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Study of nonlinear boundary value hybrid fractional integro-differential equations with infinite delay

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ABSTRACT: In this study, we establish sufficient conditions for the existence of solutions to a nonlinear boundary value hybrid fractional integro-differential equation with infinite delay, incorporating the generalized fractional proportional Caputo derivative of order $1 < \theta < 2$. To analyze the existence of solutions for the given problem, we utilize the theory of infinite delay along with Dhage's fixed point theorem. Finally, we present two illustrative examples to emphasize the key findings.

Key Words: Hybrid equation, generalized Caputo proportional fractional derivative, infinite delay, fixed point theorem.

Contents

1. Introduction

Fractional calculus is based on the generalization of classical differentiation and integration to non-integer orders, a concept that was discovered nearly 300 years ago. In 1695, Leibniz suggested the symbol for the derivative of order n, $\frac{d^n x}{dt^n}$ where n is a positive integer. L'Hôpital wrote him a letter in which he asked the following question: "What if the order n is replaced by $\frac{1}{2}$?" On September 30, 1695, Leibniz replied, "An apparent paradox from which useful consequences will one day be drawn" [4]. In 1730, in the paper [9], Euler addressed the issue of fractional differentiation and proposed a definition for the derivative of order $\alpha > 0$ of x^β. Then, in 1819, the first definition of fractional differentiation was presented by Lacroix [15]. Later, Liouville studied fractional calculus in detail in the eight papers he published between 1832 and 1837. For more historical details, we refer to [10,6].

Fractional differential equations (FDEs) naturally emerge in various scientific disciplines, including physics, engineering, medicine, electrochemistry, and control theory (see [1,21,18,7,20,22]). Their effectiveness in modeling real-world phenomena has sparked significant interest among researchers, leading to extensive studies on their quantitative and qualitative properties. Additionally, fractional differential equations with delay constitute a particularly relevant and intriguing research area. The growing interest in these equations is reflected in the increasing number of published studies exploring the existence and uniqueness of their solutions. Recent studies have explored such equations extensively, with contributions addressing both finite delay [16,2] and infinite delay [17].

Chen and Dong [5] investigated the existence and uniqueness of a specific class of two-term boundary value problems with infinite delay using standard fixed-point theorems. Additionally, they examined the stability of solutions for the following given problem by applying the Hyers–Ulam stability theorem:

$$\begin{cases} {}^C D_{0+}^{\lambda} w(t) - \chi {}^C D_{0+}^{\vartheta} w(t) = \mathcal{K}(t, w_t), & t \in \mathcal{T} := [0, b], \\ w(t) = \Theta(t), & t \in (-\infty, 0], \end{cases} \tag{1.1}$$

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where ${}^{C}D^{\lambda}_{0+}$ and ${}^{C}D^{\vartheta}_{0+}$ denote the Caputo fractional derivatives of orders $0 < \upsilon < \lambda < 1$, χ is a constant, $\mathcal{F} : \mathcal{T} \times \mathfrak{X} \to \mathbb{R}$ is a given function, $\Theta \in \mathfrak{X}$, and $\mathfrak{X}$ is called a phase space.
The authors of [24] investigated the existence of solutions for the following Hadamard-type neutral fractional integro-differential equation with infinite delay:

$$\begin{cases} {}^{H}D^{\theta}_{1^+}\left[x(t) - \sum_{i=1}^{m} {}^{H}I^{\lambda_i}_{1^+} E_i(t, x_t)\right] = \omega(t) f\left(t, x_t, \int_1^t h(t, s, x_s) ds\right), & t \in \Theta := [1, T], \\ x(t) = \zeta(t), & t \in (-\infty, 1], \end{cases} \tag{1.2}$$

where $T > 1$, $0 < \theta \leq 1$, $\lambda_i > 0$, $(i = 1, 2, ..., m)$ are some real constants, ${}^{H}D^{\theta}_{1^+}$ is the Hadamard-type fractional derivative of order θ, ${}^{H}I^{\lambda_i}_{1^+}$ stands the Hadamard fractional integral of order λ_i, $E_i : \Theta \times \mathcal{G} \to \mathbb{R}$, $f : \Theta \times \mathcal{G} \times \mathbb{R} \to \mathbb{R}$, and $h : \Theta \times \Theta \times \mathcal{G} \times \mathbb{R} \to \mathbb{R}$ are given functions, where $\mathcal{G}$ is an abstract phase space.

Recent advances in hybrid fractional differential equations have further expanded their utility in modeling non-homogeneous phenomena, such as electromagnetic wave propagation and gravity-driven flows [17,12,19,23]. These systems unify diverse dynamical behaviors and are increasingly applied in theoretical and applied contexts.

Motivated by these developments, this paper examines the existence of solutions for a novel class of nonlinear boundary value hybrid fractional integro-differential systems with infinite delay. We consider a generalized Caputo proportional fractional derivative of order $1 < \theta < 2$ and establishe the existence result of the following system:

$$\begin{cases} {}^{C}_{\gamma}D^{\theta,g}_{c^+}\left[\frac{x(t)}{\mathcal{F}(t,x(t))} - \int_c^t \sigma(s, x_s) ds\right] = \mathcal{H}(t, x_t), & t \in \Delta := [c, d], \\ x(t) = \lambda(t), & t \in (-\infty, c], \\ \left(\frac{x(t)}{\mathcal{F}(t,x(t))}\right)_{t=c} = 0, & \\ \left(\frac{x(t)}{\mathcal{F}(t,x(t))}\right)_{t=d} = \sum_{i=1}^{n} \alpha_i \frac{x(\nu_i)}{\mathcal{F}(\nu_i,x(\nu_i))}, & c < \nu_i < d, \end{cases} \tag{1.3}$$

here, ${}^{C}_{\gamma}D^{\theta,g}_{c^+}$ represents the generalized Caputo proportional fractional derivative of order $1 < \theta < 2$ with $0 < \gamma < 1$, $g : \Delta \to \mathbb{R}$, $\mathcal{H}, \sigma \in C(\Delta \times \mathfrak{X}, \mathbb{R})$, and $\mathcal{F} \in C(\Delta \times \mathfrak{X}_d, \mathbb{R}^*)$, where the phase space $\mathfrak{X}$ and the space $\mathfrak{X}_d$ are formally defined in the section 2. We consider the function $x_t : (-\infty, c] \to \mathbb{R}$ of $\mathfrak{X}$ such that $x_t(s) = x(t + s)$, $(s \leqslant 0)$ for any $u : (-\infty, d] \to \mathbb{R}$ and any $t \in \Delta$.

The rest of the paper is structured as follows. Section 2 presents fundamental preliminaries and derives the integral equation corresponding to the nonlinear boundary value hybrid fractional integro-differential system (1.3). Section 3 establishes the main existence result for the solution of problem (1.3) using Dhage's fixed point theorem. In Section 4, we offer two illustrative examples to demonstrate the core findings of this study, and finally, Section 5 provides a summary of our conclusions.

2. Preliminaries and Problem Formulation

In this section, we provide a concise overview of essential definitions, lemmas, and properties associated with the generalized Caputo proportional fractional derivative. Additionally, we present the solution formula for the nonlinear boundary value hybrid fractional integro-differential system (1.3). These fundamental concepts and results will be consistently utilized in the subsequent sections of this study.

- Let $C(\Delta, \mathbb{R})$ be the Banach space of all continuous functions with the norm $\|x\| = \sup_{t \in \Delta.} |x(t)|$.
- In this study, the space $(\mathfrak{X}, \| \, . \, \|_{\mathfrak{X}})$ is defined as a seminormed linear space consisting of functions mapping $(-\infty, c]$ into $\mathbb{R}$ and adhering to the axioms introduced by Hale and Kato in [11]:
  - (i) Let $t \in [c, d]$, $C \geq 0$ be a constant, $\varpi : [c, d] \to [0, \infty)$ be a continuous function, and $\Lambda : [c, \infty] \to [0, \infty)$ be a locally bounded function. If $x : (-\infty, d] \longrightarrow \mathbb{R}$ and $x_0 \in \mathfrak{X}$, we have
    1. $x_t \in \mathfrak{X}$,
    2. $|x(t)| \leq C \parallel x_t \parallel_{\mathfrak{X}}$,
    3. $\parallel x_t \parallel_{\mathfrak{X}} \leq \varpi(t) \parallel x_0 \parallel_{\mathfrak{X}} + \Lambda(t) \sup\limits_{\tau \in [0,t]} \{|x(\tau)|\}$,

where ϖ, Λ are independent of $x(.)$ and $\varpi^* = \sup_{t\in[c,d]} \varpi(t)$, $\Lambda^* = \sup_{t\in[c,d]} \Lambda(t)$;

(ii) x_t is a $\mathfrak{X}$-valued continuous function on $[c,d]$;

(iii) The space $\mathfrak{X}$ is complete.

- We consider the space $\mathfrak{X}_d = \big\{x : (-\infty, d] \to \mathbb{R} : x|_{(-\infty,c]} \in \mathfrak{X} \,\text{and}\, x|_{[c,d]} \in C(\Delta,\mathbb{R})\big\}$ with the semi-norm
$$\| x \|_{\mathfrak{X}_d} = \| \lambda \|_{\mathfrak{X}} + \sup_{t\in[c,d]} |x(t)|.$$

- Throughout this paper, we consider that $g : \Delta \mapsto \mathbb{R}$ be a strictly positive, increasing, and differentiable function.

Definition 2.1 *[13] Let $0 < \gamma < 1$, $\theta > 0$, $f \in L^1(\Delta,\mathbb{R})$. The left-sided generalized proportional fractional integral with respect to g of order θ of the function f is given by*
$$_{\gamma}I^{\theta,g}_{c^+} f(t) = \frac{1}{\gamma^\theta\Gamma(\theta)} \int_c^t e^{\frac{\gamma-1}{\gamma}(g(t)-g(\tau))}(g(t)-g(\tau))^{\theta-1} f(\tau)g'(\tau)d\tau, \tag{2.1}$$
where $\Gamma(\theta) = \int_0^{+\infty} e^{-x}x^{\theta-1}dx$ is the Euler gamma function.

Definition 2.2 *[13] Let $\gamma \in [0,1]$, $\zeta, \rho : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous functions such that*
$$\lim_{\gamma\to 0^+} \zeta(\gamma,t) = 0, \quad \lim_{\gamma\to 1^-} \zeta(\gamma,t) = 1, \quad \lim_{\gamma\to 0^+} \rho(\gamma,t) = 1, \quad \lim_{\gamma\to 1^-} \rho(\gamma,t) = 0,$$
and
$$\zeta(\gamma,t) + \rho(\gamma,t) \neq 0 \text{ for each } \gamma \in [0,1], \text{ and } t \in \mathbb{R}.$$
Then the proportional derivative of order γ with respect to g of the function f is given by
$$_{\gamma}D^g f(t) = \rho(\gamma,t)f(t) + \zeta(\gamma,t)\frac{f'(t)}{g'(t)}.$$
In particular, if $\zeta(\gamma,t) = \gamma$ and $\rho(\gamma,t) = 1-\gamma$, then we have
$$_{\gamma}D^g f(t) = (1-\gamma)f(t) + \gamma\frac{f'(t)}{g'(t)}.$$

Definition 2.3 *[13] Let $\gamma \in (0,1]$. The left-sided generalized Caputo proportional fractional derivative of order $n-1 < \theta < n$ is defined by*
$$\begin{aligned} {}^{C}_{\gamma}D^{\theta,g}_{c^+}(f)(t) &= {}_{\gamma}I^{n-\theta,g}_{c^+}({}_{\gamma}D^{n,g}f(t)) \\ &= \frac{1}{\gamma^{n-\theta}\Gamma(n-\theta)} \int_c^t e^{\frac{\gamma-1}{\gamma}(g(t)-g(s))}(g(t)-g(s))^{n-\theta-1}({}_{\gamma}D^{n,g}f)(s)g'(s)ds, \end{aligned} \tag{2.2}$$
where $n = [\theta]+1$, and $_{\gamma}D^{n,g} = \underbrace{{}_{\gamma}D^g_{\gamma}D^g \,...\, {}_{\gamma}D^g}_{n\text{-times}}$.

Lemma 2.1 *[14] Let $t \in \Delta$, $\gamma \in (0,1]$, $(\vartheta, \theta > 0)$, and $f \in L^1(\Delta,\mathbb{R})$. Then, we have*
$$_{\gamma}I^{\theta,g}_{c^+}(\,{}_{\gamma}I^{\vartheta,g}_{c^+} f(t)) = \,{}_{\gamma}I^{\vartheta,g}_{c^+}(\,{}_{\gamma}I^{\theta,g}_{c^+} f(t)) = \,{}_{\gamma}I^{\theta+\vartheta,g}_{c^+} f(t).$$

Lemma 2.2 *[14] Let $\gamma \in (0,1]$, $n-1 < \theta < n$, $n = [\theta]+1$. Then, we have*
$$_{\gamma}I^{\theta,g}_{c^+}(\,{}^{c}_{\gamma}D^{\theta,g}_{c^+} f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{(\,{}_{\gamma}D^{k,g}f)(c)}{\gamma^k\Gamma(k+1)} Q^g_k(t,c),$$
where $Q^g_k(t,c) = e^{\frac{\gamma-1}{\gamma}(g(t)-g(c))}(g(t)-g(c))^k$.

Theorem 2.1 *[8] Let $\mathcal{B}$ be a closed, convex and bounded subset of a Banach algebra $\mathcal{X}$ and let $\Upsilon : \mathcal{X} \to \mathcal{X}$, and $\Psi : \mathcal{B} \to \mathcal{X}$ be two operators such that*

(a) Υ is Lipschitzian with a Lipschitz constant ς.

(b) Ψ is completely continuous.

(c) $x = \Upsilon x \Psi y \Rightarrow x \in \mathcal{B}$, for all $y \in \mathcal{B}$. Then the operator $\mathcal{A}x = \Upsilon x \Psi x$ has at least a fixed point in $\mathcal{B}$, whenever $\varsigma \mathcal{W} < 1$, where $\mathcal{W} = \|\Psi(\mathcal{B})\|$.

We are now ready to introduce the definition of a solution to problem (1.3), which is central to our work. To do so, we first present the following lemma, which enables us to derive the solution's definition for problem (1.3).

Lemma 2.3 *Let $g : \Delta \longrightarrow \mathbb{R}$, $\mathcal{F} \in C(\Delta \times \mathbb{R}, \mathbb{R}^*)$, $h, \varphi \in C([c,d])$, and $x \in C(\Delta, \mathbb{R}) \cap \mathfrak{X}_d$. Then the following problem*

$$\begin{cases} {}^{C}_{\gamma}D^{\theta,g}_{c^+}\left[\dfrac{x(t)}{\mathcal{F}(t,x(t))} - \displaystyle\int_c^t \varphi(s)ds\right] = h(t), & t \in \Delta := [c,d], \\ x(t) = \lambda(t), & t \in (-\infty, c], \\ \left(\dfrac{x(t)}{\mathcal{F}(t,x(t))}\right)_{t=c} = 0, & \\ \left(\dfrac{x(t)}{\mathcal{F}(t,x(t))}\right)_{t=d} = \displaystyle\sum_{i=1}^{n} \alpha_i \frac{x(\nu_i)}{\mathcal{F}(\nu_i, x(\nu_i))}, \quad c < \nu_i < d. & \end{cases} \tag{2.3}$$

has a solution given by

$$x(t) = \begin{cases} \lambda(t), \quad t \in (-\infty, c], \\ \mathcal{F}(t,x(t))\Bigg(\displaystyle\int_c^t \varphi(s)ds + \frac{Q_1^g(t,c)}{\mathcal{Q}}\Bigg[-\sum_{i=1}^{n}\alpha_i \int_c^{\nu_i}\varphi(s)ds \\ + \displaystyle\int_c^d \varphi(s)ds + \frac{1}{\gamma^\theta \Gamma(\theta)}\Bigg(\int_c^d Q^g_{\theta-1}(d,\tau)h(\tau)g'(\tau)d\tau \\ - \displaystyle\sum_{i=1}^{n}\alpha_i \int_c^{\nu_i} Q^g_{\theta-1}(\nu_i,\tau)h(\tau)g'(\tau)d\tau\Bigg)\Bigg] \\ + \dfrac{1}{\gamma^\theta\Gamma(\theta)}\displaystyle\int_c^t Q^g_{\theta-1}(t,\tau)h(\tau)g'(\tau)d\tau\Bigg), \quad t \in \Delta, \end{cases} \tag{2.4}$$

with $\mathcal{Q} := \displaystyle\sum_{i=1}^{n}\alpha_i Q_1^g(\nu_i,c) - Q_1^g(d,c) \neq 0.$

Proof: Let $x(t)$ be a solution of the problem (2.3), applying the operator ${}_{\gamma}I^{\theta;g}_{c^+}(.)$ on both sides of the fractional differential Equation (2.3) and using Lemma 2.2. Then for $t \in \Delta$, we have

$$\frac{x(t)}{\mathcal{F}(t,x(t))} = \int_c^t \varphi(s)ds + \left(c_1 e^{\frac{\gamma-1}{\gamma}(g(t)-g(c))} + \frac{c_2}{\gamma}Q_1^g(t,c)\right) + \frac{1}{\gamma^\theta\Gamma(\theta)}\int_c^t Q^g_{\theta-1}(t,\tau)h(\tau)g'(\tau)d\tau, \tag{2.5}$$

where c_1 and c_2 are arbitrary real constants.
Then, putting $t = c$ in (2.5) and using the initial condition $\left(\frac{x(t)}{\mathcal{F}(t,x(t))}\right)_{t=c} = 0$, we obtain $c_1 = 0$, by substituting $t = d$ into equation (2.5) and utilizing the initial condition $\left(\frac{x(t)}{\mathcal{F}(t,x(t))}\right)_{t=d} = \sum_{i=1}^{n}\alpha_i \frac{x(\nu_i)}{\mathcal{F}(\nu_i,x(\nu_i))}$.

We reach the following result:

$$\sum_{i=1}^{n}\alpha_i\left(\int_c^{\nu_i}\varphi(s)ds+\frac{c_2}{\gamma}Q_1^g(\nu_i,c)+\frac{1}{\gamma^\theta\Gamma(\theta)}\int_c^{\nu_i}Q_{\theta-1}^g(\nu_i,\tau)h(\tau)g'(\tau)d\tau\right)=\int_c^d\varphi(s)ds$$
$$+\frac{c_2}{\gamma}Q_1^g(d,c)+\frac{1}{\gamma^\theta\Gamma(\theta)}\int_c^d Q_{\theta-1}^g(d,\tau)h(\tau)g'(\tau)d\tau,$$

this implies that

$$\sum_{i=1}^{n}\alpha_i\frac{c_2}{\gamma}Q_1^g(\nu_i,c)-\frac{c_2}{\gamma}Q_1^g(d,c)=-\sum_{i=1}^{n}\alpha_i\int_c^{\nu_i}\varphi(s)ds+\int_c^d\varphi(s)ds$$
$$+\frac{1}{\gamma^\theta\Gamma(\theta)}\left(\int_c^d Q_{\theta-1}^g(d,\tau)h(\tau)g'(\tau)d\tau-\sum_{i=1}^{n}\alpha_i\int_c^{\nu_i}Q_{\theta-1}^g(\nu_i,\tau)h(\tau)g'(\tau)d\tau\right).$$

Therefore,

$$c_2=\gamma\left(\sum_{i=1}^{n}\alpha_iQ_1^g(\nu_i,c)-Q_1^g(d,c)\right)^{-1}\left[-\sum_{i=1}^{n}\alpha_i\int_c^{\nu_i}\varphi(s)ds+\int_c^d\varphi(s)ds\right.$$
$$\left.+\frac{1}{\gamma^\theta\Gamma(\theta)}\left(\int_c^d Q_{\theta-1}^g(d,\tau)h(\tau)g'(\tau)d\tau-\sum_{i=1}^{n}\alpha_i\int_c^{\nu_i}Q_{\theta-1}^g(\nu_i,\tau)h(\tau)g'(\tau)d\tau\right)\right].$$

Direct computation, following the Inserting of c_1 and c_2 into Equation (2.5), yields the solution as presented in Equation (2.4). $\square$

Using the information from the previous lemma, we can now define the solution to nonlinear boundary value hybrid fractional integro-differential system (1.3).

Definition 2.4 *If x is a solution to the nonlinear boundary value hybrid fractional integro-differential system (1.3), then x is also a solution of the following equation:*

$$x(t)=\begin{cases}\lambda(t), \quad t\in(-\infty,c],\\ \mathcal{F}(t,x(t))\Bigg(\displaystyle\int_c^t\sigma(s,x_s)ds+\frac{Q_1^g(t,c)}{\mathcal{Q}}\Bigg[-\sum_{i=1}^{n}\alpha_i\int_c^{\nu_i}\sigma(s,x_s)ds\\ +\displaystyle\int_c^d\sigma(s,x_s)ds+\frac{1}{\gamma^\theta\Gamma(\theta)}\Bigg(\int_c^d Q_{\theta-1}^g(d,\tau)h(\tau)g'(\tau)d\tau\\ -\displaystyle\sum_{i=1}^{n}\alpha_i\int_c^{\nu_i}Q_{\theta-1}^g(\nu_i,\tau)\mathcal{H}(\tau,x_\tau)g'(\tau)d\tau\Bigg)\Bigg]\\ +\displaystyle\frac{1}{\gamma^\theta\Gamma(\theta)}\int_c^t Q_{\theta-1}^g(t,\tau)\mathcal{H}(\tau,x_\tau)g'(\tau)d\tau\Bigg), \quad t\in\Delta,\end{cases}$$

provided that the integrals above are finite.

3. Existence results

In this section, we explore and analyze the existence of solutions for the given nonlinear boundary value hybrid fractional integro-differential system (1.3) using Dhage's fixed point theorem.
Using Lemma 2.3, we transform the problem (2.3) into a fixed point problem by defining an operator

$\mathcal{S}:\mathfrak{X}_d\to\mathfrak{X}_d$ as follows:

$$(\mathcal{S}x)(t)=\begin{cases}\lambda(t), \quad t\in(-\infty,c],\\ \mathcal{F}(t,x(t))\Bigg(\displaystyle\int_c^t\sigma(s,x_s)ds+\frac{Q_1^g(t,c)}{\mathcal{Q}}\Bigg[-\sum_{i=1}^n\alpha_i\int_c^{\nu_i}\sigma(s,x_s)ds\\ +\displaystyle\int_c^d\sigma(s,x_s)ds+\frac{1}{\gamma^\theta\Gamma(\theta)}\Bigg(\int_c^d Q_{\theta-1}^g(d,\tau)\mathcal{H}(\tau,x_\tau)g'(\tau)d\tau\\ -\displaystyle\sum_{i=1}^n\alpha_i\int_c^{\nu_i}Q_{\theta-1}^g(\nu_i,\tau)\mathcal{H}(\tau,x_\tau)g'(\tau)d\tau\Bigg)\Bigg]\\ +\displaystyle\frac{1}{\gamma^\theta\Gamma(\theta)}\int_c^t Q_{\theta-1}^g(t,\tau)\mathcal{H}(\tau,x_\tau)g'(\tau)d\tau\Bigg), \quad t\in\Delta.\end{cases} \tag{3.1}$$

We propose that the solution $x(.)$ is a decomposition of two functions $y(t)$ and $\bar{z}(t)$: $(-\infty,d]\to\mathbb{R}$, such that $x(t)=y(t)+\bar{z}(t)$, which involves $x_t=y_t+\bar{z}_t$ for all $t\in\Delta$.
The functions y and $\bar{z}$ are defined as follows:

$$y(t)=\begin{cases}\lambda(t), \quad t\in(-\infty,c],\\ 0, \quad t\in\Delta,\end{cases} \tag{3.2}$$

and

$$\bar{z}(t)=\begin{cases}0, \quad t\in(-\infty,c],\\ z(t), \quad t\in\Delta,\end{cases} \tag{3.3}$$

where $z\in C([c,d],\mathbb{R})$ is given by

$$\begin{aligned}z(t)=\mathcal{F}(t,y(t)+\bar{z}(t))&\Bigg(\int_c^t\sigma(s,y_s+\bar{z}_s)ds+\frac{Q_1^g(t,c)}{\mathcal{Q}}\Bigg[-\sum_{i=1}^n\alpha_i\int_c^{\nu_i}\sigma(s,y_s+\bar{z}_s)ds\\ &+\int_c^d\sigma(s,y_s+\bar{z}_s)ds+\frac{1}{\gamma^\theta\Gamma(\theta)}\Bigg(\int_c^d Q_{\theta-1}^g(d,\tau)\mathcal{H}(\tau,y_\tau+\bar{z}_\tau)g'(\tau)d\tau\\ &-\sum_{i=1}^n\alpha_i\int_c^{\nu_i}Q_{\theta-1}^g(\nu_i,\tau)\mathcal{H}(\tau,y_\tau+\bar{z}_\tau)g'(\tau)d\tau\Bigg)\Bigg]\\ &+\frac{1}{\gamma^\theta\Gamma(\theta)}\int_c^t Q_{\theta-1}^g(t,\tau)\mathcal{H}(\tau,y_\tau+\bar{z}_\tau)g'(\tau)d\tau\Bigg).\end{aligned} \tag{3.4}$$

Then we have $z(c)=0$, $z_c=0$, and $x_c=\lambda$.
Now we introduce the space $\mathfrak{X}'_d=\{z\in\mathfrak{X}_d: z_c=0\}$ and define the seminorm $\|\cdot\|_{\mathfrak{X}'_d}$ on $\mathfrak{X}'_d$ as follows:

$$\| z \|_{\mathfrak{X}'_d}=\sup_{t\in\Delta}|z(t)|+\| z_c \|_{\mathfrak{X}_d}=\sup_{t\in\Delta}|z(t)|.$$

This shows that $\|\cdot\|_{\mathfrak{X}'_d}$ defines a norm on $\mathfrak{X}'_d$, and consequently, $(\mathfrak{X}'_d,\|\cdot\|_{\mathfrak{X}'_d})$ is a Banach space. We then define the operator $\mathcal{A}:\mathfrak{X}'_d\to\mathfrak{X}'_d$ as follows:

$$\begin{aligned}\mathcal{A}z(t)=\mathcal{F}(t,y(t)+\bar{z}(t))&\Bigg(\int_c^t\sigma(s,y_s+\bar{z}_s)ds+\frac{Q_1^g(t,c)}{\mathcal{Q}}\Bigg[-\sum_{i=1}^n\alpha_i\int_c^{\nu_i}\sigma(s,y_s+\bar{z}_s)ds\\ &+\int_c^d\sigma(s,y_s+\bar{z}_s)ds+\frac{1}{\gamma^\theta\Gamma(\theta)}\Bigg(\int_c^d Q_{\theta-1}^g(d,\tau)\mathcal{H}(\tau,y_\tau+\bar{z}_\tau)g'(\tau)d\tau\\ &-\sum_{i=1}^n\alpha_i\int_c^{\nu_i}Q_{\theta-1}^g(\nu_i,\tau)\mathcal{H}(\tau,y_\tau+\bar{z}_\tau)g'(\tau)d\tau\Bigg)\Bigg]\\ &+\frac{1}{\gamma^\theta\Gamma(\theta)}\int_c^t Q_{\theta-1}^g(t,\tau)\mathcal{H}(\tau,y_\tau+\bar{z}_\tau)g'(\tau)d\tau\Bigg), \; t\in\Delta.\end{aligned} \tag{3.5}$$

Clearly, we observe that the operator $\mathcal{S}$ possesses a fixed point if and only if $\mathcal{A}$ has a fixed point. The following hypotheses are necessary to derive our results:

(H_0) : The functions $\mathcal{F} : \Delta \times \mathfrak{X}_d \to \mathbb{R} \setminus \{0\}$, $\sigma, \mathcal{H} : \Delta \times \mathfrak{X} \to \mathbb{R}$ are continuous.

(H_1) : There exists a constant $\varsigma > 0$ such that

$$|\mathcal{F}(t,x) - \mathcal{F}(t,y)| \leq \varsigma |x-y|, \quad \text{for all } t \in \Delta \text{ and } x, y \in \mathbb{R}.$$

(H_2) : There exist continuous functions $\beta(t), \zeta(t), \kappa(t)$ such that

$$|\mathcal{F}(t,p)| \leq \beta(t), \quad |\sigma(t,x)| \leq \zeta(t), \quad |\mathcal{H}(t,x)| \leq \kappa(t), \quad \forall (t,x,p) \in \Delta \times \mathfrak{X} \times \mathbb{R}.$$

With $\beta^* = \sup\limits_{t\in\Delta} \beta(t)$, $\zeta^* = \sup\limits_{t\in\Delta} \zeta(t)$, and $\kappa^* = \sup\limits_{t\in\Delta} \kappa(t)$.

• We consider the subset Ω_ε of $\mathfrak{X}'_d$ defined as:

$$\Omega_\varepsilon = \{z \in \mathfrak{X}'_d : \|z\|_{\mathfrak{X}'_d} \leq \varepsilon\},$$

where

$$\begin{aligned}\varepsilon = \beta^* &\left(\zeta^*(d-c) + \frac{(g(d)-g(c))}{|\mathcal{Q}|} \left(1 + \sum_{i=1}^n |\alpha_i| \right) \right. \\ &\times \left. \left(\zeta^*(d-c) + \frac{(g(d)-g(c))^\theta \kappa^*}{\gamma^\theta \Gamma(\theta+1)} \right) + \frac{(g(d)-g(c))^\theta \kappa^*}{\gamma^\theta \Gamma(\theta+1)} \right).\end{aligned}$$

It is easy to see that Ω_ε is a convex, closed, bounded, and nonempty subset of the Banach space $\mathfrak{X}'_d$.
• To apply the Dhage fixed point theorem, we define the operators $\Upsilon : \mathfrak{X}'_d \to \mathfrak{X}'_d$ and $\Psi : \Omega_\varepsilon \to \mathfrak{X}'_d$ by

$$(\Upsilon z)(t) = \mathcal{F}(t, y(t) + \bar{z}(t)). \tag{3.6}$$

and

$$\begin{aligned}(\Psi z)(t) = &\int_c^t \sigma(s, y_s + \bar{z}_s) ds + \frac{Q_1^g(t,c)}{\mathcal{Q}} \left[-\sum_{i=1}^n \alpha_i \int_c^{\nu_i} \sigma(s, y_s + \bar{z}_s) ds \right. \\ &+ \int_c^d \sigma(s, y_s + \bar{z}_s) ds + \frac{1}{\gamma^\theta \Gamma(\theta)} \left(\int_c^d Q_{\theta-1}^g(d,\tau) \mathcal{H}(\tau, y_\tau + \bar{z}_\tau) g'(\tau) d\tau \right. \\ &\left.\left. - \sum_{i=1}^n \alpha_i \int_c^{\nu_i} Q_{\theta-1}^g(\nu_i,\tau) \mathcal{H}(\tau, y_\tau + \bar{z}_\tau) g'(\tau) d\tau \right) \right] \\ &+ \frac{1}{\gamma^\theta \Gamma(\theta)} \int_c^t Q_{\theta-1}^g(t,\tau) \mathcal{H}(\tau, y_\tau + \bar{z}_\tau) g'(\tau) d\tau.\end{aligned} \tag{3.7}$$

It is evident that the operator $\mathcal{A} : \mathfrak{X}'_d \to \mathfrak{X}'_d$, defined by (3.5), can be decomposed as:

$$\mathcal{A}z(t) = \Upsilon z(t) \Psi z(t). \tag{3.8}$$

Now, we are ready to state the main results of this work, then we have the following existence theorem.

Theorem 3.1 *Assume that hypotheses $(H_0) - (H_2)$ hold and*

$$\varsigma \frac{\varepsilon}{\beta^*} < 1. \tag{3.9}$$

Then the given nonlinear boundary value hybrid fractional integro-differential system (1.3) has a solution on $(-\infty, d]$.

Proof: We shall proceed to establish that operators Υ and Ψ satisfy the requisite conditions set forth in Theorem 2.1. Then, the proof is given in the following claims:
Claim 1. The operator $\Upsilon : \mathfrak{X}'_d \to \mathfrak{X}'_d$ is Lipschitz.
For any $x_1, x_2 \in \mathfrak{X}'_d$ and each $t \in \Delta$, then thanks to hypothesis (H_1), we get

$$\begin{aligned}|\Upsilon x_1(t) - \Upsilon x_2(t)| &= |\mathcal{F}(t, y(t) + \bar{x_1}(t)) - \mathcal{F}(t, y(t) + \bar{x_2}(t))|\\ &\leq \varsigma |x_1(t) - x_2(t)|\\ &\leq \varsigma \|x_1 - x_2\|_{\mathfrak{X}'_d}.\end{aligned}$$

Then the operator Υ is Lipschitzian on $\mathfrak{X}'_d$ with a Lipschitz constant ς.
Claim 2. The operator Ψ is completely continuous on Ω_ε.
(i) Ψ is continuous on Ω_ε.
Let $(z_m)_{m\in\mathbb{N}}$ be a sequence of Ω_ε such that $z_m \to z$ as $m \longrightarrow +\infty$ in Ω_ε.
By using the fact that $e^{\frac{\gamma-1}{\gamma}(g(t)-g(c))} < 1$, we get

$$\begin{aligned}&|(\Psi z_m)(t) - (\Psi z)(t)|\\ &= \Bigg| \int_c^t \big(\sigma(s, y_s + \bar{z}_{m_s}) - \sigma(s, y_s + \bar{z}_s)\big) ds\\ &+ \frac{Q_1^g(t,c)}{\mathcal{Q}} \Bigg[-\sum_{i=1}^n \alpha_i \int_c^{\nu_i} \big(\sigma(s, y_s + \bar{z}_{m_s}) - \sigma(s, y_s + \bar{z}_s)\big) ds + \int_c^d \big(\sigma(s, y_s + \bar{z}_{m_s}) - \sigma(s, y_s + \bar{z}_s)\big) ds\\ &+ \frac{1}{\gamma^\theta \Gamma(\theta)} \Bigg(\int_c^d Q_{\theta-1}^g(d,\tau)\big(\mathcal{H}(\tau, y_\tau + \bar{z}_{m_\tau}) - \mathcal{H}(\tau, y_\tau + \bar{z}_\tau)\big) g'(\tau) d\tau\\ &- \sum_{i=1}^n \alpha_i \int_c^{\nu_i} Q_{\theta-1}^g(\nu_i,\tau)\big(\mathcal{H}(\tau, y_\tau + \bar{z}_{m_\tau}) - \mathcal{H}(\tau, y_\tau + \bar{z}_\tau)\big) g'(\tau) d\tau \Bigg) \Bigg]\\ &+ \frac{1}{\gamma^\theta \Gamma(\theta)} \int_c^t Q_{\theta-1}^g(t,\tau)\big(\mathcal{H}(\tau, y_\tau + \bar{z}_{m_\tau}) - \mathcal{H}(\tau, y_\tau + \bar{z}_\tau)\big) g'(\tau) d\tau \Bigg|.\\ &\leq \int_c^t \big|\sigma(s, y_s + \bar{z}_{m_s}) - \sigma(s, y_s + \bar{z}_s)\big| ds\\ &+ \frac{g(d)-g(c)}{|\mathcal{Q}|} \Bigg[\sum_{i=1}^n |\alpha_i| \int_c^{\nu_i} \big|\sigma(s, y_s + \bar{z}_{m_s}) - \sigma(s, y_s + \bar{z}_s)\big| ds + \int_c^d \big|\sigma(s, y_s + \bar{z}_{m_s}) - \sigma(s, y_s + \bar{z}_s)\big| ds\\ &+ \frac{1}{\gamma^\theta \Gamma(\theta)} \Bigg(\int_c^d (g(d) - g(\tau))^{\theta-1} \big|\mathcal{H}(\tau, y_\tau + \bar{z}_{m_\tau}) - \mathcal{H}(\tau, y_\tau + \bar{z}_\tau)\big| g'(\tau) d\tau\\ &+ \sum_{i=1}^n |\alpha_i| \int_c^{\nu_i} (g(\nu_i) - g(\tau))^{\theta-1} \big|\mathcal{H}(\tau, y_\tau + \bar{z}_{m_\tau}) - \mathcal{H}(\tau, y_\tau + \bar{z}_\tau)\big| g'(\tau) d\tau \Bigg) \Bigg]\\ &+ \frac{1}{\gamma^\theta \Gamma(\theta)} \int_c^t (g(t) - g(\tau))^{\theta-1} \big|\mathcal{H}(\tau, y_\tau + \bar{z}_{m_\tau}) - \mathcal{H}(\tau, y_\tau + \bar{z}_\tau)\big| g'(\tau) d\tau.\end{aligned}$$

Subsequently, employing the continuity of σ, $\mathcal{H}$, and the Lebesgue Dominated Convergence Theorem, we obtain

$$\int_c^{(\cdot)} \big|\sigma(s, y_s + \bar{z}_{m_s}) - \sigma(s, y_s + \bar{z}_s)\big| ds \to 0, \quad \text{as} \quad m \to \infty,$$

$$\int_c^{(\cdot)} (g(\nu_i) - g(\tau))^{\theta-1} \big|\mathcal{H}(\tau, y_\tau + \bar{z}_{m_\tau}) - \mathcal{H}(\tau, y_\tau + \bar{z}_\tau)\big| g'(\tau) d\tau \to 0, \quad \text{as} \quad m \to \infty.$$

We conclude that

$$\| \Psi z_m - \Psi z \|_{\mathfrak{X}'_d} \to 0, \text{ as } m \to \infty.$$

Thus, $\Psi z_m \to \Psi z$ in $\mathfrak{X}'_d$, which proves that Ψ is a continuous operator on Ω_ε.
(ii) The operator Ψ is uniformly bounded.
Let $z \in \Omega_\varepsilon$, by hypotheses (H_1) and (H_3), for all $t \in \Delta$, we get

$$
\begin{aligned}
|\Psi z(t)| = \Bigg| & \int_c^t \sigma(s, y_s + \bar{z}_s)ds + \frac{Q_1^g(t,c)}{\mathcal{Q}}\Bigg[-\sum_{i=1}^n \alpha_i \int_c^{\nu_i} \sigma(s, y_s + \bar{z}_s)ds \\
& + \int_c^d \sigma(s, y_s + \bar{z}_s)ds + \frac{1}{\gamma^\theta \Gamma(\theta)}\bigg(\int_c^d Q_{\theta-1}^g(d,\tau)\mathcal{H}(\tau, y_\tau + \bar{z}_\tau)g'(\tau)d\tau \\
& - \sum_{i=1}^n \alpha_i \int_c^{\nu_i} Q_{\theta-1}^g(\nu_i,\tau)\mathcal{H}(\tau, y_\tau + \bar{z}_\tau)g'(\tau)d\tau \bigg)\Bigg] \\
& + \frac{1}{\gamma^\theta \Gamma(\theta)} \int_c^t Q_{\theta-1}^g(t,\tau)\mathcal{H}(\tau, y_\tau + \bar{z}_\tau)g'(\tau)d\tau \Bigg| \\
\leq & \int_c^t |\sigma(s, y_s + \bar{z}_s)|ds + \frac{g(d)-g(c)}{|\mathcal{Q}|}\Bigg[\sum_{i=1}^n |\alpha_i| \int_c^{\nu_i} |\sigma(s, y_s + \bar{z}_s)|ds \\
& + \int_c^d |\sigma(s, y_s + \bar{z}_s)|ds + \frac{1}{\gamma^\theta \Gamma(\theta)}\bigg(\int_c^d (g(d)-g(\tau))^{\theta-1}|\mathcal{H}(\tau, y_\tau + \bar{z}_\tau)|g'(\tau)d\tau \\
& + \sum_{i=1}^n |\alpha_i| \int_c^{\nu_i} (g(\nu_i)-g(\tau))^{\theta-1}|\mathcal{H}(\tau, y_\tau + \bar{z}_\tau)|g'(\tau)d\tau \bigg)\Bigg] \\
& + \frac{1}{\gamma^\theta \Gamma(\theta)} \int_c^t (g(t)-g(\tau))^{\theta-1}|\mathcal{H}(\tau, y_\tau + \bar{z}_\tau)|g'(\tau)d\tau \\
\leq & \int_c^t |\zeta(s)|ds + \frac{(g(d)-g(c))}{|\mathcal{Q}|}\Bigg[\sum_{i=1}^n |\alpha_i| \int_c^{\nu_i} |\zeta(s)|ds + \int_c^d |\zeta(s)|ds \\
& + \frac{1}{\gamma^\theta \Gamma(\theta)}\bigg(\int_c^d (g(d)-g(\tau))^{\theta-1}|\kappa(\tau)|g'(\tau)d\tau + \sum_{i=1}^n |\alpha_i| \int_c^{\nu_i} (g(\nu_i)-g(\tau))^{\theta-1}|\kappa(\tau)|g'(\tau)d\tau \bigg)\Bigg] \\
& + \frac{1}{\gamma^\theta \Gamma(\theta)} \int_c^t (g(t)-g(\tau))^{\theta-1}|\kappa(\tau)|g'(\tau)d\tau \\
\leq & \ \zeta^*(d-c) + \frac{(g(d)-g(c))}{|\mathcal{Q}|}\left(1 + \sum_{i=1}^n |\alpha_i|\right)\left(\zeta^*(d-c) + \frac{(g(d)-g(c))^\theta \kappa^*}{\gamma^\theta \Gamma(\theta+1)}\right) \\
& + \frac{(g(d)-g(c))^\theta \kappa^*}{\gamma^\theta \Gamma(\theta+1)}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\|\Psi z\|_{\mathfrak{X}'_d} \leq & \ \zeta^*(d-c) + \frac{(g(d)-g(c))}{|\mathcal{Q}|}\left(1 + \sum_{i=1}^n |\alpha_i|\right)\left(\zeta^*(d-c) + \frac{(g(d)-g(c))^\theta \kappa^*}{\gamma^\theta \Gamma(\theta+1)}\right) \\
& + \frac{(g(d)-g(c))^\theta \kappa^*}{\gamma^\theta \Gamma(\theta+1)}. \qquad (3.10)
\end{aligned}
$$

This shows that Ψ is uniformly bounded on Ω_ε.
(iii) $\Psi(\Omega_\varepsilon)$ is an equicontinuous set on $\mathfrak{X}'_d$.
Let $z \in \Omega_\varepsilon$, and $\varepsilon_1, \varepsilon_2 \in \Delta$, such that $\varepsilon_1 < \varepsilon_2$, then by using the fact that $e^{\frac{\gamma-1}{\gamma}(g(t)-g(c))} < 1$ and our

hypotheses, we get

$$
\begin{aligned}
&|(\Psi z)(\varepsilon_2)-(\Psi z)(\varepsilon_1)| \\
&= \Bigg| \int_c^{\varepsilon_2} \sigma(s,y_s+\bar{z}_s)ds - \int_c^{\varepsilon_1} \sigma(s,y_s+\bar{z}_s)ds + \frac{Q_1^g(\varepsilon_2,c)-Q_1^g(\varepsilon_1,c)}{\mathcal{Q}}\Bigg[-\sum_{i=1}^n \alpha_i \int_c^{\nu_i} \sigma(s,y_s+\bar{z}_s)ds \\
&+ \int_c^d \sigma(s,y_s+\bar{z}_s)ds + \frac{1}{\gamma^\theta\Gamma(\theta)}\Bigg(\int_c^d Q_{\theta-1}^g(d,\tau)\mathcal{H}(\tau,y_\tau+\bar{z}_\tau)g'(\tau)d\tau \\
&- \sum_{i=1}^n \alpha_i \int_c^{\nu_i} Q_{\theta-1}^g(\nu_i,\tau)\mathcal{H}(\tau,y_\tau+\bar{z}_\tau)g'(\tau)d\tau\Bigg)\Bigg] + \frac{1}{\gamma^\theta\Gamma(\theta)}\int_c^{\varepsilon_2} Q_{\theta-1}^g(\varepsilon_2,\tau)\mathcal{H}(\tau,y_\tau+\bar{z}_\tau)g'(\tau)d\tau \\
&- \frac{1}{\gamma^\theta\Gamma(\theta)}\int_c^{\varepsilon_1} Q_{\theta-1}^g(\varepsilon_1,\tau)\mathcal{H}(\tau,y_\tau+\bar{z}_\tau)g'(\tau)d\tau \Bigg| \\
&\leq \int_{\varepsilon_1}^{\varepsilon_2} |\xi(s)|ds + \frac{|g(\varepsilon_2)-g(\varepsilon_1)|}{|\mathcal{Q}|}\left(1+\sum_{i=1}^n|\alpha_i|\right)\left(\zeta^*(d-c)+\frac{(g(d)-g(c))^\theta\kappa^*}{\gamma^\theta\Gamma(\theta+1)}\right) \\
&+ \frac{1}{\gamma^\theta\Gamma(\theta)}\int_c^{\varepsilon_1} \left((g(\varepsilon_2)-g(\tau))^{\theta-1}-(g(\varepsilon_1)-g(c))^{\theta-1}\right)|\kappa(\tau)|\,g'(\tau)d\tau \\
&+ \frac{1}{\gamma^\theta\Gamma(\theta)}\int_{\varepsilon_1}^{\varepsilon_2} (g(\varepsilon_2)-g(\tau))^{\theta-1}|\kappa(\tau)|\,g'(\tau)d\tau \\
&\leq \zeta^*|\varepsilon_2-\varepsilon_1| + \frac{|g(\varepsilon_2)-g(\varepsilon_1)|}{|\mathcal{Q}|}\left(1+\sum_{i=1}^n|\alpha_i|\right)\left(\zeta^*(d-c)+\frac{(g(d)-g(c))^\theta\kappa^*}{\gamma^\theta\Gamma(\theta+1)}\right) \\
&+ \frac{\kappa^*}{\gamma^\theta\Gamma(\theta+1)}\left|(g(\varepsilon_2)-g(c))^\theta-(g(\varepsilon_1)-g(c))^\theta\right|.
\end{aligned}
$$

By using the continuity of the function g, from the above inequality, we get $|(\Psi z)(\varepsilon_2)-(\Psi z)(\varepsilon_1)| \to 0$ as $\varepsilon_1 \to \varepsilon_2$. This shows that the operator $\Psi(\Omega_\varepsilon)$ is equicontinuous in $\mathfrak{X}'_d$.
From *ii.* and *iii.*, we deduce by the Arzelà-Ascoli theorem that $\Psi(\Omega_\varepsilon)$ is a relatively compact subset of $\mathfrak{X}'_d$. Furthermore, since $\Psi:\Omega_\varepsilon \to \mathfrak{X}'_d$ is a continuous operator, then it follows that Ψ is completely continuous.

Claim 3.

we verify that condition (c) of Theorem 2.1 is satisfied. Let $x \in \mathfrak{X}'_d$ and $y \in \Omega_\varepsilon$ such that $x = \Upsilon x \Psi y$. Then, by hypothesis (H_1), we have

$$
\begin{aligned}
|x(t)| &= |\Upsilon x(t)||\Psi y(t)| \\
&\leq \beta^*\left(\zeta^*(d-c)+\frac{(g(d)-g(c))}{|\mathcal{Q}|}\left(1+\sum_{i=1}^n|\alpha_i|\right)\left(\zeta^*(d-c)+\frac{(g(d)-g(c))^\theta\kappa^*}{\gamma^\theta\Gamma(\theta+1)}\right)\right. \\
&\left.+\frac{(g(d)-g(c))^\theta\kappa^*}{\gamma^\theta\Gamma(\theta+1)}\right) \\
&=: \varepsilon,
\end{aligned}
$$

this implies that $\|x\|_{\mathfrak{X}'_d} \leq \varepsilon$, and thus $x \in \Omega_\varepsilon$.

On the other hand, for $\eta = \|\Psi(\Omega_\varepsilon)\| = \sup\{\Psi(x),\, x \in \Omega_\varepsilon\}$ and by (3.10), we obtain $\eta \leq \frac{\varepsilon}{\beta^*}$, therefore from (3.9), we get

$$\varsigma\eta < 1.$$

This shows that condition (c) of Theorem 2.1 is satisfied. Thus, for all $x \in \Omega_\varepsilon$ and $t \in \Delta$, all the conditions of Theorem 2.1 are fulfilled. Consequently, the operator $x = \Upsilon x \Psi x$ admits at least one solution in $\mathfrak{X}'_d$. It follows that the nonlinear boundary value hybrid fractional integro-differential system (1.3) has at least one solution on $(-\infty, d]$. This completes the proof. □

4. Applications

In this section, we build two practical exapmles to illustrate the applicability and consistency of our main results.

For a continuous function $\mathfrak{B} : (-\infty, 0) \to [0, \infty)$ satisfying $\int_{-\infty}^{0} \mathfrak{B}(s)ds < \infty$, consider the space $\mathfrak{X}_{\mathfrak{B}} = \left\{x \in C(-\infty, 0], \mathbb{R}) : \int_{-\infty}^{0} \mathfrak{B}(s) \parallel x \parallel_{[s,0]} ds < \infty\right\}$, where $\parallel x \parallel_{[s,0]} = \sup_{t\in[s,0]} |x(t)|$. Let $\mathfrak{B}(s) = e^{3s}$, then $\int_{-\infty}^{0} e^{3s}ds = \frac{1}{3} < \infty$, and supplement this space with the norm $\parallel x \parallel_{\mathfrak{X}_{\mathfrak{B}}} = \int_{-\infty}^{0} \mathfrak{B}(s) \parallel x \parallel_{[s,0]} ds$. The space $\mathfrak{X}_{\mathfrak{B}}$ satisfied the phase space's axioms, with $\varpi(t) = \frac{1}{3}$, $\Lambda(t) = 1$, $C = 3$. See [3].

Example 1. Let $\Delta = [0, 1]$, $\gamma = \frac{5}{3}$, $\theta = \frac{3}{2}$, $\mathcal{F}(t, x(t)) = 2 + t^2 \tan^{-1}(x(t))$, $\sigma(t, x_s) = \frac{s^2 \sin(x_s)}{10}$, $\mathcal{H}(t, x_t) = \frac{t^2 \cos(x_t)}{10}$ and $g(t) = t^2$.
We consider the following nonlinear boundary value hybrid fractional integro-differential system:

$$\begin{cases} {}^{C}_{\frac{5}{3}}D^{\frac{3}{2},t^2}_{0^+}\left[\frac{x(t)}{2+t^2\tan^{-1}(x(t))} - \int_0^t \frac{s^2\sin(x_s)}{10}ds\right] = \frac{t^2\cos(x_t)}{10}, & t \in [0,1], \\ x(t) = \lambda(t), & t \in (-\infty, 0], \\ \left(\frac{x(t)}{2+t^2\tan^{-1}(x(t))}\right)_{t=0} = 0, & \\ \left(\frac{x(t)}{2+t^2\tan^{-1}(x(t))}\right)_{t=1} = \sum_{i=1}^{2}\alpha_i \, \frac{x(\nu_i)}{2+\nu_i^2\tan^{-1}(x(\nu_i))} & \end{cases} \tag{4.1}$$

where $\nu_1 = 0.3$, $\nu_2 = 0.7$, and $\alpha_1 = \alpha_2 = 0.1$.
In the following we aim to establish the validity of the hypotheses $(H_0), (H_1), (H_2)$, and the condition (3.9). Obviously, $\mathcal{H}, \sigma$, and $\mathcal{F}$ are continuous funtions, then H_0 holds.
For all $t \in [0, 1]$ and $x, y \in \mathbb{R}$, we have

$$\begin{aligned} |\mathcal{F}(t,x) - \mathcal{F}(t,y)| &\leq t^2|\tan^{-1}(x) - \tan^{-1}(y)| \\ &\leq |x-y|. \end{aligned}$$

Hence, the hypothesis (H_1) is satisfied with $\varsigma = 1$.

$$|\mathcal{F}(t,x)| = |2 + t^2\tan^{-1}(x)| \leq \beta(t) = 2 + \frac{\pi}{2}t^2.$$

$$|\sigma(t,x_t)| = \left|\frac{t^2\sin(x_t)}{10}\right| \leq \zeta(t) = \frac{t^2}{10}.$$

$$|\mathcal{H}(t,x_t)| = \left|\frac{t^2\cos(x_t)}{10}\right| \leq \kappa(t) = \frac{t^2}{10}.$$

Therefore, the hypothesis (H_2) holds with $\beta^* = \sup_{t\in\Delta}\left(2 + \frac{\pi}{2}t^2\right) = 3.57$, and $\quad \zeta^* = \kappa^* = \sup_{t\in\Delta}\frac{t^2}{10} = 0.1.$
Calculations employing the prior parameters revealed that

$$|\mathcal{Q}| \geq |Q_1^g(1,0)| - \sum_{i=1}^{2}\alpha_i \, |Q_1^g(\nu_i,0)| \geq 1.49 - 0.06 = 1.42 \geq 1.$$

Thus, using the fact that $\frac{g(1)-g(0)}{|\mathcal{Q}|} < 1$, we obtain

$$\begin{aligned} \varsigma\frac{\varepsilon}{\beta^*} &= \varsigma\left[\zeta^* + \frac{g(1)-g(0)}{|\mathcal{Q}|}\left(1 + \sum_{i=1}^{2}|\alpha_i|\right)\left(\zeta^* + \frac{(g(1)-g(0))^\theta\kappa^*}{\gamma^\theta\Gamma(\theta+1)}\right) + \frac{(g(1)-g(0))^\theta\kappa^*}{\gamma^\theta\Gamma(\theta+1)}\right] \\ &\leq \zeta^* + \left(1 + \sum_{i=1}^{2}|\alpha_i|\right)\left(\zeta^* + \frac{\kappa^*}{\gamma^\theta\Gamma(\theta+1)}\right) + \frac{\kappa^*}{\gamma^\theta\Gamma(\theta+1)} \\ &\leq 0.1 + 1.2\left(0.1 + \frac{0.1}{\left(\frac{5}{3}\right)^{3/2}\cdot\Gamma\left(\frac{5}{2}\right)}\right) + \frac{0.1}{\left(\frac{5}{3}\right)^{3/2}\cdot\Gamma\left(\frac{5}{2}\right)} \\ &\approx 0.296 \leq 1. \end{aligned}$$

Then, the condition (3.9) holds.
We note that all the conditions of Theorem 3 are fulfilled. Therefore, the nonlinear boundary value hybrid fractional integro-differential system (4.1) possesses at least one solution on $(-\infty, 1]$.

Example 2. Let $\Delta = [1,2]$, $\gamma = \frac{7}{10}$, $\theta = \frac{4}{5}$, $\mathcal{F}(t, x(t)) = 2+10^{-2}\sin(x(t))$, $\sigma(t, x_s) = e^{-s}\cos(x_s)$, $\mathcal{H}(t, x_t) = t^2 + \cos(x_t)$ and $g(t) = t^2$.
Consider the following nonlinear boundary value hybrid fractional integro-differential system:

$$\begin{cases} {}^{C}_{\frac{7}{10}}D^{\frac{4}{5},t^2}_{1^+}\left[\frac{x(t)}{2+10^{-2}\sin(x(t))} - \int_1^t e^{-s}\cos(x_s)\,ds\right] = t^2 + \cos(x_t), & t \in [1,2], \\ x(t) = \lambda(t), & t \in (-\infty, 1], \\ \left(\frac{x(t)}{2+10^{-2}\sin(x(t))}\right)_{t=1} = 0, \\ \left(\frac{x(t)}{2+10^{-2}\sin(x(t))}\right)_{t=2} = \sum_{i=1}^{2} \alpha_i \frac{x(\nu_i)}{2+10^{-2}\sin(x(\nu_i))}, \end{cases} \tag{4.2}$$

where $\nu_1 = 1.3$, $\nu_2 = 1.7$, and $\alpha_1 = \alpha_2 = 0.05$.

In what follows, we seek to demonstrate the validity of hypotheses (H_0), (H_1), (H_2), as well as the condition given by (3.9).

It is evident that $\mathcal{H}, \sigma$ and $\mathcal{F}$ are continuous funtions, which implies that (H_0) is fulfilled.

Moreover, we have

$$|\mathcal{F}(t,x) - \mathcal{F}(t,y)| \le 10^{-2}|x-y| \le \|x-y\|_{\mathfrak{X}_2}, \quad \forall t \in [1,2],\ x, y \in \mathbb{R}.$$

Thus, $\mathcal{F}$ satisfies the Lipschitz condition with a constant $\varsigma = 10^{-2}$. then, hypothesis (H_1) is satisfied. The next step is to check the validity of hypothesis (H_2),

$$|\mathcal{F}(t, x(t))| = \left|2 + 10^{-2}\sin(x(t))\right| \le \beta(t) = 2.01.$$

$$|\sigma(s, x_s)| = \left|e^{-s}\cos(x_s)\right| \le \zeta(s) = e^{-s}.$$

$$|\mathcal{H}(t, x_t)| = |t^2 + \cos(x_t)| \le \kappa(t) = t^2 + 1.$$

Therefore, hypothesis (H_2) holds with $\beta^* = 2.01$, $\zeta^* = \sup\limits_{s\in\Delta} e^{-s} = e^{-1}$ and $\kappa^* = \sup\limits_{t\in\Delta}(t^2+1) = 5$.
Based on the previous values, we found that $Q_1^g(1.3, 1) \approx 0.5127$, $Q_1^g(1.7, 1) \approx 0.839$ and $Q_1^g(2, 1) \approx 0.829$. Substitute these into $\mathcal{Q}$ we obtain

$$|\mathcal{Q}| = \left|\sum_{i=1}^{2} \alpha_i Q_1^g(\nu_i, 1) - Q_1^g(2,1)\right| \approx 0.761.$$

From the above data, we have

$$\begin{aligned} \varsigma\frac{\varepsilon}{\beta^*} &= \varsigma\left[\zeta^* + \frac{g(2)-g(1)}{|\mathcal{Q}|}\left(1 + \sum_{i=1}^{2}|\alpha_i|\right)\left(\zeta^* + \frac{(g(2)-g(1))^\theta\kappa^*}{\gamma^\theta\Gamma(\theta+1)}\right) + \frac{(g(2)-g(1))^\theta\kappa^*}{\gamma^\theta\Gamma(\theta+1)}\right] \\ &= 10^{-2}\left[e^{-1} + \frac{3}{0.761} \times 1.1\left(e^{-1} + \frac{3^{\frac{4}{5}} \times 5}{(\frac{7}{10})^{\frac{4}{5}} \times \Gamma\left(\frac{9}{5}\right)}\right) + \frac{3^{\frac{4}{5}} \times 5}{(\frac{7}{10})^{\frac{4}{5}} \times \Gamma\left(\frac{9}{5}\right)}\right] \\ &\approx 0.937 \le 1. \end{aligned}$$

Thus, the condition (3.9) is satisfied.
We note that all the conditions of Theorem 3 are fulfilled. Therefore, the nonlinear boundary value hybrid fractional integro-differential system (4.2) possesses at least one solution on $(-\infty, 2]$.

5. Conclusion

This research investigates the existence of solutions for a class of nonlinear hybrid fractional integro-differential equations with boundary conditions, governed by the generalized Caputo proportional fractional derivative of order $\theta \in (1, 2)$. By employing the Dhage fixed point theorem, we derive rigorous criteria ensuring the existence of solutions under specified hypotheses. To validate the theoretical framework, two illustrative examples are presented, demonstrating the applicability and consistency of the established results.

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