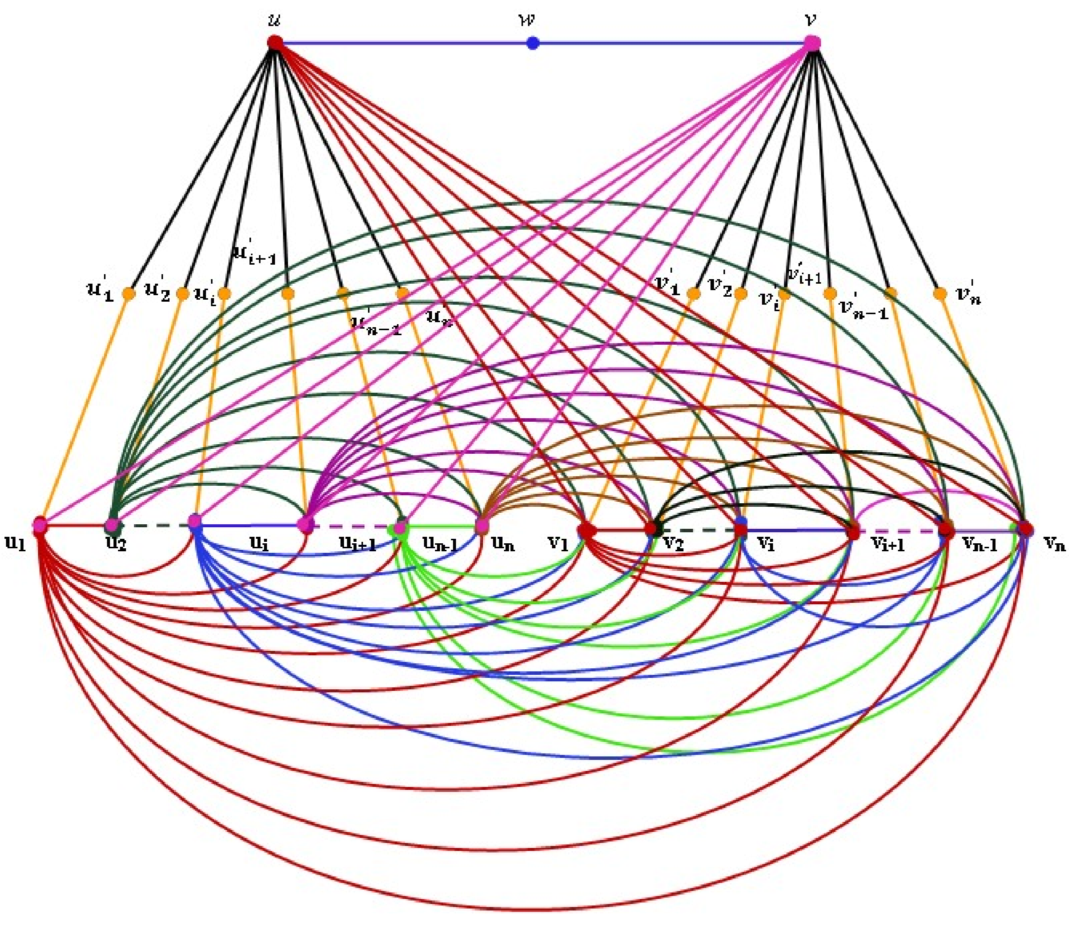

Figure 11: Labeling of Central Graph of Bistar $\mathcal{C}[\mathcal{B}_{n,n}]$.

Express $f : V(\mathcal{C}(\mathcal{B}_{n,n})) \to \{1, 3, 5, \ldots\}$
The Vertex labels are expressed by

$$\varphi(u) = 1$$

$$\varphi(v) = 9$$

$$\varphi(w) = 5$$

$$\varphi(u_\imath) = 4\imath + 9, \quad \text{for } i = 1, 2, \ldots, n$$

$$\varphi(v_\imath) = 4\imath + 4n + 9, \quad \text{for } i = 1, 2, \ldots, n$$

$$\varphi(u'_i) = 4i + 9, \quad \text{for } i = 1, 2, \ldots, n$$

$$\varphi(v'_i) = 4i + 4n + 9, \quad \text{for } i = 1, 2, \ldots, n$$

The function $\varphi^* : E(\mathcal{C}(\mathcal{B}_{n,n})) \to \{1, 3, 5, \ldots\}$ has perceptibly been induced, such that the proper coloring of G is characterized by the edge set $E(\mathcal{C}(\mathcal{B}_{n,n}))$ and defined as:

$$\varphi^*(uv) = \left\lceil \frac{\varphi(u) + \varphi(v)}{2} \right\rceil \quad \text{(i.e., } \varphi^*(uv) \neq \varphi^*(vw) \text{ for every pair of adjacent edges } uv \text{ and } vw).$$

In $\mathcal{C}(\mathcal{B}_{n,n})$, the least positive odd integer $8n + 9$ that represents the lucky-edge odd mean number of G is denoted by:

$$\eta'_{OM}(\mathcal{C}(\mathcal{B}_{n,n})) = 8n + 9$$

Hence, $\mathcal{C}(\mathcal{B}_{n,n})$ is a lucky-edge odd mean graph.

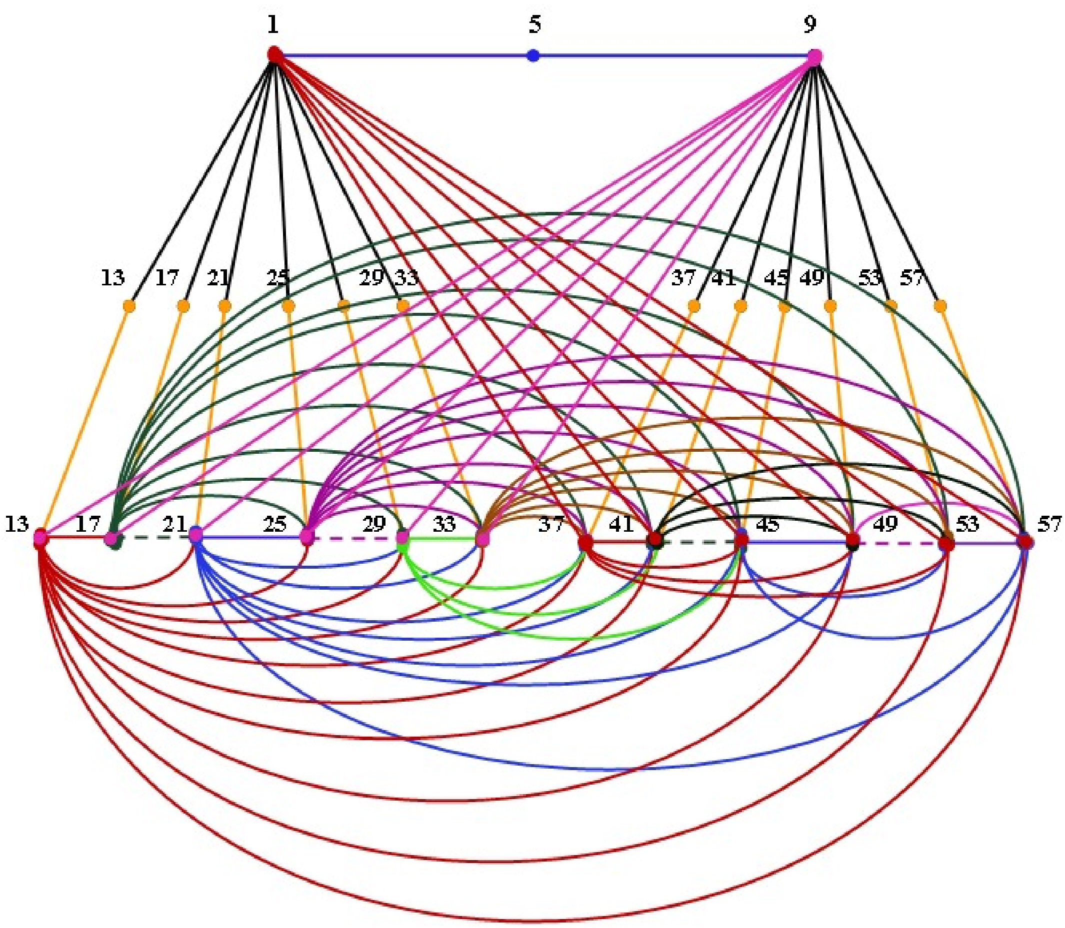

Figure 12: Lucky-edge odd mean Labeling of Central Graph of Bistar $\mathcal{C}[\mathcal{B}_{\mathbf{6,6}}]$.

4. Applications in Circuits

Circuits are indispensable for today's technology and form the backbone of almost all electronic devices and systems. From simple wiring in home appliances to computer networks and communications, circuits are crucial in transmitting and processing electrical signals and power.
A loop consists of many components, each with a specific function. Main products include:

1. Resistors: limit current and divide voltage.
2. Capacitors: Store and release electrical energy, filter signals, and control electrical changes.
3. Inductors: Store energy in a magnetic field and filter signals in power and radio frequency circuits.
4. Diode: allows current to flow in one direction and is used for corrections and adjustments.
5. Transistor: Acts as a switch or noise in an electronic circuit and is essential for logic and signal processing.
6. Power supply: Provides the voltage and current needed to run the circuit, including batteries and power supplies.

Problem 4.1
Examine the value of energy of the graph model representation from the circuit given below, and compare the value of energy with the lucky-edge odd mean number.

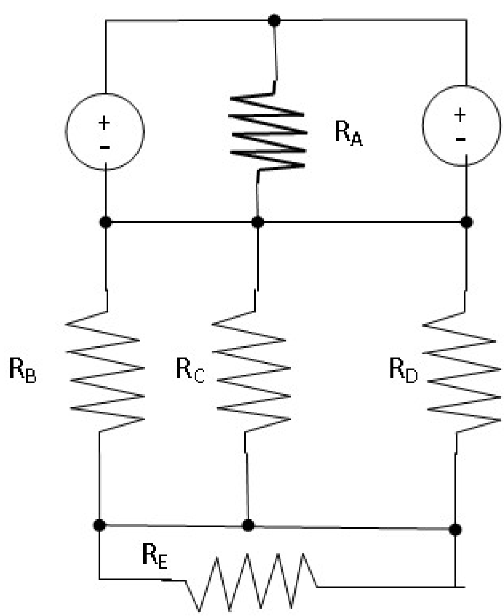

Figure 13:

Solution:
Generating a graph model representation of a given circuit involves representing the components and connections of the circuit as nodes and edges of a graph.
Step 1: Categorize Components and Connections

1. **Components:** These are the elements, like resistors, capacitors, inductors, voltage sources, etc.
2. **Connections:** These are the wires connecting the components.

Step 2: Denote Components as Nodes

1. Each component in the circuit can be represented as a node in the graph.
2. Label the nodes with the type of component and its value.

Step 3: Denote Connections as Edges

1. Each wire connecting two components is represented as an edge between two nodes.
2. Label the edges with the type of connection if necessary.

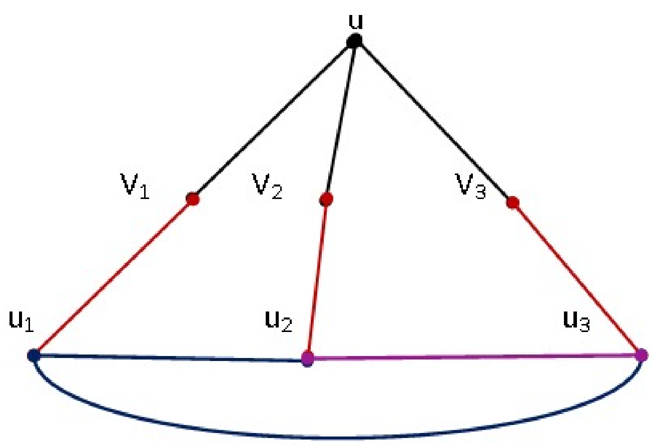

Figure 14: Graph Model representation of the given circuit.

Energy of the graph modal representation of circuit:
Step1: To find the Adjacency of the graph model representation of the circuit

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Step 2: Find the absolute value of the Sum of the eigenvalues of the matrix
Hence, the energy of the graph modal representation of the circuit 13, which is the lucky-edge odd mean number of graph modal representation of the circuit.

From this Fig. 14 we say that it is the central graph of Star $\mathcal{K}_{\mathbf{1,3}}$.

(i.e) The Graph Model representation of the given circuits is $\mathcal{C}(\mathcal{K}_{\mathbf{1,3}})$. By Theorem 3.3, we have
The Vertex labels are expressed by

$$\varphi(u) = 1$$

$$\varphi(u_i) = 4i + 1, \quad 1 \leq i \leq 3$$

Hence $\varphi(u_1) = 5$, $\varphi(u_2) = 9$, $\varphi(u_3) = 13$

$$\varphi(v_i) = 4i + 1, \quad 1 \leq i \leq 3$$

Hence $\varphi(v_1) = 5$, $\varphi(v_2) = 9$, $\varphi(v_3) = 13$

The function $\varphi^* : E(\mathcal{C}(\mathcal{K}_{1,n})) \to \{1, 3, 5, \dots\}$ has perceptibly been induced, such that the proper coloring of G is characterized by the edge set $E(\mathcal{C}(\mathcal{K}_{1,n}))$, and

$$\varphi^*(uv) = \left\lceil \frac{\varphi(u) + \varphi(v)}{2} \right\rceil$$

that is, $\varphi^*(uv) \neq \varphi^*(vw)$ whenever uv and vw are neighboring edges.
In $\mathcal{C}(\mathcal{K}_{1,n})$, the least positive odd integer $4n + 1$ that represents the lucky-edge odd mean number of G is denoted by:

$$\therefore \eta'_{OM}(\mathcal{C}(\mathcal{K}_{1,n})) = 4n + 1$$

(i.e.) The lucky-edge odd mean number of the central graph of the star graph with 3 vertices is:

$$4 \times 3 + 1 = 13$$

5. Discussion and Findings:

The study proposes that LEOMLN could serve as an indirect measure for estimating graph energy, providing a new avenue for theoretical research. The established bounds for LEOML numbers could potentially help predict the energy of similar graphs without direct computation of eigenvalues.

6. Conclusion:

The investigation of Lucky-edge Odd Mean Labelling (LEOML) within star-related graphs presents a substantial development in both theoretical graph theory and practical circuit design. This study has successfully established the existence and applications of LEOML, highlighting its potential to optimize and boost the performance of circuit layouts.
Through rigorous analysis and application, we have recognized the foundational principles of LEOML, detailing its constraints and benefits. The theoretical insights gained provide a robust framework for further research and development in this area, cheering the examination of LEOML in other types of graphs and more complex network structures. In the kingdom of circuit design, the practical implications of LEOML are profound. By applying this innovative labelling technique, we have shown improvements in circuit reliability, efficiency, and signal integrity. The case studies presented illustrate how LEOML can be leveraged to solve real-world engineering challenges, offering a pathway to more strong and efficient circuit designs.
In conclusion, the integration of LEOML into star-related graphs bridges the gap between abstract mathematical concepts and perceptible engineering applications. This research not only enriches the field of graph theory but also provides valuable tools for advancing modern circuit design. The artistic results pave the way for future innovations, underscoring the importance of interdisciplinary approaches in solving complex technical problems. In this research paper, we have computed the lucky-edge odd mean number of the Middle graph of the star, the Total graph of the star, the Central graph of the star, the Middle graph of Bistar, the Total graph of Bistar, and the Central graph of Bistar. In addition to that, we compute the energy of the graph model representation of the circuit using lucky-edge odd mean labeling. In the future, we can compute the lucky-edge odd mean number for Product graphs, Triangular graphs, Petersen graph, etc.

Theoretical implication: The study proposes that LEOML could serve as an indirect measure for estimating graph energy, providing a new avenue for theoretical research. The established bounds for LEOML numbers could potentially help predict the energy of similar graphs without direct computation of eigenvalues.

Practical implication: The practical implications of LEOML extend across a wide range of fields, from network design and circuit analysis to computational methods and infrastructure planning. By applying the insights gained from LEOML, practitioners can optimize systems for energy efficiency, robustness, and performance, leading to more sustainable and effective designs. Moreover, LEOML offers valuable contributions to both theoretical research and practical applications, making it a versatile tool in the study and application of graph theory.

Limitations and future Research: The study focused primarily on star-related graphs, which, while important, represents a limited class of graphs. Extending the analysis to other types of graphs, such as complete or bipartite graphs, could provide a more comprehensive understanding of LEOML.

Author Contributions: The study offers new insights, particularly in the context of star-related graphs, where the LEOML numbers and their relationship to graph energy have not been extensively explored before. These contributions could open up new lines of inquiry in both theoretical and applied graph theory.

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