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# Extended fixed point results for nonextensive mappings in convex metric spaces

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ABSTRACT: This paper delves into establishing common fixed point results for asymptotically regular and nonexpansive-type mappings within convex metric spaces. It extends Gornicki's contractive-type mapping theorems to include metric spaces with a convex structure, while building on Khan and Oyetubi's work on common fixed points of asymptotically regular mappings satisfying the Reich-type contractive condition in complete metric spaces. By generalizing these results to convex metric spaces, the study introduces analogous findings for enriched Ciric-Reich-Rus-type contractions, contributing significantly to the advancement of fixed point theory in structured metric environments. The paper also demonstrates a common fixed point solution for pairs of compatible maps in convex metric spaces and presents a novel fixed point result for non-expansive type mappings in Banach spaces. Furthermore, an approximation result is achieved for quasi-nonexpansive mappings through the utilization of Ishikawa iterations in uniformly convex metric spaces. An additional feature of this study is the inclusion of graphs illustrating various convergence and dynamic behaviors, which provide valuable visual insights into system responses under differing conditions. The results are framed within the scope of the Banach Contraction Principle, enriching its applicability to broader classes of mappings and spaces.

Key Words: Convex metric space, non-expensive mappings, contractions mappings, asymptotical regular mappings, unique fixed point.

## Contents



## 1. Introduction

Fixed point theory is one of the most fundamental tools in modern mathematics, offering profound applications across various disciplines including engineering, optimization theory, physics, economics, and game theory. A fixed point of a mapping $h$ is a point $a \in M$ such that $h(a) = a$. The set of fixed points of $h$ is denoted by $P(h)$. The question of whether such a point exists and under what conditions has led to the development of a rich mathematical theory.

One of the most influential results in this area is the *Banach Contraction Principle* [1], which guarantees the existence and uniqueness of a fixed point for a contraction mapping on a complete metric space. Despite its simplicity and elegance, the Banach principle has limitations; specifically, it requires continuity and cannot be directly applied to discontinuous functions in metric or Banach spaces.

In order to overcome these limitations, several generalized classes of mappings have been investigated, such as Kannan-type mappings, nonexpansive and quasi-nonexpansive mappings, and asymptotically regular mappings in both metric and uniformly convex metric spaces [2-4].

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Recent developments have further extended fixed point theory to accommodate more generalized conditions and structures. For instance, Górnicki [5] extended contractive mapping results to broader contexts, while Khan and Oyetubi [6] explored common fixed points under Reich-type conditions. Other studies have incorporated convexity structures and iterative techniques such as the Ishikawa and Mann iterations to improve convergence analysis [7, 8].

The concept of a *common fixed point*—a point that remains invariant under two or more mappings simultaneously—has also attracted considerable attention due to its relevance in the study of compatible and weakly compatible mappings. Such results have implications in the analysis of equilibrium problems and systems of nonlinear equations [9, 10].

In this paper, we extend and unify several existing fixed point results by considering asymptotically regular and nonexpansive-type mappings within convex metric spaces. Our results contribute to the ongoing efforts to generalize fixed point theory in structured environments and to explore its applicability in broader mathematical and applied settings.

## 2. Contractions in metric space

**Definition 2.1** *Let $(M, \theta)$ be a metric space.Mapping*

$$h : M \to M$$

*is a contraction mapping, or just a contraction if is constant $e(0 \leq e < 1)$ s.t.*

$$\theta(h(q), h(p)) \leq e\theta(q, p)$$

*holds $\forall q, p \in M$.*

*In case , $e = 1$, in the above definition, $h$ will become nonexpensive.*

*The conceptualization of asymptotically regular (ARM) has been proposed by Browder and Petryshyn [14] as follows.*

**Definition 2.2** *Let $(M, \theta)$ be a metric space and $S : M \to M$ be a map that mapping maps asymptotically to $q_0 \in M$ if*

$$\lim_{n\to\infty} \theta(S^n q_0, S^{n+1} q_0) = 0.$$

*S is asymptotically regular in M; If every point of M is asymptotically regular.*

**Definition 2.3** *Let $(M, \theta)$ be a metric space and let $q_0 \in M$ .*

$$\begin{aligned} h(q_0) &= q_1 \\ h(q_1) &= q_2 \\ h(q_2) &= q_3 \\ h(q_3) &= q_4 \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ h(q_n) &= h(q_{n+1}) \end{aligned}$$

*The sequence $\{q_n\}$ is called Picard iterations sequence for the mapping $h$.*

**Definition 2.4** *Let $(M, \theta)$ be a metric space and $h : M \to$ a point $q_0 \in M$ is called the fixed point $h$ if*

$$h(q_0) = q_0.$$

*The set of fixed point will be written as P(h).*

**Theorem 2.1** *[11] Let $M$ be a complete metric space and an asymptotically regular continuum and map satisfying the condition:*

$$\theta(ha, hb) \leq m\theta(a, b) + K\{\theta(a, ha) + \theta(b, hb)\} \tag{2.1}$$

*$\forall\ a, b \in M$, here, $m \in [0, 1)$ and $K \in [0, \infty)$. as has a unique fixed point $p_1 \in M$ and $h^n a \to p_1$ for each $a \in M$.*

**Definition 2.5 (12)** *If $h$ is the self-definition of the metric space $(M, \theta)$, then the set $O(h, a) = \{h^n a : n = 0, 1, 2, ...\}$ is called the orbit of $h$ in $h$ is called for any sequence $\{a_n\}$, the trajectory is continuous at point $z \in M$ $a \subset O(h, a)$ of $M$, $\lim_{n\to\infty} a_n = z$ $\lim_{n\to\infty} h_{a_n} = h_z$.*

*The continuity of operator $h$ entails orbital continuity; however, it is important to note that the converse is not true.*

**Definition 2.6 (12)** *In the context of a metric space $(M, \theta)$, a self-mapping denoted as $h$ is termed k-continuous, where k represents a positive integer (k = 1, 2, ...), if the condition $\lim_{n\to\infty} h^{k-1} a_n = z$ implies that $\lim_{n\to\infty} h^k a_n = hz$. It should be noted that 1-reliability is synonymous with reliability and for any integer k (k = 1, 2, ...), k-reliability implies (k+1)-reliability. However, it is important point out that the opposite implication is not universally true continuity $h^k$ and the k-reliability of $h$ are distinct and individual.*

**Theorem 2.2** *[2] If $(M, \theta)$ is a metric space and the mapping $h : M \to M$ is asymptotically regular that adheres to condition (2.1), it can be established that $h$ possesses a unique fixed point. This assertion holds true if and only if $h$ satisfies either the k-continuity condition for some integer $k \geq 1$ or exhibits orbital continuity. Furthermore, it can be noted that the sequence $h^n a \to p$.*
*Recently, Bisht and Singh [13, 20,21] extended this theorem as follows. A pair of self-charges $S$ and $h$ satisfying the Lipschitz-Kannan type state:*

$$\theta(Sa, hb) \leq m\theta(ha, hb) + K\{\theta(Sa, Sb) + \theta(hb, Sa)\}$$

*$\forall\ a, b \in M$.*

*We further broaden the scope of Theorem to encompass a pair of ARM that conform to a contractive condition as outlined. In addition, we present findings that pertain to mappings that fulfill the condition (2.1)" but do not exhibit asymptotic regularity.*
*In 1970, Takahashi [14, 19] introduced the concept of convex structure a metric space $(M, \theta)$ as a satisfying map $W : M^2 \times I \to M$*

$$\theta(u, W(a, b, \psi)) \leq \psi\theta(u, a) + (1 - \psi)\theta(u, b)$$

*$\forall\ a, b, u \in M$ and $\psi \in I = [0, 1]$.*

**Theorem 2.3** *(Banach Contraction Principle (BCP))*
*Consider $(M, \theta)$ as follows is a complete metric space and $\psi : M \to M$ is a contraction, so $\psi$ is: fixed point $m \in M$. Furthermore, for any $q \in M$, the sequence $\{\psi^n q\}$ converges to $m$.*

**Example 2.1** **[6]** Suppose that $M = \mathbb{R}$ and $\psi : M \to M$ is defined by $\psi(q) = -q$.

Then $\psi$ is nonexpansive but not asymptotically regular.

**Solution:**

We have:

$$|\psi(q) - \psi(p)| = |-q - (-p)| = |q - p|$$

Hence, $\psi$ is nonexpansive.

Also, $P(\psi) = \{0\}$. Let $q_0 \in M$ and $q_0 > 0$. Then:

$$\begin{aligned} q_1 &= \psi(q_0) = -q_0, \\ q_2 &= \psi(q_1) = \psi(-q_0) = q_0, \\ q_3 &= \psi(q_2) = \psi(q_0) = -q_0, \\ q_4 &= \psi(q_3) = \psi(-q_0) = q_0, \\ &\vdots \\ q_{n+1} &= \psi(q_n) = (-1)^n q_0 \end{aligned}$$

Now,

$$d(q_n, \psi(q_n)) = |q_n - \psi(q_n)| = |(-1)^n q_0 - (-1)^{n+1} q_0| = 2q_0$$

Thus,

$$\lim_{n\to\infty} d(q_n, \psi(q_n)) = \lim_{n\to\infty} 2q_0 = 2q_0 \neq 0$$

Therefore,

$$d(q_n, \psi(q_n)) \nrightarrow 0 \quad \text{as } n \to \infty$$

and

$$d(\psi^{n-1}(q_0), \psi^n(q_0)) \nrightarrow 0 \quad \text{as } n \to \infty$$

This implies that $\psi$ is not an asymptotically regular map.

## 3. Common Fixed Point in Convex Metric Space

We delve into the inquiry of the existence and uniqueness of a common fixed point for mappings that are either contractive or nonexpansive in nature. This analysis is carried out within the framework of metric spaces.

By the definition of infimum, we can construct subsequences $\{x_{n_i}\}$ or $\{y_{n_i}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, such that

$$\lim_{i\to\infty} d(x_{n_i}, T(y_{n_i})) = 0,$$

and hence,

$$\liminf_{n\to\infty} d(x_n, T(y_n)) = 0.$$

### 3.1. Common fixed point of Asymptotically Regular Maps

**Theorem 3.1** *Consider the sequences of real numbers $\{u_n\}$ and $\{w_n\}$ with $u_n \geq 0$ and $v_n \in (0,1)$, satisfying the following conditions:*

*(i)* $u_{n+1} \leq (1 - v_n)u_n + v_n w_n$

*(ii)* $\sum_{n=1}^{\infty} v_n = \infty$

*(iii) Either* $\limsup_{n\to\infty} w_n \leq 0$ *or* $\sum_{n=1}^{\infty} w_n^- < \infty$, *where* $w_n^- = \max\{-w_n, 0\}$

*Then* $\lim_{n\to\infty} u_n = 0$.

**Theorem 3.2** *[15] Assume the completeness of the metric space $(M, \theta)$. Consider two asymptotically regular self-mappings $h$ and $S$ on $M$ such that*

$$\theta(ha, Sb) \leq m\theta(a, b) + K\{\theta(a, ha) + \theta(b, Sb)\} \tag{3.1}$$

*for all $a, b \in M$, where $m \in [0, 1)$ and $K \in [0, \infty)$. Assume further that $h$ and $S$ are either $k$-continuous for some $k \geq 1$ or continuous on an orbit.*

*Then $h$ and $S$ have a unique common fixed point $p \in M$, and for all $a \in M$:*

$$\lim_{n\to\infty} h^n a = p = \lim_{n\to\infty} S^n a.$$

**Proof: Step 1:** We show that $\lim_{n\to\infty} \theta(h^n a, S^n a) = 0$.

If $h = S$, the result is trivial. Otherwise, set $a_n = h^n a$ and $b_n = S^n a$.

From (3.1), if $m = 0$:

$$\theta(ha, Sb) \leq K\{\theta(a, ha) + \theta(b, Sb)\}.$$

Substituting $a = h^n a$, $b = S^n a$, we get:

$$\theta(h^{n+1}a, S^{n+1}a) \leq K\left\{\theta(h^n a, h^{n+1}a) + \theta(S^n a, S^{n+1}a)\right\}.$$

By asymptotic regularity, both terms on the right tend to 0, so:

$$\lim_{n\to\infty} \theta(h^{n+1}a, S^{n+1}a) = 0.$$

If $m \neq 0$, again from (3.1):

$$\theta(h^{n+1}a, S^{n+1}a) \leq m\theta(h^n a, S^n a) + K\{\theta(h^n a, h^{n+1}a) + \theta(S^n a, S^{n+1}a)\}.$$

Define:

$$a_n = \theta(h^n a, S^n a), \quad \alpha_n = 1 - m, \quad \beta_n = \frac{K}{\alpha_n}\left[\theta(h^n a, h^{n+1}a) + \theta(S^n a, S^{n+1}a)\right].$$

As $h$ and $S$ are asymptotically regular, $\lim_{n\to\infty} \beta_n = 0$ and $\sum \alpha_n = \infty$. By above Lemma, we conclude that $\lim_{n\to\infty} a_n = 0$.

**Step 2:** Define $a_n = h^n a$ for any $a \in M$. Suppose $\{a_n\}$ is not Cauchy. Then there exist $\varepsilon > 0$ and sequences $\{m(k)\}, \{n(k)\}$ with $m(k) > n(k) \geq k$ such that:

$$\theta(a_{m(k)}, a_{n(k)}) \geq \varepsilon.$$

Choose $m(k)$ minimal such that:

$$\theta(a_{m(k)-1}, a_{n(k)}) < \varepsilon.$$

Then:

$$\varepsilon \leq \theta(a_{m(k)}, a_{n(k)}) \leq \theta(a_{m(k)}, a_{m(k)-1}) + \theta(a_{m(k)-1}, a_{n(k)}) < \theta(a_{m(k)}, a_{m(k)-1}) + \varepsilon. \tag{3.2}$$

By asymptotic regularity:

$$\lim_{k\to\infty} \theta(a_{m(k)}, a_{m(k)-1}) = 0 \Rightarrow \lim_{k\to\infty} \theta(a_{m(k)}, a_{n(k)}) = \varepsilon.$$

A contradiction arises from:

$$\theta(a_{m(k)}, a_{n(k)}) \leq \theta(a_{m(k)}, S^{n(k)}a) + \theta(S^{n(k)}a, a_{n(k)}).$$

Applying (3.1) and asymptotic regularity gives:

$$\varepsilon \leq m\varepsilon \Rightarrow \varepsilon(1 - m) \leq 0,$$

which contradicts $\varepsilon > 0$. Thus, $\{a_n\}$ is Cauchy. Since $M$ is complete, $a_n \to p \in M$.

From Step 1 and the triangle inequality:

$$\theta(S^n a, p) \leq \theta(S^n a, h^n a) + \theta(h^n a, p) \to 0.$$

So $S^n a \to p$ as well.

**Step 3:** Show that $p$ is a unique common fixed point of $h$ and $S$.

If $h$ is $k$-continuous and $\lim_{n\to\infty} h^{k-1} a_n = p$, then:

$$h^k a_n \to hp \Rightarrow hp = p.$$

Likewise, if $h$ is orbitally continuous, then $ha_n \to hp$, so $hp = p$. Similarly, $Sp = p$.

Suppose another point $q \neq p$ is a common fixed point: $hq = q$, $Sq = q$.

Then from (3.1):

$$\theta(p, q) \leq m\theta(p, q) \Rightarrow \theta(p, q)(1 - m) \leq 0.$$

Contradiction. Hence, $p$ is the unique common fixed point. □

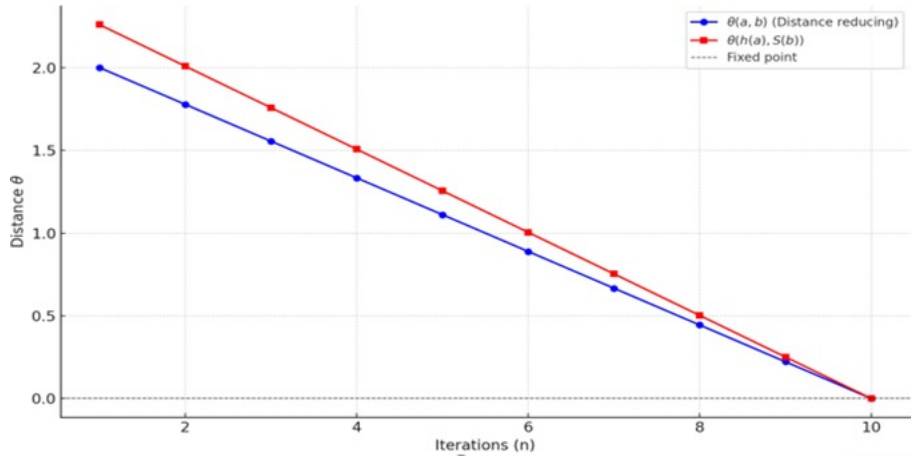


Figure 1: Convergence of Asymptotically Regular Self-Mapping to Fixed Point

**Corollary 3.1** *Assume that $(M, \theta)$ is a metric space, with $h$ and $S$ as self-maps on $M$. It is given that both $h^m$ and $S^q$ with exhibit asymptotically regular for certain $m$ and $q$ are both positive integers. Additionally, we have the following circumstances:*

$$\theta(h^m a, S^q b) \leq m\theta(a, b) + K\{\theta(a, h^m a) + \theta(b, S^q a)\} \tag{3.3}$$

*For $a, b \in M$ $\forall m \in [0, 1)$ and $K \in [0, \infty)$, then $S$ and $h$ share a singular fixed theory $p \in M$, assumes $h^m$ and $S^q$ are both or $k \geq 1$ is k-continuous or has orbital continuity.*

**Proof:** Let $f = h^m$ and $g = S^q$. Therefore, for all $a, b \in M$ (8) looks like this:

$$\theta(fa, gb) \leq m\theta(a, b) + K[\theta(a, fa) + \theta(b, gb)]. \tag{3.4}$$

According to Theorem, it can be established that f and g possess one and one contraction fixed point, denoted as p. Now, $f(hp) = h^m(hp) = h^{m+1}p = h(fp) = hp$. This means that hp is a fixed point. Likewise, it can be observed that Sp is also a fixed point of g. Using (3.4) we have that $Sp = hp$. By virtue of the individuality of the contraction fixed point of f and g, it follow that $Sp = hp = p$. Now, let's assume that z is a contraction fixed point of both S and h. Then, $fz = gz = z$. The uniqueness of the contraction fixed point of f and g entails $p = z$.

Remark: For $m + 2K < 1$, a distinct shared fixed point being present for S and h does not necessitate asymptotically regular and continuity.
Moving forward, we will explore mappings that may not necessarily possess AR but still comply with the condition defined in (3). □

**Theorem 3.3** *[16] Consider a complete metric space $(M, \theta)$, and let $h$ be a continuous mappings that satisfies the conditions specified in (3). We also assume the presence of sequence of approximate fixed point, denoted as $\{a_n\}$, which is sequence contained within h, s.t $\theta(a_n, ha_n)$ approaches to zero as n tend to infinity Under these conditions, it can be established that h possesses a singular fixed point , which we denote as p. Furthermore, it can be denote that the sequence $\{a_n\}$ converge to this fixed point p as n tends towards infinity. Let $(M, \theta)$ is a complete metric space and $h : M \to M$ is a continuous mapping performing (3). Let h has an approximate FP sequence, i.e., $\exists$ a sequence $\{a_n\}, ha$, s.t. $\theta(a_n, ha_n) \to 0$ as $n \to \infty$. Then, h has a unique fixed point p. To be specific , $a_n \to p$ as $n \to \infty$.*

**Proof:** Assume that$m > n$. Triangle inequality is then utilised and (3), we get

$$\theta(a_n, a_m) \leq \theta(u_n, hu_n) + \theta(hu_n, ha_m) + \theta(ha_m, a_m)$$

$$\theta(u_n, a_m) \leq \theta(u_n, hu_n) + m\theta(u_n, a_m) + K\{\theta(u_n, hu_n) + \theta(a_m, ha_m) + \theta(ha_m, a_n)\}$$

Now,

$$(1 - m)\theta(u_n, a_m) \leq (1 + K)\{\theta(u_n, hu_n) + \theta(a_m, ha_m)\}$$

implies that $\lim_{n,m\to\infty} \theta(u_n, a_m) = 0$. Hence, $\{u_n\}$ is Cauchy. By completeness of M, $\{u_n\}$ converge to $p \in M$. Since $\lim_{n,m\to\infty} d(u_n, Hu_n) = 0$ continuity of h indicates that p is one of its fixed points. The inequality leads to fixed point uniqueness (3).

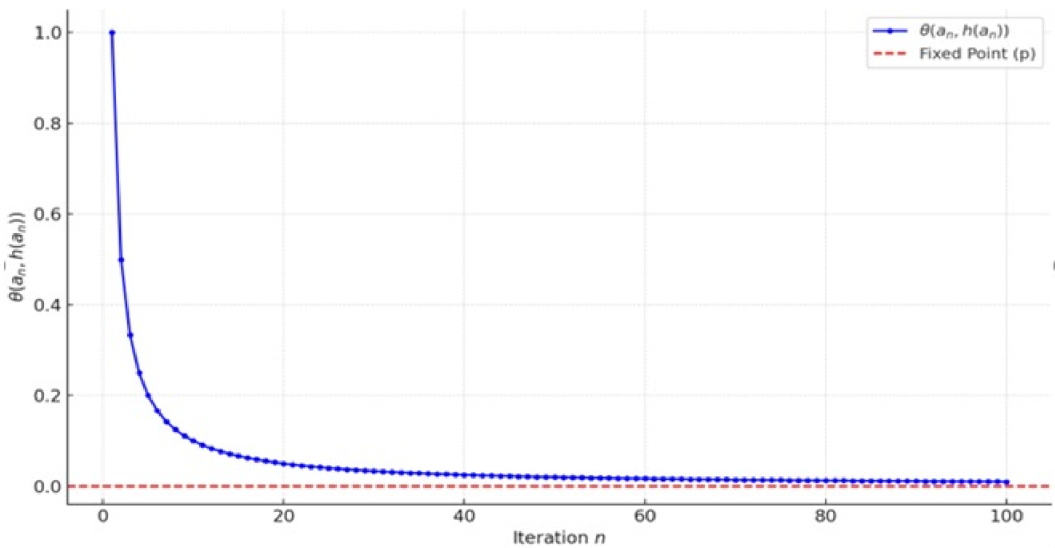


Figure 2: Convergence of $a_n$ to fixed point p

□

**Example 3.1** Suppose $M = [0, 1]$ is equipped with the standard metric, and let $h : M \to M$ be defined by $h(a) = 1 - a$, for all $a \in [0, 1]$. It is straightforward to verify that the inequality

$$|h(a) - h(b)| \leq (1 - 2a)|a - b| + a|a - h(b)| + |a - h(b)|$$

holds for all $a, b \in M$ with $a \in \left[0, \frac{1}{2}\right)$. This demonstrates that above theorem fails in this setting, since $h$ is not asymptotically regular. In particular, for any $a \neq \frac{1}{2}$, the iterates $h^n a$ do not converge to $\frac{1}{2}$, which is the unique fixed point of $h$. However, consider the sequence defined by the recurrence $a_{n+1} = (1 - \psi)a_n + \psi h(a_n)$, where $0 < \psi < 1$ and $n \geq 0$. It can be shown that $|a_n - h(a_n)| \to 0$ as $n \to \infty$, and hence, by above theorem, the mapping $h$ has a unique fixed point at $\frac{1}{2}$, and the sequence $\{a_n\}$ converges to $\frac{1}{2}$. This motivates the extension of above theorem to complete metric spaces. For a mapping $T$ on a complete metric space $M$ and $\psi \in (0, 1]$, the associated average mapping $h_\psi$ is given by $h_\psi(a) = (1 - \psi)a + \psi h(a)$ for all $a \in M$.

**Lemma 3.1** *Let $(M, \theta, w)$ be a complete metric space and $h : M \to M$ be a mappings define the mapping*

$$h_\psi a = h(a, ha, \psi), a \epsilon M$$

*then for any* $\psi\epsilon(0,1)$

$$Fix(h) = Fix(h\psi)$$

**Definition 3.1** *Give us a complete metric space,* $(M,\theta,w)$ *should a mapping* $h : M \to M$ *exist and* $a,b \geq 0$*satisfies* $a+2b<1$ *s.t. for every* $a,b \in M$*, the mapping is described as a* $(k,a,b)$ *enriched Ciric-Reich-Rus contraction.*

*Let* $(M,\theta,W)$ *be a complete metric space. It is said to be a mapping* $h : M \to M$ *a Ciric-Reich-Rus contraction enriched in* $(k,a,b)$ *if* $\exists\ k\epsilon[0,\infty)$ *and* $a,b \geq 0$ *satisfactory* $a+2b<1$ *s.t* $\forall\ a,b \in M$

$$\theta((h,ha;\psi),h(b,hb;\psi)) \leq a\theta(a,b) + b(\theta(a,h(a,ha;\psi)) + \theta(b,h(b,hb;\psi)))$$

.

**Theorem 3.4** *[17] Assume that* $h\psi$ *and* $(M,\theta,w)$ *are full complete metric space. Mapping h:* $M \to M$ *is continuous which satisfies contractive following situation*

$$\theta(h_\psi a, a_\psi b) \leq m\theta(a,b) + K\{\theta(a,h_\psi a) + \theta(b,h_\psi)K\{d(a),T_\psi,a\} + d(y_1,T_\psi,y_1)$$

*Assume that* $h_\psi$ *has an approximate fixed point sequence, that is a sequence* $\{a_n\}$ *s.t* $\theta(h_\psi,a_n) \to 0$ *as* $n \to \infty$*. Then* $h_\psi$ *has a unique fixed point p. More specifically ,* $a_n \to p$ *as* $n \to \infty$*.*

**Proof:** The map $T_\psi$ is Ciric-Reich-Rus contraction by given condition (10),
We note that $T\psi$ iterations of Picard actually serves
the krasnoseskij iterative process $\{a_n\}_{n=0}^{\infty}$

Suppose $m > n$ using triangular inequality, we have

$$\theta(a_n,a_m) \leq \theta(a_n,h_\psi,a_n) + \theta(h_\psi,a_n,h_\psi,a_m) + \theta(h_\psi,a_m,a_m)$$

$$\theta(a_n,a_m) \leq \theta(a_n,h_\psi,a_n) + m\theta(a_n,a_m) + K\{\theta(a_n,h_\psi,a_n + \theta(a_m,h_\psi,a_m\} + \theta(h_\psi,a_m,a_m)$$

Now,

$$\theta(a_n,a_m) \leq \theta(a_n,h_\psi,a_n) + m\theta(a_n,a_m) + K\{\theta(a_n,h_\psi a_m + k\theta(a_m,h_\psi a_m\} + \theta(h_\psi a_m,a_m)$$

$$(1-M)\theta(a_n,a_m) \leq (1+K)\{\theta(a_n,h_\psi a_n) + (1+K)\theta(a_m,h_\psi a_m\}$$

$$(1-M)\theta(a_n,a_m) \leq (1+K)\{\theta(a_n,h_\psi a_n) + \theta(a_m,h_\psi a_m\}$$

implies that $\lim_{n,m\to\infty}\theta(a_n,a_m) = 0$. $\{a_n\}$ is Cauchy. By completeness of M, $\{a_n\}$ converge to $p \in M$. Since $\lim_{n\to\infty}\theta(a_n,h_\psi a_n) = 0$ continuity of $h_\psi$ indicates that p is one of its fixed points of $h_\psi$ by Lemma (3.1.5) the uniqueness of this fixed point follow from(10).

□

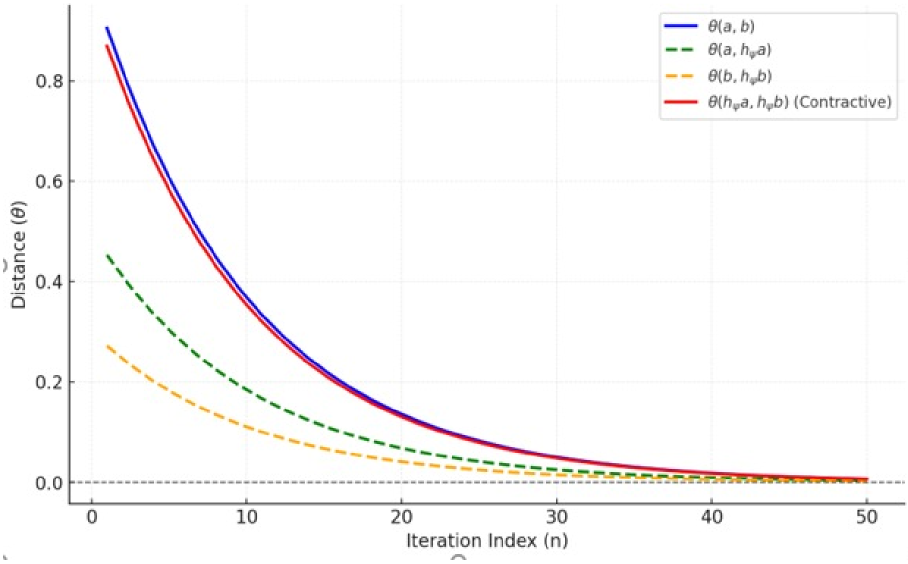


Figure 3: Contrative Mapping Dynamics

## 4. Common fixed points in convex metric space

- A **fixed point** of a mapping $f$ is a point $x$ such that $f(x) = x$.
- A **common fixed point** refers to a point that is fixed for two or more mappings simultaneously: $f(x) = g(x) = x$.
- In **convex metric spaces**, conditions like continuity and contractiveness are often used to guarantee the existence and uniqueness of such common fixed points.

**Lemma 4.1** *Consider that the metric $(S, \rho)$ are self mappings of $h$ and $I$. If the pair $(h, I)$ satisfies the inequality, is mutually continuous, and subcompatible*

$$\rho(ha, hb) \leq \alpha\rho(Ia, Ib) + (1 - \alpha)max\{\rho(Ia, Wa), \rho(Ib, hb)\} \tag{4.1}$$

$\forall\ a, b \in M$, *in which* $0 < \alpha < 1$. *concluded that $h$ and $I$ possess a special shared fixed point within $M$.*

**Theorem 4.1** *[18] Let $M$ be a non-empty q-star-shaped subset of a metric space $(M, \rho)$. Let $h$ and $I$ be continuous self-maps on $M$, such that they satisfy the (E.A.) property with respect to $q$. Suppose that the image $h(M)$ is relatively compact, $I$ is q-affine, and $h$ and $I$ are compatible. Also assume the following contractive condition holds:*

$$\rho(ha, hb) \leq \rho(Ia, Ib) + \frac{1 - K}{K} \max\{\rho(Ia, [ha, q]), \rho(Ib, [hb, q])\}, \tag{4.2}$$

*for all $a, b \in M$, where $\frac{1}{2} < K < 1$. Then $h$ and $I$ have a common fixed point in $M$.*

**Proof:** For each $n \in \mathbb{N}$, define $h_n : M \to M$ by

$$h_n(a) = h(ha, q, k_n),$$

where $\{k_n\} \subset \left(\frac{1}{2}, 1\right)$ and $k_n \to 1$.

Let $\{a_m\} \subset M$ be a sequence such that for all $\psi \in (0, 1)$,

$$\lim_{m\to\infty} Ia_m = \lim_{m\to\infty} h_\psi a_m = t \in M, \tag{4.3}$$

where $h_\psi a_m = h(ha_m, q, \psi)$.

Then for fixed $n$,

$$\lim_{m\to\infty} h_n a_m = \lim_{m\to\infty} h(ha_m, q, k_n) = t.$$

Since $I$ is $q$-affine, we obtain:

$$\rho(h_n I a_m, I h_n a_m) = \rho(h(hIa_m, q, k_n), I(h(ha_m, q, k_n))) = \rho(h(hIa_m, q, k_n), h(I(ha_m, q, k_n))).$$

Using the Lipschitz condition of $h$, we have:

$$\rho(h_n I a_m, I h_n a_m) \le k_n \rho(hIa_m, Iha_m).$$

Since $h$ and $I$ are compatible and satisfy the (E.A.) property, it follows:

$$\lim_{m\to\infty} \rho(h_n I a_m, I h_n a_m) = 0.$$

So $h_n$ and $I$ are asymptotically commuting on $\{a_m\}$.

Now from the definition of $h_n$ and contractive condition (4.2), we have:

$$\rho(h_n a, h_n b) \le k_n \rho(ha, hb),$$

$$\rho(h_n a, h_n b) \le k_n \left[\rho(Ia, Ib) + \frac{1-k_n}{k_n} \max\{\rho(Ia, [ha, q]), \rho(Ib, [hb, q])\}\right].$$

Therefore:

$$\rho(h_n a, h_n b) \le k_n \left[\rho(Ia, Ib) + (1-k_n) \max\{\rho(Ia, h_n a), \rho(Ib, h_n b)\}\right]. \tag{4.4}$$

By Lemma 3.2.1, for each $n \in \mathbb{N}$, there exists $u_n \in M$ such that:

$$u_n = Iu_n = h_n u_n.$$

Since $\overline{h(M)}$ is compact, there exists a convergent subsequence $\{hu_n\} \to z \in M$. As $a_m = h_n a_m \to z$ and $h$ is continuous, we have:

$$z = \lim_{m\to\infty} a_m = \lim_{m\to\infty} h(ha_m, q, k_n).$$

Therefore:

$$Iz = z = hz.$$

Hence, $z$ is a common fixed point of both $h$ and $I$ in $M$. □

## 5. Nonexpansive type results

In an effort to extend the results of Khan and Oyetunbi [13]concerning nonexpansive type mappings, Huang and Qian [19] established a significant generalization that broadens the applicability of such fixed point results. Their work contributes to the ongoing development of fixed point theory in convex metric spaces by addressing mappings that may not strictly satisfy contraction conditions but still guarantee the existence of fixed points under relaxed assumptions.

**Theorem 5.1** *We consider $M$ as a Banach space and $D$ as a non-empty quasi-convex subset. from $X, F, G : D \to D$ — two maps, $r \in (0,1)$, $\tau \in [0,\infty)$, $z \in D$ and*

$$\widetilde{S}a := rSa + (1-r)z^\star$$

$$\widetilde{G}a := rGa + (1-r)z^\star$$

$\forall\ a \in M$.

*Assuming follows condition are met:*

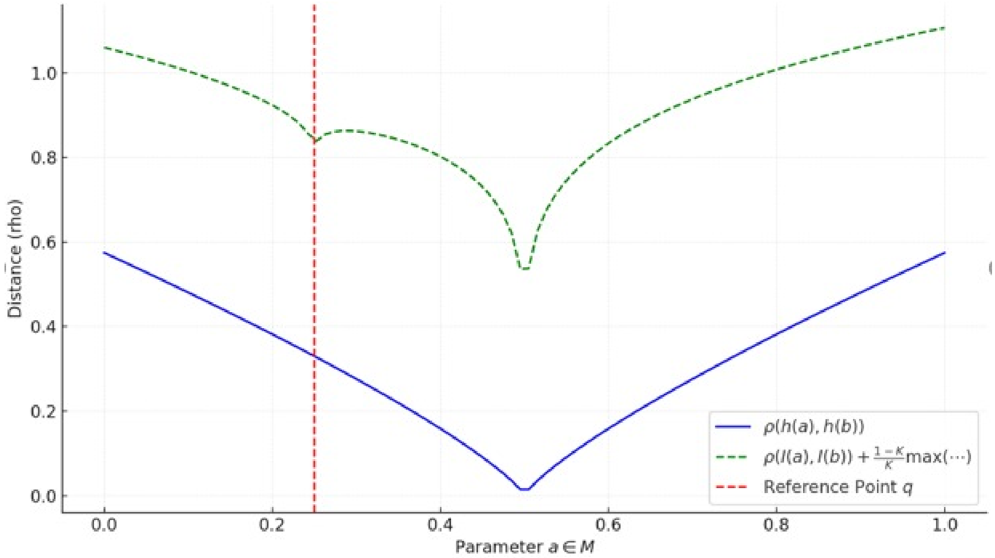


Figure 4: Inequality Dynamics

*(i)* $||Fa - Gb|| \leq ||a - b|| + \tau(||a - Fa|| + ||b - Gb||)$, $\forall\ a, b \in \theta$;
*(ii)* $\widetilde{F}$ *and* $\widetilde{G}$ *on the set D, are AR;*
*(iii)* $\lim_{n\to\infty} ||\widetilde{F}^n x - F(\widetilde{F}^n a)||$, $\lim_{n\to\infty} ||\widetilde{G}^n a - G(\widetilde{G}^n a)||$ $\forall\ a \in D$.
*(iv)* $F$ *and* $G$ *are continuous functions in the domain D.*
*In such a case, it can be concluded that F and G share a complete fixed point within.*

**Proof:**

$$\|Fa - Gb\| \leq \|a - b\| + \tau(\|a - Fa\| + \|b - Gb\|), \quad \forall a, b \in D \tag{5.1}$$

Let $a \in M$. Define $u_n = F^n a$ and $b_n = G^n a$, for all $a \in \mathbb{N}$. By (5.1),

$$\|\widetilde{F}^{n+1}a - \widetilde{G}^{n+1}a\| = r\|F(\widetilde{F}^n a) - G(\widetilde{G}^n a)\|$$

$$\|\widetilde{F}^{n+1}a - \widetilde{G}^{n+1}a\| \leq r\|\widetilde{F}^n a - \widetilde{G}^n a\| \tag{5.2}$$

$$+ \tau r(\|\widetilde{F}^n a - F(\widetilde{F}^n a)\| + \|\widetilde{G}^n a - G(\widetilde{G}^n a)\|) \tag{5.3}$$

Since $F$ and $G$ are asymptotically regular,

$$\lim_{n\to\infty} \|\widetilde{F}^n a - \widetilde{F}^{n+1}a\| = 0, \quad \lim_{n\to\infty} \|\widetilde{G}^n a - \widetilde{G}^{n+1}a\| = 0,$$

so (5.2) becomes

$$\|\widetilde{F}^n a - \widetilde{G}^n a\| \leq r\|\widetilde{F}^n a - \widetilde{G}^n a\| + \tau\|\widetilde{F}^n a - \widetilde{F}^{n+1}a\| + \tau\|\widetilde{G}^n a - \widetilde{G}^{n+1}a\| \tag{5.4}$$

From this, it follows that

$$\|F^{n+1}a - G^{n+1}a\| \leq r\|\widetilde{F}^n a - \widetilde{G}^n a\| \tag{5.5}$$

Let $\overline{a}, \overline{b} \in D$ such that $F(\overline{a}) = \overline{a}$ and $G(\overline{b}) = \overline{b}$. Assume $\overline{a} \neq \overline{b}$. By continuity of $F$ and $G$ and asymptotic regularity:

$$\lim_{n\to\infty} \|\overline{a} - F(\overline{a})\| = 0 \tag{5.6}$$

$$\lim_{n\to\infty} \|\bar{b} - G(\bar{b})\| = 0 \tag{5.7}$$

From (5.6) and (5.7), we get:

$$\bar{a} = F(\bar{a}), \quad \bar{b} = G(\bar{b})$$

Thus, $\bar{a} = \bar{b}$, which contradicts the assumption. Hence, there exists a common fixed point.

Using Lemma 2.1.1 and (5.4):

$$\lim_{n\to\infty} \|\overline{F}^n a - \overline{G}^n a\| = 0 \tag{5.8}$$

To show $\{\overline{F}^n a\}$ is Cauchy, suppose there exists $\epsilon_0 > 0$ and sequences $\{\widetilde{n}(i)\}$, $\{n(i)\}$ such that $\widetilde{n}(i) > n(i) > i$ and

$$\|\overline{F}^{\widetilde{n}(i)} a - \overline{F}^{n(i)} a\| \geq \epsilon_0 \tag{5.9}$$

Then by condition (i):

$$\|\overline{F}^{\widetilde{n}(i)} a - \overline{F}^{n(i)} a\| \leq r\|\overline{F}^{\widetilde{n}(i)-1} a - \overline{F}^{\widetilde{n}(i)} a\| + \cdots + (\text{other terms vanishing as } i \to \infty) \tag{5.10}$$

Divide both sides of (5.10) by $\epsilon_0$, and take limit to get:

$$1 \leq r(1 + \cdots) < 1, \tag{5.11}$$

which is a contradiction. Thus $\{\overline{F}^n a\}$ is Cauchy and converges to some $u \in D$. Then,

$$Fu = u = Gu,$$

so $u$ is a common fixed point of $F$ and $G$.

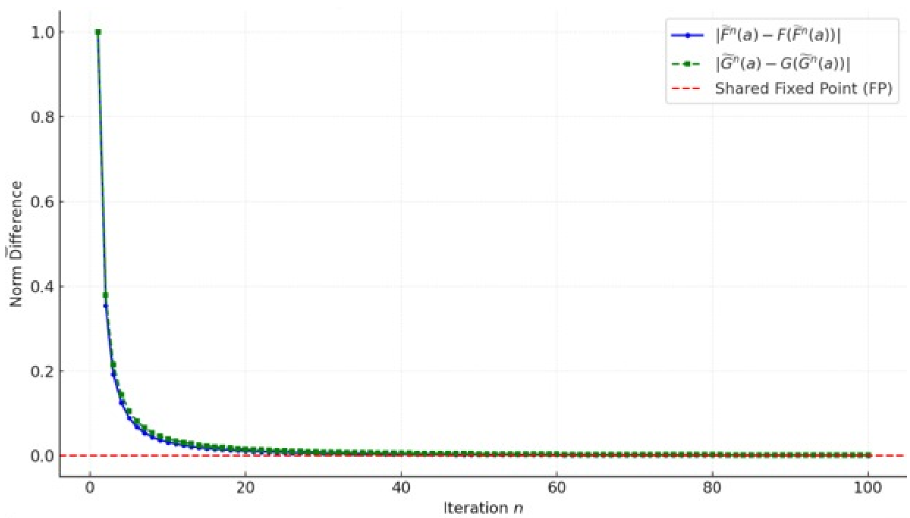


Figure 5: Convergence of Iterative Processes for F and G

□

**Example 5.1** Suppose $M = \mathbb{R}$ is equipped with the usual norm $\|a\| = |a|$, and let $D^\star = [-1, 1]$. Define mappings $F, G : D^\star \to D^\star$ by $F(a) = a$ for all $a \in D^\star$ and

$$G(a) = \begin{cases} 0, & \text{if } a \in [0, 1], \\ a, & \text{if } a \in [-1, 0). \end{cases}$$

The metric induced by the norm is $\theta(a,b) = |a-b|$ for all $a, b \in M$. We first show that $F$ and $G$ do not satisfy inequality. Assume, for contradiction, that they do:

$$\theta(Fa, Gb) \leq \theta(a,b) + \tau(\theta(a, Fa) + \theta(b, Gb)), \quad \forall a, b \in D^{\star}. \tag{5.12}$$

Choosing $a = 1$ and $b = 0$, we obtain $1 \leq \theta$, which contradicts $\theta \in [0,1)$. Hence, $F$ and $G$ do not satisfy the condition of Theorem 1.2. However, for all $a, b \in D^{\star}$, the inequality $|a-b|+(|a-Fa|+|b-Gb|)-|Fa-Gb| \geq 0$ holds, so $F$ and $G$ fulfill condition (i) of above theorem when $\tau = 1$. Define iterative mappings $\overline{F}a = \frac{3}{4}a$ and $\overline{G}a = \frac{3}{4}Ga = \frac{3}{4}a^2$ for all $a \in D^{\star}$. Then, for all $a \in D^{\star}$, we find:

$$\lim_{n\to\infty} \|\overline{F}^n a - \overline{F}^{n+1} a\| = 0, \quad \lim_{n\to\infty} \|\overline{G}^n a - \overline{G}^{n+1} a\| = 0,$$

$$\lim_{n\to\infty} \|\overline{F}^n a - F(\overline{F}^n a)\| = 0, \quad \lim_{n\to\infty} \|\overline{G}^n a - G(\overline{G}^n a)\| = 0.$$

Thus, conditions (ii) and (iii) of Theorem 2.3.2 are satisfied. Since $F$ and $G$ are continuous on $D^{\star}$, condition (iv) also holds. Therefore, all conditions of Theorem 2.3.2 are satisfied, and $F$ and $G$ share a common fixed point. It is easily verified that the unique common fixed point of $F$ and $G$ is $u = 0$. **Remark:** If we set $\tau = 0$ in Theorem 5.1, we recover a nonexpansive-type result. In subsequent analysis, $F$ represents the set of fixed points of a mapping $h$ on a subset $C_1$ of a complete metric space.

**Theorem 5.2** *Consider $(M, \theta)$ as a complete and continuously convex metric space. Let $C_1 \subset M$ be a nonempty closed convex subset, and let $h : C_1 \to C_1$ be a continuous quasi-nonexpansive mapping satisfying Condition 2.*

*If the sequence $\{u_n\}$ is defined as in (2.3.12), $\sum_{n_1=0}^{\infty} \alpha_{n_1}(1-\alpha_{n_1}) = \infty$, and $0 \leq \beta_n \leq \beta < 1$, then $\{u_n\}$ converges to a fixed point of $h$.*

**Proof:** It is easy to show that $\theta(a_{n+1}, p) \leq \theta(u_n, p)$ for any fixed point $p$ of $h$.

Therefore, $\theta(a_{n+1}, F) \leq \theta(u_n, F)$, where $F$ is the fixed point set of $h$. This implies that the sequence $\{\theta(u_n, F)\}$ is non-increasing and bounded below. Thus,

$$\lim_{n\to\infty} \theta(u_n, F) \text{ exists.}$$

Due to Condition 2, we have

$$\liminf_{n\to\infty} f(\theta(V_n, F)) \leq \liminf_{n\to\infty} \theta(h(b_n), V_n) = 0.$$

By the properties of $f$, it follows that

$$\liminf_{n\to\infty} \theta(V_n, F) = 0.$$

Since the limit exists, we conclude that

$$\lim_{n\to\infty} \theta(V_n, F) = 0.$$

Now we show that $\{V_n\}$ is a Cauchy sequence. Given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\theta(V_n, F) < \frac{\epsilon}{4}.$$

In particular,

$$\theta(a_{n_0}, F) < \frac{\epsilon}{4},$$

so there exists $p^* \in F$ such that

$$\theta(a_{n_0}, p^*) < \frac{\epsilon}{2}.$$

For $m, n \geq n_0$, we have:

$$\begin{aligned}\theta(a_{n+m}, V_n) &\leq \theta(a_{n+m}, p^*) + \theta(V_n, p^*) \\ &< 2\theta(a_{n_0}, p^*) < \epsilon.\end{aligned}$$

Thus, $\{V_n\}$ is a Cauchy sequence in $C$, and since $C$ is closed in the complete space $M$, it follows that $V_n \to q \in C$.

Next, we verify that $q$ is a fixed point of $h$. Since

$$\theta(q, F) \leq \theta(q, V_n) + \theta(V_n, F),$$

and both terms on the right tend to zero, we get $\theta(q, F) = 0$. As $F$ is closed, $q \in F$.

Finally, note that if we set $\beta_n = 0$ for all $n \geq 1$, this theorem reduces to the convergence result for a Mann-type iteration scheme. □

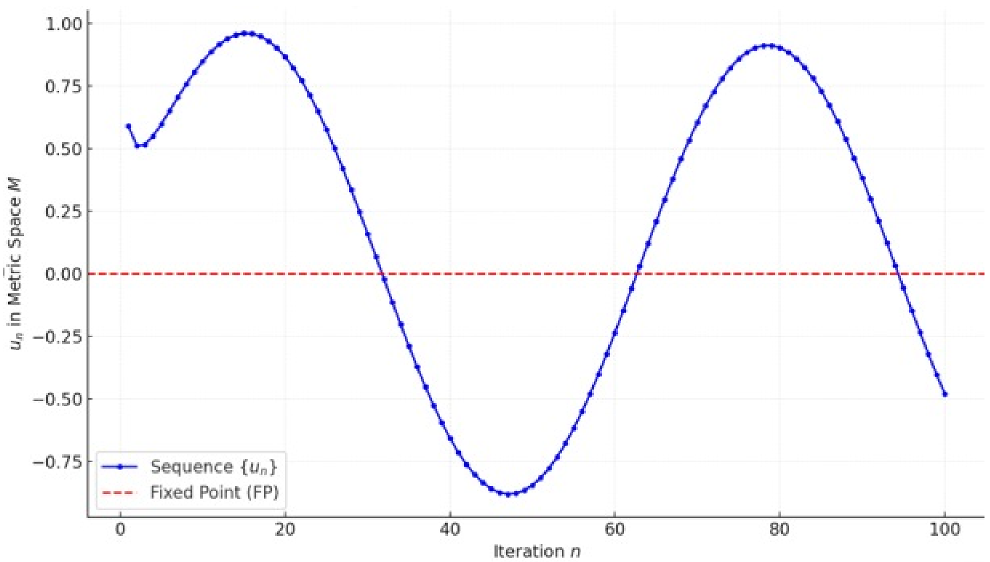


Figure 6: Convergence of Sequence $\{u_n\}$ to a Fixed Point of $h$

## 6. Conclusion

This paper significantly advances fixed point theory by extending Gornicki's contractive mapping theorem to convex metric spaces and by building upon the foundational work of Khan and Oyetubi on asymptotically regular mappings satisfying the Reich-type contractive condition. By generalizing these concepts and establishing new results for enriched Ciric–Reich–Rus-type contractions and nonexpansive mappings—as detailed in Theorem above and Theorem 5.2—this work broadens the theoretical framework of fixed point phenomena in structured metric environments. Future research can explore fixed point results for multivalued or fuzzy mappings in convex and modular metric spaces. Extending the current results to uniformly convex or probabilistic metric spaces could broaden their applicability. Additionally, developing iterative numerical schemes based on the presented theorems may aid in solving real-world nonlinear problems.

**List of Abbreviations:**

- UFP — Unique Fixed Point
- CFP — Common Fixed Point

- CS — Cauchy Sequence
- EA — Existence and Approximation Property
- FP — Fixed Point
- NS — Nonexpansive Sequence

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