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Dynamics of the nonlinear Systems in the Frame of the Caputo-Fabrizio Fractional Derivative*

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ABSTRACT: In this manuscript, efficient numerical technique called the fractional homotopy perturbation transform method (FHPTM) is used for solving linear and nonlinear systems. Caputo fabrizio derivative (CFD) is used to define fractional operator in this class to get approximate solution for four cases of the homogeneous linear system and inhomogeneous nonlinear system of equation. The FHPTM is used with the Laplace transform technique in this work. To demonstrate the capabilities and efficiency of the FHPTM approach, several applicable problems from various domains of Science and Engineering, such as Physics and Biology, are presented. We exhibit various two and three dimensional figures to demonstrate, these solutions have wave-like features.

Key Words: Homotopy perturbation transform method, Caputo fabrizio derivative, Laplace transform.

Contents

1. Introduction

Despite its origins in Newton's time, the theory and applications of arbitrary order derivatives have recently piqued the interest of many young Scientists. Fractional calculus(FC) had attracted the consideration of a large number of academics from numerous professions in fresh decades. The fractional derivatives have been found to be very useful for modelling a variety of problems, including computational fluid dynamics, Statistics, plasma Physics, Astrophysics, Chemistry, Physics, Mechanics, Quantum theory , Molecular Biology, hydro-dynamics, nonlinear optics, optics fibres, stratum water wave, bio-sciences, Economics, Geology, Probability, Chemical Physics,control theory of dynamical systems, material viscoelastic theory, dynamics of earthquakes, electromagnetic theory, engineering sciences, & so on [1- 4]. The data that there are several different type of fractional operator makes dealing with this calculus appealing. Due to the variety of fractional operators available, it is possible to select the most suited operator for achieving better outcomes. The most well-known fractional operators are the CF, the (R-L) fractional integrals & operators,the Caputo Fabrizio derivative, [5- 6]. However, the CF, which has a close relationship with fractional differentiation (RL), is used to simulate various physical difficulties. There are other

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fractional derivatives that resemble the (R-L) and Caputo Fabrizio derivative ones [7- 8]. Singular kernels are present in all of the above-mentioned derivative, which may prevent superior results from being obtained when they are applied. Scientists have sensed the necessity for fractional derivative along with non-singular kernel for this reason. In data, few researchers have been able to show fractional operators with nonsingular kernels [9- 11]. The homotopy perturbation method[12], Sine-Gordon expansion method [13], [14] Integral transform method [15], rational sine–cosine and rational sinh–cosh methods [16], Conformable Derivative [17],[18], Hybrid B-spline differential quadrature [19, 20], the additive formalism [21], the design method [22], the computer algebra method [23], the q-homotopy analysis transform method [24], [25], the natural transform iterative method [26]. Furthermore, the method's versatility allows it to be used in a wide range of analytical and numerical applications. Systems of PDEs has received a significant amount of attention.
Consider that the non-linear system of equations is given as follows [4]:

$$\begin{aligned} L_t u + \Re_1(u,v,w) + \wp_1(u,v,w) =& \Im_1, \\ L_t v + \Re_2(u,v,w) + \wp_2(u,v,w) =& \Im_2, \\ L_t w + \Re_3(u,v,w) + \wp_3(u,v,w) =& \Im_3, \end{aligned} \tag{1.1}$$

with initial conditions

$$\begin{aligned} u(x,0) =& \aleph_1(x), \\ v(x,0) =& \aleph_2(x), \\ w(x,0) =& \aleph_2(x), \end{aligned} \tag{1.2}$$

Generalizing the considered model Eq. (1) to fractional order with the aid of novel Caputo-Fabrizio fractional derivative. Hence, the generalised system of the considered model i.e. the nonlinear fractional PDEs model in the sense of Caputo-Fabrizio fractional derivative

$$\begin{aligned} {}_0^{CF}\, \text{Ł}_t^{\alpha}\ u + \Re_1(u,v,w) + \wp_1(u,v,w) =& \Im_1, \\ {}_0^{CF}\, \text{Ł}_t^{\alpha}\ v + \Re_2(u,v,w) + \wp_2(u,v,w) =& \Im_2, \\ {}_0^{CF}\, \text{Ł}_t^{\alpha}\ w + \Re_3(u,v,w) + \wp_3(u,v,w) =& \Im_3, \end{aligned} \tag{1.3}$$

such that

$$\begin{aligned} u(x,0) =& \aleph_1(x), \\ v(x,0) =& \aleph_2(x), \\ w(x,0) =& \aleph_2(x), \end{aligned} \tag{1.4}$$

Where ${}_0^{CF}\, \text{Ł}_t^{\alpha}$ is Caputo fabrizio fractional operator $\Re_j,\ 1 \leq j \leq 3$, stands for linear differential operator $\wp_j,\ 1 \leq j \leq 3$, for nonlinear differential operator, and $\Im_1, \Im_2$ and $\Im_3$ for source terms. The major steps of He's FHPTM in dealing with erudite and technical challenges are outlined below. In serval physical system, such as turbulent mixing, soil water, traffic flow, wave propagation,and nuclear fusion reactors, this equation retains the linear and nonlinear characteristics of the governing equations. Because most complex phenomena are formally modelled by systems of equations, analytical and numerical solutions for linear and non-linear differential equation of arbitrary order are crucial. [27]. The semi analytical solutions to NPDEs with stochastic [28].The novel approximation formula connected fractional order PDEs [29]. To validate the technique, a variety of test problems have been included. The present article (i) the brief introduction and few basic term of FC is relevant in section 2, (ii) the (FHPTM) is discussed in section 3, (iii) the solution for system of equation using Caputo fabrizio fractional operator by FHPTM is given in section 4, (iv) We used $2D$ and $3D$ graphics to give a physical and graphical representation of some of the solutions we found. in section 5. Finally, we have presented the detailed conclusion in section 6.

2. Preliminaries

Some fundamental definitions of the Riemann-Liouville (R-L) fractional differentiation, Laplace transform (LT) and FCD are presented [2-3].

Definition 2.1 For $\alpha > 0$ left (R-L) order fractional integral of α is defined as [2]

$$ {}_a\, I_t^{\alpha}\, u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} u(\xi) d\xi, \tag{2.1}$$

Definition 2.2 For $0 < \alpha < 1$ left (R-L) order fractional integral of α is given as [2]

$$ ({}_a\, D_t^{\alpha}\, u)(t) = \frac{d}{dt} ({}_a\, I_t^{\alpha}\, u)(t) = \frac{\frac{d}{dt}}{\Gamma(1-\alpha)} \int_a^t (t-\xi)^{-\alpha} u(\xi) d\xi, \tag{2.2}$$

Definition 2.3 For Caputo derivative is define for $\alpha \geq 0$ & $n \in N \cup 0$ is define as follows [2]

$$ {}_a^C\, D_t^{\alpha}\, u(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\xi) \frac{d^n}{dt^n} u(\xi) d\xi, \tag{2.3}$$

Definition 2.4 Consider u be a function $u \in H^1(a_1, b_1)$, $b_1 > 0$, $0 < \alpha < 1$. Then, the fractional Caputo-fabrizio factional operator is define as [2]

$$ {}_0^{CF}\, D_t^{\alpha}\, u(t) = \frac{\lambda(\alpha)}{1-\alpha)} \int_0^t exp[-\frac{\alpha(1-\xi)}{1-\alpha}] u'(\xi) d\xi, t \geq 0,\ 0 < \alpha < 1, \tag{2.4}$$

with a normalize functions $\lambda(\alpha)$ which is depend on $\alpha \in \lambda(0) = \lambda(1) = 1$.
Definition 2.5 for CFD for integer order of $0 < \alpha < 1$. is given by [2]

$$ {}_0^{CF}\, D_t^{\alpha}\, u(t) = \frac{2(1-\alpha)}{\lambda(\alpha)(2-\alpha)} u(t) + \frac{2\alpha}{\lambda(\alpha)(2-\alpha)} \int_0^t u(\xi) d\xi,\ t \geq 0, \tag{2.5}$$

where ${}_0^{CF}\, D_t^{\alpha}\, u(t) = 0$, if u is a constant function.
Definition 2.6 Laplace transform (LT) for the (CFD) of order $0 < \alpha < 1$. and $m \in N$ is gives by [2]

$$ \begin{aligned} \mathscr{L}\left[{}_0^{CF}\, D_t^{(m+\alpha)}\, u(t)\right](s) =& \frac{1}{1-\alpha} \mathscr{L}[u^{m+1}(t)] \mathscr{L}\left[exp\left(\frac{-\alpha}{(1-\alpha)}t\right)\right] \\ =& \frac{s^{m+1}\mathscr{L}[u(t)] - s^m u(0) - s^{m-1}u'(0) \ldots - u^m(0)}{s+\alpha(1-s)} \end{aligned} \tag{2.6}$$

In particular, we have

$$ \mathscr{L}\left[{}_0^{CF}\, D_t^{(m+\alpha)}\, u(t)\right](s) = \frac{s\mathscr{L}(u(t))}{s+\alpha(1-s)}, \quad m = 0, $$

$$ \mathscr{L}\left[{}_0^{CF}\, D_t^{(m+\alpha)}\, u(t)\right](s) = \frac{s^2\mathscr{L}(u(t)) - su(o) - u'(0)}{s+\alpha(1-s)}, \quad m = 1. $$

3. General description system of FHPTM using Caputo-Fabrizio type operator:

Let's as considering the following system of equation [2-3]

$$ \begin{aligned} {}_0^{CF}\, D_t^{(m+\alpha)} u(x,t) + \beta u(x,t) + \varphi u(x,t) = k(x,t), &\quad n-1 < \alpha+m \leq n, \\ {}_0^{CF}\, D_t^{(m+\alpha)} v(x,t) + av(x,t) + bv(x,t) = y(x,t), &\quad n-1 < \alpha+m \leq n, \\ {}_0^{CF}\, D_t^{(m+\alpha)} w(x,t) + cw(x,t) + dw(x,t) = z(x,t), &\quad n-1 < \alpha+m \leq n, \end{aligned} \tag{3.1}$$

such that

$$ \begin{aligned} \frac{\partial^l u(x,0)}{\partial t^l} &= f_l(x), \quad l = 0,1,2,\ldots n-1, \\ \frac{\partial^l v(x,0)}{\partial t^l} &= g_l(x), \quad l = 0,1,2,\ldots n-1, \\ \frac{\partial^l w(x,0)}{\partial t^l} &= \hbar_l(x), \quad l = 0,1,2,\ldots n-1. \end{aligned} \tag{3.2}$$

Taking Laplace transform's on both side's of Eqs. (3.1), we get

$$\begin{aligned}\mathscr{L}[u(x,t)] &= \Theta(x,s) - \left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathscr{L}[\beta u(x,t)+\varphi u(x,t)].\\ \mathscr{L}[v(x,t)] &= \Omega(x,s) - \left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathscr{L}[av(x,t)+bv(x,t)].\\ \mathscr{L}[w(x,t)] &= \Xi(x,s) - \left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathscr{L}[cw(x,t)+dw(x,t)].\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\Theta(x,s) &= \frac{1}{s^{m+1}}[s^m f_0(x)+s^{m-1}f_1(x)+\ldots+f_m(x)] + \frac{s+\alpha(1-s)}{s^{n+1}}\tilde{k}(x,s).\\ \Omega(x,s) &= \frac{1}{s^{m+1}}[s^m g_0(x)+s^{m-1}g_1(x)+\ldots+g_m(x)] + \frac{s+\alpha(1-s)}{s^{n+1}}\Phi(x,s).\\ \Xi(x,s) &= \frac{1}{s^{m+1}}[s^m g_0(x)+s^{m-1}g_1(x)+\ldots+g_m(x)] + \frac{s+\alpha(1-s)}{s^{n+1}}\Psi(x,s).\end{aligned} \tag{3.4}$$

Taking the inverse Laplace transformation the Eq.(3.3) we have

$$\begin{aligned}u(x,t) &= \Theta(x,s) - \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathscr{L}[\beta u(x,t)+\varphi u(x,t)]\right],\\ v(x,t) &= \Omega(x,s) - \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathscr{L}[av(x,t)+bv(x,t)]\right],\\ w(x,t) &= \Xi(x,s) - \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathscr{L}[cw(x,t)+dw(x,t)]\right],\end{aligned} \tag{3.5}$$

$\Theta(x,s),\ \Omega(x,s),\ \Xi(x,s)$ is the term that arises from the source term, and it specifies the initial conditions. The solution $u(x,t),\ v(x,t),\ w(x,t)$ may be extended into an infinite sequence using the regular HPTM as follows:

$$\begin{aligned}u(x,t) &= \sum_{m=0}^{\infty} p^m u_m(x,t),\\ v(x,t) &= \sum_{m=0}^{\infty} q^m v_m(x,t),\\ w(x,t) &= \sum_{m=0}^{\infty} r^m w_m(x,t).\end{aligned} \tag{3.6}$$

where $u_m(x,t), v_m(x,t), w_m(x,t)$ are known functions, we have

$$\begin{aligned}\varphi u(x,t) &= \sum_{m=0}^{\infty} p^m H_m(x,t),\\ cv(x,t) &= \sum_{m=0}^{\infty} q^m I_m(x,t),\\ dw(x,t) &= \sum_{m=0}^{\infty} r^m J_m(x,t).\end{aligned} \tag{3.7}$$

The poly. $H_m(x,t),\ I_m(x,t),\ J_m(x,t)$ are define as [12]

$$\begin{aligned} H_m(u_0,u_1,u_2,\dots u_m) &= \frac{1}{n!}\frac{\partial^m}{\partial p^m}\left[\left(\sum_{m=0}^{\infty} p^i u_i\right)\right]_{p=0}, \qquad m=0,1,2,\dots; \\ I_m(v_0,v_1,v_2,\dots v_m) &= \frac{1}{n!}\frac{\partial^m}{\partial q^m}\left[\left(\sum_{m=0}^{\infty} q^i v_i\right)\right]_{q=0}, \qquad m=0,1,2,\dots; \\ J_m(w_0,w_1,w_2,\dots w_m) &= \frac{1}{n!}\frac{\partial^m}{\partial r^m}\left[\left(\sum_{m=0}^{\infty} r^i w_i\right)\right]_{r=0}, \qquad m=0,1,2,\dots; \end{aligned} \tag{3.8}$$

substitute Eqs.(3.8- 3.7) into Eq.(3.6) we are getting

$$\begin{aligned} \sum_{m=0}^{\infty} u_m(x,t) &= \Theta(x,s) - p\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}\left[\beta\sum_{m=0}^{\infty} z^m u_m(x,t) + \sum_{m=0}^{\infty} z^m H_m\right]\right], \\ \sum_{m=0}^{\infty} v_m(x,t) &= \Omega(x,s) - q\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}\left[a\sum_{m=0}^{\infty} q^m v_m(x,t) + \sum_{m=0}^{\infty} q^m I_m\right]\right], \\ \sum_{m=0}^{\infty} w_m(x,t) &= \Xi(x,s) - r\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}\left[c\sum_{m=0}^{\infty} r^m w_m(x,t) + \sum_{m=0}^{\infty} r^m J_m\right]\right]. \end{aligned} \tag{3.9}$$

Comparing the coefficients of $p_0,\ p_1,\ p_2,\ p_3,\ q_0,\ q_1,\ q_2,\ q_3$, and $r_0,\ r_1,\ r_2,\ r_3$we get

$$
\begin{aligned}
p_0 &: u_0(x,t) = \Theta(x,s),\\
q_0 &: v_0(x,t) = \Omega(x,s),\\
r_0 &: w_0(x,t) = \Xi(x,s),\\
p_1 &: u_1(x,t) = -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[\beta u_0(x,t)+H_0(u)]\right],\\
q_1 &: v_1(x,t) = -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[a v_0(x,t)+I_0(v)]\right],\\
r_1 &: w_1(x,t) = -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[c v_0(x,t)+J_0(v)]\right],\\
p_2 &: u_2(x,t) = -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[\beta u_1(x,t)+H_1(u)]\right],\\
q_2 &: v_2(x,t) = -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[a v_1(x,t)+I_1(v)]\right],\\
r_2 &: u_2(x,t) = -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[c w_1(x,t)+J_1(w)]\right],\\
p_3 &: u_3(x,t) = -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[\beta u_2(x,t)+H_2(u)]\right],\\
q_3 &: v_3(x,t) = -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[a v_2(x,t)+H_2(v)]\right],\\
r_3 &: w_3(x,t) = -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[c w_2(x,t)+H_2(w)]\right],\\
&\vdots\\
p_{m+1} &: u_{m+1}(x,t) = -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[\beta u_{m+1}(x,t)+H_{m+1}(u)]\right],\\
q_{m+1} &: v_{m+1}(x,t) = -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[a v_{m+1}(x,t)+I_{m+1}(v)]\right],\\
r_{m+1} &: w_{m+1}(x,t) = -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[c w_{m+1}(x,t)+J_{m+1}(w)]\right].
\end{aligned}
\tag{3.10}
$$

As a result, approximate solutions is

$$
\begin{aligned}
u(x,t) &= \sum_{m=0}^{\infty} u_m(x,t),\\
v(x,t) &= \sum_{m=0}^{\infty} v_m(x,t),\\
w(x,t) &= \sum_{m=0}^{\infty} w_m(x,t).
\end{aligned}
\tag{3.11}
$$

Equation (19) represents series solution, which converges very fast.

4. Numerical experiments

Some basic terminology and terms are included to help the reader comprehend the technique and make the article more readable.

4.1. Homogeneous linear system

We first considering homogeneous linear system is given by [4]

$$\begin{aligned}
{}_0^{CF}\, u_t^{\alpha} &- v_x + (u+v) = 0,\\
{}_0^{CF}\, v_t^{\alpha} &- u_x + (u+v) = 0,
\end{aligned} \tag{4.1}$$

such that initial condition (I.C)

$$u(x,0) = \sinh x, \quad v(x,0) = \cosh x. \tag{4.2}$$

Laplace transformation for Eq. (4.1), we have

$$\begin{aligned}
\mathscr{L}[u(x,t)] &= \frac{1}{s}u(x,0) + \left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\frac{\partial v}{\partial x} - (u+v)\right],\\
\mathscr{L}[v(x,t)] &= \frac{1}{s}v(x,0) + \left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\frac{\partial u}{\partial x} - (u+v)\right],
\end{aligned} \tag{4.3}$$

Apply the inverse Laplace transform for Eq.(4.3)

$$\begin{aligned}
u(x,t) &= \sinh x + \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right) L\left[\frac{\partial v}{\partial x} - (u+v)\right]\right],\\
v(x,t) &= \cosh x + \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right) L\left[\frac{\partial u}{\partial x} - (u+v)\right]\right].
\end{aligned} \tag{4.4}$$

Now, we applying FHPTM

$$\begin{aligned}
\sum_{m=0}^{\infty} u_m(x,t) &= \sinh x + p\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\left(\sum_{m=0}^{\infty} p^m v_m(x,t)\right)_x - (u+v)\right]\right],\\
\sum_{m=0}^{\infty} v_m(x,t) &= \cosh x + q\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\left(\sum_{m=0}^{\infty} q^m u_m(x,t)\right)_x - (u+v)\right]\right],
\end{aligned} \tag{4.5}$$

Firmly facing the above conditions, we get

$$\begin{aligned}
p_0 : u_0(x,t) &= \sinh x,\\
q_0 : v_0(x,t) &= \cosh x,\\
u_1(x,t) &= -(1-\alpha+t\alpha)\text{cosh}x,\\
v_1(x,t) &= -(1-\alpha+t\alpha)\text{sinh}x,
\end{aligned}$$

$$\begin{aligned}
p_2 : u_2(x,t) &= \left(-1+2\alpha-\alpha^2-\frac{t^2\alpha^2}{2}+2t\left(-\alpha+\alpha^2\right)\right)\text{sinh}x,\\
q_2 : v_2(x,t) &= \left(1-2\alpha+\alpha^2+\frac{t^2\alpha^2}{2}-2t\left(-\alpha+\alpha^2\right)\right)\text{cosh}x,\\
p_3 : u_3(x,t) &= \left(-1+3\alpha-3\alpha^2+\alpha^3-\frac{t^3\alpha^3}{6}-3t\left(\alpha-2\alpha^2+\alpha^3\right)+\frac{3}{2}t^2\left(-\alpha^2+\alpha^3\right)\right)\text{sinh}x,\\
q_3 : v_3(x,t) &= \left(-1+3\alpha-3\alpha^2+\alpha^3-\frac{t^3\alpha^3}{6}-3t\left(\alpha-2\alpha^2+\alpha^3\right)+\frac{3}{2}t^2\left(-\alpha^2+\alpha^3\right)\right)\text{cosh}x,
\end{aligned} \tag{4.6}$$

$\vdots$

we can get the remaining components of the iteration formula by continuing in this manner. As a result, approximate solutions is follows

$$\begin{aligned} u(x,t) &= \sum_{m=0}^{\infty} u_m(x,t) \\ v(x,t) &= \sum_{m=0}^{\infty} v_m(x,t) \end{aligned} \tag{4.7}$$

4.2. Inhomogeneous linear system

Let's considering inhomogeneous linear system is given as [4]

$$\begin{aligned} {}_0^{CF}\ u_t^{\alpha}\ - v_x - (u-v) &= -2, \\ {}_0^{CF}\ v_t^{\alpha}\ + u_x - (u-v) &= -2, \end{aligned} \tag{4.8}$$

such that initial condition

$$u(x,0) = 1 + e^x, \quad v(x,0) = -1 + e^x. \tag{4.9}$$

Laplace transformation on both sides Eq. (4.8) , we get

$$\begin{aligned} \mathscr{L}[u(x,t)] &= \frac{1}{s}u(x,0) + \left(\frac{s+\alpha(1-s)}{s}\right)\left[\frac{-2}{s}\right] + \left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\frac{\partial v}{\partial x} + (u-v)\right], \\ \mathscr{L}[v(x,t)] &= \frac{1}{s}v(x,0) + \left(\frac{s+\alpha(1-s)}{s}\right)\left[\frac{-2}{s}\right] - \left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\frac{\partial u}{\partial x} + (u-v)\right], \end{aligned} \tag{4.10}$$

Apply the inverse Laplace transform for Eq.(4.10)

$$\begin{aligned} u(x,t) &= u(x,0) - 2(1-\alpha+t\alpha) + \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right) L\left[\frac{\partial v}{\partial x} + (u-v)\right]\right], \\ v(x,t) &= v(x,0) - 2(1-\alpha+t\alpha) - \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right) L\left[\frac{\partial u}{\partial x} + (u-v))\right]\right], \end{aligned} \tag{4.11}$$

Now, we apply FHPTM

$$\begin{aligned} \sum_{m=0}^{\infty} u_m(x,t) &= u(x,0) - 2(1-\alpha+t\alpha) + p\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\left(\sum_{m=0}^{\infty} p^m v_m(x,t)\right)_x + (u-v)\right]\right], \\ \sum_{m=0}^{\infty} v_m(x,t) &= v(x,0) - 2(1-\alpha+t\alpha) - q\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\left(\sum_{m=0}^{\infty} q^m u_m(x,t)\right)_x + (u-v))\right]\right], \end{aligned} \tag{4.12}$$

Firmly facing the above conditions, we get

$$\begin{aligned}
p_0 : u_0(x,t) =& 1 + e^x - 2(1-\alpha+t\alpha),\\
q_0 : v_0(x,t) =& -1 + e^x - 2(1-\alpha+t\alpha),\\
p_1 : u_1(x,t) =& (2+e^x)(1-\alpha+t\alpha),\\
q_1 : v_1(x,t) =& -(2+e^x)(1-\alpha+t\alpha),\\
p_2 : u_2(x,t) =& \frac{1}{2}(4+e^x)\left(2-4\alpha+4t\alpha+2\alpha^2-4t\alpha^2+t^2\alpha^2\right),\\
q_2 : v_2(x,t) =& -\frac{3}{2}(2+e^x)\left(2-4\alpha+4t\alpha+2\alpha^2-4t\alpha^2+t^2\alpha^2\right),\\
p_3 : u_3(x,t) =& \frac{1}{2}(4+e^x)\left(\frac{t^3\alpha^3}{3}-2\left(-1+3\alpha-3\alpha^2+\alpha^3\right)+6t\left(\alpha-2\alpha^2+\alpha^3\right)-3t^2\left(-\alpha^2+\alpha^3\right)\right),\\
q_3 : v_3(x,t) =& 5(2+e^x)(-1+\alpha)^3-15(2+e^x)t(-1+\alpha)^2\alpha+\frac{15}{2}(2+e^x)t^2(-1+\alpha)\alpha^2-\frac{5}{6}(2+e^x)t^3.\\
&\vdots
\end{aligned} \tag{4.13}$$

we can get the remaining components of the iteration formula by continuing in this manner. As a result, approximate solutions is follows

$$\begin{aligned}
u(x,t) &= \sum_{m=0}^{\infty} u_m(x,t)\\
v(x,t) &= \sum_{m=0}^{\infty} v_m(x,t)
\end{aligned} \tag{4.14}$$

4.3. Inhomogeneous nonlinear system

We now considering inhomogeneous nonlinear system as follow [4]

$$\begin{aligned}
{}_0^{CF}\ u_t^{\alpha}\ + vu_x + u = 1,\\
{}_0^{CF}\ v_t^{\alpha}\ - uv_x - v = 1,
\end{aligned} \tag{4.15}$$

such that Initial Condition

$$u(x,0) = e^x, \quad v(x,0) = e^{-x}. \tag{4.16}$$

Laplace transformation on both sides Eq. (4.15) , we get

$$\begin{aligned}
\mathscr{L}[u(x,t)] &= \frac{1}{s}u(x,0) + \left(\frac{s+\alpha(1-s)}{s}\right)\left[\frac{1}{s}\right] - \left(\frac{s-\alpha(1-s)}{s}\right)\mathscr{L}\left[v\frac{\partial u}{\partial x}+u\right],\\
\mathscr{L}[v(x,t)] &= \frac{1}{s}v(x,0) + \left(\frac{s+\alpha(1-s)}{s}\right)\left[\frac{1}{s}\right] + \left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[u\frac{\partial v}{\partial x}+v\right].
\end{aligned} \tag{4.17}$$

Inverse Laplace transform for Eq.(4.17)

$$\begin{aligned}
u(x,t) &= u(x,0) + (1-\alpha+t\alpha) - \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)L\left[v\frac{\partial u}{\partial x}+u\right]\right],\\
v(x,t) &= v(x,0) + (1-\alpha+t\alpha) + \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)L\left[u\frac{\partial v}{\partial x}+v)\right]\right].
\end{aligned} \tag{4.18}$$

Now, we apply FHPTM

$$\begin{aligned}
\sum_{m=0}^{\infty} u_m(x,t) &= u(x,0) + (1-\alpha+t\alpha) - p\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\left(\sum_{m=0}^{\infty} p^m H_m(x,t)\right)+u\right]\right],\\
\sum_{m=0}^{\infty} v_m(x,t) &= v(x,0) + (1-\alpha+t\alpha) + q\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\left(\sum_{m=0}^{\infty} q^m I_m(x,t)\right)+v\right]\right],
\end{aligned} \tag{4.19}$$

where

$$\sum_{m=0}^{\infty} p^m H_m(x,t) = vu_x$$
$$\sum_{m=0}^{\infty} q^m I_m(x,t) = uv_x.$$

Homotopy polynomials' first few components are written as

$$\begin{aligned} H_0(u) &= v_0 \frac{\partial u_o}{\partial x}, \\ I_0(v) &= u_0 \frac{\partial v_o}{\partial x}, \\ H_1(u) &= v_0 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_o}{\partial x}, \\ I_1(v) &= u_0 \frac{\partial v_1}{\partial x} + u_1 \frac{\partial v_o}{\partial x}. \\ &\vdots \end{aligned} \tag{4.20}$$

Firmly facing the above conditions, we get

$$\begin{aligned} p_0 : u_0(x,t) =& e^x + (1 - \alpha + t\alpha), \\ q_0 : v_0(x,t) =& e^{-x} + (1 - \alpha + t\alpha), \\ p_1 : u_1(x,t) =& -\frac{1}{2}(1+e^x) t^2\alpha^2 - e^{-x}(-1+\alpha)\left(-1 - e^x - 2e^{2x} + e^x\alpha + e^{2x}\alpha\right) \\ & + e^{-x} t\alpha \left(-1 - 2e^x - 3e^{2x} + 2e^x\alpha + 2e^{2x}\alpha\right), \\ q_1 : v_1(x,t) =& \frac{1}{2} e^{-x}(-1+e^x) t^2\alpha^2 + e^{-x} t\alpha\left(-1 + 2e^x + e^{2x} + 2\alpha - 2e^x\alpha\right) - e^{-x}(-1+\alpha)\left(e^x + e^{2x} + \alpha\right), \\ p_2 : u_2(x,t) =& \frac{1}{8} e^x t^4\alpha^4 + \frac{1}{2} e^{-x} t^2(-1+\alpha)\alpha^2\left(1 - 15e^{2x} + 8e^{2x}\alpha\right) - \frac{1}{6} e^{-x} t^3\alpha^3\left(1 - 11e^{2x} + 8e^{2x}\alpha\right) \\ & + e^{-x} t\alpha \left(-1 + e^x + 10e^{2x} - e^{3x} + \alpha - 3e^x\alpha - 23e^{2x}\alpha + e^{3x}\alpha - \alpha^2 + 2e^x\alpha^2\right. \\ & \left. + 17e^{2x}\alpha^2 - 4e^{2x}\alpha^3)\right) \\ & - e^{-x}(-1+\alpha)\left(-1 + e^x + 3e^{2x} - e^{3x} + \alpha - 3e^x\alpha - 6e^{2x}\alpha + e^{3x}\alpha - \alpha^2 + 2e^x\alpha^2\right. \\ & \left. + 4e^{2x}\alpha^2 - e^{2x}\alpha^3\right), \\ q_2 : v_2(x,t) =& \frac{1}{6} e^{-x} t^3\left(5 + e^{2x} - 8\alpha\right)\alpha^3 + \frac{1}{8} e^{-x} t^4\alpha^4 \\ & - \frac{1}{2} e^{-2x} t^2\alpha^2\left(-1 - 4e^x - 5e^{2x} - 4e^{3x} + 11e^x\alpha + 4e^{2x}\alpha + 3e^{3x}\alpha - 8e^x\alpha^2\right) \\ & - e^{-x}(-1+\alpha)\left(-1 + e^{2x} + \alpha + \alpha^2 - \alpha^3\right) \\ & + e^{-2x} t\alpha\left(1 + e^x + 5e^{2x} + 4e^{3x} - \alpha - 5e^x\alpha - 9e^{2x}\alpha - 5e^{3x}\alpha + 8e^x\alpha^2 + 4e^{2x}\alpha^2 + 2e^{3x}\alpha^2\right). \\ & \vdots \end{aligned} \tag{4.21}$$

we can get the remaining components of the iteration formula by continuing in this manner. As a result, approximate solutions is follows

$$\begin{aligned} u(x,t) &= \sum_{m=0}^{\infty} u_m(x,t) \\ v(x,t) &= \sum_{m=0}^{\infty} v_m(x,t) \end{aligned} \tag{4.22}$$

4.4. Homogeneous nonlinear system

Let's considering homogeneous nonlinear system as follow [4]

$$
\begin{aligned}
{}_0^{CF}\ u_t^{\alpha}\ + u_x v_x + u_y v_y + u = 0,\\
{}_0^{CF}\ v_t^{\alpha}\ + v_x w_x - v_y w_y - v = 0,\\
{}_0^{CF}\ w_t^{\alpha}\ + w_x u_x + w_y u_y - w = 0,
\end{aligned} \tag{4.23}
$$

such that initial condition

$$
u(x,y,0) = e^{x+y}, \quad v(x,y,0) = e^{x-y}, \quad w(x,y,0) = e^{-x+y}. \tag{4.24}
$$

Laplace transformation on both sides Eqs. (4.23) , we get

$$
\begin{aligned}
\mathscr{L}[u(x,y,t)] &= \frac{1}{s}u(x,0) - \left(\frac{s-\alpha(1-s)}{s}\right)\mathscr{L}\left[u_x v_x + u_y v_y + u\right],\\
\mathscr{L}[v(x,y,t)] &= \frac{1}{s}v(x,0) + \left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[v_x w_x - v_y w_y + v\right],\\
\mathscr{L}[w(x,y,t)] &= \frac{1}{s}w(x,0) - \left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[w_x u_x + w_y u_y + w\right].
\end{aligned} \tag{4.25}
$$

Applying inverse Laplace transform for Eq.(4.25)

$$
\begin{aligned}
u(x,y,t) &= u(x,0) - \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)L\left[u_x v_x + u_y v_y + u\right]\right],\\
v(x,y,t) &= v(x,0) + \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)L\left[v_x w_x - v_y w_y + v\right]\right],\\
w(x,y,t) &= w(x,0) - \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)L\left[w_x u_x + w_y u_y + w\right]\right].
\end{aligned} \tag{4.26}
$$

Now, we apply the FHPTM

$$
\begin{aligned}
\sum_{m=0}^{\infty} u_m(x,y,t) &= u(x,0) + (1-\alpha+t\alpha) - p\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\left(\sum_{m=0}^{\infty} p^m H_m(x,t)\right) + u\right]\right],\\
\sum_{m=0}^{\infty} v_m(x,y,t) &= v(x,0) + (1-\alpha+t\alpha) + q\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\left(\sum_{m=0}^{\infty} q^m I_m(x,t)\right) + v\right]\right],\\
\sum_{m=0}^{\infty} w_m(x,y,t) &= w(x,0) + (1-\alpha+t\alpha) + r\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\left(\sum_{m=0}^{\infty} r^m J_m(x,t)\right) + w\right]\right],
\end{aligned} \tag{4.27}
$$

where

$$
\begin{aligned}
\sum_{m=0}^{\infty} p^m H_m(x,t) &= u_x v_x + u_y v_y,\\
\sum_{m=0}^{\infty} q^m I_m(x,t) &= v_x w_x - v_y w_y,\\
\sum_{m=0}^{\infty} r^m J_m(x,t) &= w_x u_x + w_y u_y.
\end{aligned}
$$

Homotopy polynomials' first few components are written as

$$\begin{aligned}H_0(u) &= v_0(u_0)_x(v_0)_x + (u_0)_y(v_0)_y,\\ I_0(v) &= u_0(v_0)_x(w_0)_x - (v_0)_y(w_0)_y,\\ J_0(w) &= w_0(w_0)_x(u_0)_x + (w_0)_y(u_0)_y,\\ H_1(u) &= v_0(u_1)_x(v_0)_x + (u_0)_x(v_1)_x + (u_1)_y(v_0)_y + (u_0)_y(v_1)_y,\\ I_1(v) &= u_0(v_1)_x(w_0)_x + (v_0)_x(w_1)_x - \{(v_1)_y(w_0)_y + (v_0)_y(w_1)_y\}\\ J_1(v) &= u_0(w_1)_x(u_0)_x + (w_0)_x(u_1)_x + (w_1)_y(u_0)_y + (w_0)_y(u_1)_y\\ &\vdots\end{aligned} \tag{4.28}$$

Firmly facing the above conditions, we get

$$\begin{aligned}p_0 &: u_0(x,y,t) = e^{x+y},\\ q_0 &: v_0(x,y,t) = e^{x-y},\\ r_0 &: w_0(x,y,t) = e^{-x+y},\\ p_1 &: u_1(x,y,t) = -e^{x+y}(1-\alpha+t\alpha),\\ q_1 &: v_1(x,y,t) = e^{x-y}(1-\alpha+t\alpha),\\ r_1 &: w_1(x,y,t) = e^{-x+y}(1-\alpha+t\alpha)\\ p_2 &: u_2(x,y,t) = e^{x+y}\left(1-2\alpha+\alpha^2+\frac{t^2\alpha^2}{2}-2t\left(-\alpha+\alpha^2\right)\right),\\ q_2 &: v_2(x,t) = e^{x-y}\left(1-2\alpha+\alpha^2+\frac{t^2\alpha^2}{2}-2t\left(-\alpha+\alpha^2\right)\right)\\ r_2 &: w_2(x,t) = -e^{-x+y}\left(1-2\alpha+\alpha^2+\frac{t^2\alpha^2}{2}-2t\left(-\alpha+\alpha^2\right)\right)\\ p_3 &: u_3(x,y,t) = -e^{x+y}\left(1-3\alpha+3\alpha^2-\alpha^3+\frac{t^3\alpha^3}{6}+3t\left(\alpha-2\alpha^2+\alpha^3\right)-\frac{3}{2}t^2\left(-\alpha^2+\alpha^3\right)\right)\\ q_3 &: v_3(x,y,t) = e^{x-y}\left(1-3\alpha+3\alpha^2-\alpha^3+\frac{t^3\alpha^3}{6}+3t\left(\alpha-2\alpha^2+\alpha^3\right)-\frac{3}{2}t^2\left(-\alpha^2+\alpha^3\right)\right)\\ r_3 &: w_3(x,y,t) = e^{-x+y}\left(1-3\alpha+3\alpha^2-\alpha^3+\frac{t^3\alpha^3}{6}+3t\left(\alpha-2\alpha^2+\alpha^3\right)-\frac{3}{2}t^2\left(-\alpha^2+\alpha^3\right)\right).\\ &\vdots\end{aligned} \tag{4.29}$$

we can get the remaining components of the iteration formula by continuing in this manner.
As a result, approximate solutions is follows

$$\begin{aligned}u(x,t) &= \sum_{m=0}^{\infty} u_m(x,t)\\ v(x,t) &= \sum_{m=0}^{\infty} v_m(x,t)\\ w(x,t) &= \sum_{m=0}^{\infty} w_m(x,t)\end{aligned} \tag{4.30}$$

Table 1: Numerically study regard to $u(x,t)$ by FHPTM along with Caputo-Fabrizio derivative for system of equation (4.1) with diverse x,t as well as its separate values as α.

x	t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
0.2	0.2	-0.845818	-0.523989	-0.24726	-0.00697252
	0.4	-0.931392	-0.669015	-0.433717	-0.224945
	0.6	-1.01865	-0.819476	-0.62946	-0.454193
	0.8	-1.10762	-0.975575	-0.83517	-1.92876
0.4	0.2	-0.925581	-0.469008	-0.104443	0.185775
	0.4	-1.05016	-0.664879	-0.33484	-0.0589182
	0.6	-1.17819	-0.871841	-0.584183	-0.326614
	0.8	-1.3097	-1.0903	-0.853859	-0.620597
0.6	0.2	-1.04249	-0.432849	0.0341818	0.385979
	0.4	-1.21108	-0.687427	-0.249402	0.104744
	0.6	-1.38501	-0.959195	-0.562352	-0.212143
	0.8	-1.56434	-1.24879	-0.906815	-0.569776
0.8	0.2	-1.20124	-0.414063	0.174179	0.601673
	0.4	-1.42061	-0.737565	-0.173973	0.27261
	0.6	-1.64741	-1.08505	-0.563089	-0.106186
	0.8	-1.88177	-1.45739	-0.996166	-0.541821

Table 2: Numerically study regard to $v(x,t)$ by FHPTM along with Caputo-Fabrizio derivative for system of equation (4.1) with diverse x,t as well as its separate values as α.

x	t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
0.2	0.2	0.991267	1.04955	1.03746	0.998841
	0.4	0.966706	1.04608	1.05966	1.01026
	0.6	0.938703	1.03547	1.08072	1.04615
	0.8	0.90713	1.0167	1.09719	1.09837
0.4	0.2	0.89265	0.993899	1.02055	1.0191
	0.4	0.856752	0.970481	1.01448	0.991726
	0.6	0.817205	0.939496	1.00719	0.990295
	0.8	0.773875	0.899862	0.995039	1.00616
0.6	0.2	0.829858	0.978132	0.0341818	1.08026
	0.4	0.781182	0.933829	1.01001	1.013
	0.6	0.728504	0.881228	0.974085	0.954332
	0.8	0.800371	0.819143	0.932822	0.671678
0.8	0.2	0.736963	1.00162	1.11059	1.18478
	0.4	0.669041	0.934654	1.04608	1.07492
	0.6	0.669041	0.858326	0.980071	0.997162
	0.8	0.596438	0.771299	0.908043	0.940802

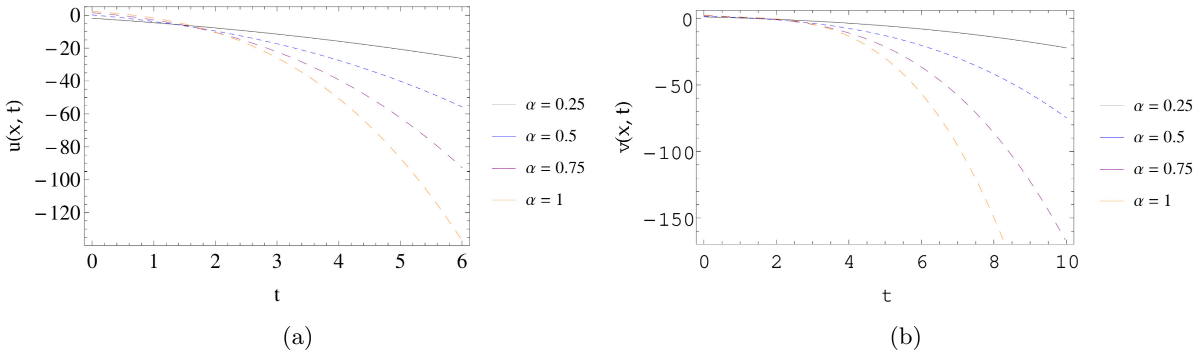


Figure 1: Comparison of approximate solution $u(x,t)$ & $v(x,y)$ for separate values of $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$ and $\alpha = 1$ for 4.1.

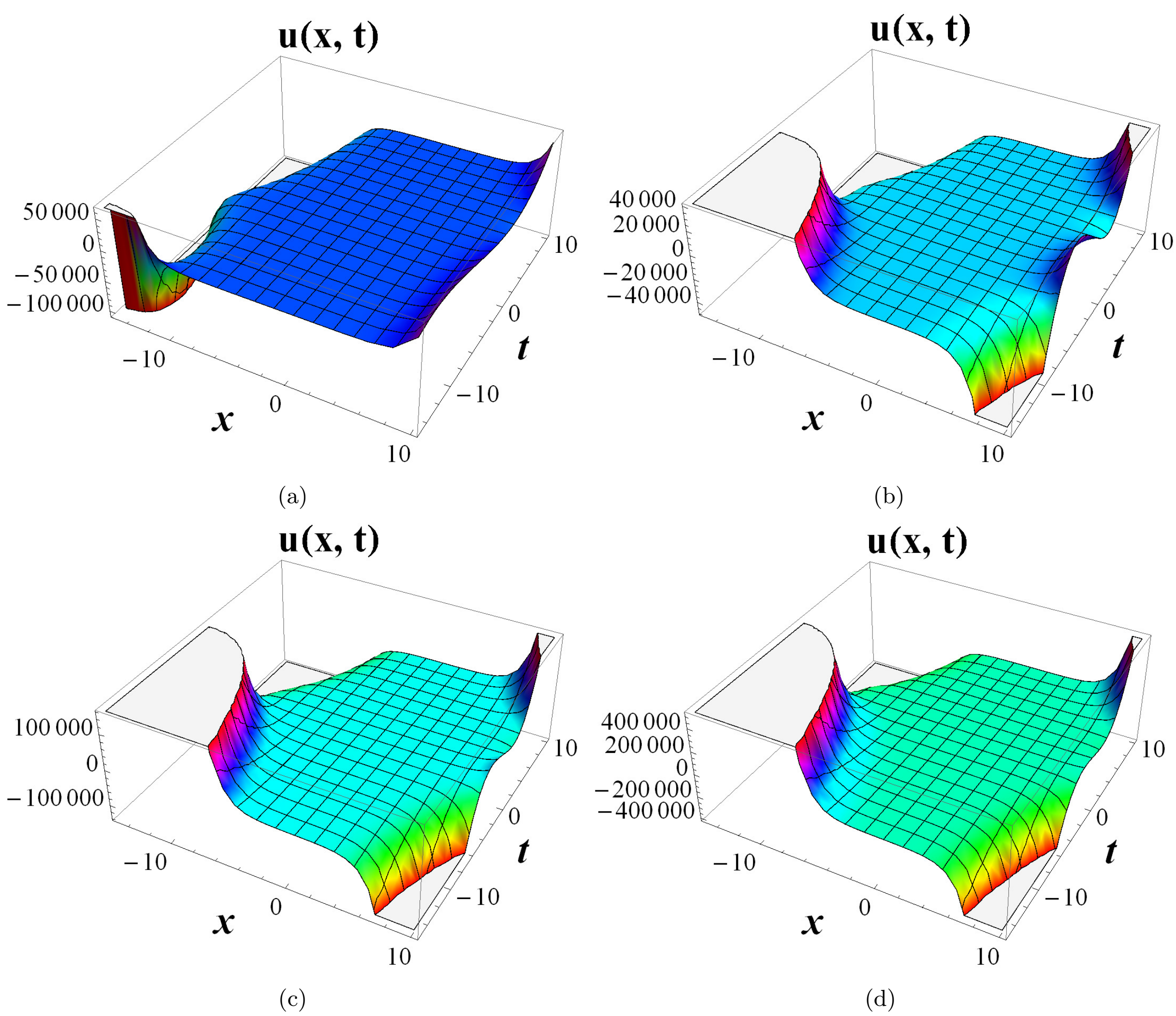
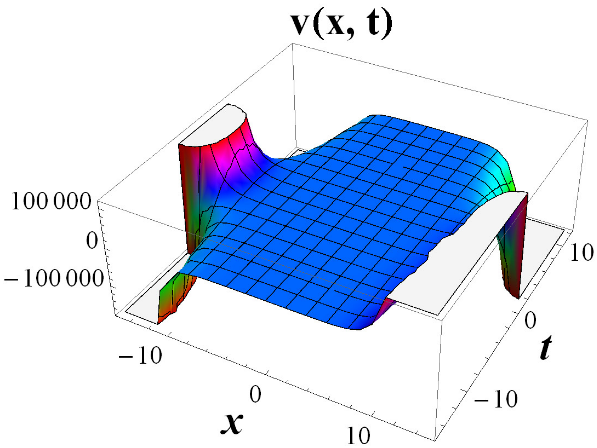
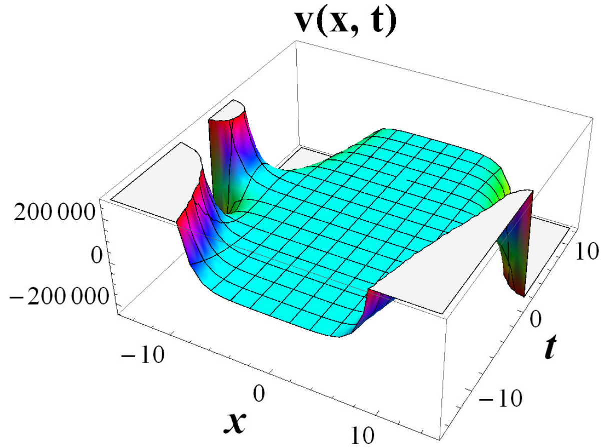


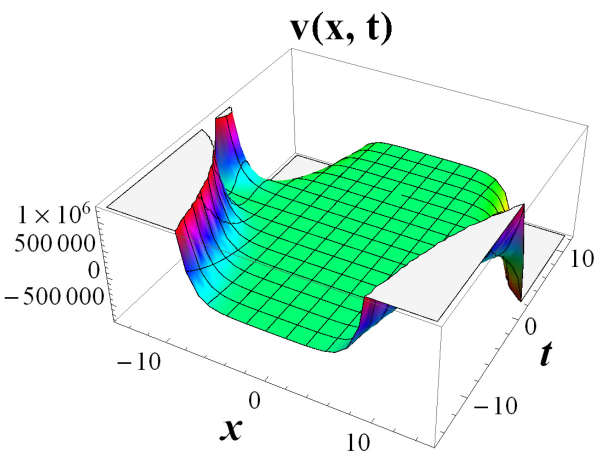
Figure 2: Surface show the 3D wave function $u(x,t)$ for (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$ and (d) $\alpha = 1$.



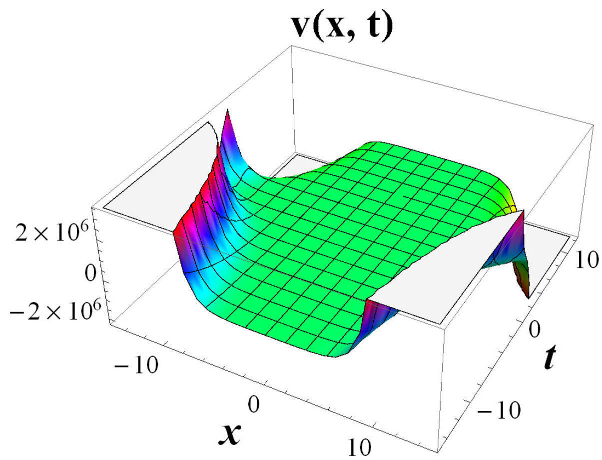
(a)



(b)



(c)



(d)

Figure 3: Surface show the 3D wave function $u(x,t)$ for (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$ and (d) $\alpha = 1$.

Table 3: Numerically study regard to $u(x,t)$ by FHPTM along with Caputo-Fabrizio derivative for system of equation (4.2) with diverse $x,\ t$ as well as its separate values as α.

x	t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
0.2	0.2	1.18072	1.18846	1.11908	0.995969
	0.4	1.11337	1.1503	1.12395	1.31699
	0.6	1.0064	1.04012	1.00466	1.20624
	0.8	0.85473	0.838169	0.765124	0.840033
0.4	0.2	1.8562	1.66124	1.47107	1.24642
	0.4	1.8481	1.68736	1.54652	1.71883
	0.6	1.80229	1.64397	1.48374	1.68489
	0.8	1.71363	1.5118	1.29184	1.34194
0.6	0.2	2.81425	2.32852	1.9613	1.58829
	0.4	2.89124	2.44974	2.14254	2.27525
	0.6	2.93355	2.50592	2.16857	2.36
	0.8	2.93595	2.47855	2.05555	2.06714
0.8	0.2	4.18282	3.27752	2.65006	2.05953
	0.4	4.38276	3.53966	2.9897	3.05273
	0.6	4.5527	3.74429	3.15372	3.31955
	0.8	4.68737	3.87415	3.16789	3.12024

Table 4: Numerically study regard to $v(x,t)$ by FHPTM along with Caputo-Fabrizio derivative for system of equation (4.2) with diverse $x,\ t$ and separate values of α.

x	t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
0.2	0.2	-18.3285	-9.68712	-4.15435	-1.03764
	0.4	-20.848	-12.9099	-6.91269	-2.8121
	0.6	-23.4861	-16.5031	-10.269	-5.23085
	0.8	-26.2451	-20.4829	-14.2776	-8.42273
0.4	0.2	-19.481	-10.1477	-4.1841	-0.839329
	0.4	-22.2035	-13.6242	-7.1488	-2.72917
	0.6	-25.0548	-17.5023	-10.7617	-5.31738
	0.8	-28.037	-21.7994	-15.0816	-8.74363
0.6	0.2	-20.8886	-10.7103	-4.22043	-0.597113
	0.4	-23.8592	-14.4968	-7.43719	-2.62788
	0.6	-26.9707	-18.7227	-11.3634	-5.42308
	0.8	-30.2255	-23.4073	-16.0636	-9.13558
0.8	0.2	-22.6078	-11.3975	-4.26481	-0.30127
	0.4	-25.8814	-15.5625	-7.78943	-2.50417
	0.6	-29.3108	-20.2134	-12.0984	-5.55217
	0.8	-32.8986	-25.3713	-17.2631	-9.61431

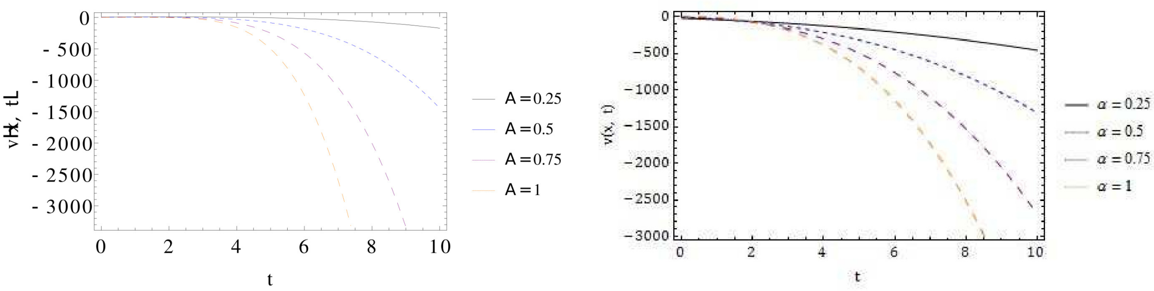
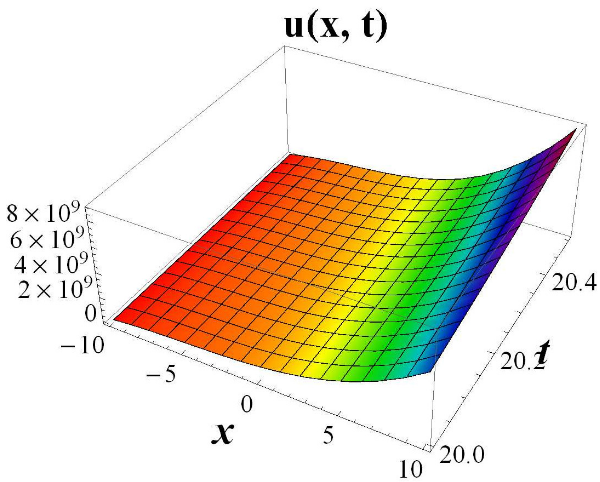
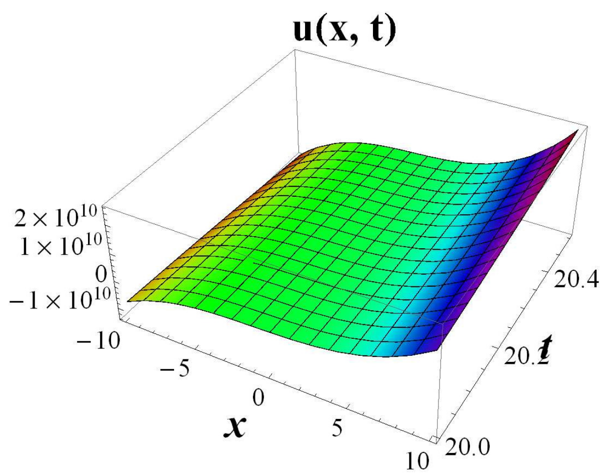


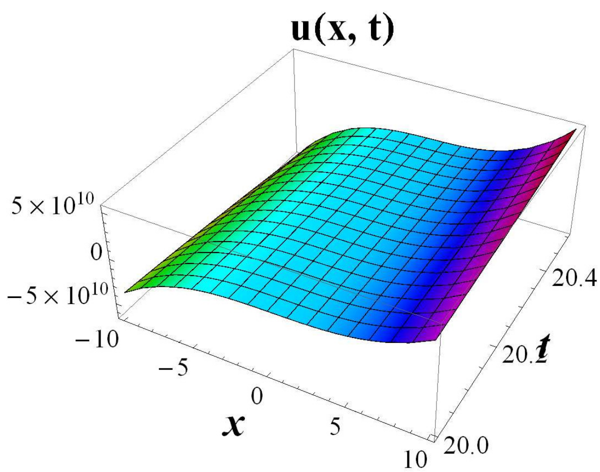
Figure 4: Comparison of approximate solution $u(x,t), \& v(x,y)$ for separate values of $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$ and $\alpha = 1$ for 4.2.



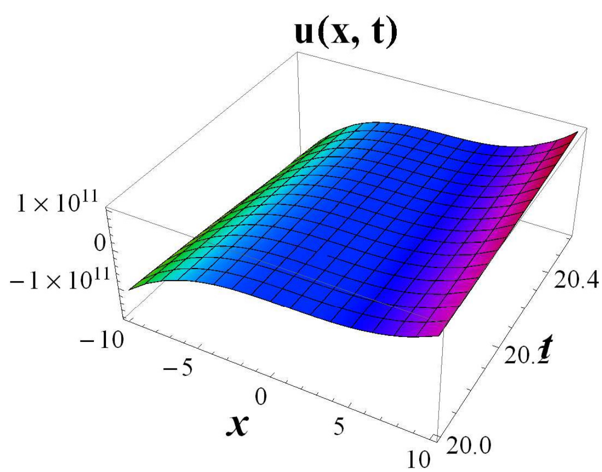
(a)



(b)



(c)



(d)

Figure 5: Surface show the 3D wave function $u(x,t)$ for (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$ and (d) $\alpha = 1$.

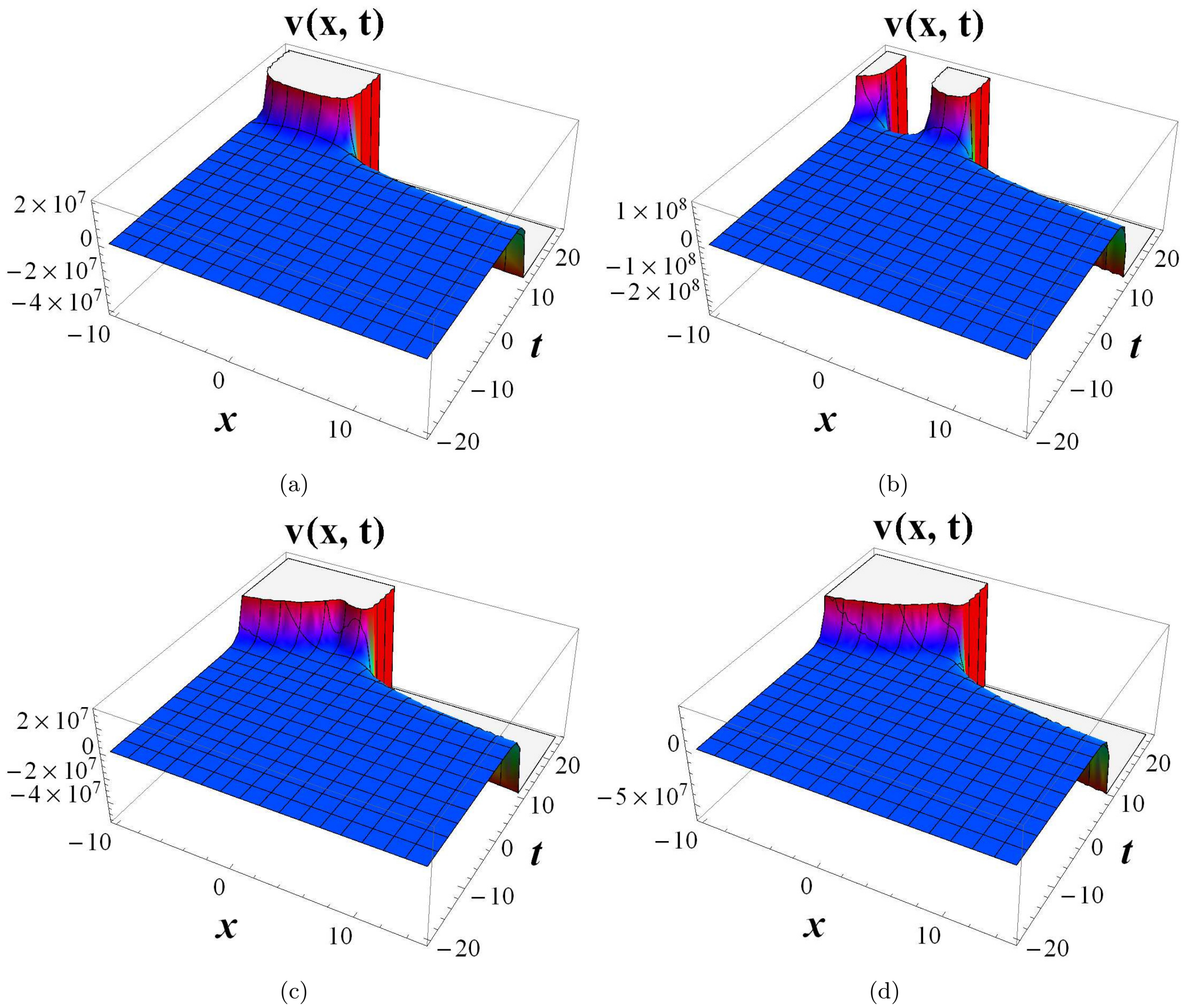


Figure 6: Surface show the 3D wave function $u(x,t)$ for (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$ and (d) $\alpha = 1$.

Table 5: Numerically study regard to $u(x,t)$ by FHPTM along with Caputo-Fabrizio derivative for system of equation (4.3) with diverse $x,\ t$ and separate values of α.

x	t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
0.2	0.2	-0.233168	-0.165227	0.256011	0.80924
	0.4	-0.166093	-0.282008	-0.0937316	0.334405
	0.6	-0.0737442	-0.341694	-0.396668	-0.171544
	0.8	0.0452078	-0.335638	-0.630311	-0.671188
0.4	0.2	-0.150615	-0.0300356	0.467924	1.08087
	0.4	-0.0507517	-0.128971	0.114349	0.604855
	0.6	0.0812357	-0.153649	-0.172995	0.10497
	0.8	0.247012	-0.0931781	-0.365531	-0.370443
0.6	0.2	-0.255543	0.0126551	0.663385	1.38865
	0.4	-0.128665	-0.0853809	0.280701	0.886737
	0.6	0.0385772	-0.0886147	-0.0127652	0.368842
	0.8	0.24825	0.0165262	-0.1812	-0.10383
0.8	0.2	-0.632294	-0.0755065	0.836885	1.74494
	0.4	-0.488098	-0.19623	0.39364	1.19136
	0.6	-0.293578	-0.197401	0.0670816	0.630661
	0.8	-0.0461838	-0.06222	-0.0983003	0.139352

Table 6: Numerically study regard to $v(x,t)$ by FHPTM along with Caputo-Fabrizio derivative for system of equation (4.3) with diverse $x,\ t$ as well as its separate values as α.

x	t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
0.2	0.2	4.22198	3.62816	2.76111	1.74739
	0.4	4.78497	4.49035	3.7681	2.82413
	0.6	5.37454	5.43028	4.89726	4.04497
	0.8	5.9912	6.45059	6.15306	5.40986
0.4	0.2	4.57674	3.81364	2.80398	1.67938
	0.4	5.18818	4.75396	3.90028	2.84381
	0.6	5.82752	5.77746	5.12929	4.16428
	0.8	6.49524	6.8868	6.49632	5.64469
0.6	0.2	5.05335	4.11024	2.93613	1.67023
	0.4	5.72825	5.15353	4.15354	2.95953
	0.6	6.43312	6.28774	5.51804	4.42205
	0.8	7.16843	7.51563	7.03601	6.0658
0.8	0.2	5.67023	4.52887	3.16197	1.71922
	0.4	6.4254	5.70295	4.53586	3.17449
	0.6	7.21341	6.97806	6.07525	4.82539
	0.8	8.03473	8.35724	7.78784	6.68433

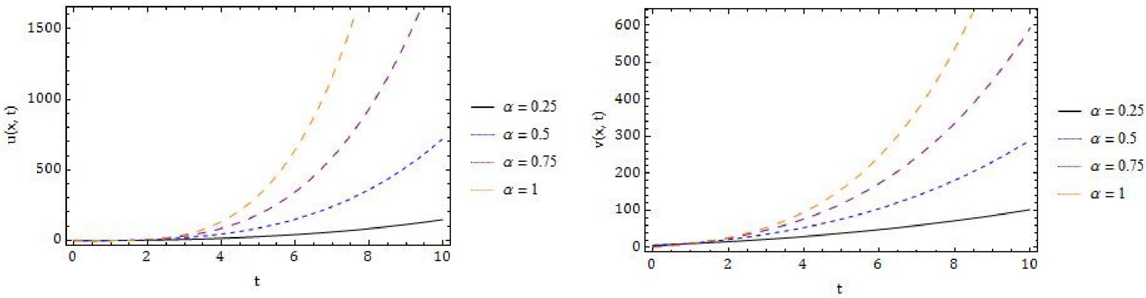


Figure 7: Comparison of approximate solution $u(x,t), \& v(x,t)$ for separate values of $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$ and $\alpha = 1$ for 4.3.

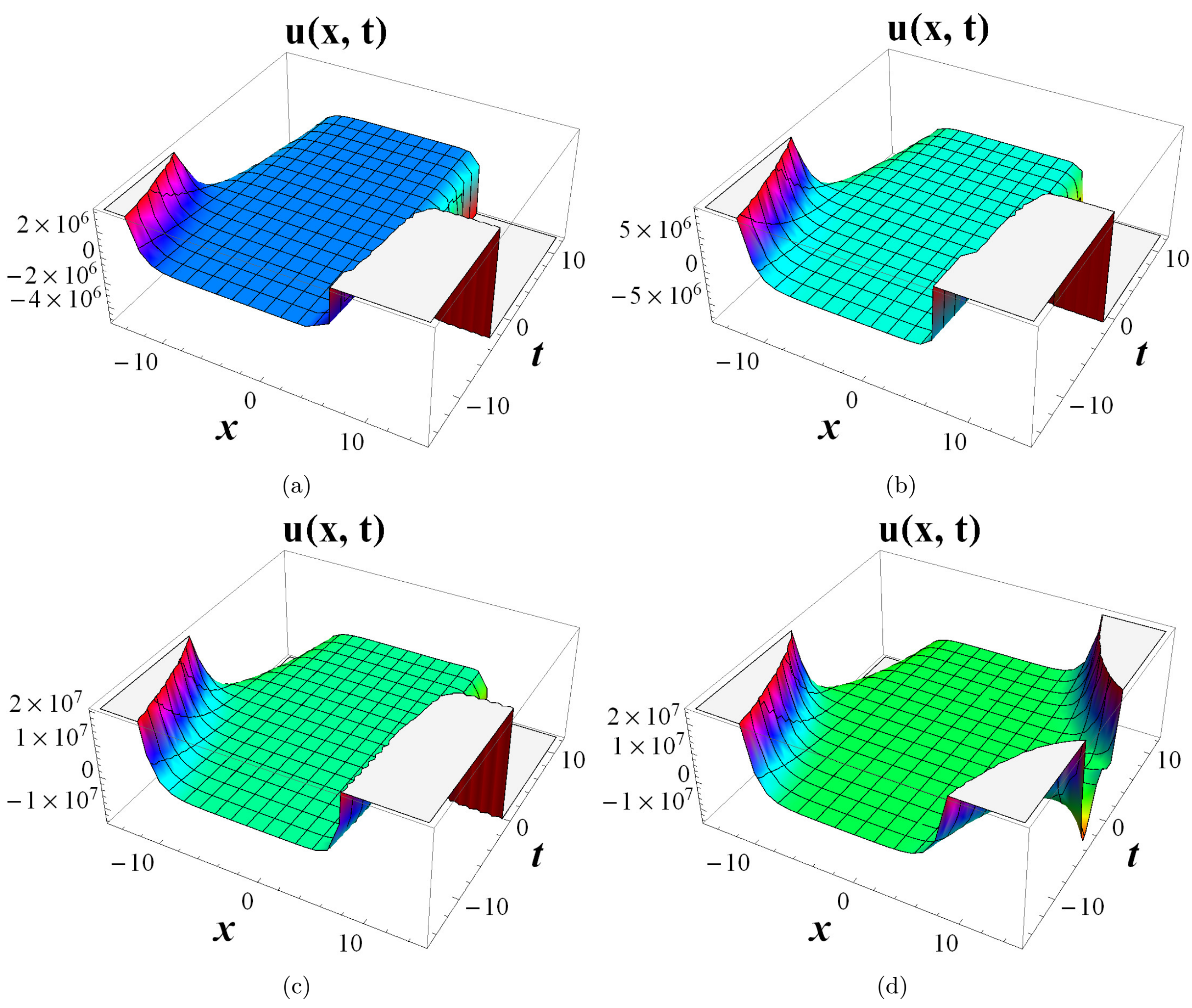


Figure 8: Surface show the 3D wave function $u(x,t)$ for (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$ and (d) $\alpha = 1$.

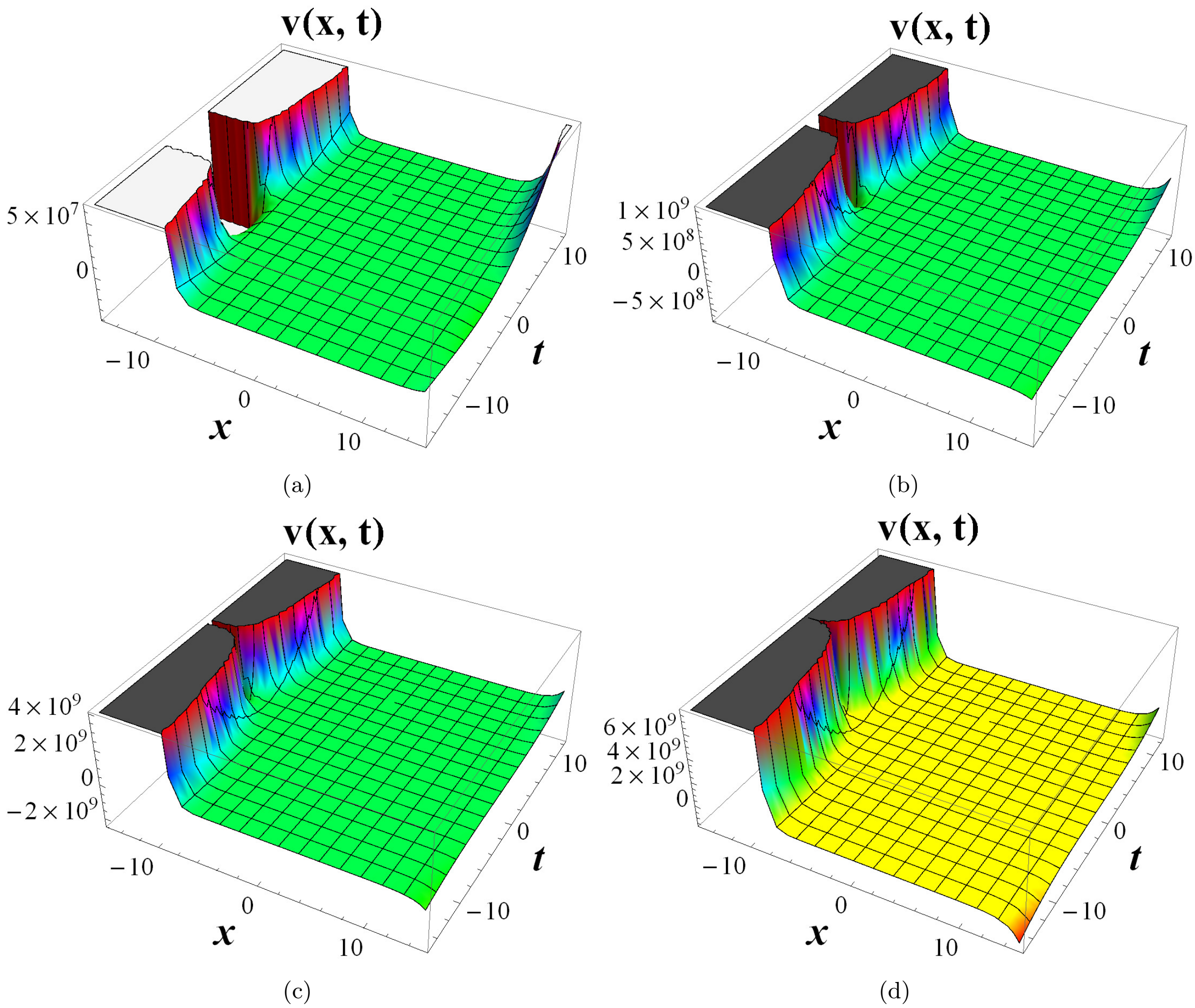


Figure 9: Surface show the 3D wave function $u(x,t)$ for (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$ and (d) $\alpha = 1$.

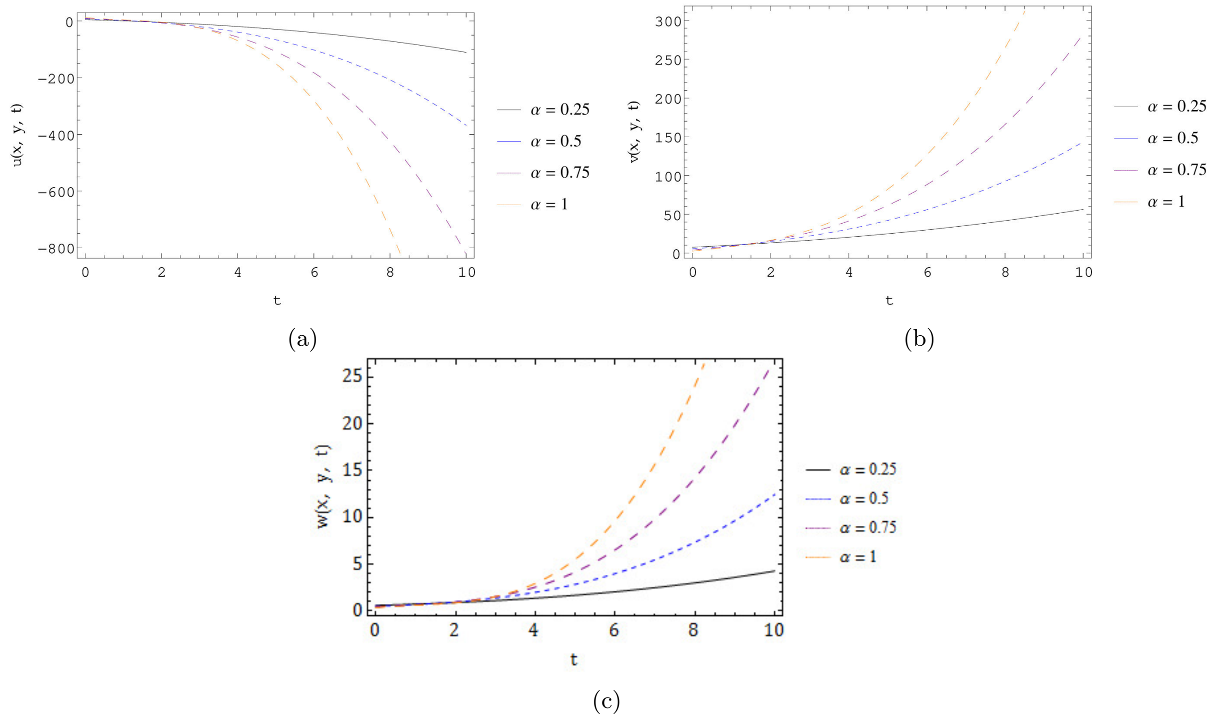
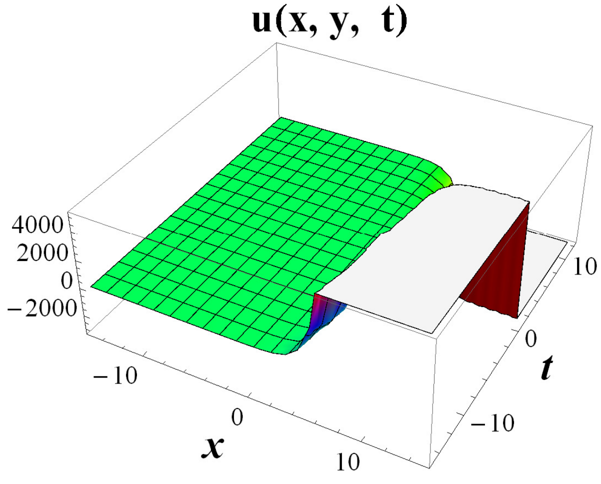
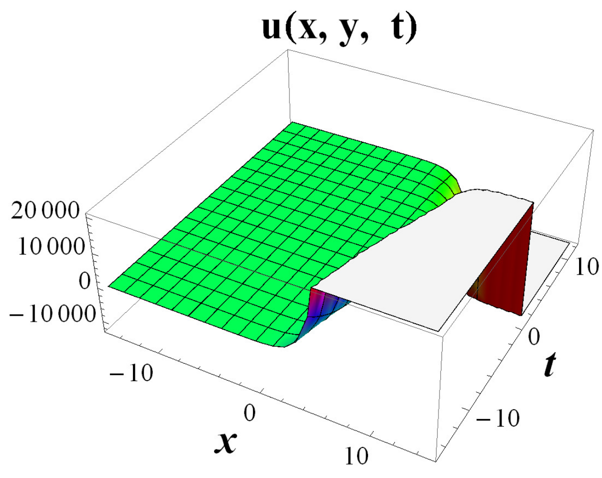


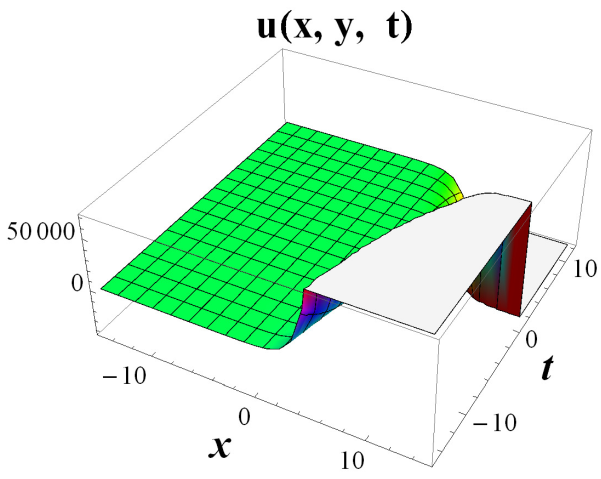
Figure 10: Comparison of approximate solution $u(x,y,t), v(x,y,t)$ & $w(x,y,t)$ for separate values of $\alpha = 0.25,\ \alpha = 0.5,\ \alpha = 0.75$ and $\alpha = 1$ for 4.4.



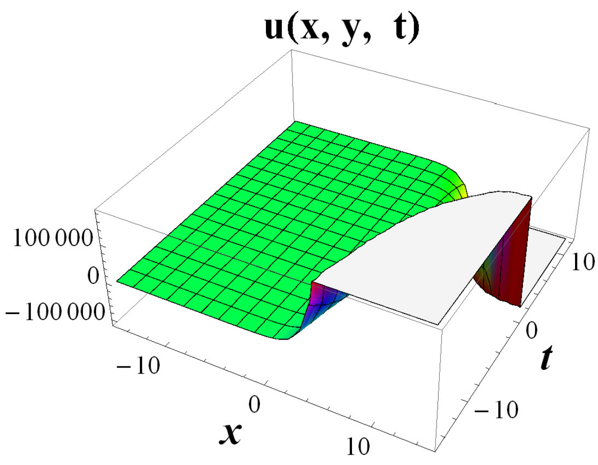
(a)



(b)



(c)



(d)

Figure 11: Surface show the 3D wave function $u(x, y, t)$ for (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$ and (d) $\alpha = 1$.

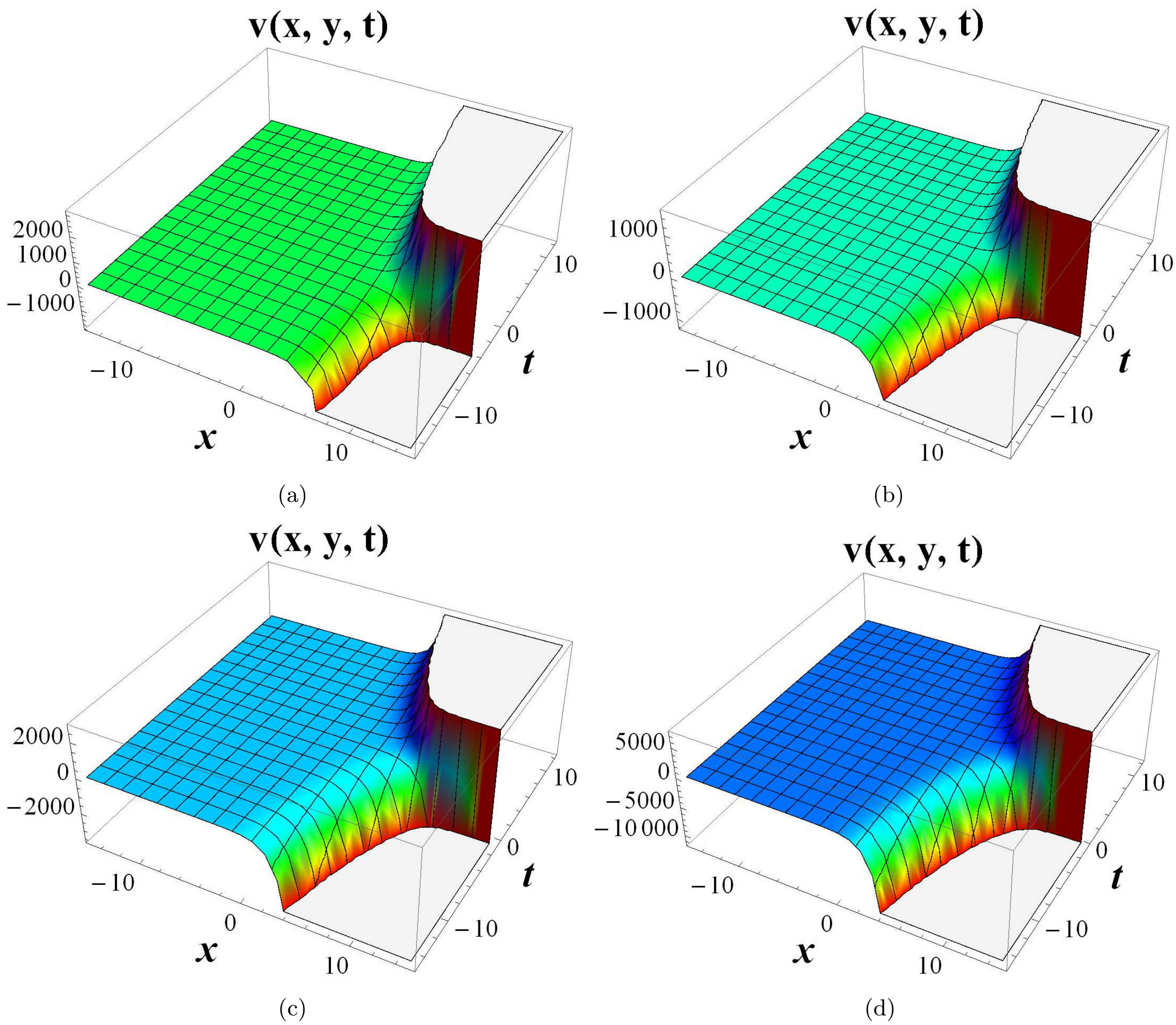


Figure 12: Surface show the 3D wave function $u(x,t)$ for (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$ and (d) $\alpha = 1$.

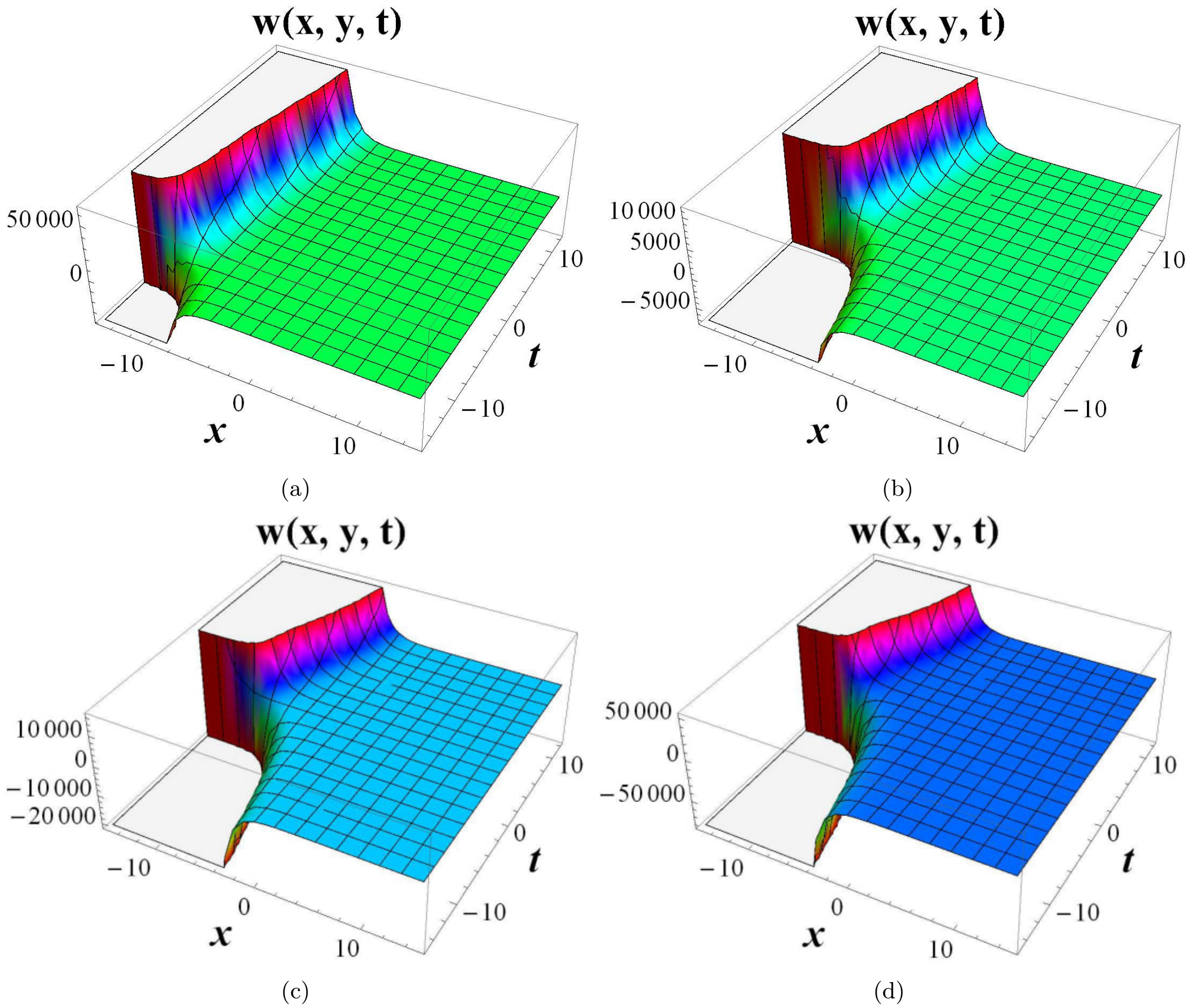


Figure 13: Surface show the 3D wave function $u(x,t)$ for (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$ and (d) $\alpha = 1$.

Table 7: Numerically study regard to $u(x,\ y,\ t)$ by FHPTM along with Caputo-Fabrizio derivative for system of equation (4.4) with diverse $x,\ y,\ t$ and separate values of α.

x	t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
0.2	0.2	1.80458	2.99607	3.80987	4.48134
	0.4	1.45310	2.53809	3.26999	3.66389
	0.6	1.08316	2.04178	2.72397	2.97783
	0.8	0.694051	1.50169	2.15331	2.37934
0.4	0.2	2.20412	3.65941	4.65338	5.47352
	0.4	1.77483	3.10003	3.99399	4.47509
	0.6	1.32297	2.49384	3.32707	3.63713
	0.8	0.847716	1.83416	2.63006	2.90614
0.6	0.2	2.69211	4.46962	5.68365	6.68537
	0.4	2.16778	3.78638	4.87827	5.46589
	0.6	1.61588	3.04598	4.06369	4.44239
	0.8	1.03540	2.24025	3.21237	8.16553
0.8	0.2	3.28816	5.45920	6.94203	6.67605
	0.4	2.64773	4.62469	5.95833	6.67605
	0.6	1.97364	3.72037	4.96340	5.42596
	0.8	1.26464	2.73625	3.92359	4.33544

Table 8: Numerically study regard to $v(x,y,t)$ by FHPTM along with Caputo-Fabrizio derivative for system of equation (4.4) with diverse $x,\ y,\ t$ as well as its separate values as α.

x	t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
0.2	0.2	0.84732	0.65616	0.50493	0.38190
	0.4	0.89991	0.73756	0.602329	0.49346
	0.6	0.95409	0.82359	0.70616	0.60938
	0.8	1.00992	0.91452	0.81735	0.73183
0.4	0.2	1.03492	0.80144	0.616727	0.466463
	0.4	1.09915	0.90086	0.735687	0.602719
	0.6	1.16534	1.00594	0.862511	0.74429
	0.8	1.23352	1.11700	0.998322	0.89387
0.6	0.2	1.26406	0.978884	0.753272	0.56974
	0.4	1.34251	1.10031	0.89857	0.736162
	0.6	1.42335	1.22865	1.05347	0.90909
	0.8	1.50663	1.36431	1.21935	1.09178
0.8	0.2	1.54393	1.19561	0.920048	0.695882
	0.4	1.63975	1.34393	1.09752	0.89915
	0.6	1.73848	1.50068	1.28671	1.11036
	0.8	1.84020	1.66637	1.48932	1.33349

Table 9: Numerically study regard to $w(x, y, t)$ by FHPTM along with Caputo-Fabrizio derivative for system of equation (4.4) with diverse $x,\ y,\ t$ as well as its separate values as α.

x	t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
0.2	0.2	1.36892	1.19053	1.06869	0.968162
	0.4	1.42154	1.25910	1.14952	1.09055
	0.6	1.47693	1.33341	1.23127	1.19326
	0.8	1.53519	1.41427	1.31671	1.28287
0.4	0.2	1.12078	0.974726	0.616727	0.792664
	0.4	1.16386	1.03087	0.874972	0.892865
	0.6	1.20921	1.091701	0.941148	0.976962
	0.8	1.25691	1.15791	1.00808	1.05032
0.6	0.2	0.917616	0.798038	0.716366	0.648978
	0.4	0.952889	0.844001	0.770547	0.731016
	0.6	0.990017	0.89381	0.825346	0.799869
	0.8	1.02907	0.948013	0.882617	0.859933
0.8	0.2	0.75128	0.653378	0.586511	0.531338
	0.4	0.78016	0.69101	0.630871	0.598505
	0.6	0.810557	0.73179	0.675736	0.654878
	0.8	0.842529	0.776168	0.722625	0.704053

5. Results and discussion

In this section, using the homotopy perturbation transform technique and using the initial condition (1) and (2), we solved the fractional order system of equation (1). By setting $0 < \alpha \leq 1$. For $x = \frac{\pi}{2}$, Fig.1 provides a comparison of third-order approximate solutions for separate value of $\alpha = 0.25, \alpha = 0.5, \alpha = 0.75$, and $\alpha = 1$. Fig.(2, 3) (a)- (d) shows the profile of the third order approximation solution for homogeneous linear system. Here the numerical results obtained by FHPTM, table.(1,2) at different valve $x,\ t$ and $\alpha = 1$. For $x = \frac{\pi}{2}$,Fig.4 provides a comparison of third-order approximate solutions for separate value of $\alpha = 0.25, \alpha = 0.5, \alpha = 0.75$, and $\alpha = 1$. Fig.(5, 6) (a)- (d) shows the profile of the third order approximation solution for inhomogeneous linear system. Here the numerical results obtained by FHPTM, table.(3,4) at different valve $x,\ t$ and $\alpha = 1$. For $x = 1$, Fig.7 provides a comparison of third-order approximate solutions for separate value of $\alpha = 0.25, \alpha = 0.5, \alpha = 0.75$,and $\alpha = 1$. Fig.(8, 9)(a)-(d) shows the profile of the third order approximation solution for inhomogeneous non-linear system. Here the numerical results obtained by FHPTM, Table.(5,6) at different valve $x,\ t$ and $y = 1.5,\ \alpha = 1$. For $x = 1,\ y = 1.5$, Fig.10 provides a comparison of third-order approximate solutions for separate value of $\alpha = 0.25, \alpha = 0.5, \alpha = 0.75$, and $\alpha = 1$. Fig.(11, 12, 13) (a)- (d) shows the profile of the third order approximation solution for inhomogeneous non-linear system. Here the numerical results obtained by FHPTM, table.(7, 8, 9) at separate valve $x,\ t$ and $y = 0.05\ \alpha = 1$. It's also worth noting that the solution is Caputo Fabrizio derivative dependent. Increasing the number of iterations can improve accuracy and efficiency.

6. Conclusion

In this article, we introduced a homotopy perturbation transform method to solve Fractional differential equations with dynamic analysis for a nonlinear system in this paper. Using initial condition, this approach produces an approximation solution to the problem. The findings show that FHPTM is an effective and computationally appealing method for investigating nonlinear fractional models. As a result, FHPTM can be consumed to solve a variety of linear and nonlinear fractional models that arise in science and engineering.

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