Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.77007

Pythagorean Fuzzy Topological Polygroups

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ABSTRACT: In this paper, we introduce and study the notion of Pythagorean fuzzy topological subpolygroup and Pythagorean fuzzy topological polygroup. Moreover we investigate some of a Pythagorean fuzzy subpolygroup and Pythagorean fuzzy normal subpolygroup interesting properties.

Keywords: Pythagorean fuzzy sets, Pythagorean fuzzy topological polygroup, Pythagorean fuzzy topological subpolygroup.

Contents

1. Introduction

In order to account for the uncertainty in decision-making, Zadeh [1] proposed the concept of a fuzzy group, which has a membership function that assigns a number from the unit interval [0, 1] to every element of the universe of discourse to indicate the degree of belongingness to the set under consideration. Fuzzy sets are a generalization of classical sets theory that allows for intermediate states between the whole and nothing. A membership function is described in a fuzzy set to describe the degree of membership of an element to a class. The membership value ranges from 0 to 1, with 0 denoting that the element does not belong to a class, 1 denoting that it does, and other values denoting the degree of class membership. The membership function in fuzzy sets has taken the place of the characteristic function in crisp sets.

Despite, the idea of fuzzy sets theory seems inconclusive due to the exclusion of non-membership functions and the disregard for the probability of hesitation margin. Atanassov critically studied these shortcomings and proposed a concept called intuitionistic fuzzy sets in [2]. The construct intuitionistic fuzzy sets incorporates membership function, μ and non-membership function, ν with hesitation margin, π such that $\mu + \nu \leqslant 1$ and $\mu + \nu + \pi = 1$. Atanassov [12] introduced intuitionistic fuzzy sets of second type with the property that the sum of the square of the membership and non-membership degrees is less than or equal to one. The idea of intuitionistic fuzzy sets provides a flexible framework to elaborate uncertainty and vagueness. There are situations where $\mu+\nu \geq 1$ unlike the cases capture in intuitionistic fuzzy sets. Also the propertiesand applications of bitopological spaces and soft bitopological spaces, soft ditopological spaces have been studied increasingly [13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

This limitation in intuitionistic fuzzy set naturally led to a construct, called Pythagorean fuzzy sets. Pythagorean fuzzy set proposed in [3–6] is a new tool to deal with vagueness considering the membership grade, μ and non-membership grade, ν satisfying the conditions $\mu+\nu \leqslant 1$ or $\mu+\nu \geq 1$, and also, it follows that $\mu^2 + \nu^2 + \pi^2 = 1$, where π is the Pythagorean fuzzy set index. As a generalized set, Pythagorean

2020 *Mathematics Subject Classification:* 20N20, 20N25, 03E72.
Submitted May 23, 2025. Published March 22, 2026

fuzzy set has close relationship with intuitionistic fuzzy set. The construct of Pythagorean fuzzy sets can be used to characterize uncertain information more sufficiently and accurately than intuitionistic fuzzy set.Murat, Mehmet and Yardimci in [7] was introduced the concept of Pythagorean fuzzy topological spaces.

Marty introduced the concept of hypergroup [8] in 1934, and thus the hyperstructure theory was born. The hypergroup theory is a natural extension of the group theory. The composition of two elements in a group is an element, while the composition of two elements in a hypergroup is a set. Let H be a non-empty set. Then a mapping $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ is called a hyperoperation, where $\mathcal{P}^*(H)$ is the collection of non-empty subsets of H. The couple $(H, \circ)$ is called a hypergroupoid. If A and B two non-empty subsets of H and $x \in H$, then we define

$$A \circ B \bigcup_{a\in A, b\in B} a \circ b,$$

$$x \circ A = \{x\} \circ A \quad and \quad A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a semihypergroup if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ and it is called a quasihypergroup if for every $x \in H$, we have $x \circ H = H = H \circ x$. This condition is called the reproduction axiom. The couple $(H, \circ)$ is called a hypergroup if it is a semihypergroup and a quasihypergroup [8, 9].
A special subclass of hypergroups is the class of polygroups. We recall the following definition from [10]. A polygroup is a system $P =< P, \circ, e, ^{-1} >$, where $\circ : P \times P \longrightarrow \mathcal{P}^*(P), e \in P, ^{-1}$ is a unitary operation on P and the following axioms hold for all $x, y, z \in P$:
(1) $(x \circ y) \circ z = x \circ (y \circ z)$,
(2) $e \circ x = x = x \circ e$,
(3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

The following elementary facts about polygroups follow easily from the axioms:

$$e \in x \circ x^{-1} \cap x^{-1} \circ x, e^{-1} = e, (x^{-1})^{-1} = x$$

and

$$(x \circ y)^{-1} = y^{-1} \circ x^{-1}.$$

A non-empty subset K of a polygroup P is a subpolygroup of P if and only if $a, b \in K$ implies $a \circ b \subseteq K$ and $a \in K$ implies $a^{-1} \in K$. The subpolygroup N of P is normal in P if and only if $a^{-1} \circ N \circ a \subseteq N$ for all $a \in P$. For a subpolygroup K of P and $x \in P$, denote the right coset of K by $K \circ x$ and let P/K be the set of all right cosets of K in P. If N is a normal subpolygroup of P, then $(P/N, \odot, N, ^{-1})$ is a polygroup, where $N \circ x \odot N \circ y = \{N \circ z : z \in N \circ x \circ y\}$ and $(N \circ x)^{-1} = N \circ x^{-1}$.

Let $P =< P, \circ, e, ^{-1} >$ be a polygroup and (P, τ) be a topological space. Then, the system $P =< P, \circ, e, ^{-1}, \tau >$ is called a topological polygroup if the mapping $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$ and $^{-1} : P \longrightarrow P$ are continuous (see [11]).

The main purpose of this paper is to extend the notions of fuzzy topological space and intuitionistic fuzzy topological space by introducing the notion of Pythagorean fuzzy topological subpolygroup and Pythagorean fuzzy topological polygroup. we define the concept of Pythagorean fuzzy topological space and Pythagorean fuzzy topological subspace. We also study the continuity and open of a function defined among Pythagorean fuzzy topological spaces. After we define the concept of Pythagorean fuzzy subpolygroup and we study some of properties of Pythagorean fuzzy subpolygroup. Moreover we introduce the concept of Pythagorean fuzzy topological polygroup. We also study some of properties of Pythagorean fuzzy topological polygroup.
In the futureworks, categorical properties of Pythagorean fuzzy topological polygroup, applications of Pythagorean fuzzy topological polygroup on decision making theory and Pythagorean fuzzy soft topological polygroupmay be studied.

2. Preliminaries

Definition 2.1 [1] *Let $X \neq \emptyset$. A fuzzy set A in X is an object having the form:*

$$A = \{< x, \mu_A(x) >: x \in X\}$$

where the function

$$\mu_A(x) : X \longrightarrow [0,1]$$

defines the degree of membership of the element, $x \in X$.

The closer the membership value $\mu_A(x)$ to 1, the more x belongs to A, where the grades 1 and 0 represent full membership and full nonmembership.

Definition 2.2 [2] *Let $X \neq \emptyset$. An intuitionistic fuzzy set A in X is an object having the form:*

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$$

where the functions

$$\mu_A(x) : X \longrightarrow [0,1] \quad \textit{and} \quad \nu_A(x) : X \longrightarrow [0,1]$$

define the degree of membership and the degree of nonmembership, respectively, of the element $x \in X$ to A, which is a subset of X, and for every $x \in X$:

$$0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1$$

For every A in X:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

is the intuitionistic fuzzy set index or hesitation margin of x in X. The hesitation margin $\pi_A(x)$ is the degree of nondeterminacy of $x \in X$ to the set A and $\pi_A(x) \in [0,1]$. The hesitation margin is the function that expresses lack of knowledge of whether $x \in X$ or $x \notin X$. Thus:

$$\pi_A(x) + \mu_A(x) + \nu_A(x) = 1.$$

Definition 2.3 $[3-5]$ *Let X be a universal set. Then, a Pythagorean fuzzy set A (shortly PFS), which is a set of ordered pairs over X, is defined by the following:*

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$$

where the functions

$$\mu_A(x) : X \longrightarrow [0,1] \quad \textit{and} \quad \nu_A(x) : X \longrightarrow [0,1]$$

define the degree of membership and the degree of nonmembership, respectively, of the element $x \in X$ to A, which is a subset of X, and for every $x \in X$:

$$0 \leqslant (\mu_A(x))^2 + (\nu_A(x))^2 \leqslant 1.$$

Supposing $(\mu_A(x))^2 + (\nu_A(x))^2 \leqslant 1$, then there is a degree of indeterminacy of $x \in X$ to A defined by

$$\pi_A(x) = \sqrt{1 - [(\mu_A(x))^2 + (\nu_A(x))^2]}$$

and $\pi_A(x) \in [0,1]$. In what follows

$$(\mu_A(x))^2 + (\nu_A(x))^2 + (\pi_A(x))^2 = 1.$$

Otherwise, $\pi_A(x) = 0$ whenever

$$(\mu_A(x))^2 + (\nu_A(x))^2 = 1.$$

We denote the set of all PFSs over X by PFS(X).

Theorem 2.1 *Let $X = \{x_i\}$ be a universal set, for $i = 1, ..., n$ and $A \in PFS(X)$. Suppose that $\pi_A(x_i) = 0$, and then, the following hold:*

1. $|\mu_A(x_i)| = \sqrt{|(\nu_A(x_i)+1)(\nu_A(x_i)-1)|}$.
2. $|\nu_A(x_i)| = \sqrt{|(\mu_A(x_i)+1)(\mu_A(x_i)-1)|}$.

Proof: *See* [6]. □

Definition 2.4 [5] *Let $A, B \in PFS(X)$. Then*

1. *The complement of A denoted by $A^c = \{< x, \nu_A(x), \mu_A(x) >: x \in X\}$.*
2. $A \cup B = \{< x, max(\mu_A(x), \mu_B(x)), min(\nu_A(x), \nu_B(x)) >: x \in X\}$.
3. $A \cap B = \{< x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_B(x)) >: x \in X\}$.
4. $A \subseteq B$ *iff* $\mu_A(x) \leqslant \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$ *for every* $x \in X$.
5. $A = B$ *iff* $A \subseteq B$ *and* $B \subseteq A$.

Definition 2.5 [5]

$$\mu_{1_X} = 1 \quad and \quad \nu_{1_X} = 0$$

and

$$\mu_{0_X} = 0 \quad and \quad \nu_{0_X} = 1.$$

Definition 2.6 [5] *Let $A, B \in PFS(X)$. Then*

1. $A \oplus B = \{< x, \sqrt{(\mu_A(x))^2 + (\mu_B(x))^2 - (\mu_A(x))^2(\mu_B(x))^2}, \nu_A(x)\nu_B(x) >: x \in X\}$.
2. $A \otimes B = \{< x, \mu_A(x)\mu_B(x), \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} >: x \in X\}$.

Definition 2.7 [6] *Let X and Y be sets and let f be a function from X to Y. Suppose that A and B are Pythagorean fuzzy sets of X and Y, respectively. Then*

1. *The image of A under f , denoted by $f(A)$, is a Pythagorean fuzzy set of Y defined by the following:*
   $$\mu_{f(A)}(y) = \begin{cases} sup_{x \in f^{-1}(y)} \mu_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & otherwise \end{cases}$$
   and
   $$\nu_{f(A)}(y) = \begin{cases} inf_{x \in f^{-1}(y)} \nu_A(x), & f^{-1}(y) \neq \emptyset \\ 1, & otherwise \end{cases}$$
   for each $y \in Y$.
2. *The inverse image of B under f , denoted by $f^{-1}(B)$, is a Pythagorean fuzzy set of X defined by the following:*
   $$\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$$
   and
   $$\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$$
   for every $x \in X$.

Proposition 2.1 [7] *Let X and Y be two non-empty sets and let $f : X \longrightarrow Y$ be a function. Then, we have*

1. $f^{-1}[B^c] = f^{-1}[B]^c$ *for any Pythagorean fuzzy subset B of Y.*

2. $f[A]^c \subset f[A^c]$ *for any Pythagorean fuzzy subset* A *of* X.
3. *if* $B_1 \subset B_2$ *then* $f^{-1}[B_1] \subset f^{-1}[B_2]$ *where* B_1 *and* B_2 *are Pythagorean fuzzy subsets of* Y.
4. *if* $A_1 \subset A_2$ *then* $f[A_1] \subset f[A_2]$ *where* A_1 *and* A_2 *are Pythagorean fuzzy subsets of* X .
5. $f[f^{-1}[B]] \subset B$ *for any Pythagorean fuzzy subset* B *of* Y.
6. $A \subset f^{-1}[f[A]]$ *for any Pythagorean fuzzy subset* A *of* X.

Proof: *See* [7] □

Definition 2.8 [6] *Let X and Y be two nonempty sets. A Pythagorean fuzzy relation (PFR), R, from X to Y is a PFS of* $X \times Y$ *characterized by the membership function,* μ_R *and nonmembership function,* ν_R. *A PF relation or PFR from X to Y is denoted by* $R(X \longrightarrow Y)$.

Definition 2.9 [6] *Let* $A \in PFS(X)$. *Then, the max–min–max composition of* $R(X \longrightarrow Y)$ *with A is a PFS B of Y denoted by* $B = R \circ A$, *such that its membership and nonmembership functions are defined by the following:*

$$\mu_B(y) = \bigvee_x (min[\mu_A(x), \mu_R(x, y)])$$

and

$$\nu_B(y) = \bigwedge_x (max[\nu_A(x), \nu_R(x, y)])$$

for every $x \in X$ *and* $y \in Y$.

Definition 2.10 [6] *Let* $Q(X \longrightarrow Y)$ *and* $R(Y \longrightarrow Z)$ *be two PFRs. Then, the max–min–max composition* $R \circ Q$ *is a PFR from X to Z, such that its membership and nonmembership functions are defined by the following:*

$$\mu_{R\circ Q}(x, z) = \bigvee_y (min[\mu_Q(x, y), \mu_R(y, z)])$$

and

$$\nu_{R\circ Q}(x, z) = \bigwedge_y (max[\nu_Q(x, y), \nu_R(y, z)])$$

for every $(x, z) \in X \times Z$ *and* $y \in Y$.

3. Pythagorean Fuzzy Topological Spaces

In this section, we define the concept of Pythagorean fuzzy topological space and Pythagorean fuzzy topological subspace. We also study the continuity and open of a function defined among Pythagorean fuzzy topological spaces.

Definition 3.1 [7] *Let X be a non-empty set and let* $\tau \subseteq PFS(X)$. *If*

1. $1_X, 0_X \in \tau$.
2. $A_1 \cap A_2 \in \tau$ *for any* $A_1, A_2 \in \tau$.
3. $\bigcup_{i\in I} A_i \in \tau$ *for any* $\{A_i\}_{i\in I} \in \tau$.

Then τ *is called a Pythagorean fuzzy topology on X.*

In this case the pair (X, τ) *is called a Pythagorean fuzzy topological space (shortly, PFTS). Each member of* τ *is called a Pythagorean fuzzy open set (shortly, PFOS). The complement of a Pythagorean fuzzy open set is called a Pythagorean fuzzy closed set (shortly, PFCS).*

Example 3.1 1. The family $\{0_X, 1_X\}$ is Pythagorean fuzzy topological space on X.

2. The family of all Pythagorean fuzzy sets in X is a Pythagorean fuzzy topological space on X.

Example 3.2 Let $X = \{a, b, c\}$ and $A =< x, (\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.7}), (\frac{a}{0.8}, \frac{b}{0.5}, \frac{c}{0.3}) >$. Then the family $\tau = \{\{0_X, A, 1_X\}$ of PFSs in X is a PFT on X.

Definition 3.2 *Let (X, τ) be a PFTS and let $A \in PFS(X)$. Then the family $\tau_A = \{U \cap A \in PFS(X) : U \in \tau\}$ is called the induced Pythagorean fuzzy topology (shortly, IPFT) on A. The pair (A, τ_A) is called a Pythagorean fuzzy subspace (shortly,PFSP) of (X, τ).*

Definition 3.3 [7] *Let (X, τ) be a PFTS and let $A, B \in PFS(X)$. Then B is called a neighbourhood of A if there exists a Pythagorean fuzzy open set U such that $A \subseteq U \subseteq B$.*

Definition 3.4 *Let (X, τ_X) and (Y, τ_Y) be two PFTSs and let $f : X \longrightarrow Y$ be a mapping.*

1. *f is called Pythagorean fuzzy continuous if for every $V \in \tau_Y, f^{-1}(V) \in \tau_X$.*
2. *f is called Pythagorean fuzzy open if for every $U \in \tau_X, f(U) \in \tau_Y$.*

Definition 3.5 *Let (A, τ_A) and (B, τ_B) be PFSPs of PFTSs (X, τ_X) and (Y, τ_Y) respectively and let $f : X \longrightarrow Y$ be a mapping.*

1. *f is called a mapping of (A, τ_A) into (B, τ_B) denoted by $f : (A, \tau_A) \longrightarrow (B, \tau_B)$ if $f(A) \subseteq B$.*
2. *A mapping $f : (A, \tau_A) \longrightarrow (B, \tau_B)$ is called relatively Pythagorean fuzzy continuous if $f^{-1}(V) \cap A \in \tau_A$ for every $V \in \tau_B$.*
3. *A mapping $f : (A, \tau_A) \longrightarrow (B, \tau_B)$ is called relatively Pythagorean fuzzy open if for every $U \in \tau_A, f(U) \in \tau_B$.*

Proposition 3.1 *Let (A, τ_A) and (B, τ_B) be PFSPs of PFTSs (X, τ_X) and (Y, τ_Y) respectively and let $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ be a Pythagorean fuzzy continuous mapping such that $f(A) \subseteq B$. Then $f : (A, \tau_A) \longrightarrow (B, \tau_B)$ is relatively Pythagorean fuzzy continuous.*

Proof: *Let $V^{'} \in \tau_B$. Then there exists a $V \in \tau_Y$ such that $V^{'} = V \cap B$. Since $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is Pythagorean fuzzy continuous and $V \in \tau_Y, f^{-1}(V) \in \tau_X$. On the other hand, $f^{-1}(V^{'}) \cap A = f^{-1}(V) \cap f^{-1}(B) \cap A$. Since $f(A) \subseteq B$, then $A \subseteq f^{-1}(B)$. Thus $f^{-1}(V^{'}) \cap A = f^{-1}(V) \cap A$. So $f^{-1}(V^{'}) \cap A \in \tau_A$. Hence f is relatively Pythagorean fuzzy continuous.* □

Proposition 3.2 *Let $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ and $g : (Y, \tau_Y) \longrightarrow (Z, \tau_Z)$ be Pythagorean fuzzy continuous (Pythagorean fuzzy open). Then $g \circ f : (X, \tau_X) \longrightarrow (Z, \tau_Z)$ is Pythagorean fuzzy continuous (Pythagorean fuzzy open).*

Definition 3.6 *Let (X, τ) be a PFTS and let $\beta \subseteq \tau$. Then β is called a base for τ if for every $U \in \tau$ either $U = 0_X$ or there exists a $\beta^{'} \subseteq \beta$ such that $U = \bigcup \beta^{'}$.*

Definition 3.7 *Let (X, τ) be a PFTS, let $A \in PFS(X)$ and let $\beta \subseteq \tau_A$. Then β is called a base for τ_A if for every $U \in \tau_A$ either $U = 0_X$ or there exists a $\beta^{'} \subseteq \beta$ such that $U = \bigcup \beta^{'}$.*

Definition 3.8 *Let $f : X \longrightarrow Y$ be a mapping and let τ_Y an PFT on Y. Then the family $\tau_{f^{-1}} = \{f^{-1}(U) \in PFS(X) : U \in \tau_Y\}$ is called the inverse image of τ_Y under f.*

Definition 3.9 *Let $f : X \longrightarrow Y$ be a mapping and let τ_X an PFT on Y. Then the family $\tau_f = \{U \in PFS(X) : f^{-1}(U) \in \tau_X\}$ is called the inverse image of τ_X under f.*

Definition 3.10 *Let $\{(X_i, \tau_i)\}_{i \in I}$ be a family of PFTSs, let $X = \prod_{i \in I} X_i$, let (X, τ_X) an PFTS and let τ the coarest PFT on X for which $\pi_i : (X, \tau) \longrightarrow (X_i, \tau_i)$ is Pythagorean fuzzy continuous for every $i \in I$, where π_i is usual projection. Then τ is called Pythagorean fuzzy product topology (shortly, PFPT) on X and denoted by $\prod_{i \in I} \tau_i$ and (X, τ) a Pythagorean fuzzy product space (shortly, PFPS).*

Proposition 3.3 *Let $\{(X_i, \tau_i)\}_{i \in I}$ be a family of PFTSs and let (X, τ) the PFPS. Then τ has as a base the set of finite intersection of PFSs in X of the from $\pi_i^{-1}(U_i)$ where $U_i \in \tau_i$ for every $i \in I$.*

4. Pythagorean Fuzzy Subpolygroups

In this section, we define the concept of Pythagorean fuzzy subpolygroup and we study some of properties of Pythagorean fuzzy subpolygroup.

Definition 4.1 *Let P be a polygroup and let $A \in PFS(P)$. Then A is called Pythagorean fuzzy subpolygroup (shortly, PFSP) of P if it satisfies the following conditions:*

1. $\mu_A(x \circ y) \geq \mu_G(x) \wedge \mu_G(y)$ *and* $\nu_A(x \circ y) \leqslant \nu_G(x) \vee \nu_G(y)$ *for every* $x, y \in P$.
2. $\mu_A(x^{-1}) \geq \mu_A(x)$ *and* $\nu_A(x^{-1}) \leqslant \nu_A(x)$ *for every* $x \in P$.

We will denote the set of all PFSPs of P as PFSP(P).

Example 4.1 Let $P = \{e, a, b\}$. Then, P together with the following hyperoperation

$\circ$	e	a	b
e	e	a	b
a	a	e	b
b	b	b	{e,a}

is a polygroup. Let $A =< x, (\frac{e}{0.9}, \frac{a}{0.4}, \frac{b}{0.6}), (\frac{e}{0.2}, \frac{a}{0.7}, \frac{b}{0.3}) >$. Then, A is a PFSP of P.

Proposition 4.1 *Let P be a polygroup and let $A \in PFS(P)$. Then A is PFSP of P iff*

$$\mu_A(x \circ y^{-1}) \geq \mu_A(x) \wedge \mu_A(y^{-1}) \quad and \quad \nu_G(x \circ y^{-1}) \leqslant \nu_A(x) \vee \nu_A(y^{-1})$$

for every $x, y \in P$.

Proof: *It is straightforward.* □

Definition 4.2 *Let $< P, \cdot, e, ^{-1} >$ and $< P^{'}, *, e^{'}, ^{-1} >$ be two polygroups. Let f be a mapping from P to $P^{'}$ such that $f(e) = e^{'}$. Then, f is called a polygroup homomorphism if $f(x \cdot y) = f(x) * f(y)$, for all $x, y \in P$.*

Definition 4.3 *Let $A \in PFSP(P)$. Then A is said to have the sup property if for any $T \in \mathcal{P}^*(P)$, there exists a $t_0 \in T$ such that*

$$A(t_0) = \bigcup_{t \in T} A(t).$$

Proposition 4.2 *Let $f : P \longrightarrow P^{'}$ be a group polygroup homomorphism and let $A \in PFSP(P), B \in PFSP(P^{'})$. Then*

1. *If A has the sup property then, $f(A) \in PFSP(P^{'})$.*
2. $f^{-1}(B) \in PFSP(P)$.

Proof: *1. Let $u, v \in P^{'}$. Suppose neither $f^{-1}(u)$ nor $f^{-1}(v)$ is empty. Let $r_0 \in f^{-1}(u), s_0 \in f^{-1}(v)$ such that*

$$\mu_A(r_0) = \bigvee_{t \in f^{-1}(u)} \mu_A(t), \quad \nu_A(r_0) = \bigwedge_{t \in f^{-1}(u)} \nu_A(t)$$

and

$$\mu_A(s_0) = \bigvee_{t \in f^{-1}(v)} \mu_A(t), \quad \nu_A(s_0) = \bigwedge_{t \in f^{-1}(v)} \nu_A(t).$$

Then

$$\mu_{f(A)}(uv^{-1}) = \bigvee_{w \in f^{-1}(uv^{-1})} \mu_A(w) \geq \mu_A(r_0) \wedge \mu_A(s_0) = \mu_{f(A)}(u) \wedge \mu_{f(A)}(v)$$

and

$$\nu_{f(A)}(uv^{-1}) = \bigwedge_{w\in f^{-1}(uv^{-1})} \nu_A(w) \leqslant \nu_A(r_0) \vee \nu_A(s_0) = \nu_{f(A)}(u) \vee \nu_{f(A)}(v).$$

Hence $f(A) \in PFSP(P^{'})$.

2. For every $x, y \in P$,

$$\mu_{f^{-1}(B)}(xy^{-1}) = \mu_B(f(xy^{-1})) = \mu_B(f(x)(f(y))^{-1}) \geq \mu_B(f(x)) \wedge \mu_B(f(y)) = \mu_{f^{-1}(B)}(x) \wedge \mu_{f^{-1}(B)}(y)$$

and

$$\nu_{f^{-1}(B)}(xy^{-1}) = \nu_B(f(xy^{-1})) = \nu_B(f(x)(f(y))^{-1}) \leqslant \nu_B(f(x)) \vee \nu_B(f(y)) = \nu_{f^{-1}(B)}(x) \vee \nu_{f^{-1}(B)}(y).$$

Hence $f^{-1}(B) \in PFSP(P)$.. □

Definition 4.4 *Let* $f : P \longrightarrow P^{'}$ *be a polygroup homomorphism and let* $A \in PFSP(P)$. *A is called PF-invariant if for any* $x_1, x_2 \in P$,

$$f(x_1) = f(x_2) \Rightarrow \mu_A(x_1) = \mu_A(x_2) \quad and \quad \nu_A(x_1) = \nu_A(x_2).$$

Clearly, if A is PF-invariant, then $f(A) \in PFSP(P^{'})$.

Proposition 4.3 *If* $A \in PFSP(P)$. *Then* $\mu_A(x^{-1}) = \mu_A(x), \nu_A(x^{-1}) = \nu_A(x)$ *and* $\mu_A(e) \geq \mu_A(x), \nu_A(e) \leqslant \nu_A(x)$ *for every* $x \in P$, *where e is the identity element of P.*

Proposition 4.4 *Let* $\{A_j\}_{j\in J} \subseteq PFSP(P)$. *Then* $\bigcap_{j\in J} A_j \in PFSP(P)$.
Proof: *It is straightforward.* □

Proposition 4.5 *If* $A \in PFSP(P)$. *Then*

$$P_A = \{x \in P : \mu_A(x) = \mu_A(e), \nu_A(x) = \nu_A(e)\}$$

is a subpolygroup of P.

Proof: *We have to show that:*

1. $x \circ y \subseteq P_A$ *for every* $x, y \in P$.

2. *If* $x \in P_A$ *then* $x^{-1} \in P_A$.

Let $x, y \in P_A$. *Then* $\mu_A(x) = \mu_A(e), \nu_A(x) = \nu_A(e)$ *and* $\mu_A(y) = \mu_A(e), \nu_A(y) = \nu_A(e)$. *Since* $A \in PFSP(P)$. *Then*

$$\mu_A(x \circ y) \geq \mu_A(x) \wedge \mu_A(y) = \mu_A(e) \wedge \mu_A(e) = \mu_A(e),$$

and

$$\nu_A(x \circ y) \leqslant \nu_A(x) \wedge \nu_A(y) = \nu_A(e) \wedge \nu_A(e) = \nu_A(e).$$

So $\mu_A(x \circ y) \geq \mu_A(e)$ *and* $\nu_A(x \circ y) \leqslant \nu_A(e)$. *Then* $\mu_A(x \circ y) = \mu_A(e)$ *and* $\nu_A(x \circ y) = \nu_A(e)$, *that is,* $x \circ y \in P_A$.
Now, if $x \in P_A$ *then,* $\mu_A(x) = \mu_A(e)$ *and* $\nu_A(x) = \nu_A(e)$. *Since* $\mu_A(x^{-1}) = \mu_A(x) = \mu_A(e)$ *and* $\nu_A(x^{-1}) = \nu_A(x) = \nu_A(e)$, *that is,* $x^{-1} \in P_A$. *Hence* P_A *is a subpolygroup of P.* □

Proposition 4.6 *If* $A, B \in PFSP(P)$. *Then* $A \cap B \in PFSP(P)$.
Proof: *It is straightforward.* □

Proposition 4.7 *If A and B be a PFSP of polygroups P_1 and P_2 respectively. Then $A \times B$ is also PFSP of polygroup $P_1 \times P_2$.*

Proof: *It is straightforward.* □

Proposition 4.8 *Let A and B be a PFS of polygroups P_1 and P_2 respectively, such that $\mu_A(x) \leqslant \mu_B(e_2)$ and $\nu_A(x) \geq \nu_B(e_2)$ hold for $x \in P_1$, e_2 being the identity element of P_2. If $A \times B$ is a PFSP of $P_1 \times P_2$. Then A is PFSP of polygroup P_1.*

Proof: *Let $x, y \in P_1$, we have*

$\mu_A(x \circ y) = \mu_A(x \circ y) \wedge \mu_B(e_2) = \mu_{A \times B}(x \circ y, e_2) \geq \mu_{A \times B}(x, e_2) \wedge \mu_{A \times B}(y, e_2) = [\mu_A(x) \wedge \mu_B(e_2)] \wedge [\mu_A(y) \wedge \mu_B(e_2)] = \mu_A(x) \wedge \mu_A(y)$.
Then $\mu_A(x \circ y) \geq \mu_A(x) \wedge \mu_A(y)$. Also,

$\nu_A(x \circ y) = \nu_A(x \circ y) \vee \nu_B(e_2) = \nu_{A \times B}(x \circ y, e_2) \leqslant \nu_{A \times B}(x, e_2) \vee \nu_{A \times B}(y, e_2) = [\nu_A(x) \vee \nu_B(e_2)] \vee [\nu_A(y) \vee \nu_B(e_2)] = \nu_A(x) \vee \nu_A(y)$.
Hence, $\nu_A(x \circ y) \leqslant \nu_A(x) \vee \nu_A(y)$. Therefore A is an PFSP of P_1. □

Proposition 4.9 *Let A and B be a PFS of polygroups P_1 and P_2 respectively, such that $\mu_B(x) \leqslant \mu_A(e_1)$ and $\nu_B(x) \geq \nu_A(e_1)$ hold for $x \in P_2$, e_1 being the identity element of P_1. If $A \times B$ is a PFSP of $P_1 \times P_2$. Then B is PFSP of polygroup P_2.*

Proof: *The proof is similar to the proof of Proposition 4.7.* □

Corollary 4.1 *Let A and B be a PFS of the polygroups P_1 and P_2 respectively. If $A \times B$ is an PFSP of $P_1 \times P_2$. Then, either A is PFSP of P_1 or B is PFSP of polygroup P_2*

Definition 4.5 *Let A be a PFSP in a polygroup P. Then A is called a Pythagorean fuzzy normal subpolygroup (in short, PFNSP) of P if for all $x, y \in P$*

$$\mu_A(x \circ y) = \mu_A(y \circ x) \quad \text{and} \quad \nu_A(x \circ y) = \nu_A(y \circ x).$$

Theorem 4.1 *Let $A \in PFSP(P)$. Then A is a Pythagorean fuzzy normal subpolygroup iff*

1. *$\mu_A(x \circ y \circ x^{-1}) = \mu_A(y)$ and*

2. *$\nu_A(x \circ y \circ x^{-1}) = \nu_A(y)$, for every $x, y \in P$.*

Proof: *It is straightforward.* □

Proposition 4.10 *If A is a PFNSP of P, then P_A is a normal subpolygroup of P.*

Proof: *It is straightforward.* □

Proposition 4.11 *Let A and B be a PFNSP of polygroups P_1 and P_2 respectively. Then $A \times B$ is also PFNSP of polygroup $P_1 \times P_2$.*

Proof: *It is straightforward.* □

5. Pythagorean Fuzzy Topological Polygroups

In this section, we introduce the concept of Pythagorean fuzzy topological polygroup. We also study some of properties of Pythagorean fuzzy topological polygroup.

Definition 5.1 *Let (P, τ) be a PFTS and let $A, B \in PFS(P)$. Then we define AB and A^{-1} by the respective formula:*

1. $$\mu_{AB}(x) = \begin{cases} \bigvee_{(x_1,x_2)\in X\times X}(\mu_A(x_1)\wedge \mu_B(x_2)) & if \quad x \in x_1 \circ x_2 \\ 0 & otherwise \end{cases}$$

and

$$\nu_{AB}(x) = \begin{cases} \bigwedge_{(x_1,x_2)\in X\times X}(\mu_A(x_1)\vee \mu_B(x_2)) & if \quad x \in x_1 \circ x_2 \\ 1 & otherwise \end{cases}$$

2. $\mu_{A^{-1}}(x) = \mu_A(x^{-1})$ *and* $\nu_{A^{-1}}(x) = \nu_A(x^{-1})$.

Definition 5.2 *Let $P =< P, \circ, e, ^{-1} >$ be a polygroup and let (P, τ) be a PFTS. A triad $(P, \circ, \tau)$ is called a Pythagorean fuzzy topological polygroup (shortly, PFTP) if:*

1. *For every $x, y \in P$ and every Pythagorean fuzzy open Q-neighborhood W of a Pythagorean fuzzy point $z_{(r,s)}$ of $x \circ y$, there are Pythagorean fuzzy open Q-neighborhoods U of $x_{(r,s)}$ and V of $y_{(r,s)}$ such that $UV \subseteq W$.*
2. *For every $x \in P$ and every Pythagorean fuzzy open Q-neighborhood V of a Pythagorean fuzzy point $x^{-1}_{(r,s)}$ there exists a Pythagorean fuzzy open Q-neighborhood U of $x_{(r,s)}$ such that $U^{-1} \subseteq V$.*

Evidently, every Pythagorean fuzzy topological group is a Pythagorean fuzzy topological polygroup.

Example 5.1 Let $P =< P, \circ, e, ^{-1} >$ ba a polygroup and let τ be the collection of all constant Pythagorean fuzzy sets in P. Then (P, τ) is a Pythagorean fuzzy topological polygroup.

Example 5.2 Let $P = \{e, a, b\}$. Then, P together with the following hyperoperation

$\circ$	e	a	b
e	e	a	b
a	a	e	b
b	b	b	{e,a}

is a polygroup. It is clear that $a^{-1} = a, b^{-1} = b$. Consider on P the Pythagorean fuzzy topology $\tau = \{0_X, 1_X, A\}$, where $A =< x, (\frac{e}{0.9}, \frac{a}{0.4}, \frac{b}{0.6}), (\frac{e}{0.2}, \frac{a}{0.7}, \frac{b}{0.3}) >$. Then (P, τ) is PFTP.

Theorem 5.1 *Let (X, τ_X) be a PFTS, $(Y, \circ, \tau_Y)$ be a PFTP, and $f, g \in PFC(X, Y)$. Then the maps $f * g$ and f^{-1} from the Pythagorean fuzzy topological space X into the Pythagorean fuzzy topological space Y with*

$$(f * g)(x) = f(x) \circ g(x)$$

and

$$f^{-1}(x) = (f(x))^{-1}$$

Proof: *Let $x \in X$ and $x_{(r,s)}$ be a Pythagorean fuzzy point. Let W be a Pythagorean fuzzy open Q-neighborhood of $(f * g)(x_{(r,s)})$. Since $(f * g)(x) = f(x) \circ g(x)$, it follows that W is a Pythagorean fuzzy open Q-neighborhood of any Pythagorean fuzzy point of $f(x) \circ g(x)$. Since $(Y, \circ, \tau_Y)$ is a PFTP, there exist a Pythagorean fuzzy open Q-neighborhoods U and V of $f(x)_{(r,s)}$ and $g(x)_{(r,s)}$ respectively, such that*

$UV \subseteq W$.
Now, since the maps f and g are Pythagorean fuzzy continuous, there exist a Pythagorean fuzzy open Q-neighborhoods U_1 and V_1 of the Pythagorean fuzzy point $x_{(r,s)}$ in X such that $f(U_1) \subseteq U$ and $g(V_1) \subseteq V$. Clearly, the Pythagorean fuzzy set $U_1 \cap V_1$ is a Pythagorean fuzzy open Q-neighborhood of $x_{(r,s)}$ in X. We prove that $(f*g)(U_1 \cap V_1) \subseteq W$. Hence, $(f*g)(U_1 \cap V_1) \subseteq W$, and so the map $f*g$ is Pythagorean fuzzy continuous.

Now, we prove that the map f^{-1} is Pythagorean fuzzy continuous. Let $x_{(r,s)}$ be a Pythagorean fuzzy point of X and W be a Pythagorean fuzzy open Q-neighborhood of $f^{-1}(x_{(r,s)}) = f^{-1}(x)_{(r,s)} = (f(x)_{(r,s)})^{-1}$ in Y. Since $(Y, \circ, \tau_Y)$ is a PFTP, there exists a Pythagorean fuzzy open Q-neighborhood U of $f(x)_{(r,s)}$ in Y such that $U^{-1} \subseteq W$. Now, since the map f is Pythagorean fuzzy continuous in the Pythagorean fuzzy point $x_{(r,s)}$ in X, there exists a Pythagorean fuzzy open Qneighborhood U_1 of $x_{(r,s)}$ such that $f(U_1) \subseteq U$. For the Pythagorean fuzzy open Q-neighborhood U_1 of $x_{(r,s)}$ in X, we have $f^{-1}(U_1) \subseteq W$. Therefore, the map f^{-1} is Pythagorean fuzzy continuous. □

Theorem 5.2 *Let (X, τ_X) be a fully stratified Pythagorean fuzzy topological space, $(Y, \circ, \tau_Y)$ a PFTP, and e the identity element of the polygroup $(Y, \circ)$. Then, the map e' from the Pythagorean fuzzy topological space X into the Pythagorean fuzzy topological space Y where $e'(x) = e$ for every $x \in X$ is Pythagorean fuzzy continuous.*

Proof: *Let $U \in \tau_Y$. We show that $(e')^{-1}(U) \in \tau_X$ We have $((e')^{-1}(U))(x) = U(e'(x)) = U(e)$, for every $x \in X$. Since the Pythagorean fuzzy space (X, τ_X) is fully stratified, it follows that $(e')^{-1}(U) \in \tau_X$. Thus, the map e' is Pythagorean fuzzy continuous.* □

Theorem 5.3 *Let (P, τ) is a fully stratified space. Let $(P, \circ, \tau)$ be a Pythagorean fuzzy topological polygroup. Then the mapping $f : x \longrightarrow x^{-1}$ is Pythagorean fuzzy homeomorphic function of (P, τ) onto itself.*

Proof: *It is seen that f is invertible. Hence the only thing which needs to be proved that f is Pythagorean fuzzy continuous. Let $(P, \circ, \tau)$ be a Pythagorean fuzzy topological polygroup and V be a Pythagorean fuzzy open Q-neighbourhood of Pythagorean fuzzy point $x^{-1}_{(r,s)}$. Then, there exists a Pythagorean fuzzy open Q-neighbourhood U of $x_{(r,s)}$ such that $U^{-1} \subseteq V$. Then that $x_{(r,s)} q U^{-1}$. Hence U^{-1} is a Pythagorean fuzzy open Q-neighbourhood of $x_{(r,s)}$. Thus $f(U) = U^{-1} \subseteq V$. Then f is a Pythagorean fuzzy continuous function at the Pythagorean fuzzy point $x_{(r,s)}$. Therefore, f is a Pythagorean fuzzy continuous function.* □

Proposition 5.1 *Let $(P, \circ, \tau)$ be a PFTP.*

1. *If U is a Pythagorean fuzzy compact subset of P then, U^{-1} is a Pythagorean fuzzy compact.*
2. *If U is a Pythagorean fuzzy open in τ then, U^{-1} is a Pythagorean fuzzy open in τ.*

Proof: *It is straightforward.* □

Definition 5.3 *Let $P =< P, \circ, e, ^{-1} >$ ba a polygroup, let $A \in PFSP(P)$ and let $a \in P$ be a fixed element. Then the set $aA =< \mu_{aA}, \nu_{aA} >$ where*

$$\mu_{aA}(x) = \bigvee_{z \in a^{-1} \circ x} \mu_A(z) \quad \forall x \in P,$$

and

$$\nu_{aA}(x) = \bigwedge_{z \in a^{-1} \circ x} \nu_A(z) \quad \forall x \in P,$$

is called Pythagorean fuzzy left coset of P determined by A and a. Similarly, the set $Aa =< \mu_{Aa}, \nu_{Aa} >$ *where*

$$\mu_{Aa}(x) = \bigvee_{z \in x \circ a^{-1}} \mu_A(z) \quad \forall x \in P,$$

and

$$\nu_{Aa}(x) = \bigwedge_{z \in x \circ a^{-1}} \nu_A(z) \quad \forall x \in P,$$

is called Pythagorean fuzzy right coset of P determined by A and a.

Proposition 5.2 *Let* (P, τ) *be a Pythagorean fuzzy topological polygroup. Then the family* $\beta = \{\tilde{A} \in PFS(\mathcal{P}^*(P)) : A \in \tau\}$, *where*

$$\mu_{\tilde{A}}(X) = \bigvee_{x \in X} \mu_A(x) \quad \text{and} \quad \nu_{\tilde{A}}(X) = \bigwedge_{x \in X} \nu_A(x).$$

Is a base for a Pythagorean fuzzy topology τ^* *on* $\mathcal{P}^*(P)$.

Proof: β *is a base for a Pythagorean fuzzy topology on* $\mathcal{P}^*(P)$ *because:*

1. *For every* $\tilde{A}_1, \tilde{A}_2 \in \beta$ *with* $A_1, A_2 \in \tau$, *it follows that* $\tilde{A}_1 \cap \tilde{A}_2 \in \beta$, *since* $\tilde{A}_1 \cap \tilde{A}_2 = \widetilde{A_1 \cap A_2}$ *and* $A_1 \cap A_2 \in \tau$.

Indeed, for every $X \in \mathcal{P}^*(P)$, *we have*
$\mu_{\widetilde{A_1 \cap A_2}}(X) = \bigvee_{x \in X} \mu_{(A_1 \cap A_2)}(x) = \bigvee_{x \in X}(\mu_{A_1}(x) \wedge \mu_{A_2}(x)) = (\bigvee_{x \in X} \mu_{A_1}(x)) \wedge (\bigvee_{x \in X} \mu_{A_2}(x)) = \mu_{\tilde{A_1}}(X) \wedge \mu_{\tilde{A_2}}(X) = \mu_{(\tilde{A_1} \cap \tilde{A_2})}(X)$,
and
$\nu_{\widetilde{A_1 \cap A_2}}(X) = \bigwedge_{x \in X} \nu_{(A_1 \cap A_2)}(x) = \bigwedge_{x \in X}(\nu_{A_1}(x) \vee \nu_{A_2}(x)) = (\bigwedge_{x \in X} \nu_{A_1}(x)) \vee (\bigwedge_{x \in X} \nu_{A_2}(x)) = \nu_{\tilde{A_1}}(X) \vee \nu_{\tilde{A_2}}(X) = \nu_{(\tilde{A_1} \cap \tilde{A_2})}(X)$.

2. *Since* $1_X \in \tau$ *it is follows that* $\tilde{1_X}(X) = 1$ *for every* $X \in \mathcal{P}^*(P)$ *and thus* $\bigcup_{\tilde{A} \in \beta} \tilde{A} = 1$. □

Lemma 5.1 *Let U be a Pythagorean fuzzy open subset of a Pythagorean fuzzy topological polygroup* (P, τ). *Then, aU and Ua are Pythagorean fuzzy open subsets of* (P, τ) *for every* $a \in P$.

Proof: *Let U be a Pythagorean fuzzy open subset of* (P, τ). *Then*

$$\mu_{(a^{-1}\phi^{-1}(\tilde{U}))}(z) = \mu_{\tilde{U}}(a^{-1}\phi(z)) = \mu_{\tilde{U}}(a^{-1} \circ z) = \bigvee_{t \in a^{-1} \circ z} \mu_U(t) = \mu_{aU}(z),$$

and

$$\nu_{(a^{-1}\phi^{-1}(\tilde{U}))}(z) = \nu_{\tilde{U}}(a^{-1}\phi(z)) = \nu_{\tilde{U}}(a^{-1} \circ z) = \bigwedge_{t \in a^{-1} \circ z} \nu_U(t) = \nu_{aU}(z).$$

Since the mapping $a^{-1}\phi^{-1} : P \longrightarrow \mathcal{P}^*(P), x \longrightarrow a^{-1} \circ x$, *is Pythagorean fuzzy continuous, thus aU is Pythagorean fuzzy open. Similarly, we can prove that Ua is Pythagorean fuzzy open.* □

Proposition 5.3 *Let* (P, τ) *be a fully stratified space. Let* $(P, \circ, \tau)$ *be a Pythagorean fuzzy topological polygroup and U be a Pythagorean fuzzy set of P. If cl(U) is a Pythagorean fuzzy closed set, then acl(U) and cl(U)a are Pythagorean fuzzy closed sets, where* $a \in P$ *is a definite point.*

Proof: *It is straightforward.* □

Proposition 5.4 *Let* (P, τ) *be a fully stratified space. Let* $(P, \circ, \tau)$ *be a Pythagorean fuzzy topological polygroup and U be a Pythagorean fuzzy set of P. If cl(U) is a Pythagorean fuzzy closed set, then* $a_{(r,s)}cl(U)$, $cl(U)a_{(r,s)}$ *and* $cl(U)^{-1}$ *are Pythagorean fuzzy closed sets.*

Theorem 5.4 *In a Pythagorean fuzzy topological polygroup* (P, τ)*, U is a Q-neighbourhood of* $e_{(r,s)}$ *if and only if* U^{-1} *is a Q-neighbourhood of* $e_{(r,s)}$.

Proof: *Let U be a Q-neighbourhood of* $e_{(r,s)}$*. Then there exists Pythagorean fuzzy open set A such that* $e_{(r,s)} q A \subseteq U$ *that is,*

$$\mu_A(e) + r > 1, A \subseteq U,$$

$$\nu_A(e) + r < 1, A \subseteq U.$$

For every $x \in P, \mu_A(x^{-1}) \leqslant \mu_U(x^{-1})$ *and* $\nu_A(x^{-1}) \geq \nu_U(x^{-1})$*, so* $\mu_{A^{-1}}(x) \leqslant \mu_{U^{-1}}(x)$ *and* $\nu_{A^{-1}}(x) \geq \nu_{U^{-1}}(x)$*. Then* $A^{-1} \subseteq U^{-1}$.
Now,

$$\mu_{A^{-1}}(e) + \mu_{e_{(r,s)}}(e) = \mu_{A^{-1}}(e) + r > 1,$$

$$\nu_{A^{-1}}(e) + \nu_{e_{(r,s)}}(e) = \nu_{A^{-1}}(e) + s < 1.$$

Hence $e_{(r,s)} q A^{-1}$ *and* $A^{-1} \subseteq U^{-1}$*. Therefore,* U^{-1} *is a Q-neighbourhood of* $e_{(r,s)}$.
Conversely, let U^{-1} *be a Q-neighbourhood of* $e_{(r,s)}$*. Then there exists Pythagorean fuzzy open set A such that* $e_{(r,s)} q A \subseteq U^{-1}$*. As above,* $A^{-1} \subseteq U$ *and* $e_{(r,s)} q A^{-1}$*. That is, U is a Q-neighbourhood of* $e_{(r,s)}$. □

Proposition 5.5 *Let* (P, τ) *be a fully stratified space. Let* $(P, \circ, \tau)$ *be a Pythagorean fuzzy topological polygroup and U be a Pythagorean fuzzy set of P. If U is a Q-neighbourhood of* $e_{(r,s)}$*, then* $x_{(1,0)}U$ *is a Q-neighbourhood of* $x_{(r,s)}$.

Proof: *Since U is a Q-neighbourhood of* $e_{(r,s)}$*, there exists a Pythagorean fuzzy open set A such that* $r + \mu_A(e) > 1$ *and* $s + \nu_A(e) < 1$*,* $A \subseteq U$*. So,*

$$\mu_{x_{(1,0)}A}(x) = \bigvee_{x \in xy} (\mu_{x_{(1,0)}}(x) \wedge \mu_A(y)) \geq 1 \wedge \mu_A(e) = \mu_A(e),$$

and

$$r + \mu_{x_{(1,0)}A}(x) \geq r + \mu_A(e) > 1.$$

Also

$$\nu_{x_{(1,0)}A}(x) = \bigwedge_{x \in xy} (\nu_{x_{(1,0)}}(x) \vee \nu_A(y)) \leqslant 1 \vee \nu_A(e) = \nu_A(e),$$

and

$$r + \mu_{x_{(1,0)}A}(x) \leqslant r + \mu_A(e) < 1.$$

Hence $x_{(r,s)} q x_{(1,0)} A \subseteq x_{(1,0)} U$ *and since* $x_{(1,0)}A$ *is a Pythagorean fuzzy open set, Therefore* $x_{(1,0)}U$ *is a Q-neighbourhood of* $x_{(r,s)}$. □

6. Conclusion

In this paper, we introduce the notion of fuzzy topological space and intuitionistic fuzzy topological space by introducing the notion of Pythagorean fuzzy topological subpolygroup and Pythagorean fuzzy topological polygroup. we define the concept of Pythagorean fuzzy topological space and Pythagorean fuzzy topological subspace. We also study the continuity and open of a function defined among Pythagorean fuzzy topological spaces. After we define the concept of Pythagorean fuzzy subpolygroup and we study some of properties of Pythagorean fuzzy subpolygroup. Moreover we introduce the concept of Pythagorean fuzzy topological polygroup. We also study some of properties of Pythagorean fuzzy topological polygroup.

Acknowledgments

We are grateful to the reviewers for their valuable comments and suggestions, which greatly helped improve the quality of the paper.

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