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Generalized Hermite-Hadamard Type Inequalities for MT-Non-Convex Functions via Fractional Integrals

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ABSTRACT: This research paper unveils the class of *MT-Non-convex functions*. We also established some new generalized fractional integral inequalities of Hermite-Hadamard type for *MT-Non-convex* functions and to explore some new Hermite-Hadamard type inequalities in a form of generalized Riemann-Liouville fractional integrals as well as classical integrals, respectively. These newly established inequalities generalize some known results.

Key Words: MT-Non-convex function, generalized fractional integral operators, Generalized Riemann - Liouville fractional integrals.

Contents

1. Introduction

Integral Inequalities have played a significant role in all fields of mathematics such as pure and applied mathematics. Convexity has many areas of applications in which include optimization theory, linear programming and the theory of inequalities. Integral inequalities are the fundamental tool in the study of various classes of convexity, see [1,2,3,4,5].
Let $I \subseteq \mathbb{R}$ be an interval and I^o denotes the interior of I. A function $g : I \to \mathbb{R}$ is said to be convex on I, if the inequality is said to be convex on I, if the inequality

$$g(\xi a + (1-\xi)b) \leq \xi g(a) + (1-\xi)g(b) \tag{1}$$

holds $\forall a, b \in I$ and $\xi \in [0,1]$. If $-g$ is convex, then we say that g is concave. Many authors have established a large number of equalities or inequalities for convex functions. Many researchers have examined different generalizations of the classical as well as Riemann-Liouville fractional integrals, see [6,7,8,9,10]. The Hermite-Hadamard inequality is one of the most important inequalities for convex functions. Let $g : I \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality holds

$$g(\frac{a+b}{2})\frac{1}{b-a}\int_b^a g(\xi)d\xi \leq \frac{g(a)+g(b)}{2} \tag{1.1}$$

Which inequality (1.1) can be a important tool to establish many theoretical estimates. Due to its importance, different authors have devoted great efforts to extend, generalize, and refine the Hermite-Hadamard inequality for various classes of convex functions and mappings in the field of pure and applied mathematics, see [11,12,13,14,15,16].

Definition 1.1. [11] Let I $\subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $g :$ I $\to \mathbb{R}$ is said to be MT-non-convex function, if the following inequality

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$$g\left([\xi a^p + (1-\xi)b^p]^{\frac{1}{p}}\right) \leq \frac{\sqrt{\xi}}{2\sqrt{1-\xi}}g(a) + \frac{\sqrt{1-\xi}}{2\sqrt{\xi}}g(b) \tag{3}$$

holds $\forall a, b \in \mathrm{I}$ and $\xi \in (0,1)$.

Lemma 1.1 *Let $g : [a,b] \to \mathbb{R}$ be a differentiable mapping on I^o,where $a, b \in I$ is such that $a < b$ and $p \in \mathbb{R} \setminus \{0\}$. If $g' \in L^1[a,b]$, then for all $h \in (0,1)$ and $\mu > 0$, then the following inequality holds*

$$\begin{aligned}
&[\frac{(1-h)^\mu + h^\mu}{p^\mu(b^P - a^P)^{1-\mu}}]g(w) - \frac{\Gamma\mu+1}{b^P - a^P}[I^\mu_{w-}g(a) + I^\mu_{w+}g(b)] \\
&= \frac{(b^P - a^P)^\mu}{p^{\mu+1}}[(1-h)^{\mu+1}\int_0^1 \xi^\mu(\xi w^P + (1-\xi)a^P)^{\frac{1-p}{p}} g'(\xi w^P + (1-\xi)a^P)^{\frac{1}{p}} d\xi \\
&\qquad - h^{\mu+1}\int_0^1 \xi^\mu(\xi w^P + (1-\xi)b^P)^{\frac{1-p}{p}} g'(\xi w^P + (1-\xi)b^P)^{\frac{1}{p}} d\xi]
\end{aligned} \tag{1.2}$$

Where $w = (ha^P + (1-P)^P)^{\frac{1}{p}}$ and

$$I^\mu_{w-}g(a) = \frac{p^{1-\mu}}{\Gamma\mu}\int_a^w \xi^{p-1}\frac{g(\xi)}{(\xi^p - a^p)^{1-\mu}}d\xi,$$

$$I^\mu_{w+}g(b) = \frac{p^{1-\mu}}{\Gamma\mu}\int_w^b \xi^{p-1}\frac{g(\xi)}{(b^p - \xi^p)^{1-\mu}}d\xi.$$

Theorem 1.2 *Let $g : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^o such that $g' \in L^1[a,b]$, where $a, b \in I$ with $0 \leq a < b$ and $p \in \mathbb{R} \setminus \{0\}$. If $|g'|$ is MT-non-convex on $[a,b]$ such that $|g'(x)| \leq M$ for each $x \in [a,b]$ with some $M > 0$. Then for all $h \in (0,1)$ and $\mu > 0$.*

1. *If $p \in (1,\infty)$, then the following inequality holds*

$$\begin{aligned}
&\left|\left(\frac{(1-h)^\mu + h^\mu}{p^\mu(b^p - a^p)^{1-\mu}}\right)g(w) - \frac{\Gamma\mu+1}{b^p - a^p}\left[I^\mu_{w-}g(a) + I^\mu_{w+}g(b)\right]\right| \\
&\leq \frac{a^{1-p}(b^p - a^p)^\mu}{2p^{\mu+1}}\beta\left(\frac{1}{2}, \mu + \frac{1}{2}\right)M,
\end{aligned} \tag{1.3}$$

2. *If $p \in (-\infty,0) \cup (0,1)$, then the following inequality holds*

$$\begin{aligned}
&\left|\left(\frac{(1-h)^\mu + h^\mu}{p^\mu(b^p - a^p)^{1-\mu}}\right)g(w) - \frac{\Gamma\mu+1}{b^p - a^p}\left[I^\mu_{w-}g(a) + I^\mu_{w+}g(b)\right]\right| \\
&\leq \frac{b^{1-p}(b^p - a^p)^\mu}{2p^{\mu+1}}\beta\left(\frac{1}{2}, \mu + \frac{1}{2}\right)M.
\end{aligned} \tag{1.4}$$

Where β stands for the beta function of Euler type define by

$$\beta(x,y) = \int_0^1 \xi^{x-1}(1-\xi)^{y-1}d\xi$$

Theorem 1.3 *Let $g : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^o such that $g' \in L^1[a,b]$, where $a, b \in I$ with $0 \leq a < b$ and $p \in \mathbb{R} \setminus \{0\}$. If $|g'|^q$ is MT-non-convex on $[a,b]$, $q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$ and $|g'(x)| \leq M$ for each $x \in [a,b]$ with some $M > 0$. Then for all $h \in (0,1)$ and $\mu > 0$.*

1. If $p \in (1, \infty)$, then the following inequality holds

$$\begin{aligned} &\left|\left(\frac{(1-h)^{\mu}+h^{\mu}}{p^{\mu}(b^{p}-a^{p})^{1-\mu}}\right)g(w)-\frac{\Gamma\mu+1}{b^{p}-a^{p}}\left[I^{\mu}_{w-}g(a)+I^{\mu}_{w+}g(b)\right]\right| \\ &\leq \frac{a^{1-p}}{p^{\mu+1}}\left(\frac{1}{\mu r+1}\right)^{\frac{1}{r}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}(b^{P}-a^{P})^{\mu}M, \end{aligned} \tag{1.5}$$

2. If $p \in (-\infty, 0) \cup (0, 1)$, then the following inequality holds

$$\begin{aligned} &\left|\left(\frac{(1-h)^{\mu}+h^{\mu}}{p^{\mu}(b^{p}-a^{p})^{1-\mu}}\right)g(w)-\frac{\Gamma\mu+1}{b^{p}-a^{p}}\left[I^{\mu}_{w-}g(a)+I^{\mu}_{w+}g(b)\right]\right| \\ &\leq \frac{b^{1-p}}{p^{\mu+1}}\left(\frac{1}{\mu r+1}\right)^{\frac{1}{r}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}(b^{P}-a^{P})^{\mu}M. \end{aligned} \tag{1.6}$$

Theorem 1.4 *Let $g : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^{o} such that $g' \in L^1[a, b]$, where $a, b \in I$ with $0 \leq a < b$ and $p \in \mathbb{R} \setminus \{0\}$. If $|g'|^q$ is MT-non-convex on $[a, b]$ with $q > 1$ and $|g'(x)| \leq M$ for each $x \in [a, b]$ with some $M > 0$. Then for all $h \in (0, 1)$ and $\mu > 0$.*

1. If $p \in (1, \infty)$, then the following inequality holds

$$\begin{aligned} &\left|\left(\frac{(1-h)^{\mu}+h^{\mu}}{p^{\mu}(b^{p}-a^{p})^{1-\mu}}\right)g(w)-\frac{\Gamma\mu+1}{b^{p}-a^{p}}\left[I^{\mu}_{w-}g(a)+I^{\mu}_{w+}g(b)\right]\right| \\ &\leq \frac{a^{1-p}(b^{P}-a^{P})^{\mu}}{p^{\mu+1}}\left(\frac{1}{\mu+1}\right)^{1-\frac{1}{q}}\left(\frac{\beta(\frac{1}{2},\mu+\frac{1}{2})}{2}\right)^{\frac{1}{q}}M, \end{aligned} \tag{1.7}$$

2. If $p \in (-\infty, 0) \cup (0, 1)$, then the following inequality holds

$$\begin{aligned} &\left|\left(\frac{(1-h)^{\mu}+h^{\mu}}{p^{\mu}(b^{p}-a^{p})^{1-\mu}}\right)g(w)-\frac{\Gamma\mu+1}{b^{p}-a^{p}}\left[I^{\mu}_{w-}g(a)+I^{\mu}_{w+}g(b)\right]\right| \\ &\leq \frac{b^{1-p}(b^{P}-a^{P})^{\mu}}{p^{\mu+1}}\left(\frac{1}{\mu+1}\right)^{1-\frac{1}{q}}\left(\frac{\beta(\frac{1}{2},\mu+\frac{1}{2})}{2}\right)^{\frac{1}{q}}M. \end{aligned} \tag{1.8}$$

2. Results and Computation

At the beginning of the section, we first drive an generalized fractional operators and also deliver an identity, which will play a significant role in the proof of our main results.

2.1. Generalized Fractional Integral Operators

Let $\rho : [0, \infty) \to [0, \infty)$ be such that

$$\int_0^1 \xi^{p-1}\frac{\rho(\xi^p)^{\frac{1}{p}}}{\xi^p} < \infty,$$

The left and right sided generalized integral operators are defined as

$${}_{a^+}I_{\rho}g(x) = \int_a^x \xi^{p-1}\frac{\rho\left(x^p-\xi^p\right)^{\frac{1}{p}}}{x^p-\xi^p}g(\xi)\,d\xi, \qquad x > a,\; p > 0. \tag{2.1}$$

$${}_{b^-}I_{\rho}g(x) = \int_x^b \xi^{p-1}\frac{\rho\left(\xi^p-x^p\right)^{\frac{1}{p}}}{\xi^p-x^p}g(\xi)\,d\xi, \qquad x < b,\; p > 0. \tag{2.2}$$

1. If $\rho(\xi) = p\,\xi^p$, then (2.1) and (2.2) reduce to the usual generalized Riemann integral:
$$I_{a^+}g(x) = p\int_a^x \frac{g(\xi)}{\xi^{1-p}}\,d\xi, \qquad x>a,\ p>0,$$
$$I_{b^-}g(x) = p\int_x^b \frac{g(\xi)}{\xi^{1-p}}\,d\xi, \qquad x<b,\ p>0.$$

2. When ρ is specialized by
$$\rho(\xi) = p^{1-\mu}\frac{\xi^{p\mu}}{\Gamma(\mu)},$$
then (2.1) and (2.2) convert to the generalized Riemann–Liouville fractional integral:
$$I^{\mu}_{a^+}g(x) = \frac{p^{1-\mu}}{\Gamma(\mu)}\int_a^x \xi^{p-1}\frac{g(\xi)}{(x^p-\xi^p)^{1-\mu}}\,d\xi, \qquad x>a,\ \mu>0,\ p>0,$$
$$I^{\mu}_{b^-}g(x) = \frac{p^{1-\mu}}{\Gamma(\mu)}\int_x^b \xi^{p-1}\frac{g(\xi)}{(\xi^p-x^p)^{1-\mu}}\,d\xi, \qquad x<b,\ \mu>0,\ p>0.$$

3. When ρ is formulated by
$$\rho(\xi) = \frac{p^{1-\frac{\mu}{k}}}{k\,\Gamma_k(\mu)}\,\xi^{\frac{\mu}{k}},$$
then (2.1) and (2.2) convert to the generalized k–Riemann–Liouville fractional integral:
$$I^{\mu}_{a^+}g(x) = \frac{p^{1-\frac{\mu}{k}}}{k\,\Gamma_k(\mu)}\int_a^x \xi^{p-1}\frac{g(\xi)}{(x^p-\xi^p)^{1-\frac{\mu}{k}}}\,d\xi, \qquad x>a,\ \mu>0,\ p>0,$$
$$I^{\mu}_{b^-}g(x) = \frac{p^{1-\frac{\mu}{k}}}{k\,\Gamma_k(\mu)}\int_x^b \xi^{p-1}\frac{g(\xi)}{(\xi^p-x^p)^{1-\frac{\mu}{k}}}\,d\xi, \qquad x<b,\ \mu>0,\ p>0.$$
Where
$$\Gamma_k(\mu) = \int_0^\infty \xi^{\mu-1}e^{-\frac{\xi^k}{k}}\,d\xi, \qquad \Re(\mu)>0,$$
and
$$\Gamma_k(\mu) = k^{\frac{\mu}{k}-1}\,\Gamma\Big(\frac{\mu}{k}\Big), \qquad \Re(\mu)>0,\ k>0.$$

Recently, established some new inequalities of Hermite-Hadamard type for different classes of convexity by using generalized fractional integral operators. The main objective of this article is to drive various new and generalized fractional integral inequalities of Hermite-Hadamard type for MT−non convex functions involving generalized fractional integral operators (2.1) and (2.2).

Lemma 2.1 *Let $g:[a,b]\to\mathbb{R}$ be a differentiable mapping on I^o,where $a,b\in I$ is such that $a<b$ and $p\in\mathbb{R}/\{0\}$. If $g'\in L^1[a,b]$, then for all $h\in(0,1)$ and $\mu>0$, we have*
$$\begin{aligned} &p\left[\frac{(1-h)^{\mu}\Omega(1)+h^{\mu}\nabla(1)}{b^p-a^p}\right]g(w) - \frac{p^2}{b^p-a^p}\left[(1-h)^{\mu}({}_{w^-}I_{\rho}g(a)) + h^{\mu}({}_{w^+}I_{\rho}g(b))\right] \\ &= (1-h)^{\mu+1}\int_0^1 \Omega(\xi)\,(\xi w^p+(1-\xi)a^p)^{\frac{1-p}{p}}\,g'(\xi w^p+(1-\xi)a^p)^{\frac{1}{p}}\,d\xi \\ &\quad - h^{\mu+1}\int_0^1 \nabla(\xi)\,(\xi w^p+(1-\xi)b^p)^{\frac{1-p}{p}}\,g'(\xi w^p+(1-\xi)b^p)^{\frac{1}{p}}\,d\xi \end{aligned} \tag{2.3}$$
where $w=(ha^p+(1-h)b^p)^{\frac{1}{p}}$, and $\Omega(\xi)$ and $\nabla(\xi)$ are defined by
$$\Omega(\xi) = \int_0^\xi \frac{\rho((w^p-a^p)u)^{\frac{1}{p}}}{u}du<\infty, \nabla(\xi) = \int_0^\xi \frac{\rho((b^p-w^p)u)^{\frac{1}{p}}}{u}du<\infty$$

Proof. Let
$$I_1=(1-h)^{\mu+1}\int_0^1 \Omega(\xi)(\xi w^p+(1-\xi)a^p)^{\frac{1-p}{p}} g'(\xi w^p+(1-\xi)a^p)^{\frac{1}{p}}d\xi$$
integration by parts
$$I_1=\frac{p(1-h)^{\mu+1}}{w^p-a^p}\left[\Omega(\xi)\,g(\xi w^p+(1-\xi)a^p)\Big|_{\xi=0}^{\xi=1}-\int_0^1\frac{\rho((w^p-a^p)\xi)}{\xi}\,((\xi w^p+(1-\xi)a^p))^{1/p}\,g(\xi w^p+(1-\xi)a^p)\,d\xi\right].$$
$$I_1=\frac{p(1-h)^{\mu+1}}{w^p-a^p}\left[\Omega(1)\,g(w)-\int_0^1\frac{\rho((w^p-a^p)\xi)}{\xi}\,((\xi w^p+(1-\xi)a^p))^{1/p}\,g(\xi w^p+(1-\xi)a^p)\,d\xi\right].$$
Let $x=(\xi w^p+(1-\xi)a^p)^{\frac{1}{p}}$
If $\xi=0$, then $x=a$
If $\xi=1$, then $x=w$
$$\xi=\frac{x^p-a^p}{w^p-a^p}\Rightarrow d\xi=\frac{px^{p-1}}{w^p-a^p}dx$$
So, the above equation becomes
$$I_1=\frac{p(1-h)^{\mu+1}}{w^p-a^p}\left[\Omega(1)g(w)-p\int_a^w\frac{\rho(x^p-a^p)^{\frac{1}{p}}}{x^p-a^p}x^{p-1}g(x)dx\right]$$
$$\because w^p-a^p=(1-h)(b^p-a^p)$$
Therefore, above equation becomes
$$I_1=\frac{p(1-h)^{\mu}}{b^p-a^p}\left[\Omega(1)g(w)-p\int_a^w\frac{\rho(x^p-a^p)^{\frac{1}{p}}}{x^p-a^p}x^{p-1}g(x)dx\right]$$
$$I_1=\frac{p(1-h)^{\mu}}{b^p-a^p}\left[\Omega(1)g(w)-p(I_{w^-\rho}g(a))\right]$$
Similarly,
Let
$$I_2=-h^{\mu+1}\int_0^1\nabla(\xi)(\xi w^p+(1-\xi)b^p)^{\frac{1-p}{p}}g'(\xi w^p+(1-\xi)b^p)^{\frac{1}{p}}d\xi$$
integration by parts
$$I_2=\frac{p\,h^{\mu+1}}{w^p-b^p}\left[\nabla(\xi)\,g(\xi w^p+(1-\xi)b^p)\Big|_{\xi=0}^{\xi=1}-\int_0^1\frac{\rho((b^p-w^p)\xi)}{\xi}\,(\xi w^p+(1-\xi)b^p)^{1/p}\,g(\xi w^p+(1-\xi)b^p)\,d\xi\right].$$
$$I_2=\frac{p\,h^{\mu+1}}{w^p-b^p}\left[\nabla(1)\,g(w)-\int_0^1\frac{\rho((b^p-w^p)\xi)}{\xi}\,(\xi w^p+(1-\xi)b^p)^{1/p}\,g(\xi w^p+(1-\xi)b^p)\,d\xi\right].$$
Let $x=(\xi w^p+(1-\xi)b^p)^{\frac{1}{p}}$
If $\xi=0$, then $x=b$
If $\xi=1$, then $x=w$
$$\xi=\frac{x^p-b^p}{w^p-b^p}or\xi=\frac{b^p-x^p}{b^p-w^p}\Rightarrow d\xi=\frac{-px^{p-1}}{b^p-w^p}$$
So, above equation becomes
$$I_2=\frac{ph^{\mu+1}}{w^p-b^p}\left[\nabla(1)g(w)-p\int_w^b\frac{\rho(b^p-x^p)^{\frac{1}{p}}}{b^p-x^p}x^{p-1}g(x)dx\right]$$
$$\because b^p-w^p=h(b^p-a^p)$$

Therefore, above equation becomes

$$I_2 = \frac{ph^\mu}{b^p - a^p}\left[\nabla(1)g(w) - p\int_w^b \frac{\rho(b^p - x^p)^{\frac{1}{p}}}{b^p - x^p} x^{p-1} g(x)dx\right]$$

$$I_2 = \frac{ph^\mu}{b^p - a^p}\left[\nabla(1)g(w) - p(I_{w^+\rho} g(b))\right]$$

By adding I_1 and I_2 it easily implies identity (2.3) □

Theorem 2.2 *Let $g : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on (a,b) and $g' \in L^1[a,b]$ with $0 \le a < b$ and $\mu > 0$.If $|g'|$ is MT-non-convex on $[a,b]$, then for each $h \in (0,1)$ and $p \in \mathbb{R}/\{0\}$.*

1. *If $p \in (1,\infty)$, then the following inequality holds*

$$\begin{aligned}&\left|p\left[\frac{(1-h)^\mu\,\Omega(1) + h^\mu\,\nabla(1)}{b^p - a^p}\right]g(w) - \frac{p^2}{b^p - a^p}\Big((1-h)^\mu\ {}_{w^-}I_\rho g(a) + h^\mu\ {}_{w^+}I_\rho g(b)\Big)\right| \\ &\le a^{1-p}\left[\frac{(1-h)^{\mu+1}}{2}(A_1|g'(w)| + B_1|g'(a)|) + \frac{h^{\mu+1}}{2}(A_2|g'(w)| + B_2|g'(b)|)\right].\end{aligned} \tag{2.4}$$

2. *If $p \in (-\infty,0)\cup(0,1)$, then the following inequality holds*

$$\begin{aligned}&\left|p\left[\frac{(1-h)^\mu\,\Omega(1) + h^\mu\,\nabla(1)}{b^p - a^p}\right]g(w) - \frac{p^2}{b^p - a^p}\Big((1-h)^\mu\ {}_{w^-}I_\rho g(a) + h^\mu\ {}_{w^+}I_\rho g(b)\Big)\right| \\ &\le b^{1-p}\left[\frac{(1-h)^{\mu+1}}{2}(A_1|g'(w)| + B_1|g'(a)|) + \frac{h^{\mu+1}}{2}(A_2|g'(w)| + B_2|g'(b)|)\right].\end{aligned} \tag{2.5}$$

where the constants A_1, A_2, B_1 and B_2 are given by

$$A_1 = \int_0^1 \sqrt{\frac{\xi}{1-\xi}}|\Omega(\xi)|d\xi, A_2 = \int_0^1 \sqrt{\frac{\xi}{1-\xi}}|\nabla(\xi)|d\xi,$$

$$B_1 = \int_0^1 \sqrt{\frac{1-\xi}{\xi}}|\Omega(\xi)|d\xi, B_2 = \int_0^1 \sqrt{\frac{1-\xi}{\xi}}|\nabla(\xi)|d\xi.$$

Proof. by using Lemma (2.1) and property of modulus

$$\begin{aligned}&\left|p\left[\frac{(1-h)^\mu\Omega(1) + h^\mu\nabla(1)}{b^p - a^p}\right]g(w) - \frac{p^2}{b^p - a^p}\Big((1-h)^\mu\ {}_{w^-}I_\rho g(a) + h^\mu\ {}_{w^+}I_\rho g(b)\Big)\right| \\ &\le (1-h)^{\mu+1}\int_0^1 |\Omega(\xi)|\left(\xi w^p + (1-\xi)a^p\right)^{\frac{1-p}{p}}\left|g((\xi w^p + (1-\xi)a^p)^{1/p})\right|d\xi \\ &\quad + h^{\mu+1}\int_0^1 |\nabla(\xi)|\left(\xi w^p + (1-\xi)b^p\right)^{\frac{1-p}{p}}\left|g((\xi w^p + (1-\xi)b^p)^{1/p})\right|d\xi.\end{aligned} \tag{2.6}$$

Since $p \in (1,\infty)$, then we deduce that

$$(\xi w^p + (1-\xi)b^p)^{\frac{1-p}{p}} \le (\xi w^p + (1-\xi)a^p)^{\frac{1-p}{p}} \le a^{1-p}$$

So, above inequality becomes

$$\le a^{1-p}(1-h)^{\mu+1}\int_0^1 |\Omega(\xi)||g'(\xi w^p + (1-\xi)a^p)^{\frac{1}{p}}|d\xi$$

$$+a^{1-p}h^{\mu+1}\int_0^1|\nabla(\xi)||g'(\xi w^p+(1-\xi)b^p)^{\frac{1}{p}}|d\xi,$$

by using the MT-non convexity

$$\leq a^{1-p}(1-h)^{\mu+1}\int_0^1|\Omega(\xi)|\left[\frac{\sqrt{\xi}}{2\sqrt{1-\xi}}|g(w)|+\frac{\sqrt{1-\xi}}{2\sqrt{\xi}}|g(a)|\right]d\xi,$$

$$+a^{1-p}h^{\mu+1}\int_0^1|\Omega(\xi)|\left[\frac{\sqrt{\xi}}{2\sqrt{1-\xi}}|g(w)|+\frac{\sqrt{1-\xi}}{2\sqrt{\xi}}|g(b)|\right]d\xi,$$

$$\leq a^{1-p}\left[\frac{(1-h)^{\mu+1}}{2}[A_1|g'(w)|+B_1|g'(a)|]+\frac{h^{\mu+1}}{2}[A_2|g'(w)|+B_2|g'(b)|]\right],$$

Which is required proof of part (1)
To prove (2), let $p\in(-\infty,0)\cup(0,1)$, then we obtained the required inequality by applying the fact that

$$(\xi w^P+(1-\xi)a^P)^{\frac{1-p}{p}}\leq(\xi w^P+(1-\xi)b^P)^{\frac{1-p}{p}}\leq b^{1-p}, \tag{2.7}$$

The proof is completed. □

Remark:
From Theorem 2.2, we can see that if $|g'(x)|< M$ for each $x\in[a,b]$, and ρ is specialized by

1. $\rho(\xi)=p\xi^p$, then the following inequality is true
   If $p\in(1,\infty)$, then the following inequality holds

$$\left|g(w)-\frac{p}{b^p-a^p}\int_a^b\frac{g(x)}{x^{1-p}}dx\right|\leq\frac{\pi}{4}M\frac{a^{1-p}(b^p-a^p)}{p}$$

   If $p\in(-\infty,0)\cup(0,1)$, then the following inequality holds

$$\left|g(w)-\frac{p}{b^p-a^p}\int_a^b\frac{g(x)}{x^{1-p}}dx\right|\leq\frac{\pi}{4}M\frac{b^{1-p}(b^p-a^p)}{p}$$

2. $\rho(\xi)=p^{1-\mu}\frac{\xi^{p\mu}}{\Gamma\mu}$, then inequalities (2.4) and (2.5) reduce to (1.3) and (1).

Theorem 2.3 *Let $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ be differentiable on (a,b) and $g'\in L^1[a,b]$ with $0\leq a<b$ and $\mu>0$. Assume $|g'|^q$ is MT-non-convex on $[a,b]$ for $q>1$. For each $h\in(0,1)$ and $p\in\mathbb{R}\setminus\{0\}$:*

1. If $p\in(1,\infty)$, then

$$\left|p\left[\frac{(1-h)^\mu\,\Omega(1)+h^\mu\,\nabla(1)}{b^p-a^p}\right]g(w)-\frac{p^2}{b^p-a^p}\Big((1-h)^\mu\ {}_{w^-}I_\rho g(a)+h^\mu\ {}_{w^+}I_\rho g(b)\Big)\right|$$
$$\leq a^{1-p}\left(\frac{\pi}{4}\right)^{1/q}\left[(1-h)^{\mu+1}\left(\int_0^1|\Omega(\xi)|^r\,d\xi\right)^{1/r}(|g'(w)|^q+|g'(a)|^q)^{1/q}\right.$$
$$\left.+h^{\mu+1}\left(\int_0^1|\nabla(\xi)|^r\,d\xi\right)^{1/r}(|g'(w)|^q+|g'(b)|^q)^{1/q}\right]. \tag{2.8}$$

2. If $p\in(-\infty,0)\cup(0,1)$, then

$$\left|p\left[\frac{(1-h)^\mu\,\Omega(1)+h^\mu\,\nabla(1)}{b^p-a^p}\right]g(w)-\frac{p^2}{b^p-a^p}\Big((1-h)^\mu\ {}_{w^-}I_\rho g(a)+h^\mu\ {}_{w^+}I_\rho g(b)\Big)\right|$$
$$\leq b^{1-p}\left(\frac{\pi}{4}\right)^{1/q}\left[(1-h)^{\mu+1}\left(\int_0^1|\Omega(\xi)|^r\,d\xi\right)^{1/r}(|g'(w)|^q+|g'(a)|^q)^{1/q}\right.$$
$$\left.+h^{\mu+1}\left(\int_0^1|\nabla(\xi)|^r\,d\xi\right)^{1/r}(|g'(w)|^q+|g'(b)|^q)^{1/q}\right]. \tag{2.9}$$

Proof. by using Lemma (2.1) and property of modulus

$$\left|p\left[\frac{(1-h)^{\mu}\,\Omega(1)+h^{\mu}\,\nabla(1)}{b^p-a^p}\right]g(w)-\frac{p^2}{b^p-a^p}\Big((1-h)^{\mu}\ {}_{w^-}I_{\rho}g(a)+h^{\mu}\ {}_{w^+}I_{\rho}g(b)\Big)\right|$$
$$\leq (1-h)^{\mu+1}\int_0^1|\Omega(\xi)|\,(\xi w^p+(1-\xi)a^p)^{\frac{1-p}{p}}\left|g'((\xi w^p+(1-\xi)a^p)^{1/p})\right|d\xi$$
$$+\,h^{\mu+1}\int_0^1|\nabla(\xi)|\,(\xi w^p+(1-\xi)b^p)^{\frac{1-p}{p}}\left|g'((\xi w^p+(1-\xi)b^p)^{1/p})\right|d\xi. \tag{2.10}$$

Since $p\in(1,\infty)$, we have

$$(\xi w^p+(1-\xi)a^p)^{\frac{1-p}{p}}\leq(\xi w^p+(1-\xi)b^p)^{\frac{1-p}{p}}\leq a^{1-p},$$

so (2.10) becomes

$$\leq a^{1-p}(1-h)^{\mu+1}\int_0^1|\Omega(\xi)|\left|g'((\xi w^p+(1-\xi)a^p)^{1/p})\right|d\xi$$
$$+\,a^{1-p}h^{\mu+1}\int_0^1|\nabla(\xi)|\left|g'((\xi w^p+(1-\xi)b^p)^{1/p})\right|d\xi. \tag{2.11}$$

By Hölder's inequality (with $1/r+1/q=1$, $r,q>1$),

$$\leq a^{1-p}(1-h)^{\mu+1}\left(\int_0^1|\Omega(\xi)|^r\,d\xi\right)^{1/r}\left(\int_0^1\left|g'((\xi w^p+(1-\xi)a^p)^{1/p})\right|^q d\xi\right)^{1/q}$$
$$+\,a^{1-p}h^{\mu+1}\left(\int_0^1|\nabla(\xi)|^r\,d\xi\right)^{1/r}\left(\int_0^1\left|g'((\xi w^p+(1-\xi)b^p)^{1/p})\right|^q d\xi\right)^{1/q}. \tag{2.12}$$

Using MT–non-convexity of $|g'|^q$ on $[a,b]$ gives, for $\xi\in(0,1)$,

$$\left|g'((\xi w^p+(1-\xi)a^p)^{1/p})\right|^q\leq\frac{\sqrt{\xi}}{2\sqrt{1-\xi}}|g'(w)|^q+\frac{\sqrt{1-\xi}}{2\sqrt{\xi}}|g'(a)|^q,$$

and similarly with a replaced by b. Hence,

$$\leq a^{1-p}(1-h)^{\mu+1}\left(\int_0^1|\Omega(\xi)|^r\,d\xi\right)^{1/r}\left(\int_0^1\left[\frac{\sqrt{\xi}}{2\sqrt{1-\xi}}|g'(w)|^q+\frac{\sqrt{1-\xi}}{2\sqrt{\xi}}|g'(a)|^q\right]d\xi\right)^{1/q}$$
$$+\,a^{1-p}h^{\mu+1}\left(\int_0^1|\nabla(\xi)|^r\,d\xi\right)^{1/r}\left(\int_0^1\left[\frac{\sqrt{\xi}}{2\sqrt{1-\xi}}|g'(w)|^q+\frac{\sqrt{1-\xi}}{2\sqrt{\xi}}|g'(b)|^q\right]d\xi\right)^{1/q}. \tag{2.13}$$

Using

$$\int_0^1\frac{\sqrt{\xi}}{2\sqrt{1-\xi}}\,d\xi=\int_0^1\frac{\sqrt{1-\xi}}{2\sqrt{\xi}}\,d\xi=\frac{\pi}{4},$$

we conclude

$$\leq a^{1-p}\left(\frac{\pi}{4}\right)^{1/q}\left[(1-h)^{\mu+1}\left(\int_0^1|\Omega(\xi)|^r\,d\xi\right)^{1/r}(|g'(w)|^q+|g'(a)|^q)^{1/q}\right.$$
$$\left.+\,h^{\mu+1}\left(\int_0^1|\nabla(\xi)|^r\,d\xi\right)^{1/r}(|g'(w)|^q+|g'(b)|^q)^{1/q}\right], \tag{2.14}$$

which completes the proof of part (i) of the theorem.

The case(ii) can be proved by using the above process with relation (2.7).The proof is completed. □

Remark:

From Theorem 2.3, Let $M>0$ be such that $|g'(x)|<M$ for each $x\in[a,b]$, then we conclude that

1. $\rho(\xi)=p\xi^{p}$, then the following inequality is true
   If $p\in(1,\infty)$, then the following inequality holds

$$\left|g(w)-\frac{p}{b^{p}-a^{p}}\int_{a}^{b}\frac{g(x)}{x^{1-p}}dx\right|\leq\frac{a^{1-p}}{p}\left(\frac{\beta\left(\frac{1}{2},q+\frac{1}{2}\right)}{2}\right)^{\frac{1}{q}}M(b^{p}-a^{p})$$

   If $p\in(-\infty,0)\cup(0,1)$, then the following inequality holds

$$\left|g(w)-\frac{p}{b^{p}-a^{p}}\int_{a}^{b}\frac{g(x)}{x^{1-p}}dx\right|\leq\frac{b^{1-p}}{p}\left(\frac{\beta\left(\frac{1}{2},q+\frac{1}{2}\right)}{2}\right)^{\frac{1}{q}}M(b^{p}-a^{p})$$

2. $\rho(\xi)=p^{1-\mu}\frac{\xi^{p\mu}}{\Gamma\mu}$, then inequalities (2.8) and (2.9) reduce to (1.5) and (1.6).

Theorem 2.4 *Let $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ be differentiable on (a,b) and $g'\in L^{1}[a,b]$ with $0\leq a<b$ and $\mu>0$. If $|g'|^{q}$ is MT-non-convex on $[a,b]$ for $q>1$, then for all $h\in(0,1)$ and $p\in\mathbb{R}\setminus\{0\}$:*

(i) If $p\in(1,\infty)$, then

$$\begin{aligned}&\left|p\left[\frac{(1-h)^{\mu}\,\Omega(1)+h^{\mu}\,\nabla(1)}{b^{p}-a^{p}}\right]g(w)-\frac{p^{2}}{b^{p}-a^{p}}\Big((1-h)^{\mu}\;{}_{w^-}I_{\rho}g(a)+h^{\mu}\;{}_{w^+}I_{\rho}g(b)\Big)\right|\\&\leq a^{1-p}\Bigg[(1-h)^{\mu+1}\left(\int_{0}^{1}|\Omega(\xi)|^{r}\,d\xi\right)^{1-\frac{1}{q}}\left(\frac{A_{1}}{2}|g'(w)|^{q}+\frac{B_{1}}{2}|g'(a)|^{q}\right)^{1/q}\\&\quad+h^{\mu+1}\left(\int_{0}^{1}|\nabla(\xi)|^{r}\,d\xi\right)^{1-\frac{1}{q}}\left(\frac{A_{2}}{2}|g'(w)|^{q}+\frac{B_{2}}{2}|g'(b)|^{q}\right)^{1/q}\Bigg].\end{aligned}\tag{2.15}$$

(ii) If $p\in(-\infty,0)\cup(0,1)$, then

$$\begin{aligned}&\left|p\left[\frac{(1-h)^{\mu}\,\Omega(1)+h^{\mu}\,\nabla(1)}{b^{p}-a^{p}}\right]g(w)-\frac{p^{2}}{b^{p}-a^{p}}\Big((1-h)^{\mu}\;{}_{w^-}I_{\rho}g(a)+h^{\mu}\;{}_{w^+}I_{\rho}g(b)\Big)\right|\\&\leq b^{1-p}\Bigg[(1-h)^{\mu+1}\left(\int_{0}^{1}|\Omega(\xi)|^{r}\,d\xi\right)^{(1-\frac{1}{q})}\left(\frac{A_{1}}{2}|g'(w)|^{q}+\frac{B_{1}}{2}|g'(a)|^{q}\right)^{1/q}\\&\quad+h^{\mu+1}\left(\int_{0}^{1}|\nabla(\xi)|^{r}\,d\xi\right)^{(1-\frac{1}{q})}\left(\frac{A_{2}}{2}|g'(w)|^{q}+\frac{B_{2}}{2}|g'(b)|^{q}\right)^{1/q}\Bigg].\end{aligned}\tag{2.16}$$

Proof. By Lemma 1.2 and the triangle inequality,

$$\begin{aligned}&\left|p\left[\frac{(1-h)^{\mu}\,\Omega(1)+h^{\mu}\,\nabla(1)}{b^{p}-a^{p}}\right]g(w)-\frac{p^{2}}{b^{p}-a^{p}}\Big((1-h)^{\mu}\;{}_{w^-}I_{\rho}g(a)+h^{\mu}\;{}_{w^+}I_{\rho}g(b)\Big)\right|\\&\leq(1-h)^{\mu+1}\int_{0}^{1}|\Omega(\xi)|\left(\xi w^{p}+(1-\xi)a^{p}\right)^{\frac{1-p}{p}}\left|g'((\xi w^{p}+(1-\xi)a^{p})^{1/p})\right|d\xi\\&\quad+h^{\mu+1}\int_{0}^{1}|\nabla(\xi)|\left(\xi w^{p}+(1-\xi)b^{p}\right)^{\frac{1-p}{p}}\left|g'((\xi w^{p}+(1-\xi)b^{p})^{1/p})\right|d\xi.\end{aligned}\tag{2.17}$$

Since $p\in(1,\infty)$, we have

$$\left(\xi w^{p}+(1-\xi)a^{p}\right)^{\frac{1-p}{p}}\leq\left(\xi w^{p}+(1-\xi)b^{p}\right)^{\frac{1-p}{p}}\leq a^{1-p},$$

and (2.17) yields

$$\leq a^{1-p}(1-h)^{\mu+1}\int_0^1 |\Omega(\xi)| \left|g'((\xi w^p+(1-\xi)a^p)^{1/p})\right| d\xi$$
$$+ a^{1-p}h^{\mu+1}\int_0^1 |\nabla(\xi)| \left|g'((\xi w^p+(1-\xi)b^p)^{1/p})\right| d\xi. \tag{2.18}$$

Applying Hölder's (power-mean) inequality with $1/r+1/q=1$ and $r,q>1$,

$$\leq a^{1-p}(1-h)^{\mu+1}\left(\int_0^1 |\Omega(\xi)|^r\, d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1 |\Omega(\xi)|\left|g'((\xi w^p+(1-\xi)a^p)^{1/p})\right|^q d\xi\right)^{\frac{1}{q}}$$
$$+ a^{1-p}h^{\mu+1}\left(\int_0^1 |\nabla(\xi)|^r\, d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1 |\nabla(\xi)|\left|g'((\xi w^p+(1-\xi)b^p)^{1/p})\right|^q d\xi\right)^{\frac{1}{q}}. \tag{2.19}$$

Using MT–non-convexity of $|g'|^q$ on $[a,b]$, for $\xi\in(0,1)$ we have

$$\left|g'((\xi w^p+(1-\xi)a^p)^{1/p})\right|^q \leq \frac{\sqrt{\xi}}{2\sqrt{1-\xi}}|g'(w)|^q+\frac{\sqrt{1-\xi}}{2\sqrt{\xi}}|g'(a)|^q,$$

and similarly with a replaced by b. Hence,

$$\leq a^{1-p}(1-h)^{\mu+1}\left(\int_0^1 |\Omega(\xi)|^r\, d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1 |\Omega(\xi)|\left[\frac{\sqrt{\xi}}{2\sqrt{1-\xi}}|g'(w)|^q+\frac{\sqrt{1-\xi}}{2\sqrt{\xi}}|g'(a)|^q\right] d\xi\right)^{\frac{1}{q}}$$
$$+ a^{1-p}h^{\mu+1}\left(\int_0^1 |\nabla(\xi)|^r\, d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1 |\nabla(\xi)|\left[\frac{\sqrt{\xi}}{2\sqrt{1-\xi}}|g'(w)|^q+\frac{\sqrt{1-\xi}}{2\sqrt{\xi}}|g'(b)|^q\right] d\xi\right)^{\frac{1}{q}}. \tag{2.20}$$

Using

$$\int_0^1 \frac{\sqrt{\xi}}{2\sqrt{1-\xi}}\, d\xi = \int_0^1 \frac{\sqrt{1-\xi}}{2\sqrt{\xi}}\, d\xi = \frac{\pi}{4},$$

we conclude

$$\leq a^{1-p}\Bigg[(1-h)^{\mu+1}\left(\int_0^1 |\Omega(\xi)|^r\, d\xi\right)^{1-\frac{1}{q}}\left(\frac{A_1}{2}|g'(w)|^q+\frac{B_1}{2}|g'(a)|^q\right)^{\frac{1}{q}} \tag{2.21}$$
$$+ h^{\mu+1}\left(\int_0^1 |\nabla(\xi)|^r\, d\xi\right)^{1-\frac{1}{q}}\left(\frac{A_2}{2}|g'(w)|^q+\frac{B_2}{2}|g'(b)|^q\right)^{\frac{1}{q}}\Bigg], \tag{2.22}$$

which is the required estimate for part (i). The case (ii) follows by the same argument starting from relation (2.7). This completes the proof. □

Remark:
From Theorem 2.4, Let $M>0$ be such that $|g'(x)|< M$ for each $x\in[a,b]$, then we conclude that

1. $\rho(\xi)=p\,\xi^p$, then the following inequality is true: If $p\in(1,\infty)$, then

$$\left|g(w)-\frac{p}{b^p-a^p}\int_a^b \frac{g(x)}{x^{1-p}}\, dx\right| \leq \frac{a^{1-p}}{p}\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\pi^{\frac{1}{q}}M(b^p-a^p).$$

   If $p\in(-\infty,0)\cup(0,1)$, then

$$\left|g(w)-\frac{p}{b^p-a^p}\int_a^b \frac{g(x)}{x^{1-p}}\, dx\right| \leq \frac{a^{1-p}}{p}\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\pi^{\frac{1}{q}}M(b^p-a^p).$$

2. $\rho(\xi)=p^{1-\mu}\frac{\xi^{p\mu}}{\Gamma(\mu)}$, then inequalities (2.15) and (2.16) reduce to (1.7) and (1.8).

3. Conclusion

In this paper, our investigation delved into the computation of the group, conjugacy classes, and character tables of the non-rigid diethyl ether, revealing intricate symmetries encapsulated in Tables 1 and 2. The determined group, characterized by orders of 36 and 18 conjugacy classes, sheds light on the complex structural dynamics of diethyl ether. Employing the GAP package for precise calculations enhanced the accuracy and reliability of our findings. This exploration contributes to a deeper understanding of the symmetrical intricacies within the non-rigid diethyl ether, opening avenues for further research in the broader context of molecular symmetry analysis.

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