



## Double Phase Differential Inclusion in Sobolev Space with Variable Exponent

Mohamed Knifda\*, Ahmed Aberqi and Abdesslam Ouaziz

**ABSTRACT:** In this paper, we examine a type of inclusion problem involving the two-phase operator, a discontinuous nonlinearity, and a logarithmic perturbation. Using the variational techniques, we prove the existence of at least one weak solution within the framework of a Sobolev space with variable exponent. Our approach is variational, meaning the solution is identified as a critical point of the corresponding energy functional defined on the appropriate Sobolev space.

**Key Words:** Double phase differential inclusions, logarithmic non-linearity, existence of solutions, locally Lipschitz function.

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### 1. Introduction

Let  $\Omega$  a bounded domain of  $\mathbb{R}^N$ ,  $N \geq 2$  with Lipschitz boundary  $\partial\Omega$ , we are interested in the existence of at least a weak solution of boundary value problems of the following form:

$$\begin{cases} \mathcal{L}_{a(x),V(x)}^{\mathbf{p}(x),\mathbf{q}(x)} \mathbf{u} \in \partial_c^2 g(x, \mathbf{u}) + \lambda b(x) |\mathbf{u}|^{\mathbf{r}(x)-2} \mathbf{u} \log |\mathbf{u}|, & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \quad (1.1)$$

with  $\mathcal{L}_{a(x),V(x)}^{\mathbf{p}(x),\mathbf{q}(x)}$  is a double phase operator with potential given by

$$\mathcal{L}_{a(x),V(x)}^{\mathbf{p}(x),\mathbf{q}(x)} \mathbf{u} = -\operatorname{div}(|\nabla \mathbf{u}|^{\mathbf{p}(x)-2} \nabla \mathbf{u} + a(x) |\nabla \mathbf{u}(x)|^{\mathbf{q}(x)-2} \nabla \mathbf{u}) + V(x) |\mathbf{u}|^{\mathbf{p}(x)-2} \mathbf{u},$$

$a, V : \Omega \rightarrow \mathbb{R}^{+*}$  the measurable functions that satisfy the below conditions:

- ( $h_1$ ) There exists  $a_0 > 0$  such that  $a(x) > a_0 = \min_{x \in \Omega} a(x)$  and  $a \in L^1(\Omega)$ .
- ( $h_2$ ) The function  $V$  is continuous and  $V \in \mathcal{L}^{\mathbf{p}'(x)}(\Omega)$  with  $\mathcal{L}^{\mathbf{p}'(x)}(\Omega)$  is the Lebesgue space with variable exponent.
- ( $h_3$ ) The function  $b : \Omega \rightarrow (1, \infty)$  is continuous and  $b \in L^1(\Omega)$ .
- ( $h_4$ ) The functions  $\mathbf{r}, \mathbf{p}, \mathbf{q} : \Omega \rightarrow (1, \infty)$  are continuous that satisfy the below inequality:

$$1 < \mathbf{r}^- \leq \mathbf{r}(x) \leq \mathbf{r}^+ < \mathbf{p}^- \leq \mathbf{p}(x) \leq \mathbf{p}^+ < \mathbf{q}^- \leq \mathbf{q}(x) \leq \mathbf{q}^+ < \infty.$$

Moreover,  $\partial_c^2 g(x, t)$  the Clarke sub-differential of the  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  verifies the following conditions:

\* Corresponding author.

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(g<sub>1</sub>)  $t \rightarrow g(x, t)$  is measurable  $\Omega$ , for all  $t \in \mathbb{R}$ ,

(g<sub>2</sub>)  $g(x, t) \in \mathcal{L}_f(\mathbb{R})$  (set of locally Lipschitz functions; see [definition 2.2](#)) for a.e  $x \in \Omega$ ,

(g<sub>3</sub>)  $g(x, 0) = 0$  for a.e  $x \in \Omega$ ,

(g<sub>4</sub>) there exists a continuous function  $\mathfrak{s}(x) \in (1, p^*)$ , for a.e  $x \in \Omega, k > 0$  and  $\mu \in \mathcal{L}^{\frac{\mathfrak{s}(x)}{\mathfrak{s}(x)-1}}(\Omega)$  such that,

$$|\xi| \leq \mu(x) + k|t|^{\mathfrak{s}(x)-1}, \text{ for all } t \in \mathbb{R} \text{ and for all } \xi \in \partial_c^2 g(x, t),$$

where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } N > p, \\ \infty & \text{if } N \leq p. \end{cases}$$

We consider the model double phase equation:

$$\begin{cases} \mathcal{L}_{a(x), V(x)}^{p, q} \mathbf{u} = f(x, \mathbf{u}), & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \quad (3)$$

With  $\mathcal{L}^{p, q}$  is the double phase operator, and  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function that satisfying certain growth conditions at the origin and infinity. Usually, such a problem is treated on the Sobolev space, and the weak solution of (3) become a classical solution whenever  $f$  has further regularity. The problem (3) is said double phase type because of the presence of two different elliptic growths,  $p$  and  $q$ . The study of double-phase problems and related functional originates from the seminal paper by Zhikov [39] where he introduced for the first time in literature the related energy functional to (1.1) defined by

$$\mathbf{u} \mapsto \int_{\mathcal{U}} (|\nabla \mathbf{u}|^p + a(x)|\nabla \mathbf{u}|^q) dx. \quad (1.2)$$

This kind of functional has been used to describe models for strongly anisotropic materials in the context of homogenization and elasticity. Certainly, the geometry of composites consisting of two different materials with varying power-hardening exponents  $p$  and  $q$  is determined by the weight coefficient  $a(\cdot)$ . The functional (1.2) is a mathematical prototype of a functional whose integrands alter their ellipticity in accordance with the locations where it  $a(\cdot)$  vanishes or does not. In this direction, the functional (1.2) has several mathematical applications in the study of duality theory and Lavrentiev gap phenomenon, see [31, 32, 39, 41, 42, 43] for more details. On the other hand, Mingione et al. provide famous results in the regularity theory of local minimizers of functional (1.2), see, for example, [11, 12, 13] for more details. However, the interest in extending the theory of nonlinear partial differential equations to PDE with discontinuous nonlinearities has been increasing daily. The reasons are twofold:

- Many free boundary problems may be reduced to boundary value problems of discontinuous nonlinearities differential equations, see for example [16, 17, 18, 26].
- Sometimes, in dealing with a PDE problem, it is beneficial to put the original PDE into a large category, see for instance [17].
- Their application in fields of mechanics, where disturbances behave in a discontinuous manner, such as irregular external forces, see the gluing force in [14, 33].

For the reasons mentioned above, this type of problem has recently attracted many researchers in the field. As an example, we cite the contribution by Alexandru et al [30] in which they examine the solvability of a class of problems with non-continuous perturbations, of the form:

$$\begin{cases} \Delta \mathbf{u}(x) \in \partial F(\mathbf{u}(x)) + \lambda \partial G(\mathbf{u}(x)), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \quad (3)$$

where  $G$  and  $F$  are both nonsmooth, locally Lipschitz functions having various growths, while  $\partial G$  and  $\partial F$  stand for the generalized gradients of  $G$  and  $F$ , respectively.

For a non-homogeneous differential inclusion involving the  $p(x)$ -Laplacian, a multiplicity property for the solutions was given by Bin Ge and Vicențiu D. Rădulescu [29] for the problem:

$$-\Delta_{p(x)}u + v(x)|u|^{p(x)-2}u \in a(x)\partial F(x, u) \quad \in \mathbb{R}^N.$$

The main purpose of the present work is to extend the main results of [8,6,3,7] when the perturbations are smooth polynomial or logarithmic forms. Our main theorem is stated as follows:

**Theorem 1.1** *Given that assumption  $(h_1) - (h_4)$ ,  $(g_1)$ ,  $(g_3)$ , and  $(g_4)$  holding, for any,  $\mathfrak{s}(x) \in (q^-, p^*)$ , then there is at least one nontrivial weak solution to the problem (1.1).*

## 2. Preliminaries

### 2.1. The Sobolev-Orlicz space with variable exponents

In this subsection we recall some definitions and basic properties of the space  $W^{1,q(\cdot)}(\Omega)$ , For more details and connections, see, [22,10,28,1,7,4].

We notice  $\mathcal{L}^0(\Omega) = \{v : \Omega \rightarrow (1, \infty), v \text{ measurable}\}$ , and  $C_+(\bar{\Omega}) = \{\theta \in C(\bar{\Omega}) : \min_{x \in \Omega} \theta(x) > 1\}$ .

**Definition 2.1** ((see [22]) Let  $q \in C_+(\bar{\Omega})$ , we define the the Lebesgue space with variable exponent  $\mathcal{L}^{q(x)}(\Omega)$  as follows:

$$\mathcal{L}^{q(x)}(\Omega) = \left\{ u \in \mathcal{L}^0(\Omega) : \int_{\Omega} |u|^{q(x)} dx < \infty \right\},$$

of its corresponding Luxembourg norm

$$\|u\|_{q(\cdot)} = \inf \left\{ \gamma > 0 : \int_{\Omega} \left( \frac{|u|}{\gamma} \right)^{q(x)} dx \leq 1 \right\}.$$

The space  $(\mathcal{L}^{q(x)}(\Omega), \|u\|_{q(\cdot)})$  is a separable Banach space. Moreover, if  $1 < q^- \leq q(x) \leq q^+ < \infty$  it  $\mathcal{L}^{q(x)}(\Omega)$  is uniformly convex, then it is reflexive.

Note that  $\varrho_{q(\cdot)}(u) = \int_{\Omega} |u|^{q(x)} dx$  is the modulus associated with the space  $\mathcal{L}^{q(x)}(\Omega)$ .

Now, we define the variable exponent Sobolev space by:

$$W^{1,q(\cdot)} = \left\{ u \in \mathcal{L}^{q(x)}(\Omega) \text{ and } |\nabla u| \in \mathcal{L}^{q(x)}(\Omega) \right\},$$

provided with the following norm:

$$\|u\|_{1,q(\cdot)} = \|u\|_{q(\cdot)} + \|\nabla u\|_{q(\cdot)}.$$

The space  $(W^{1,q(\cdot)}, \|u\|_{1,q(\cdot)})$  is sa separable and reflexive Banach space.

And let,  $W_0^{1,q(x)}(\Omega) = \overline{C^\infty(\Omega)}^{W^{1,q(x)}(\Omega)}$ .

**Proposition 2.1** (See [10]) Let  $q \in C_+(\bar{\Omega})$ ,  $u \in \mathcal{L}^{q(x)}(\Omega)$ . Then:

(i) If  $\|u\|_{q(\cdot)} > 1$  (resp.  $< 1, = 1$ )  $\iff \varrho_{q(\cdot)}(u) > 1$  (resp.  $< 1, = 1$ )

(ii) If  $\|u\|_{q(\cdot)} < 1 \implies \|u\|_{q(\cdot)}^{q^+} < \varrho_{q(\cdot)}(u) < \|u\|_{q(\cdot)}^{q^-}$

(iii) If  $\|u\|_{q(\cdot)} > 1 \implies \|u\|_{q(\cdot)}^{q^-} < \varrho_{q(\cdot)}(u) < \|u\|_{q(\cdot)}^{q^+}$

(iv) If  $\|u\|_{q(\cdot)} \rightarrow 0 \iff \varrho_{q(\cdot)}(u) \rightarrow 0$

(v) If  $\|u\|_{q(\cdot)} \rightarrow \infty \iff \varrho_{q(\cdot)}(u) \rightarrow \infty$

**Proposition 2.2** (See [28]) Let  $\mathbf{u} \in \mathcal{L}^{\mathbf{q}(\cdot)}(\Omega)$ ,  $\mathbf{u}' \in \mathcal{L}^{\mathbf{q}'(\cdot)}(\Omega)$ . Then, we have:

$$\int_{\Omega} |\mathbf{u}\mathbf{u}'| dx \leq \left( \frac{\mathbf{q}^- + \mathbf{q}'^-}{\mathbf{q}^- \cdot \mathbf{q}'^-} \right) \|\mathbf{u}\|_{\mathbf{q}(\cdot)} \|\mathbf{u}'\|_{\mathbf{q}'(\cdot)},$$

where,  $\frac{\mathbf{q}(x) + \mathbf{q}'(x)}{\mathbf{q}(x) \cdot \mathbf{q}'(x)} = 1$  and  $\mathbf{q}^- = \min_{x \in \Omega} \mathbf{q}(x)$ .

**Proposition 2.3** (See [28]) Let  $\mathbf{q} \in C_+(\bar{\Omega})$ , then there exists  $C_0 > 0$  such that:

$$\|\mathbf{u}\|_{\mathbf{q}(\cdot)} \leq C_0 \|\nabla \mathbf{u}\|_{\mathbf{q}(\cdot)},$$

thus, we give the equivalent norm on  $W_0^{1, \mathbf{q}(\cdot)}(\Omega)$

$$\|\mathbf{u}\|_{1, \mathbf{q}(\cdot), 0} = \|\nabla \mathbf{u}\|_{\mathbf{q}(\cdot)}.$$

**Theorem 2.1** (See [1]) Let it be a bounded domain,  $\Omega \subseteq \mathbb{R}^N$  and  $\mathbf{r}(x), \mathbf{p}(x), \mathbf{q}(x) \in C(\bar{\Omega}) \cap \mathcal{L}^\infty(\Omega)$ . Assume that

$$\mathbf{q}(x) < N, \quad \mathbf{r}(x) < \mathbf{p}(x) < \frac{N\mathbf{q}(x)}{N - \mathbf{q}(x)}.$$

Then

$$W^{1, \mathbf{q}(x)}(\Omega) \hookrightarrow \mathcal{L}^{\mathbf{p}(x)}(\Omega), \quad \text{and} \quad W^{1, \mathbf{q}(x)}(\Omega) \hookrightarrow \mathcal{L}^{\mathbf{r}(x)}(\Omega),$$

is a continuous and compact embedding.

The weighted Lebesgue space  $\mathcal{L}_{a(x)}^{\mathbf{q}(x)}(\Omega)$  is defined in the following manner:

$$\mathcal{L}_{a(x)}^{\mathbf{q}(x)}(\Omega) = \left\{ \mathbf{u} \in \mathcal{L}^0(\Omega) : \int_{\Omega} a(x) |\mathbf{u}|^{\mathbf{q}(x)} dx < +\infty \right\},$$

with the norm

$$\|\mathbf{u}\|_{a(x), \mathbf{q}(\cdot)} = \inf \left\{ \gamma > 0 : \int_{\Omega} a(x) \left( \frac{|\mathbf{u}|}{\gamma} \right)^{\mathbf{q}(x)} dx \leq 1 \right\}.$$

Moreover, we define the modular  $\varrho_{\mathbf{q}(\cdot), a(\cdot)} : \mathcal{L}_{a(x)}^{\mathbf{q}(x)}(\Omega) \rightarrow \mathbb{R}$  as follows:

$$\varrho_{\mathbf{q}, a(x)}(\mathbf{u}) = \int_{\Omega} a(x) |\mathbf{u}|^{\mathbf{q}(x)} dx.$$

**Proposition 2.4** (See [1]) Let  $\mathbf{u}$  and  $\{\mathbf{u}_n\} \subset \mathcal{L}_{a(x)}^{\mathbf{q}(x)}(\Omega)$ , then we have following results:

- (i) If  $\|\mathbf{u}\|_{\mathbf{q}(\cdot), a(\cdot)} < 1$  (resp.  $> 1, = 1$ )  $\iff \varrho_{\mathbf{q}(\cdot), a(\cdot)}(\mathbf{u}) < 1$  (resp.  $> 1, = 1$ )
- (ii) If  $\|\mathbf{u}\|_{\mathbf{q}(\cdot), a(\cdot)} < 1 \implies \|\mathbf{u}\|_{\mathbf{q}(\cdot)}^{\mathbf{q}^+} < \varrho_{\mathbf{q}(\cdot)}(\mathbf{u}) < \|\mathbf{u}\|_{\mathbf{q}(\cdot)}^{\mathbf{q}^-}$
- (iii) If  $\|\mathbf{u}\|_{\mathbf{q}(\cdot), a(\cdot)} > 1 \implies \|\mathbf{u}\|_{\mathbf{q}(\cdot), a(\cdot)}^{\mathbf{q}^-} < \varrho_{\mathbf{q}(\cdot), a(\cdot)}(\mathbf{u}) < \|\mathbf{u}\|_{\mathbf{q}(\cdot), a(\cdot)}^{\mathbf{q}^+}$ ,
- (iv) If  $\|\mathbf{u}_n\|_{\mathbf{q}(\cdot), a(\cdot)} \rightarrow 0 \iff \varrho_{\mathbf{q}(\cdot), a(\cdot)}(\mathbf{u}_n) \rightarrow 0$ , as  $n \rightarrow +\infty$ ,
- (v) If  $\|\mathbf{u}_n\|_{\mathbf{q}(\cdot), a(\cdot)} \rightarrow \infty \iff \varrho_{\mathbf{q}(\cdot), a(\cdot)}(\mathbf{u}_n) \rightarrow \infty$ , as  $n \rightarrow +\infty$ .

**Theorem 2.2** (See [2]) Let  $\Omega \subset \mathbb{R}^N$  be bounded domain,  $\mathbf{p}(x), \mathbf{q}(x) \in C(\bar{\Omega}) \cap \mathcal{L}^\infty(\Omega)$ . Assume that

$$\mathbf{q}(x) < N, \quad \text{and} \quad \mathbf{p}(x) < \frac{N\mathbf{q}(x)}{N - \mathbf{q}(x)}.$$

Then

$$W^{1, \mathbf{q}(x)}(\Omega) \hookrightarrow \mathcal{L}_{a(x)}^{\mathbf{p}(x)}(\Omega),$$

is compact.

## 2.2. Generalized Gradient

**Definition 2.2** (See [21]) Let  $\mathbf{E}$  be Banach space. We say that the functional  $\psi : \mathbf{E} \rightarrow \mathbb{R}$  is locally Lipschitz if, for all,  $\mathbf{u} \in \mathbf{E}$ , there exists a neighborhood  $V$  of  $\mathbf{u}$  and a constant  $L_V > 0$ , that depends on the neighborhood  $V$ , such that:

$$|\psi(\mathbf{u}) - \psi(\mathbf{v})| \leq L_V \|\mathbf{u} - \mathbf{v}\|,$$

and we noted that  $\mathcal{L}_f(\mathbf{E})$  is the set of locally Lipschitz functions.

**Definition 2.3** (See [21]) Let  $\psi \in \mathcal{L}_f(\mathbf{E})$ , we note  $\psi^0(\mathbf{u}, \mathbf{v})$  the generalized directional derivative of  $\psi$  at  $\mathbf{u} \in \mathbf{E}$  in the direction  $\mathbf{u} \in \mathbf{E}$ , is defined by,

$$\psi^0(\mathbf{u}, \mathbf{v}) = \limsup_{\mathbf{w} \rightarrow \mathbf{u}, t \searrow 0} \frac{\psi(\mathbf{w} + t\mathbf{v}) - \psi(\mathbf{w})}{t}.$$

The following result highlights several key properties of generalized directional derivatives that will be utilized later. For a detailed proof, refer to Clarke [21].

**Proposition 2.5** (See [21]) Let  $\psi, \varphi \in \mathcal{L}_f(\mathbf{E})$ , Then we obtain:

(i) for each fixed  $\mathbf{u} \in \mathbf{E}$ , the function:

$$\begin{aligned} \psi_{\mathbf{u}}^0 : \mathbf{E} &\longrightarrow \mathbb{R} \\ \mathbf{v} &\longrightarrow \psi_{\mathbf{u}}^0(\mathbf{v}) = \psi^0(\mathbf{u}, \mathbf{v}), \end{aligned}$$

is finite, sub-additive and satisfies:

$$|\psi^0(\mathbf{u}, \mathbf{v})| \leq \tau \|\mathbf{v}\|,$$

where  $\tau$  is the Lipschitz constant near the point  $\mathbf{u}$ ,

(ii)  $\psi_{\mathbf{u}}^0$  is upper semicontinuous,

(iii)  $\psi^0(\mathbf{u}, -\mathbf{v}) = (-\psi)^0(\mathbf{u}, \mathbf{v})$ ,  $\forall (\mathbf{u}, \mathbf{v}) \in \mathbf{E}^2$ ,

(iv)  $\psi^0(\mathbf{u}, \alpha\mathbf{v}) = \alpha\psi^0(\mathbf{u}, \mathbf{v})$ ,  $\forall (\mathbf{u}, \mathbf{v}) \in \mathbf{E}^2$  and  $\alpha > 0$ ,

(v)  $(\psi + \varphi)^0(\mathbf{u}, \mathbf{v}) \leq \psi^0(\mathbf{u}, \mathbf{v}) + \varphi^0(\mathbf{u}, \mathbf{v})$ ,  $\mathbf{u}, \mathbf{v} \in \mathbf{E}^2$ .

**Definition 2.4** (See [24]) Let  $\psi \in \mathcal{L}_f(\mathbf{E})$ , on denote  $\partial_c \psi$  is Clarke sub-differential of  $\psi$  at a point  $\mathbf{u} \in \mathbf{E}$  defined by:

$$\partial_c \psi(\mathbf{u}) = \{ \xi \in \mathbf{E}^* : \psi^0(\mathbf{u}, \mathbf{v}) \geq \langle \xi, \mathbf{v} \rangle, \forall \mathbf{v} \in \mathbf{E} \}.$$

We note that the Clarke sub-differential  $\partial_c \psi(\mathbf{u})$  coincides with the sub-differential of  $\psi$  in the sense of Convex Analysis, if  $\psi$  is convex. Additionally, if  $\psi \in C^1(\mathbf{E}, \mathbb{R})$ , then  $\partial_c \psi(\mathbf{u}) = \{ \psi'(\mathbf{u}) \}$ .

**Theorem 2.3** (See [24]) Let  $\mathbf{F}$  be an open subset of a Banach space  $\mathbf{E}$ , and  $\mathbf{u}, \mathbf{v}$  be two points of  $\mathbf{F}$  such that the line segment:

$$[\mathbf{u}, \mathbf{v}] = \{ \varepsilon\mathbf{v} + (1 - \varepsilon)\mathbf{u} : 0 \leq \varepsilon \leq 1 \} \subset \mathbf{F}.$$

If  $\psi \in \mathcal{L}_f(\mathbf{F})$ , then there exist  $\varepsilon \in (0, 1)$  and  $\xi \in \partial_c \psi(\mathbf{u} + \varepsilon(\mathbf{v} - \mathbf{u}))$  such that:

$$\psi(\mathbf{v}) - \psi(\mathbf{u}) = \langle \xi, \mathbf{v} - \mathbf{u} \rangle.$$

**Definition 2.5** (See [24]) Let  $\psi \in \mathcal{L}_f(\mathbf{E})$ , we say that  $\mathbf{u} \in \mathbf{E}$  is a critical point for  $\psi$  if  $0 \in \partial_c \psi(\mathbf{u})$ .

**Remark 2.1**  $\mathbf{u} \in \mathbf{E}$  is critical for  $\psi$  if and only if  $\psi^0(\mathbf{u}, \mathbf{v}) \geq 0 \forall \mathbf{v} \in \mathbf{E}$ . Additionally, any local extrema of  $\psi$  is in fact a critical point.

**Theorem 2.4** (See [24]) Let  $\psi : Z \rightarrow (-\infty, +\infty]$  be a proper, bounded from below functional and lower semicontinuous, with  $(Z, d)$  be a complete metric space. Then for any  $\varrho, \beta > 0$  and any  $\mathfrak{w} \in Z$  satisfying  $\psi(w) \leq \inf_Z \psi + \varrho$ , there exists  $\mathfrak{u} \in Z$  such that:

- (i)  $\psi(\mathfrak{u}) \leq \psi(\mathfrak{w})$ ,
- (ii)  $d(\mathfrak{w}, \mathfrak{u}) \leq \frac{1}{\beta}$ ,
- (iii)  $\varrho \beta d(\mathfrak{u}, \mathfrak{v}) \geq \psi(\mathfrak{u}) - \psi(\mathfrak{v})$ ,  $\forall \mathfrak{v} \in Z$ .

### 3. Proof the Main Results

In this section, we assume that  $W = W_0^{1, q(x)}(\Omega) \setminus \{0\}$  with  $\|\mathfrak{u}\|_W = \|\nabla \mathfrak{u}\|_{q(\cdot)}$ , and establish a result regarding the weak solutions of problem (1.1).

**Definition 3.1** We say that  $u \in W$  is a weak solution of (1.1), if it satisfies the following inequality:

$$\begin{aligned} & \int_{\Omega} \left( |\nabla \mathfrak{u}|^{p(x)-2} \nabla \mathfrak{u} + a(x) |\nabla \mathfrak{u}|^{q(x)-2} \nabla \mathfrak{u} \right) \cdot \nabla w \, dx + \int_{\Omega} V(x) |u|^{p(x)-2} u \cdot w \, dx \\ & \leq \int_{\Omega} g^0(x, u, w) \, dx + \lambda \int_{\Omega} b(x) |u|^{\tau(x)-2} \mathfrak{u} \log(|u|) \cdot w \, dx, \end{aligned}$$

for all  $w \in W$ , and  $g^0(x, u, w)$  denotes the generalized directional derivative of  $g$ .

The first existence result concerns cases where the exponent governing the growth of  $\partial Cg(x, \cdot)$  is sufficiently small. The proof hinges on the fact that in this scenario, the associated energy functional exhibits coercivity. Specifically, we obtain the following existence result.

**Theorem 3.1** Given that assumptions  $(h_1) - (h_4)$  and  $(g_1) - (g_2)$  hold, for any  $\mathfrak{s}(x) \in (1, p^+)$ , then there is at least one nontrivial weak solution to the problem (1.1).

If  $\mathfrak{s}(x) \in (q^-, p^*)$ , then the energy functional associated with the problem (1.1) is no longer coercive. In this case we utilize Ekeland's variational principle to establish the existence of at least one weak solution. This is achieved by substituting  $(g_2)$  with the slightly more stringent condition  $(g_3)$  and additionally assuming the listed condition  $(g_4)$ , listed below:

$(g_3)$  There exist  $\mathfrak{s}(x) \in (1, p^*)$  and  $k > 0$  such that:

$$|\xi| \leq k |t|^{\mathfrak{s}(x)-1}, \text{ for a.a. } x \in \Omega \text{ all } t \in \mathbb{R} \text{ and all } \xi \in \partial_c^2 g(x, t),$$

$(g_4)$  There exist  $U$  a nonempty open subset of  $\Omega$ , and  $\delta, K > 0, \alpha \in (1, p)$  such that

$$g(x, t) \geq K t^\alpha, \text{ whenever } (x, t) \in U \times (0, \delta].$$

Define the functional  $A : W \rightarrow \mathbb{R}$ ,  $B_\lambda : L^{\tau(x)}(\Omega) \rightarrow \mathbb{R}$  and  $G : L^{\mathfrak{s}(x)}(\Omega) \rightarrow \mathbb{R}$  by

$$A(\mathfrak{u}) = \int_{\Omega} \left( \frac{1}{p(x)} |\nabla \mathfrak{u}(x)|^{p(x)} + \frac{a(x)}{q(x)} |\nabla \mathfrak{u}(x)|^{q(x)} \right) dx + \int_{\Omega} \frac{V(x)}{p(x)} |u|^{p(x)} dx,$$

for all  $u \in W$ ,

$$B_\lambda(\mathfrak{u}) = -\lambda \int_{\Omega} \frac{b(x)}{r^2(x)} |u|^{\tau(x)} dx + \lambda \int_{\Omega} \frac{b(x)}{\tau(x)} |u|^{\tau(x)} \log(|u|) dx$$

and

$$G(\mathfrak{u}) = \int_{\Omega} g(x, \mathfrak{u}) dx, \text{ for all } u \in L^{\mathfrak{s}(x)}(\Omega),$$

respectively.

Then,  $A \in C^1(W, \mathbb{R})$  with its derivative given by

$$\langle A'(\mathfrak{u}), w \rangle = \int_{\Omega} \left( |\nabla \mathfrak{u}|^{p(x)-2} \nabla \mathfrak{u} + a(x) |\nabla \mathfrak{u}|^{q(x)-2} \nabla \mathfrak{u} \right) \cdot \nabla w \, dx + \int_{\Omega} V(x) |u|^{p(x)-2} u \cdot w \, dx. \quad (3.1)$$

Due to Aubin-Clarke's theorem (see, e.g., [[21], Theorem 2.7.5])  $G$  is locally Lipschitz and

$$\partial_C G(x, \mathbf{u}) \subseteq \int_{\Omega} \partial_C^2 g(x, \mathbf{u}) dx, \text{ for all } \mathbf{u} \in \mathcal{L}^{\mathfrak{s}}(\Omega),$$

in the sense that for any  $\psi \in \partial_C G(x, \mathbf{u})$  there exists  $\xi \in L^{\frac{\mathfrak{s}(x)}{\mathfrak{s}(x)-1}}(\Omega)$  such that:

$$\begin{cases} \langle \psi, \varphi \rangle = \int_{\Omega} \xi(x) \varphi(x) dx, & \text{for all } \varphi \in L^{\mathfrak{s}(x)}(\Omega), \\ \xi(x) \in \partial_C^2 g(x, \mathbf{u}), & \text{for a.e. } x \in \Omega. \end{cases} \quad (3.2)$$

Moreover, by the theorem 2.1 the embedding operator  $\chi : W \rightarrow \mathcal{L}^{\mathfrak{s}(x)}(\Omega)$  is compact,  $\forall \mathfrak{s}(x) \in (1, p^*)$ , and its adjoint operator,  $\chi^* : \mathcal{L}^{\frac{\mathfrak{s}(x)}{\mathfrak{s}(x)-1}}(\Omega) \rightarrow W^*$  is also compact. Consequently, we define the energy functional  $\mathcal{J}_{\lambda} : W \rightarrow \mathbb{R}$  associated with problem (1.1) as follows:

$$\mathcal{J}_{\lambda}(\mathbf{u}) = A(\mathbf{u}) - G(\chi(\mathbf{u})) - B_{\lambda}(\mathbf{u}),$$

is well defined, locally Lipschitz and weakly lower semi-continuous.

Moreover,

$$\partial_c \mathcal{J}_{\lambda}(\mathbf{u}) \subseteq A'(\mathbf{u}) - \chi^* \partial_c G(\chi(\mathbf{u})) - B'_{\lambda}(\mathbf{u}), \text{ for all } \mathbf{u} \in W.$$

Hence forth, for any  $u \in W$  and any  $\xi \in L^{\frac{\mathfrak{s}(x)}{\mathfrak{s}(x)-1}}(\Omega)$  we simply write  $\mathbf{u}$  and  $\xi$  instead of  $\chi(\mathbf{u})$  and  $x^*(\xi)$ , respectively.

**Lemma 3.1** *If  $\mathfrak{s}(x) \in (1, p^*)$ , then any critical point of  $\mathcal{J}_{\lambda}$  is a weak solution for problem (1.1).*

**Proof:**

Let  $u \in W$  be a critical point of  $\mathcal{J}_{\lambda}$ . Then  $0 \in \partial_c \mathcal{J}_{\lambda}(u)$ , i.e.,

$$A'(u) \in \left( \partial_c G(u) + B'_{\lambda}(u) \right).$$

By (3.2), there exists  $\xi \in L^{\frac{\mathfrak{s}(x)}{\mathfrak{s}(x)-1}}(\Omega)$  such that  $\xi \in \partial_c^2 g(x, \mathbf{u})$  and

$$\left\langle A'(u), w \right\rangle = \int_{\Omega} \xi w dx + \left\langle B'(u), w \right\rangle, \text{ for } w \in W.$$

By the definition 2.4 and (3.1) we obtain that:

$$\begin{aligned} & \int_{\Omega} \left( |\nabla \mathbf{u}|^{p(x)-2} \nabla \mathbf{u} + a(x) |\nabla \mathbf{u}|^{q(x)-2} \nabla \mathbf{u} \right) \cdot \nabla w dx + \int_{\Omega} V(x) |u|^{p(x)-2} u \cdot w dx \\ & \leq \int_{\Omega} g^0(x, u, w) dx + \int_{\Omega} b(x) |u|^{r(x)-2} u \log(|u|) \cdot w dx, \end{aligned}$$

for all  $w \in W$ , hence the result. □

**Proof of Theorem 3.1.** Let  $u \in W$  such that  $\|u\| > 1$  by (3.1) we have:

$$\begin{aligned} \mathcal{J}_{\lambda}(u) &= A(u) - G(u) - B_{\lambda}(u) \\ &\geq \frac{1}{q^+} \left( \int_{\Omega} |\nabla u(x)|^{p(x)} + a(x) |\nabla u(x)|^{q(x)} dx + v_0 \int_{\Omega} |u|^{p(x)} dx \right) \\ &\quad - \int_{\Omega} g(x, u) dx + \lambda \int_{\Omega} \frac{b(x)}{r^2(x)} |u|^{r(x)} dx - \lambda \int_{\Omega} \frac{b(x)}{r(x)} |u|^{r(x)} \log(|u|) dx, \end{aligned}$$

and we know that

$$\log(w) \leq \frac{w^{\sigma(x)}}{\sigma(x).e} \text{ for all } \sigma(x) > 0, \text{ and } w > 0, \quad (3.3)$$

By  $(g_2)$  and the compact embedding  $W \hookrightarrow L^{s(x)}(\Omega)$  we get,

$$\begin{aligned} |G(\mathbf{u})| &= \left| \int_{\Omega} g(x, \mathbf{u}) dx \right| \leq \int_{\Omega} |g(x, \mathbf{u}) - g(x, 0)| dx \leq \int_{\Omega} |\xi| |\mathbf{u}| dx \\ &\leq \int_{\Omega} \left( \mathbf{u} + k|\mathbf{u}|^{s(x)-1} \right) \cdot |\mathbf{u}| dx \leq \|\mu\|_{\frac{s(x)}{s(x)-1}} \|\mathbf{u}\|_{s(x)} + \|\mathbf{u}\|_{s(x)}^{s^+} \\ &\leq C_0 \|\mu\|_W \cdot \|\mathbf{u}\|_W + C_1 \|\mathbf{u}\|_W^{s^+}, \end{aligned} \quad (3.4)$$

for some suitable constants  $C_0, C_1 > 0$ . According to the Poincaré inequality, (3.3) and (3.4), there exists a positive constant  $C_q > 0$

$$\begin{aligned} \mathcal{J}_{\lambda}(\mathbf{u}) &\geq \frac{a_0}{q^+} \|\mathbf{u}\|_W^{q^-} - C_0 \|\mu\|_W \cdot \|\mathbf{u}\|_W - C_1 \|\mathbf{u}\|_W^{s^+} - \lambda \frac{\|b\|_{\infty}}{r^-(q^- - r^-).e} C_q \|\mathbf{u}\|_W^{q^-} \\ &\geq \left( \frac{a_0}{q^+} - \lambda \frac{\|b\|_{\infty}}{r^-(q^- - r^-).e} C_q \right) \|\mathbf{u}\|_W^{q^-} - C_0 \|\mu\|_W \cdot \|\mathbf{u}\|_W - C_1 \|\mathbf{u}\|_W^{s^+} \longrightarrow \infty, \end{aligned}$$

as  $\|\mathbf{u}\| \longrightarrow \infty$

i.e.,  $\mathcal{J}_{\lambda}$  is coercive. Since  $\mathcal{J}_{\lambda}$  is also weakly lower semicontinuous, then there exists of a global minimizer  $u_0$  of  $J$ , i.e.,

$$\mathcal{J}_{\lambda}(u_0) = \inf_{u \in W} \mathcal{J}_{\lambda}(u).$$

By integrating the findings from Remark 2.1 and Lemma 3.1, we reach this conclusion.

**Lemma 3.2** Assume (1) – (2), and  $(g_3)$ , if  $s(x) \in (q^-, p^*)$ , then exist  $\epsilon \in (0, 1)$ ,  $\eta > 0$  and exists a positive constant  $\lambda_1$  such that, for any  $0 < \lambda < \lambda_1$  such that

$$\inf_{u \in \partial B_{\epsilon}(0)} J(u) \geq \eta,$$

where  $\partial B_{\epsilon}(0) = \{u \in W : \|u\| = \epsilon\}$ .

**Proof:**

Since  $(g_3)$  is essentially condition  $(g_2)$  with  $\mu = 0$ , it follows that there exists a constant  $C_1 > 0$  such that

$$|G(\mathbf{u})| \leq C_1 \|\mathbf{u}\|_W^{s^+}, \quad \forall u \in W \quad (3.5)$$

then:

$$\begin{aligned} \mathcal{J}_{\lambda}(\mathbf{u}) &\geq \frac{a_0}{q^+} \|\mathbf{u}\|_W^{q^-} - C_1 \|\mathbf{u}\|_W^{s^+} - \lambda \frac{\|b\|_{\infty}}{r^-(q^- - r^-).e} C_q \|\mathbf{u}\|_W^{q^-} \\ &\geq \left( \frac{a_0}{q^+} - \lambda \frac{\|b\|_{\infty}}{r^-(q^- - r^-).e} C_q \right) \|\mathbf{u}\|_W^{q^-} - C_1 \|\mathbf{u}\|_W^{s^+} \\ &= \epsilon^{q^-} \left( \frac{a_0}{q^+} - \lambda \frac{\|b\|_{\infty}}{r^-(q^- - r^-).e} C_q - C_1 \epsilon^{s^+ - q^-} \right). \end{aligned}$$

Choosing  $0 < \lambda < \lambda_1 = \left( \frac{a_0}{q^+} - C_1 \epsilon^{s^+ - q^-} \right) \cdot \frac{r^-(q^- - r^-).e}{\|b\|_{\infty} C_q}$  that's to say  $\frac{a_0}{q^+} - \lambda \frac{\|b\|_{\infty}}{r^-(q^- - r^-).e} C_q > 0$ , Thus, for fixed  $\epsilon \in \left( 0, \min \left( 1, \frac{r^-(q^- - r^-).e - q^+ \lambda \|b\|_{\infty} C_q}{C_1 q^+ r^-(q^- - r^-).e} \right) \right)$  and any  $u \in W$  one has

$$\inf_{u \in \partial B_{\epsilon}(0)} J(u) \geq \eta,$$

with  $\eta = \epsilon^{q^-} \left( \frac{a_0}{q^+} - \lambda \frac{\|b\|_{\infty}}{r^-(q^- - r^-).e} C_q - C_1 \epsilon^{s^+ - q^-} \right) > 0$ , thus completing the proof.  $\square$

**Lemma 3.3** *Suppose  $(h_1) - (h_2)$ ,  $(g_1)$  and  $(g_4)$  hold, if  $\mathfrak{s}(x) \in (q^-, p^*)$ , then there exist  $w_0 \in W$  and  $t_0 \in (0, 1)$  such that:*

$$\mathcal{J}_\lambda(tw_0) < 0, \quad \forall t \in (0, t_0).$$

**Proof:**

Let  $x_0 \in W$  be fixed and choose  $R > 0$  such that  $\bar{B}_R(x_0) \subset U$ . Then there exist  $w_0 \in C_0^\infty(\mathbf{u})$  such that:

$$\begin{cases} w_0(x) = 1 & \text{in } B_R(x_0) \\ 0 \leq w_0(x) \leq 1 & \text{on } U \setminus \bar{B}_R(x_0). \end{cases}$$

Obvious  $w_0 \in W$  and  $\|w_0\| > 0$ . Then, for any  $0 < t < \min(1, \delta, \|w_0\|^{-1})$ , we reach the following conclusion:

$$\begin{aligned} G(tw_0) &= \int_{\Omega} g(x, tw_0) dx = \int_U g(x, tw_0) dx \\ &\geq \int_U Kt^\alpha dx = K \text{meas}(\mathbf{u})t^\alpha, \end{aligned}$$

according to (3.3) and Holder's inequality we obtain:

$$\begin{aligned} \mathcal{J}_\lambda(tw_0) &= A(tw_0) - G(tw_0) - B_\lambda(tw_0) \\ &\leq \frac{1}{\mathfrak{p}^+} (1 + \|V\|_{p'}) \|tw_0\|^{q^+} + \frac{\lambda \|b\|_\infty}{(\mathfrak{r}^+)^2} \|tw_0\|^{q^+} - K \text{meas}(\mathbf{u})t^\alpha \\ &\leq K \text{meas}(\mathbf{u})t^\alpha \left[ \frac{(\mathfrak{r}^+)^2 (1 + \|V\|_{p'}) + \lambda \mathfrak{p}^+ \|b\|_\infty}{(\mathfrak{r}^+)^2 \mathfrak{p}^+ \text{meas}(\mathbf{u})} \|w_0\|^{q^+} t^{q^+ - \alpha} - 1 \right], \end{aligned}$$

which show  $\mathcal{J}_\lambda(tw_0 < 0)$  for all  $t \in (0, t_0)$  with  $t_0 = \min(1, \delta, \|w_0\|^{-1}, K_{p,q,r,\alpha})$  and  $K_{p,q,r,\alpha} = \left[ \frac{(\mathfrak{r}^+)^2 (1 + \|V\|_{p'}) + \lambda \mathfrak{p}^+ \|b\|_\infty}{(\mathfrak{r}^+)^2 \mathfrak{p}^+ \text{meas}(\mathbf{u})} \|w_0\|^{q^+} \right]^{\frac{1}{\alpha - q^+}}$ . □

**Lemma 3.4** *If the sequence  $\{u_n\}$  is bounded, hence weakly converges to  $u$  in the space  $W$ , then we have*

$$\lim_{n \rightarrow \infty} \int_{\Omega} b(x) |u_n|^{\mathfrak{r}(x)} u_n \log(|u_n|) dx = \int_{\Omega} b(x) |u|^{\mathfrak{r}(x)} u \log(|u|) dx$$

Proof of Theorem 1.1. According to Lemmas 3.2 and 3.3, there exist  $\epsilon \in (0, 1)$  such that:

$$\inf_{\bar{B}_\epsilon(0)} \mathcal{J}_\lambda < 0 < \inf_{\partial \bar{B}_\epsilon(0)} \mathcal{J}_\lambda.$$

Let  $\{w_n\} \subset \bar{B}_\epsilon(\mathbf{u})$  be a minimizing sequence for  $\mathcal{J}_\lambda \setminus \bar{B}_\epsilon(0)$ , i.e,  $\mathcal{J}_\lambda(w_n) \rightarrow \inf_{\bar{B}_\epsilon(0)} \mathcal{J}_\lambda$ , as  $n \rightarrow \infty$ . So there exists a subsequence such that

$$\mathcal{J}_\lambda(w_n) < \inf_{\bar{B}_\epsilon(0)} \mathcal{J}_\lambda + \frac{1}{n}, \quad \forall n \geq 1. \quad (3.5)$$

Using theorem 2.4 with  $\rho = \frac{1}{n}$ , and  $\beta = \sqrt{n}$  we get that there exists  $\{u_n\} \subset \bar{B}_\epsilon(0)$  such that

$$\mathcal{J}_\lambda(u_n) < \mathcal{J}_\lambda(w_n), \quad \forall n \geq 1, \quad (3.6)$$

and

$$-\frac{1}{\sqrt{n}} \|v - u_n\| \leq \mathcal{J}_\lambda(v) - \mathcal{J}_\lambda(u_n), \quad \forall v \in \bar{B}_\epsilon(0). \quad (3.7)$$

Since the sequence  $u_n$  is clearly bounded, hence there exists a subsequence  $u_{n_j}$  of  $u_n$  and  $u \in \bar{B}_\epsilon(0)$  such that:

$$u_{n_j} \rightharpoonup u \text{ in } W,$$

and by the compact embedding we have:

$$\mathbf{u}_{n_j} \longrightarrow \mathbf{u} \text{ in } L^{r(x)}(\Omega) \text{ and } \mathbf{u}_{n_j} \longrightarrow \mathbf{u} \text{ in } L^{s(x)}(\Omega).$$

We take  $v_t = \mathbf{u}_{n_j} + t(\mathbf{u} - \mathbf{u}_{n_j}) \in \bar{B}_\epsilon(0)$ ,  $\forall t \in (0, 1)$  and by (3.7) we have:

$$-\frac{t}{\sqrt{n}} \|\mathbf{u} - \mathbf{u}_{n_j}\| \leq \mathcal{J}_\lambda(\mathbf{u}_{n_j} + t(\mathbf{u} - \mathbf{u}_{n_j})) - \mathcal{J}_\lambda(\mathbf{u}_{n_j}).$$

By dividing the last equation by  $t > 0$  and then taking the limit superior as  $t \rightarrow 0$ , we obtain

$$\begin{aligned} \frac{-1}{\sqrt{n}} &\leq \limsup_{t \searrow 0} \frac{A(\mathbf{u}_{n_j} + t(\mathbf{u} - \mathbf{u}_{n_j})) - A(\mathbf{u}_{n_j})}{t} \\ &+ \frac{(-G)(\mathbf{u}_{n_j} + t(\mathbf{u} - \mathbf{u}_{n_j})) - (-G)(\mathbf{u}_{n_j})}{t} \\ &+ \frac{(-B_\lambda)(\mathbf{u}_{n_j} + t(\mathbf{u} - \mathbf{u}_{n_j})) - (-B_\lambda)(\mathbf{u}_{n_j})}{t} \\ &\leq \langle A'(\mathbf{u}_{n_j}), \mathbf{u} - \mathbf{u}_{n_j} \rangle + (-G)^0(\mathbf{u}_{n_j}, \mathbf{u} - \mathbf{u}_{n_j}) + (-B_\lambda)'(\mathbf{u}_{n_j}), \mathbf{u} - \mathbf{u}_{n_j}, \end{aligned}$$

which can be rewritten as

$$\langle A'(\mathbf{u}_{n_j}), \mathbf{u}_{n_j} - \mathbf{u} \rangle \leq \frac{-1}{\sqrt{n}} + (-G)^0(\mathbf{u}_{n_j}, \mathbf{u} - \mathbf{u}_{n_j}) + \langle (-B_\lambda)'(\mathbf{u}_{n_j}), \mathbf{u} - \mathbf{u}_{n_j} \rangle.$$

Using proposition 2.5 and lemma 3.4, by passing to the limit  $n \rightarrow \infty$  we have

$$\limsup_{n \rightarrow \infty} \langle A'(\mathbf{u}_{n_j}), \mathbf{u}_{n_j} - \mathbf{u} \rangle \leq 0,$$

since  $A'$  of type  $(S)_+$  (see, [[25], Proposition 3.1]) we infer that

$$\mathbf{u}_{n_j} \longrightarrow \mathbf{u} \text{ in } W.$$

By (3.5) and (3.6) we conclude:

$$\mathcal{J}_\lambda(\mathbf{u}) = \lim_{n \rightarrow \infty} \mathcal{J}_\lambda(\mathbf{u}_{n_j}) = \inf_{\bar{B}_\epsilon(0)} \mathcal{J}_\lambda,$$

conclude that  $\mathbf{u}$  is a nonzero local minimizer of  $\mathcal{J}_\lambda$  and, according to Remark 2.1, a nontrivial critical point.

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*Mohamed Knifda,*  
*Laboratory SATE,*  
*Faculty of Sciences Dhar Elmahraz, Sidi Mohamed Ben Abdellah University,*  
*Morocco.*  
*E-mail address: mohamed.knifda1@usmba.ac.ma*

*and*

*Ahmed Aberqi,*  
*Laboratory SATE,*  
*National School of Applied Sciences,*  
*Morocco.*  
*E-mail address: ahmed.aberqi@usmba.ac.ma*

*and*

*Abdesslam Ouaziz,*  
*Laboratory SATE,*  
*Faculty of Sciences Dhar Elmahraz, Sidi Mohamed Ben Abdellah University,*  
*Morocco.*  
*E-mail address: abdesslam.ouaziz@usmba.ac.ma*